<table>
<thead>
<tr>
<th>Title</th>
<th>Some contractions and the Poncelet property of their numerical ranges (Research on structure of operators by order and geometry with related topics)</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>Departmental Bulletin Paper</td>
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</tbody>
</table>

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Some contractions and the Poncelet property of their numerical ranges

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1. A special class of contractions

In 1814, a French mathematician Jean-Victor Poncelet [26] described his famous closure theorem: Let $C$ and $D$ be two conics on the complex projective plane. If there exists a closed $n$-gon inscribed in $D$ and circumscribed to $C$ then, starting at an arbitrary point of $D$, there is a closed $n$-gon inscribed in $D$ and circumscribed to $C$ (cf. [13]). A rigid proof of Poncelet’s closure theorem was given by Jacobi [20] based on the elliptic function theory (cf. [27]). For a pair of two conics $C$ and $D$ on the plane, a point $P \in C$ and a point $Q \in D$ have a relation $P \sim Q$ if there is a tangent line of $C$ at $P$ passing through $Q$. By this relation the space curve

$$L = \{(P,Q) \in C \times D : P \sim Q\}$$

has a parametrization by elliptic functions with common modular invariants (cf. [4]). In this sense, $L$ is an elliptic curve. From a matrix theoretic viewpoint, the Poncelet property arises in the boundary of the numerical range of some contraction matrices. Let $A$ be an $n \times n$ matrix. The numerical range of $A$ is defined as

$$W(A) = \{ \langle Ax, x \rangle : x \in \mathbb{C}^n, ||x|| = 1 \},$$

(1.1)

and the rank-$k$ numerical range of $A$ is introduced and defined in [7] as the set

$$\Lambda_k(A) = \{ \lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P \},$$

$1 \leq k \leq n$. In the case $k = 1$, $\Lambda_k(A)$ reduces to $W(A)$. The rank-$k$ numerical range $\Lambda_k(A)$ is a compact convex set and $\Lambda_k(A) \neq \emptyset$ if $3k \leq n+2$
(cf. [3, 7, 8, 22]). If $A$ is a contraction, i.e., $||Ax|| \leq ||x||$ for any $x \in \mathbb{C}^n$, then its numerical range $W(A)$ is contained in the closed unit disc. Mirman [23] found an important class $S_n$ of $n \times n$ matrices for which the boundary $C = \partial W(A)$ of the numerical range of a matrix $A \in S_n$ and the unit circle $D = \{z \in \mathbb{C} : |z| = 1\}$ form a Poncelet pair. Gau-Wu [15] independently found the Poncelet property for the $S_n$ class. For a survey on numerical range and the Poncelet property, see for instance [17], and for recent works on the Poncelet property, see [10, 16, 24]. A formulation of the Poncelet property of a matrix $A \in S_n$ using the complex algebraic geometry was given in [2, 25]. An $n \times n$ matrix $A$ is in $S_n$ if $A$ is a contraction, $A$ has no eigenvalue with modulus 1, and $\text{rank}(I_n - A^*A) = 1$.

![Figure 1](image)

In Figure 1, we provide an example of the boundary of the numerical range of a matrix $A$ in $S_3$. We present two quadrilaterals inscribed in the unit circle and circumscribed to $\partial W(A)$.

The following result characterizes the class of $S_n$ matrices.

**Proposition 1.1**[Mirman ; Gau, P. Y. Wu] Let $A_0 \in S_n$. Then there exists an $n \times n$ unitary matrix $U$ so that $UA_0U^* = (a_{ij})$ is an upper triangular
matrix given by

$$a_{ij} = \begin{cases} a_j, & \text{if } i = j; \\ (1 - |a_i|^2)^{1/2}(1 - |a_j|^2)^{1/2}, & \text{if } i = j - 1; \\ \prod_{k=i+1}^{j-1}(-\overline{a_k})(1 - |a_i|^2)^{1/2}(1 - |a_j|^2)^{1/2}, & \text{if } i < j - 1; \\ 0, & \text{if } i > j; \end{cases} \quad (1.2)$$

for some $|a_j| < 1$, $j = 1, 2, \ldots, n$.

The numerical range of a matrix $A \in S_n$ can be expressed as

$$W(A) = \bigcap \{W(U) : U \text{ is an } (n+1)-\text{dimensional unitary dilation of } A\}$$

(cf. [15, 23]) which also gives a partial answer to Halmos' conjecture, namely,

$$\text{closure}(W(T)) = \bigcap \{\text{closure}(W(U)) : U \text{ is a unitary dilation of } T\},$$

for a contraction operator $T$ on a complex Hilbert space (cf. [1]). A general answer is given by Choi and Li [9]. Moreover, it is shown in [14, Theorem 1.2] that an $n \times n$ contraction $A$ with rank$(I_n - A^*A) = k$ has a general consequence:

$$\Lambda_k(A) = \bigcap \{W(U) : U \text{ is an } (n + k)-\text{dimensional unitary dilation of } A\}.$$

**Proposition 1.2** [Gau, Wu]. Let $A = (a_{ij})$ be a $S_n$ matrix (1.2). Then any $(n+1) \times (n+1)$ unitary dilation of $A$ is unitarily equivalent to a member of a one-parameter family of unitary matrices $B(\lambda) = (b_{ij}(\lambda))$ given by

$$b_{ij}(\lambda) = \begin{cases} a_{ij}, & \text{if } 1 \leq i, j \leq n; \\ \lambda(1 - |a_j|^2)^{1/2}, & \text{if } i = n + 1, j = 1; \\ \lambda\left(\prod_{k=1}^{j-1}(-\overline{a_k})\right)(1 - |a_j|^2)^{1/2}, & \text{if } i = n + 1, 2 \leq j \leq n; \\ (1 - |a_i|^2)^{1/2}, & \text{if } j = n + 1, i = n; \\ \left(\prod_{k=i+1}^{n}(-\overline{a_k})\right)(1 - |a_i|^2)^{1/2}, & \text{if } j = n + 1, 1 \leq i \leq n - 1; \\ \lambda \prod_{k=1}^{n}(-\overline{a_k}), & \text{if } i = j = n + 1; \end{cases} \quad (1.3)$$

where $\lambda$ is a parameter on the unit circle $|z| = 1$. 85
2. The algorithm generating new Poncelet pairs

In [2], a complex algebraic formulation was given for \( A \in S_n \). In [6], new Poncelet pairs are found. Let \( A \) be a \( S_n \) matrix (1.2) and \( B(\lambda) \) its unitary dilation matrix (1.3). We present an algorithm that computes the defining polynomial \( L(X,Y) \) which produces a new part \( C_P : L(X,Y) = 0 \) of the new Poncelet curve with respect to the boundary generating curve of \( W(A) \).

**Algorithm**

- **Step 1** Compute \( F_{B(\lambda)}(t, x, y) \) associated with the matrix \( B(\lambda) \) of the form (1.3).
- **Step 2** Substitute \( y = -1/Y - xX/Y \) into \( F_{B(\lambda)}(t, x, y) \) and define a polynomial
  \[
  H(x, X, Y : \lambda) = \ Y^{n+1}F_{B(\lambda)}(1, x, -1/Y - xX/Y) = F_{B(\lambda)}(Y, xY, -1 - xX) = \ c_{n+1}(X, Y)x^{n(n+1)} + \cdots + c_0(X, Y).
  \]
- **Step 3** Take the resultant \( R(X, Y : \lambda) \) of \( H(x, X, Y : \lambda) \) and \( H_x(x, X, Y : \lambda) \) with respect to \( x \).
- **Step 4** Find a factor polynomial \( K(X, Y : \lambda) \) of the resultant \( R(X, Y : \lambda) \) of total degree \( (n+1)n/2 \) in \( X, Y \) with multiplicity 2.
- **Step 5** Substitute $\lambda = ((1 - s^2) + 2is)/(1 + s^2)$ into $K(X, Y; \lambda)$ and $K_X(X, Y; \lambda)$.

- **Step 6** Take the respective numerators $\tilde{K}(X, Y; s)$ and $\tilde{K}_X(X, Y; s)$ of $K(X, Y; s)$ and $K_X(X, Y; s)$.

- **Step 7** Compute the Sylvester’s resultant $S(X, Y)$ of $\tilde{K}(X, Y; s)$ and $\tilde{K}_X(X, Y; s)$ with respect to $s$.

- **Step 8** Find a factor $L(X, Y)$ of $S(X, Y)$ with multiplicity 2.

In Figure, we present the graphic of a new Poncelet curve for a matrix $A$ in $S_4$. The union of the curve labeled 4 and the curve labeled 2 is $L(X, Y) = 0$. The curve labeled 1 is $\partial \Lambda_2(A)$. The curve labeled 3 is $\partial W(A)$.

Example. Let $n = 3$ and

$$B(\lambda) = \begin{pmatrix}
a & 1 - a^2 & -a\sqrt{1 - a^2} & a^2\sqrt{1 - a^2} \\
0 & a & 1 - a^2 & -a\sqrt{1 - a^2} \\
0 & 0 & a & \sqrt{1 - a^2} \\
\lambda\sqrt{1 - a^2} & -\lambda a\sqrt{1 - a^2} & \lambda a^2\sqrt{1 - a^2} & -\lambda a^3
\end{pmatrix}$$

for $a$ is a positive real number less than 1. Then the polynomial $L(X, Y)$ which gives the equation $L(X, Y) = 0$ of the new Poncelet curve is given by

$$L(X, Y) = 6a(-a^2 + 1)XY^2 + (a^6 + 3a^2 - 4)Y^2 + 2a(a^2 + 3)X^3$$

$$+(a^6 - 21a^2 - 4)X^2 + 6a(-a^4 + 3a^2 + 2)X + (a^6 - 9a^2).$$

3. Matrices unitarily similar to complex symmetric matrices

In this section we present a result related with the inverse problem for the shape of a numerical range (cf. [18]).

**Theorem 3.1** (cf.[5]). Every matrix in $S_n$ is unitarily similar to a complex symmetric matrix.

**Proof.** Let $A \in S_n$. Then by [16, Corollary 1.3] (see also [23] [Theorem 4]), the matrix $A$ has a canonical upper triangular form. The matrix $A$ also dilates to an $(n + 1) \times (n + 1)$ unitary matrix $W$ with distinct eigenvalues (cf. [15, Lemma 2.2]). We assume the distinct eigenvalues of $W$ are
given by $c_1, c_2, \ldots, c_{n+1}$, and their respective corresponding eigenvectors are $f_1, f_2, \ldots, f_{n+1}$. Let $P$ be the $n$-dimensional orthogonal projection satisfying $A = (PWP)|_{C^n}$. By replacing $f_j$ by $\exp(i\theta_j)f_j$ for some angles $\theta_1, \ldots, \theta_{n+1}$, the space $C^n = P(C^{n+1})$ is expressed as

$$C^n = \{z_1f_1 + z_2f_2 + \cdots + z_{n+1}f_{n+1} : (z_1, \ldots, z_{n+1}) \in C^{n+1}, b_1z_1 + b_2z_2 + \cdots + b_{n+1}z_{n+1} = 0\}$$

for some non-negative real numbers $b_1, b_2, \ldots, b_{n+1}$. Since the modulus of any eigenvalue of $A$ is strictly less than 1, the numbers $b_j$ are positive. Then the space $C^n = P(C^{n+1})$ consists of the linear spans of

$$\{b_1f_2 - b_2f_1, b_1f_3 - b_3f_1, \ldots, b_1f_{n+1} - b_{n+1}f_1\}.$$  

(3.1)

Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be an orthonormal basis of $C^n = P(C^{n+1})$ obtained by the Gram-Schmidt orthonormalization of $n$ independent vectors in (3.1). The vectors $\xi_j$ are expressed as

$$\xi_j = \xi_{j,1}f_1 + \xi_{j,2}f_2 + \cdots + \xi_{j,n+1}f_{n+1}$$

for some real numbers $\xi_{j,k}$ with $\xi_{j,j+1} > 0$ and $\xi_{j,j+2} = \xi_{j,j+3} = \cdots = 0$, $j = 1, 2, \ldots, n$. With respect to the orthonormal basis $\{\xi_1, \ldots, \xi_n\}$, the operator $A$ on the $n$-dimensional Hilbert space $C^n$ satisfies the property

$$\langle A\xi, \xi \rangle = \sum_{j=1}^{n+1} c_j \xi_{j,j} \xi_{j,j} = \sum_{j=1}^{n+1} c_j \xi_{j,j} \xi_{j,j} = \langle A\xi, \xi \rangle.$$

Thus the operator $A$ has a symmetric matrix representation with respect to this orthonormal basis $\{\xi_1, \ldots, \xi_n\}$. □

We are interested in matrices unitarily similar to complex symmetric matrices. In [18] Helton and Spitkovsky proved that every $n \times n$ complex matrix $A$ has an $n \times n$ complex symmetric matrix $B$ satisfying $W(A) = W(B)$. This result follows from the followin theorem.

**Theorem 3.2** (Helton and Vinnikov [19]). Suppose that $F(x, y, z)$ is a degree $n$ ternary homogeneous polynomial with real coefficients for which the equation $F(\cos \theta, \sin \theta, z) = 0$ in $z$ has $n$ real solutions for every angle $0 \leq \theta \leq 2\pi$ and $F(0, 0, 1) = 1$. Then there exist $n \times n$ real symmetric matrices $G, H$ satisfying

$$F(x, y, z) = \det(xH + yG + zI_n).$$
This result proved that the conjecture posed by P. Lax [21](page 184) is true. In [11], page 95, M. Fiedler posed a similar conjecture by relaxing $H, G$ by Hermitian matrices. In [12], Fiedler proved the assertion of Theorem 3.2 in the case $F(x, y, z) = 0$ is a rational curve.

In [28] T. Takagi proved that every Toeplitz matrix is unitarily symmetric to a complex symmetric matrix.

References


