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POSITIVE CONES AND GGV

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ABSTRACT. We show that a gyrometric preserving transformations between connected open subsets of the positive cones are uniquely extended to a gyrometric preserving surjection between the underlying cones.

1. INTRODUCTION

A celebrated Mazur-Ulam theorem states that a surjective isometry from a normed real-linear space onto a normed real-linear space is affine in the sense that it is a real-linear isometry followed by a translation (cf.[9]).. Mankiewicz [6] generalized in the way that any surjective isometry between connected open subsets of normed real-linear spaces is extended to a surjective isometry between underlying normed real-linear spaces. As the main motivation to our present investigation we mention that in the paper [1] we prove that a gyrometric preserving surjection from a GGV onto a GGV is a gyrometric preserving isomorphism followed by the translation. The idea of the proof in [1] emploies the one in the proof of the celebrated Mazur-Ulam theorem due to Väisälä [9]. The following is expected to be true: a gyrometric preserving surjection between connected open subsets of GGV's are extended to a gyrometric preserving surjection between the underlying GGV's. We have not yet prove the above conjecture. There is an obstruction to prove the conjecture by applying simply the way similar to the proof of Mankiewicz. We explain it; we show how to extend the given isometry between connected open subsets to a global isometry. The essential part of the proof of Mankiewicz [6] is as follows. Suppose that X_j is a normed real-linear space for $j = 1, 2$. Suppose that $B_j = \{x \in X_j : \|x\| < \varepsilon\}$ for $\varepsilon > 0$. Suppose that $T : B_1 \to B_2$ is a surjective isometry. We give $\tilde{T} : X_1 \to X_2$ by $\tilde{T}(z) = rT(\frac{1}{r}z)$ for a $z \in X_1$, where $r > 0$ is a large enough such that $\frac{1}{r}z \in B_1$. To show an idea we omit preciseness: we omit to prove $\tilde{T}$ is well defined in the

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sense that it is independent of the choice of r. We infer that $\tilde{T}$ is a surjective isometry from X_1 onto X_2. We show that it is an isometry by applying the distributive law:

$$\begin{aligned} \|z-w\| &= \|r(\frac{1}{r}z-\frac{1}{r}w)\| = r\|\frac{1}{r}z-\frac{1}{r}w\| \\ &= r\|T(\frac{1}{r}z)-T(\frac{1}{r}w)\| = \|r(T(\frac{1}{r}z)-T(\frac{1}{r}w)\| \\ &= \|rT(\frac{1}{r}z)-rT(\frac{1}{r}w)\| = \|\tilde{T}(z)-\tilde{T}(w)\|, \quad z,w\in X_1 \end{aligned}$$

A distributive law such as $r\otimes(a\oplus b)=(r\otimes a)\oplus(r\otimes b)$ is not assumed for GGV's. Hence it is an obstruction for extending a gyrometric preserving map between connected open sets to a global one.

2. Generalized gyrovector spaces

The definition of a generalized gyrovector spaces (GGV) is the following.

Definition 1 (A generalized gyrovector space [1]). Let $(G,\oplus)$ be a gyrocommutative gyrogroup with the map $\otimes:\mathbb{R}\times G\to G$. Let ϕ be an injection from G into a real normed space $(\mathbb{V},\|\cdot\|)$. We say that $(G,\oplus,\otimes,\phi)$ (or $(G,\oplus,\otimes)$ just for a simple notation) is a generalized gyrovector space or a GGV in short if the following conditions (GGV0) to (GGV8) are fulfilled:

(GGV0) $\|\phi(\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a})\|=\|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{a}\in G$;
(GGV1) $1\otimes\boldsymbol{a}=\boldsymbol{a}$ for every $\boldsymbol{a}\in G$;
(GGV2) $(r_1+r_2)\otimes\boldsymbol{a}=(r_1\otimes\boldsymbol{a})\oplus(r_2\otimes\boldsymbol{a})$ for any $\boldsymbol{a}\in G,\ r_1,r_2\in\mathbb{R}$;
(GGV3) $(r_1r_2)\otimes\boldsymbol{a}=r_1\otimes(r_2\otimes\boldsymbol{a})$ for any $\boldsymbol{a}\in G,\ r_1,r_2\in\mathbb{R}$;
(GGV4) $(\phi(|r|\otimes\boldsymbol{a}))/\|\phi(r\otimes\boldsymbol{a})\|=\phi(\boldsymbol{a})/\|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a}\in G\setminus\{\boldsymbol{e}\}$, $r\in\mathbb{R}\setminus\{0\}$, where $\boldsymbol{e}$ denotes the identity element of the gyrogroup $(G,\oplus)$;
(GGV5) $\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](r\otimes\boldsymbol{a})=r\otimes\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a}$ for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{a}\in G,\ r\in\mathbb{R}$;
(GGV6) $\mathrm{gyr}[r_1\otimes\boldsymbol{v},r_2\otimes\boldsymbol{v}]=id_G$ for any $\boldsymbol{v}\in G,\ r_1,r_2\in\mathbb{R}$;
(GGVV) $\|\phi(G)\|=\{\pm\|\phi(\boldsymbol{a})\|\in\mathbb{R}:\boldsymbol{a}\in G\}$ is a real one-dimensional vector space with vector addition $\oplus'$ and scalar multiplication $\otimes'$;
(GGV7) $\|\phi(r\otimes\boldsymbol{a})\|=|r|\otimes'\|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a}\in G$, $r\in\mathbb{R}$;
(GGV8) $\|\phi(\boldsymbol{a}\oplus\boldsymbol{b})\|\le\|\phi(\boldsymbol{a})\|\oplus'\|\phi(\boldsymbol{b})\|$ for any $\boldsymbol{a},\boldsymbol{b}\in G$.

Definition 2. Let $(G,\oplus,\otimes)$ be a GGV. Let $\varrho(\boldsymbol{a},\boldsymbol{b})=\|\phi(\boldsymbol{a}\ominus\boldsymbol{b})\|$ for all $\boldsymbol{a},\boldsymbol{b}\in G$, where $\boldsymbol{a}\ominus\boldsymbol{b}=\boldsymbol{a}\oplus(\ominus\boldsymbol{b})$. We call ϱ the gyrometric on G on a GGV.

Suppose that ρ is a gyrometric for a GGV G. We proved the equation [1, (4)] of the form

$$\rho(z,w) = \rho(\ominus z, \ominus w) = \rho(w,z), \quad z,w \in G. \tag{1}$$

By Proposition 15 in [1] we have

$$\rho(x \oplus z, x \oplus w) = \rho(z,w), \quad x,z,w \in G. \tag{2}$$

We call the inequality (GGV8) the gyrotriangle inequality. It is not the triangle inequality. In general the gyrometric does not satisfy the triangle inequality.

3. A positive cone of the positive invertible elements

The GGV is a generalization of a gyrovector spaces [8], which is a generalization of an inner product space. The Einstein gyrovector space and the Möbius gyrovector space are examples of a gyrovector space [8]. We exhibited that the positive cone of all positive invertible elements is an example of a GGV [1].

Example 3 ([1]). Suppose that A is a unital C^*-algebra with the norm $\|\cdot\|$ and A_+^{-1} is the set of all positive invertible elements of A. Let t be a positive real number. Put

$$a \oplus_t b = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}$$

for all $a, b \in A_+^{-1}$. Then $(A_+^{-1}, \oplus_t)$ is a gyrocommutative gyrogroup. The identity element 1 of A as the C^*-algebra is the identity element of the gyrogroup. The inverse element $\ominus a$ is a^{-1}, the inverse of a in A. For $a, b \in A_+^{-1}$ put

$$X = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}.$$

Then X is a unitary element in A and

$$\mathrm{gyr}[a,b]c = XcX^*, \quad a,b,c \in A_+^{-1}.$$

is the gyroautomorphism generated by a and b.

Put $r \otimes a = a^r$ for every $a \in A_+^{-1}$, $r \in \mathbb{R}$. Define $\phi = \log : A_+^{-1} \to A_S$, the real-linear subspace of all self-adjoint elements in A. The vector space $(\|\log(A_+^{-1})\|, \oplus', \otimes') = (\mathbb{R}, +, \times)$ is the usual 1 dimensional real vector space of the real line; $\oplus'$ is the addition of real numbers and $\otimes'$ is the scalar multiplication of real numbers. Then $(A_+^{-1}, \oplus_t, \otimes, \log)$ is a GGV. In fact, (GGV0) holds since $\mathrm{gyr}[a,b]$ is a unitary transform for every pair $a, b \in A_+^{-1}$. Simple calculations confirm that the conditions from (GGV0) to (GGV6) and (GGV7) hold. The condition (GGVV) is trivial by the definition of $\oplus'$ and $\otimes'$. The condition (GGV8) is also satisfied; see [1].

The gyrometric ϱ is given by the equation

$$\varrho(a,b) = \|\log(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})^{\frac{1}{t}}\|$$

for $a, b \in A_+^{-1}$. Note that in the case where $t = 1$ the metric $\varrho(a,b)$ is the Thompson metric itself. The gyromidpoint of a and b is given by the equation

$$\boldsymbol{p}(a,b) = (a^{\frac{t}{2}}(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})^{-\frac{1}{2}}a^{\frac{t}{2}})^{\frac{1}{t}},$$

which coincides with the geometric mean of a and b for the case of $t = 1$.

In the case of the GGV of the positive cone, the gyrometric satisjfes the usual triangle inequality (cf. [1, p. 399]).

4. Extension of a gyrometric preserving surjection between connected open subsets of the positive cones

For the case of the GGV of all invertible positive elements in a unital C^*-algebra we have that a surjective gyrometric preserving map between connected open subsets of the positive cones is uniquely extended to a surjective gyrometric preserving map between the whole of the positive cones. Note that the topology induced by the gyrometric coincides with that induced by the metric induced by the original norm of the C^*-algebra. For $\varepsilon > 0$, let $B_j^\varepsilon(1) = \{a \in A_{j+}^{-1} : \varrho_j(a,1) < \varepsilon\}$.

Lemma 4. *Let $T : B_1^\varepsilon(1) \to B_2^\varepsilon(1)$ be a bijection with $T(1) = 1$. Suppose that T is gyrometric preserving. Then*

$$T(\frac{1}{2} \otimes a) = \frac{1}{2} \otimes T(a), \quad a \in B_1^\varepsilon(1).$$

By induction we have that

$$T(\frac{1}{2^n} \otimes a) = \frac{1}{2^n} \otimes T(a), \quad a \in B_1^\varepsilon(1)$$

for every positive integer n. Then we have

Lemma 5. *Then there exists a Jordan *-isomorphism J from A_1 onto A_2 and a central projection p in A_2 such that*

$$T(a) = pJ(a) + (1-p)J(a)^{-1}, \quad a \in B_1^\varepsilon(1).$$

Lemma 5 is the essential point where the given gyrometric preserving map on a connected open subset of the GGV of the positive cone is extended to the whole of the GGV of A_{1+}^{-1}: $z \mapsto pJ(z) + (1-p)J(z)^{-1}$ defines a gyrometric preserving surjection from A_{1+}^{-1} onto A_{2+}^{-1}.

Lemma 6. *Suppose that T and S are surjective gyrometric preserving maps from A_{1+}^{-1} onto A_{2+}^{-1} such that $T = S$ on an open ball. Then $T = S$ on A_{1+}^{-1}.*

By the above lemma the extension is unique.

Theorem 7. *Let U_j be a non-empty connected open subset of A_{j+}^{-1} for $j = 1, 2$. Let $T : U_1 \to U_2$ be a surjection. Then T is gyrometric preserving if and only if there exists a Jordan *-isomorphism J form A_1 onto A_2 and a central projection $p \in A_2$ such that*

$$T(z) = (b(pJ(z) + (1-p)J(z)^{-1})^t b)^{\frac{1}{t}}, \quad z \in U_1, \tag{3}$$

where $b = (pJ(a_0) + (1-p)J(a_0)^{-1})^{-t} \# T(a_0)^t$ for an and any $a_0 \in U_1$; b is unique for any $a_0 \in U$. In this case T is extended to a gyrometric preserving surjection from A_{1+}^{-1} onto A_{2+}^{-1}.

We show a sketch proof. Let $a_0 \in U_1$ be arbitrary. Choose a sufficiently small $\varepsilon > 0$ with $\{b \in A_{1+}^{-1} : \varrho_1(a_0, b) < \varepsilon\} \subset U_1$. Then the induced map $T' : B_1^\varepsilon(1) \to B_2^\varepsilon(1)$ defined by

$$T'(z) = \ominus_t T(a_0) \oplus_t T(a_0 \oplus_t z), \quad z \in B_1^\varepsilon(1)$$

is a surjective gyrometric preserving map from $B_1^\varepsilon(1)$ onto $B_2^\varepsilon(1)$. Then by Lemma 5 T' is extended to a surjective gyrometric preserving map from A_{1+}^{-1} onto A_{2+}^{-1}. So is T. As U_1 is connected the extension is unique up to given point a_0. As a_0 is arbitrary there is an $\varepsilon_0 >$ and a surjective gyrometric preserving map T_0 from A_{1+}^{-1} onto A_{2+}^{-1} which is an extension of T on $\{b \in A_{1+}^{-1} : \varrho_1(a_0, b) < \varepsilon\}$. Let a_0 and a_1 be a pair of points in U_1. As U_1 is connected and open, there is a continuous map $\gamma : [0,1] \to U_1$ with $\gamma(0) = a_0$ and $\gamma(1) = a_1$. By compactness of $\gamma([0,1]$ there is a sequence $t_0, \ldots, t_n \in [0,1]$ with $t_0 = 0$, $t_n = 1$ and $\varepsilon_j > 0$ such that

$$\gamma([0,1]) \subset \cup_j \{z \in A_{1+}^{-1} : \varrho_1(\gamma(t_j), z) < \varepsilon_j\} \subset U_j$$

and

$$\{z \in A_{1+}^{-1} : \varrho_1(\gamma(t_j), z) < \varepsilon_j\} \cap \{z \in A_{1+}^{-1} : \varrho_1(\gamma(t_{j+1}), z) < \varepsilon_j\} \neq \emptyset$$

for $j = 0, \ldots, n-1$. By Lemma 6 we see that $T_j = T_{j+1}$ on A_{1+}^{-1} as $T_j = T_{j+1}$ on $\{z \in A_{1+}^{-1} : \varrho_1(\gamma(t_j), z) < \varepsilon_j\} \cap \{z \in A_{1+}^{-1} : \varrho_1(\gamma(t_{j+1}), z) < \varepsilon_j\}$ for $j = 0, \ldots, n-1$. Hence we have $T_0 = T_n$. Therefore T_0 is a unique extension of T. This does not give a complete proof, refer for the precise proof in [2].

Note that Hatori and Molnár [3] proved the case where $U_j = A_{j+}^{-1}$ and $t = 1$. Honma and Nogawa [4] proved the case of general t. Note also that the connectivity of the open sets in Theorem 7 is essential.

Let $U = \{a \in A_{1+}^{-1} : \varrho_1(a,1) < 1\} \cup \{a \in A_{1+}^{-1} : \varrho_1(a,10) < 1\}$. Suppose that J is a Jordan *-isomorphism from A_1 onto itself which is not the identity transformation. Then the map $T : U \to U$ defined by

$$T(a) = \begin{cases} J(a), & \varrho_1(a,1) < 1 \\ a, & \varrho_1(a,10) < 1. \end{cases}$$

is a surjective gyrometric preserving map while it is not extended to a surjective gyrometric preserving map from A_{1+}^{-1} by Lemma 6.

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