数理解析研究所講究録
第 1996 巻 2016 年 54-65

RESEARCH OF WEIGHTED OPERATOR MEANS FROM TWO POINTS OF VIEW

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ABSTRACT. In the recent year, Pálfia and Petz have given to make a weighted operator mean from an arbitrary operator mean. In this report, we shall give concrete formulae of the dual, orthogonal and adjoint of weighted operator means. Then the characterization of operator interpolational means is obtained. We shall show that the operator interpolational means is only the weighted power means.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator $A\in\mathcal{B}(\mathcal{H})$ is said to be positive definite (resp. positive semi-definite) if and only if $\langle Ax,x\rangle>0$ (resp. $\langle Ax,x\rangle\geq 0$) for all non-zero vectors $x\in\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})_+$ be the set of all positive definite operators in $\mathcal{B}(\mathcal{H})$. If an operator A is positive semi-definite, then we write $A\geq 0$. For self-adjoint operators $A,B\in\mathcal{B}(\mathcal{H})$, $A\leq B$ means $B-A$ is positive semi-definite. A map $\mathfrak{M}:\mathcal{B}(\mathcal{H})_+^2\to\mathcal{B}(\mathcal{H})_+$ is called an *operator mean* [6] if the operator $\mathfrak{M}(A,B)$ satisfies the following four conditions; for $A,B,C,D\geq 0$,

(i) $A\leq C$ and $B\leq D$ implies $\mathfrak{M}(A,B)\leq\mathfrak{M}(C,D)$,
(ii) $X\left(\mathfrak{M}(A,B)\right)X\leq\mathfrak{M}(XAX,XBX)$ for all self-adjoint $X\in\mathcal{B}(\mathcal{H})$,
(iii) $A_n\searrow A$ and $B_n\searrow B$ imply $\mathfrak{M}(A_n,B_n)\searrow\mathfrak{M}(A,B)$ in the strong topology,
(iv) $\mathfrak{M}(I,I)=I$.

We remark that by the above condition (iii), we may assume $A,B\in\mathcal{B}(\mathcal{H})_+$. It is known many examples of operator means, for instance, the weighted geometric mean, the weighted power mean and the logarithmic mean. In particular, the weighted power mean has been studied by many researchers (cf. [3, 5, 9]) because of its usefulness, for instance, the weighted power means interpolate weighted arithmetic, geometric and harmonic means. One of the fact is that weighted power means derive power difference means by integrating their weight [9]. However, we do not know any explicit formula of the weighted operator means except the weighted power means. For the problem, Pálfia-Petz [7] has given an algorithm to get weighted operator means from arbitrary operator means. On the other hand, J.I. Fujii-Kamei have considered another algorithm to get weighted operator means from arbitrary symmetric operator means, and they have considered about the operator interpolational means [2]. It is

2010 *Mathematics Subject Classification.* Primary 47A64. Secondary 47A63.
Key words and phrases. Positive definite operator; operator mean; operator monotone function; interpolational mean; power mean.

a family of weighted operator means $\{\mathfrak{M}_t\}_{t\in[0,1]}$ with the weight t such that

$$\mathfrak{M}_{(1-\lambda)\alpha+\lambda\beta}(A,B) = \mathfrak{M}_\lambda\left(\mathfrak{M}_\alpha(A,B), \mathfrak{M}_\beta(A,B)\right)$$

holds for all $\alpha, \beta, \lambda \in [0,1]$ and $A, B \in \mathcal{B}(\mathcal{H})_+$. In [1], a characterization of the operator interpolational means have been obtained. But it has not been given any concrete example of the operator interpolational means.

In this report, we shall study about weighted operator means. In Section 2, we shall introduce the algorithm to get weighted operator means due to Pálfia-Petz [7], and introduce some properties of weighted operator means. In Section 3, we will give the formulae of the dual, adjoint and orthogonal of weighted operator means, they have very intuitive forms. In Section 4, we shall give another characterization of the operator interpolational means. It says that the operator interpolational means are just only the weighted power means.

2. Weighted operator means

A function $f(x)$ defined on an interval $I \subseteq \mathbb{R}$ is called an *operator monotone function*, provided for $A \leq B$ implies $f(A) \leq f(B)$ for every self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ whose spectral $\sigma(A)$ and $\sigma(B)$ lie in I.

The next theorem is so important to study operator means.

Theorem A ([6]). *For any operator mean $\mathfrak{M}$, there uniquely exists an operator monotone function f on $(0,\infty)$ with $f(1) = 1$ such that*

$$f(x)I = \mathfrak{M}(I, xI), \quad x > 0. \tag{2.1}$$

The function f satisfying (2.1) is called the representing function of $\mathfrak{M}$. The following hold:

(i) *The map $\mathfrak{M} \mapsto f$ is a one-to-one onto affine mapping from the set of all operator means to the set of all non-negative operator monotone functions on $(0,\infty)$ with $f(1) = 1$. Moreover, $\mathfrak{M} \mapsto f$ preserves the order, i.e., let $\mathfrak{M}$ and $\mathfrak{N}$ be operator means with representing functions f and g, respectively, then*

$$\mathfrak{M}(A,B) \leq \mathfrak{N}(A,B)\ (A,B \geq 0) \iff f(x) \leq g(x)\ (x > 0).$$

(ii) *If $A \in \mathcal{B}(\mathcal{H})_+$, then*

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}.$$

In this paper, The symbol $\mathcal{OM}$ and $\mathcal{RF}$ denote the sets of all operator means and representing functions, respectively. Especially, for $\mathfrak{M} \in \mathcal{OM}$, we use the symbol $\mathfrak{m} \in \mathcal{RF}$ as the representing function of $\mathfrak{M}$, i.e., $\mathfrak{m}$ is an operator monotone function on $(0,\infty)$ with $\mathfrak{m}(1) = 1$, s.t.,

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} \mathfrak{m}\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}$$

holds for all $A, B \in \mathcal{B}(\mathcal{H})_+$.

For operator means $\mathfrak{M}, \mathfrak{N}_1, \mathfrak{N}_2 \in \mathcal{OM}$ with the representing functions $\mathfrak{m}, \mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{RF}$, respectively. We can obtain a representing function of a composition mean

$\mathfrak{M}(\mathfrak{N}_1, \mathfrak{N}_2)$ as follows: For $x > 0$,

$$\begin{aligned}\mathfrak{M}(\mathfrak{N}_1(I, xI), \mathfrak{N}_2(I, xI)) &= \mathfrak{M}(\mathfrak{n}_1(x)I, \mathfrak{n}_2(x)I) \\ &= \mathfrak{n}_1(x)\mathfrak{m}(\mathfrak{n}_1(x)^{-1}\mathfrak{n}_2(x))I.\end{aligned}$$

In what follows, we will use the symbol $\mathfrak{M}(\mathfrak{n}_1(x), \mathfrak{n}_2(x))$ by the representing function of $\mathfrak{M}(\mathfrak{N}_1, \mathfrak{N}_2)$, i.e.,

$$\mathfrak{M}(\mathfrak{n}_1(x), \mathfrak{n}_2(x)) = \mathfrak{n}_1(x)\mathfrak{m}(\mathfrak{n}_1(x)^{-1}\mathfrak{n}_2(x)). \tag{2.2}$$

For the following discussion, we shall define the t-weighted operator means as follows.

Definition 1. Let $\mathfrak{M} \in \mathcal{OM}$. Then $\mathfrak{M}$ is said to be a *t-weighted operator mean* if and only if its representing function $\mathfrak{m} \in \mathcal{RF}$ satisfies $\mathfrak{m}'(1) = t$.

We remark that if $\mathfrak{m} \in \mathcal{RF}$, then $\mathfrak{m}'(1) \in [0, 1]$ by [7].

Pálfia-Petz [7] suggested an algorithm for making a t-weighted operator mean from given an operator mean, recently. It can be regarded as a kind of binary search algorithm:

Definition 2 ([7]). Let $\mathfrak{M} \in \mathcal{OM}$ with the representing function $\mathfrak{m}(x)$. For $A, B \in \mathcal{B}(\mathcal{H})_+$ and $t \in [0, 1]$, let $a_0 = 0$ and $b_0 = 1$, $A_0 = A$ and $B_0 = B$. Define a_n, b_n and A_n, B_n recursively by the following procedure defined inductively for all $n = 0, 1, 2, \ldots$

- (i) If $a_n = t$, then $a_{n+1} := a_n$ and $b_{n+1} := a_n$, $A_{n+1} := A_n$ and $B_{n+1} := A_n$,
- (ii) if $b_n = t$, then $a_{n+1} := b_n$ and $b_{n+1} := b_n$, $A_{n+1} := B_n$ and $B_{n+1} := B_n$,
- (iii) if $(1 - \mathfrak{m}'(1))a_n + \mathfrak{m}'(1)b_n \leq t$, then $a_{n+1} := (1 - \mathfrak{m}'(1))a_n + \mathfrak{m}'(1)b_n$ and $b_{n+1} := b_n$, $A_{n+1} := \mathfrak{M}(A_n, B_n)$ and $B_{n+1} := B_n$,
- (iv) if $(1 - \mathfrak{m}'(1))a_n + \mathfrak{m}'(1)b_n > t$, then $b_{n+1} := (1 - \mathfrak{m}'(1))a_n + \mathfrak{m}'(1)b_n$ and $a_{n+1} := a_n$, $B_{n+1} := \mathfrak{M}(A_n, B_n)$ and $A_{n+1} := A_n$.

For $A, B \in \mathcal{B}(\mathcal{H})_+$, the Thompson metric $d(A, B)$ is defined by

$$d(A, B) = \max\{\log M(A/B), \log M(B/A)\},$$

where $M(A/B) = \sup\{\alpha > 0 \mid \alpha A \leq B\}$. It is known that $\mathcal{B}(\mathcal{H})_+$ is complete respected to the Thompson metric [8].

Theorem B ([7]). *The operator sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ defined in Definition 2 converge to the same limit point in the Thompson metric. In what follows, we shall denote $\mathfrak{M}_t(A, B)$ by the limit point of $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$.*

Proposition C ([7]). *For $\mathfrak{M}, \mathfrak{N} \in \mathcal{OM}$, $A, B \in \mathcal{B}(\mathcal{H})_+$ and $t \in [0, 1]$, $\mathfrak{M}_t(A, B)$ and $\mathfrak{N}_t(A, B)$ fulfill the following properties:*

- (i) *if $\mathfrak{N}(A, B) \leq \mathfrak{M}(A, B)$ then $\mathfrak{N}_t(A, B) \leq \mathfrak{M}_t(A, B)$,*
- (ii) *$\mathfrak{M}_{\mathfrak{m}'(1)}(A, B) = \mathfrak{M}(A, B)$,*
- (iii) *$\mathfrak{M}_t(A, B)$ is continuous in t on the norm topology.*

Corollary D ([7]). *For a nontrivial operator mean $\mathfrak{M}$, there is a corresponding one parameter family of weighted means $\{\mathfrak{M}_t\}_{t\in[0,1]}$. Let $\mathfrak{m}(x)$ be the representing function of $\mathfrak{M}$. Then similarly we have a one parameter family of operator monotone functions $\{\mathfrak{m}_t(x)\}_{t\in[0,1]}$ corresponding to $\{\mathfrak{M}_t\}_{t\in[0,1]}$. The family $\{\mathfrak{m}_t(x)\}_{t\in[0,1]}$ is continuous in*

t, and $\mathfrak{m}_0(x)=1$ and $\mathfrak{m}_1(x)=x$ are two extremal points which correspond to the two trivial means, so actually $\mathfrak{m}_t(x)$ interpolates between these two points.

It is easy to see that $\left.\dfrac{d}{dx}\mathfrak{M}_t(x)\right|_{x=1}=t.$

3. The dual, adjoint and orthogonal of weighted operator means

In this section, we shall give concrete formulae of the dual, adjoint and orthogonal of weighted operator means.

Definition 3. Let $\mathfrak{M}\in\mathcal{OM}$ and $\mathfrak{m}(x)$ be the representing function of $\mathfrak{M}$. The dual, adjoint and orthogonal of $\mathfrak{M}$ are defined by the representing functions $x\mathfrak{m}(x)^{-1}$, $\mathfrak{m}(x^{-1})^{-1}$ and $x\mathfrak{m}(x^{-1})$, respectively.

We remark that if $\mathfrak{m}'(1)=t$, then $\frac{d}{dx}\mathfrak{m}(x^{-1})^{-1}|_{x=1}=t$, $\frac{d}{dx}x\mathfrak{m}(x)^{-1}|_{x=1}=1-t$ and $\frac{d}{dx}x\mathfrak{m}(x^{-1})|_{x=1}=1-t$.

Proposition 1. *Let $\mathfrak{M}\in\mathcal{OM}$ and its representing function $\mathfrak{m}\in\mathcal{RF}$. Let $g(x)=\mathfrak{m}(x^s)^{\frac{1}{s}}$ $(s\in[-1,1]\setminus\{0\})$. Then for $t\in[0,1]$,*

$$\mathfrak{m}_t(x^s)^{\frac{1}{s}}=g_t(x)\ \left(:=\left(\mathfrak{m}(x^s)^{\frac{1}{s}}\right)_t\right)$$

holds for all $x>0$.

Proposition 1 says that the maps $\mathfrak{m}(x)\mapsto\mathfrak{m}(x^s)^{\frac{1}{s}}$ and $\mathfrak{m}(x)\mapsto\mathfrak{m}_t(x)$ are commutative like the following diagram.

$$\begin{array}{ccc}\mathfrak{m}(x) & \xrightarrow{\ g\ } & g(x)=\mathfrak{m}(x^s)^{\frac{1}{s}}\\ {\scriptstyle t}\big\downarrow & & {\scriptstyle t}\big\downarrow\\ \mathfrak{m}_t(x) & \xrightarrow{\ g\ } & g_t(x)=\mathfrak{m}_t(x^s)^{\frac{1}{s}}\end{array}$$

Remark 1. The function $\mathfrak{m}(x^s)^{\frac{1}{s}}$ in Proposition 1 is an operator monotone function since $\mathfrak{m}(x)$ is an operator monotone function. Especially, by putting $s=-1$, we get $g(x)=\mathfrak{m}(x^{-1})^{-1}$ and $\mathfrak{m}_t(x^{-1})^{-1}=g_t(x)$, namely, we obtain a relation between $\mathfrak{m}_t(x)$ and the adjoint of $\mathfrak{m}_t(x)$.

Before proving Proposition 1, we would like to define some notations which will be used in the proof. Let $\mathfrak{M}\in\mathcal{OM}$, and $\mathfrak{m}\in\mathcal{RF}$ be the representing function of $\mathfrak{M}$. For $t\in[0,1]$, we define the sequences $\{a^{(t)}_{\mathfrak{m},n}\}_{n=0}^{\infty}$, $\{b^{(t)}_{\mathfrak{m},n}\}_{n=0}^{\infty}\subset[0,1]$ as in Definition 2. For $A,B\in\mathcal{B}(\mathcal{H})_+$, we define the corresponding operator sequences $\{A^{(t)}_{\mathfrak{m},n}\}_{n=0}^{\infty}$ and $\{B^{(t)}_{\mathfrak{m},n}\}_{n=0}^{\infty}$ to $\{a^{(t)}_{\mathfrak{m},n}\}_{n=0}^{\infty}$ and $\{b^{(t)}_{\mathfrak{m},n}\}_{n=0}^{\infty}$, respectively, by Definition 2. We remark that each $A^{(t)}_{\mathfrak{m},n}$ (resp. $B^{(t)}_{\mathfrak{m},n}$) is an $a^{(t)}_{\mathfrak{m},n}$-weighted operator mean (resp. $b^{(t)}_{\mathfrak{m},n}$-weighted operator mean). Then we give a representing function of $A^{(t)}_{\mathfrak{m},n}$ (resp. $B^{(t)}_{\mathfrak{m},n}$), and denote $\mathfrak{m}^{(t)}_{L,n}(x)\in\mathcal{RF}$ (resp. $\mathfrak{m}^{(t)}_{R,n}(x)\in\mathcal{RF}$).

Proof of Proposition 1. For a given $\mathfrak{M} \in \mathcal{OM}$ with the representing function $\mathfrak{m}(x)$, and $t \in [0,1]$, let $\{a^{(t)}_{\mathfrak{m},n}\}, \{b^{(t)}_{\mathfrak{m},n}\} \subset [0,1]$ be sequences constructed by Definition 2. Then, since $\mathfrak{m}'(1) = g'(1)$, we have

$$a^{(t)}_{\mathfrak{m},n} = a^{(t)}_{g,n} \text{ and } b^{(t)}_{\mathfrak{m},n} = b^{(t)}_{g,n} \quad (n = 0,1,2,...).$$

Then we shall show

$$\mathfrak{m}^{(t)}_{L,n}(x^s)^{\frac{1}{s}} = g^{(t)}_{L,n}(x) \text{ and } \mathfrak{m}^{(t)}_{R,n}(x^s)^{\frac{1}{s}} = g^{(t)}_{R,n}(x) \tag{3.1}$$

hold for $n = 0,1,2,...$ by mathematical induction on n. The case $n = 0$ is clear. Assume that (3.1) holds in the case $n = k$. If $(1 - \mathfrak{m}'(1))a^{(t)}_{\mathfrak{m},k} + \mathfrak{m}'(1)b^{(t)}_{\mathfrak{m},k} \leq t$ (equivalently $(1 - g'(1))a^{(t)}_{g,k} + g'(1)b^{(t)}_{g,k} \leq t$), then $b^{(t)}_{\mathfrak{m},k+1} = b^{(t)}_{\mathfrak{m},k}$ and $b^{(t)}_{g,k+1} = b^{(t)}_{g,k}$, i.e., $\mathfrak{m}^{(t)}_{R,k+1}(x) = \mathfrak{m}^{(t)}_{R,k}(x)$ and $g^{(t)}_{R,k+1}(x) = g^{(t)}_{R,k}(x)$. So

$$\mathfrak{m}^{(t)}_{R,k+1}(x^s)^{\frac{1}{s}} = \mathfrak{m}^{(t)}_{R,k}(x^s)^{\frac{1}{s}} = g^{(t)}_{R,k}(x) = g^{(t)}_{R,k+1}(x)$$

hold from the assumption. On the other hand, by (2.2), we have

$$\begin{aligned} \mathfrak{m}^{(t)}_{L,k+1}(x) &= \mathfrak{M}(\mathfrak{m}^{(t)}_{L,k}(x), \mathfrak{m}^{(t)}_{R,k}(x)) \\ &= \mathfrak{m}^{(t)}_{L,k}(x)\mathfrak{m}\left(\mathfrak{m}^{(t)}_{L,k}(x)^{-1}\mathfrak{m}^{(t)}_{R,k}(x)\right), \\ g^{(t)}_{L,k+1}(x) &= g^{(t)}_{L,k}(x)g\left(g^{(t)}_{L,k}(x)^{-1}g^{(t)}_{R,k}(x)\right). \end{aligned} \tag{3.2}$$

By (3.2), we have

$$\begin{aligned} g^{(t)}_{L,k+1}(x) &= g^{(t)}_{L,k}(x)\mathfrak{m}\left(\left(g^{(t)}_{L,k}(x)^{-1}g^{(t)}_{R,k}(x)\right)^s\right)^{\frac{1}{s}} \\ &= \mathfrak{m}^{(t)}_{L,k}(x^s)^{\frac{1}{s}}\mathfrak{m}\left(\left(\mathfrak{m}^{(t)}_{L,k}(x^s)^{\frac{1}{s}}\right)^{-s}\left(\mathfrak{m}^{(t)}_{R,k}(x^s)^{\frac{1}{s}}\right)^s\right)^{\frac{1}{s}} \\ &= \left\{\mathfrak{m}^{(t)}_{L,k}(x^s)\mathfrak{m}\left(\mathfrak{m}^{(t)}_{L,k}(x^s)^{-1}\mathfrak{m}^{(t)}_{R,k}(x^s)\right)\right\}^{\frac{1}{s}} = \mathfrak{m}^{(t)}_{L,k+1}(x^s)^{\frac{1}{s}}. \end{aligned}$$

Similarly, we can also show $\mathfrak{M}^{(t)}_{R,n}(x^s)^{\frac{1}{s}} = g^{(t)}_{R,n}(x)$ in (3.1) for the case $n = k+1$. If $(1 - \mathfrak{m}'(1))a^{(t)}_{\mathfrak{m},k} + \mathfrak{m}'(1)b^{(t)}_{\mathfrak{m},k} > t$ (equivalently $(1 - g'(1))a^{(t)}_{g,k} + g'(1)b^{(t)}_{g,k} > t$). From the above, we get

$$\mathfrak{m}^{(t)}_{L,n}(x^s)^{\frac{1}{s}} = g^{(t)}_{L,n}(x), \ \mathfrak{m}^{(t)}_{R,n}(x^s)^{\frac{1}{s}} = g^{(t)}_{R,n}(x)$$

hold for $n = 0,1,2,...$ Since $g^{(t)}_{L,n}(x)$ and $\mathfrak{m}^{(t)}_{L,n}(x^s)^{\frac{1}{s}}$ converge point wise to $g_t(x)$ and $\mathfrak{m}_t(x^s)^{\frac{1}{s}}$, respectively, on $(0,\infty)$,

$$\begin{aligned} 0 &\leq \left|g_t(x) - \mathfrak{m}_t(x^s)^{\frac{1}{s}}\right| \\ &= \left|g_t(x) - g^{(t)}_{L,n}(x) + \mathfrak{m}^{(t)}_{L,n}(x^s)^{\frac{1}{s}} - \mathfrak{m}_t(x^s)^{\frac{1}{s}}\right| \\ &\leq \left|g_t(x) - g^{(t)}_{L,n}(x)\right| + \left|\mathfrak{m}^{(t)}_{L,n}(x^s)^{\frac{1}{s}} - \mathfrak{m}_t(x^s)^{\frac{1}{s}}\right| \to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

$\square$

In the next theorem, we obtain intriguing results for the relations among the dual and the orthogonal of weighted operator means. It complements our intuitive understanding of the weighted operator means.

Theorem 2. *Let $\mathfrak{M} \in \mathcal{OM}$, and let $\mathfrak{m} \in \mathcal{RF}$ be the representing function of $\mathfrak{M}$. Define $k(x) := x\mathfrak{m}(x)^{-1}$ and $l(x) := x\mathfrak{m}(x^{-1})$. Then for $t \in [0,1]$,*

$$k_{1-t}(x) = x\mathfrak{m}_t(x)^{-1} \quad \text{and} \quad l_{1-t}(x) = x\mathfrak{m}_t(x^{-1}).$$

Theorem 2 gives similar consequences to Proposition 1 as the following diagram.

$$\begin{tikzcd}
\mathfrak{m}(x) \arrow[r, "k"] \arrow[d, "1-t"'] & k(x) = x\mathfrak{m}(x^{-1}) \arrow[d, "1-t"'] \\
\mathfrak{m}_{1-t}(x) \arrow[r, "k"] & k_{1-t}(x) = x\mathfrak{m}_t(x^{-1})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{m}(x) \arrow[r, "l"] \arrow[d, "1-t"'] & l(x) = x\mathfrak{m}(x)^{-1} \arrow[d, "1-t"'] \\
\mathfrak{m}_{1-t}(x) \arrow[r, "l"] & l_{1-t}(x) = x\mathfrak{m}_t(x)^{-1}
\end{tikzcd}$$

Proof. First we shall show $l_{1-t}(x) = x\mathfrak{m}_t(x^{-1})$. Let $\{a_{l,n}^{(1-t)}\}_{n=0}^{\infty}$, $\{b_{l,n}^{(1-t)}\}_{n=0}^{\infty} \subset [0,1]$ be the sequences constructed by Definition 2 for a given function $l(x)$ and a constant $1-t \in [0,1]$, and let $\{a_{\mathfrak{m},n}^{(t)}\}_{n=0}^{\infty}$, $\{b_{\mathfrak{m},n}^{(t)}\}_{n=0}^{\infty}$ be so for $t \in [0,1]$. Then, since $\mathfrak{m}'(1) = 1 - l'(1)$, we have

$$a_{\mathfrak{m},n}^{(t)} = 1 - b_{l,n}^{(1-t)} \text{ and } b_{\mathfrak{m},n}^{(t)} = 1 - a_{l,n}^{(1-t)} \quad (n = 0, 1, 2, ...).$$

To prove $l_{1-t}(x) = x\mathfrak{m}_t(x^{-1})$, it is enough to show

$$x\mathfrak{m}_{L,n}^{(t)}(x^{-1}) = l_{R,n}^{(1-t)}(x) \text{ and } x\mathfrak{m}_{R,n}^{(t)}(x^{-1}) = l_{L,n}^{(1-t)}(x) \tag{3.3}$$

hold for $n = 0, 1, 2, ...$ by mathematical induction on n as in the proof of Proposition 1. The case $n = 0$ is clear. Assume that (3.3) holds in the case $n = k$. If $(1 - \mathfrak{m}'(1))a_{\mathfrak{m},k}^{(t)} + \mathfrak{m}'(1)b_{\mathfrak{m},k}^{(t)} \leq t$ (equivalently $(1 - l'(1))a_{l,k}^{(1-t)} + l'(1)b_{l,k}^{(1-t)} \geq 1 - t$), then $\mathfrak{m}_{R,k+1}^{(t)}(x) = \mathfrak{m}_{R,k}^{(t)}(x)$ and $l_{L,k+1}^{(1-t)}(x) = l_{L,k}^{(1-t)}(x)$. Therefore $x\mathfrak{m}_{R,k+1}^{(t)}(x^{-1}) = l_{L,k+1}^{(1-t)}(x)$ holds from the assumption. On the other hand, by (2.2), we have

$$\begin{aligned}
\mathfrak{m}_{L,k+1}^{(t)}(x) &= \mathfrak{M}(\mathfrak{m}_{L,k}^{(t)}(x), \mathfrak{m}_{R,k}^{(t)}(x)) \\
&= \mathfrak{m}_{L,k}^{(t)}(x)\mathfrak{m}\left(\mathfrak{m}_{L,k}^{(t)}(x)^{-1}\mathfrak{m}_{R,k}^{(t)}(x)\right), \\
l_{R,k+1}^{(1-t)}(x) &= l_{L,k}^{(1-t)}(x)\, l\left(l_{L,k}^{(1-t)}(x)^{-1} l_{R,k}^{(1-t)}(x)\right).
\end{aligned}$$

From the assumption, we get

$$\begin{aligned}
l_{R,k+1}^{(1-t)}(x) &= l_{R,k}^{(1-t)}(x)\, l\left(l_{R,k}^{(1-t)}(x)^{-1} l_{L,k}^{(1-t)}(x)\right) \\
&= x\mathfrak{m}_{R,k}^{(t)}(x^{-1})\, l\left(\mathfrak{m}_{R,k}^{(t)}(x^{-1})^{-1}\mathfrak{m}_{L,k}^{(t)}(x^{-1})\right) \\
&= x\mathfrak{m}_{R,k}^{(t)}(x^{-1})\left(\mathfrak{m}_{R,k}^{(t)}(x^{-1})^{-1}\mathfrak{m}_{L,k}^{(t)}(x^{-1})\right)\mathfrak{m}\left(\mathfrak{m}_{R,k}^{(t)}(x^{-1})\mathfrak{m}_{L,k}^{(t)}(x^{-1})^{-1}\right) \\
&= x\mathfrak{m}_{L,k}^{(t)}(x^{-1})\mathfrak{m}\left(\mathfrak{m}_{R,k}^{(t)}(x^{-1})\mathfrak{m}_{L,k}^{(t)}(x^{-1})^{-1}\right) \\
&= x\mathfrak{m}_{L,k+1}^{(t)}(x^{-1}).
\end{aligned}$$

Likewise, we can also show the case

$$(1-\mathfrak{m}'(1))a_{\mathfrak{m},k}^{(t)}+\mathfrak{m}'(1)b_{\mathfrak{m},k}^{(t)}>t \text{ (equivalently } (1-l'(1))a_{l,k}^{(1-t)}+l'(1)b_{l,k}^{(1-t)}<1-t).$$

From the above, we obtain

$$x\mathfrak{m}_{L,n}^{(t)}(x^{-1})=l_{R,n}^{(1-t)}(x) \text{ and } x\mathfrak{m}_{R,n}^{(t)}(x^{-1})=l_{L,n}^{(1-t)}(x) \quad (n=0,1,2,\ldots).$$

We can show $k_{1-t}(x)=x\mathfrak{m}_t(x)^{-1}$ by the same way. □

Example 1. Let $f(x)=\dfrac{1+x}{2}$ (Arithmetic mean) and $t=\dfrac{1}{4}$. Then applying the Definition 2 implies

$$f_{\frac{1}{4}}(x)=\frac{3}{4}+\frac{1}{4}x$$

and we have

$$xf_{\frac{1}{4}}(x)^{-1}=\left[\frac{1}{4}+\frac{3}{4}x^{-1}\right]^{-1}.$$

On the other hand, $k(x)=xf(x)^{-1}=\dfrac{2x}{1+x}$ (Harmonic mean) and

$$k_{\frac{3}{4}}(x)=\left[\frac{1}{4}+\frac{3}{4}x^{-1}\right]^{-1}$$

from the algorithm of Definition 2. So we obtain

$$k_{1-\frac{1}{4}}(x)=xf_{\frac{1}{4}}(x)^{-1}.$$

4. Interpolational means

In this section, characterizations of interpolatinal means will be obtained. We shall consider them in the cases of numerical and operator interpolatinal means, separately.

Definition 4 (Interpolational mean, [2]).

(i) For each $t\in[0,1]$, let $m_t:(0,\infty)^2\to(0,\infty)$ be a continuous function. Assume m_t is point wise continuous on $t\in[0,1]$. The family of continuous functions $\{m_t\}_{t\in[0,1]}$ is said to be an *interpolatinal mean* if and only if the following condition is satisfied;

$$m_\delta\left(m_\alpha(a,b),m_\beta(a,b)\right)=m_{(1-\delta)\alpha+\delta\beta}(a,b)$$

for all $\alpha,\beta,\delta\in[0,1]$ and $a,b\in(0,\infty)$.

(ii) Let $\{\mathfrak{M}_\alpha\}_{\alpha\in[0,1]}$ be a family of weighted operator means. If $\mathfrak{M}_\alpha$ is continuous on $\alpha\in[0,1]$ and satisfies the following condition, then $\{\mathfrak{M}_\alpha\}_{\alpha\in[0,1]}$ is said to be an *operator interpolational mean*;

$$\mathfrak{M}_\delta\left(\mathfrak{M}_\alpha(A,B),\mathfrak{M}_\beta(A,B)\right)=\mathfrak{M}_{(1-\delta)\alpha+\delta\beta}(A,B)$$

for all $\alpha,\beta,\delta\in[0,1]$ and all $A,B\in\mathcal{B}(\mathcal{H})_+$.

A typical example of operator interpolational mean is the weighted power mean whose representing function is

$$P_{s,\alpha}(x) = [(1-\alpha) + \alpha x^s]^{\frac{1}{s}} \quad (s \in [-1,1] \setminus \{0\}).$$

(The case $s = 0$ is considered as $\lim_{s \to 0} P_{s,\alpha}(x) = x^{\alpha}$.)

Firstly, we think about the numerical case.

Theorem 3. *For each $t \in [0,1]$, let $m_t : (0,\infty)^2 \to (0,\infty)$ be a continuous function. Assume that m_t is point wise continuous on $t \in [0,1]$, and is satisfying the following conditions*

(i) *$m_0(a,b) = a$, $m_1(a,b) = b$ and $m_t(a,a) = a$ for all $a,b \in (0,\infty)$ and $t \in [0,1]$,*
(ii) *if $m_{\frac{1}{2}}(a,b) = a$ or b, then $a = b$ for all $a,b \in (0,\infty)$.*

Then the following assertions are equivalent:

(1) *$\{m_t\}_{t\in[0,1]}$ is an interpolational mean,*
(2) *there exists a real-valued function f such that*

$$m_t(a,b) = f^{-1}[(1-t)f(a) + tf(b)]$$

for each $t \in [0,1]$ and $a,b \in (0,\infty)$.

Proof. (2) $\Longrightarrow$ (1) is clear. We shall prove (1) $\Longrightarrow$ (2). For fixed $a,b \in (0,\infty)$, let $m_t(a,b) := M_{a,b}(t)$. We may assume $a \neq b$. First we shall prove $M_{a,b}(t)$ is a one-to-one onto mapping on $t \in [0,1]$. Assume that there exists $\alpha, \beta \in [0,1]$ such that $\alpha < \beta$ and $M_{a,b}(\alpha) = M_{a,b}(\beta) = \mu$. For any $\gamma \in [\alpha,\beta]$ there uniquely exists $\delta \in [0,1]$ such that $\gamma = (1-\delta)\alpha + \delta\beta$, so we have

$$\begin{aligned} M_{a,b}(\gamma) &= M_{a,b}\left((1-\delta)\alpha + \delta\beta\right) \\ &= m_{(1-\delta)\alpha+\delta\beta}(a,b) \\ &= m_\delta\left(m_\alpha(a,b), m_\beta(a,b)\right) \qquad \text{(by (1))} \\ &= m_\delta(\mu,\mu) = \mu, \qquad \text{(by (i))} \end{aligned}$$

namely,

$$M_{a,b}(\gamma) = \mu \tag{4.1}$$

holds for all $\gamma \in [\alpha,\beta]$. Moreover, for each $\varepsilon > 0$ such that $[\alpha-\varepsilon, \alpha+\varepsilon] \subseteq [0,\beta]$, we have

$$\begin{aligned} \mu &= M_{a,b}(\alpha) \\ &= m_{\frac{1}{2}}\left(M_{a,b}(\alpha-\varepsilon), M_{a,b}(\alpha+\varepsilon)\right) \qquad \text{(by (1))} \\ &= m_{\frac{1}{2}}\left(M_{a,b}(\alpha-\varepsilon), \mu\right) \qquad \text{(by (4.1))}. \end{aligned}$$

By (ii), it implies $M_{a,b}(\alpha-\varepsilon) = \mu$, so we have $M_{a,b}(\gamma) = \mu$ for all $\gamma \in [\alpha-\varepsilon, \beta]$. By using this way several times, we get $M_{a,b}(\gamma) = \mu$ for all $\gamma \in [0,\beta]$. Thus

$$\mu = M_{a,b}(0) = a.$$

Likewise we can show $\mu = M_{a,b}(1) = b$, and hence we get $a = b$. It is a contradiction to $a \neq b$. Therefore $M_{a,b}(t)$ is a one-to-one mapping on $t \in [0,1]$. Also we can show $M_{a,b}(t)$ is an onto mapping because $M_{a,b}(0) = a$, $M_{a,b}(1) = b$ and $M_{a,b}(t)$ is continuous on $t \in [0,1]$. From the above, $M_{a,b}(t)$ is a one-to-one onto mapping. Hence, for fixed $a, b \in (0,\infty)$ such that $a \neq b$, there exists a function $f_{a,b}$ defined on $[a,b]$ such that

$$f_{a,b}(m_t(a,b)) = (1-t)f_{a,b}(a) + tf_{a,b}(b). \tag{4.2}$$

Here we may assume $a < b$. Next we shall prove that this function $f_{a,b}$ is independent of the interval $[a,b]$ and unique up to affine transformations of $f_{a,b}$. Because for any $M, N \in \mathbb{R}$ $(M \neq 0)$, let $g(x) = Mf_{a,b}(x) + N$. Then we can easily obtain

$$f_{a,b}^{-1}\left[(1-t)f(a) + tf(b)\right] = g^{-1}\left[(1-t)g(a) + tg(b)\right].$$

Case 1. $[a,b] \subset [a',b']$;

Let $M_{a,b}^{-1} : [a,b] \to [0,1]$ be the inverse function of $M_{a,b}(t)(= m_t(a,b))$. From $[a,b] \subset [a',b']$, it is clear that there exists $\delta_1, \delta_2 \in [0,1]$ satisfying $m_{\delta_1}(a',b') = a$, $m_{\delta_2}(a',b') = b$. Since $\{m_t\}_{t\in[0,1]}$ is an interpolational mean, we have

$$\begin{aligned} m_t(a,b) &= m_t\left(m_{\delta_1}(a',b'), m_{\delta_2}(a',b')\right) \\ &= m_{(1-t)\delta_1 + t\delta_2}(a',b') \\ &= M_{a',b'}\left((1-t)\delta_1 + t\delta_2\right). \end{aligned}$$

It is equivalent to

$$M_{a',b'}^{-1}\left(m_t(a,b)\right) = (1-t)\delta_1 + t\delta_2.$$

Put $x = m_t(a,b) \in [a,b]$, then $M_{a,b}^{-1}(x) = t$. We have

$$M_{a',b'}^{-1}(x) = \left(1 - M_{a,b}^{-1}(x)\right)\delta_1 + M_{a,b}^{-1}(x)\delta_2,$$

hence we have

$$M_{a,b}^{-1}(x) = \frac{1}{M_{a',b'}^{-1}(b) - M_{a',b'}^{-1}(a)} M_{a',b'}^{-1}(x) - \frac{M_{a',b'}^{-1}(a)}{M_{a',b'}^{-1}(b) - M_{a',b'}^{-1}(a)}.$$

Here by putting

$$\omega_1 = \frac{1}{M_{a',b'}^{-1}(b) - M_{a',b'}^{-1}(a)} \text{ and } \omega_2 = \frac{M_{a',b'}^{-1}(a)}{M_{a',b'}^{-1}(b) - M_{a',b'}^{-1}(a)}$$

we have

$$M_{a,b}^{-1}(x) = \omega_1 M_{a',b'}^{-1}(x) + \omega_2 \quad (x \in [a,b]).$$

For $M_{a,b}^{-1}(x)$ and $M_{a',b'}^{-1}(x)$, let

$$k(x) = \begin{cases} M_{a,b}^{-1}(x) & (x \in [a,b]) \\ \omega_1 M_{a',b'}^{-1}(x) + \omega_2 & (x \in [a',b'] \setminus [a,b]) \end{cases}.$$

Then $k(x) = \omega_1 M_{a',b'}^{-1}(x) + \omega_2$ holds for $x \in [a',b']$. This result, (4.2) and putting $x = m_t(a,b)$ (or $m_t(a',b')$) imply $t = M_{a,b}^{-1}(x) = k(x)$,

$$f_{a,b}(x) = k(x)\left(f_{a,b}(b) - f_{a,b}(a)\right) + f_{a,b}(a)$$

and

$$f_{a',b'}(x) = \frac{k(x) - \omega_1}{\omega_2}\left(f_{a',b'}(b') - f_{a',b'}(a')\right) + f_{a',b'}(a').$$

From the above, we can find that there exists $\omega_1', \omega_2' \in \mathbb{R}$ such that

$$f_{a',b'}(x) = \omega_1' f_{a,b}(x) + \omega_2'.$$

Case 2. $[a,b],[c,d](a<b<c<d)$;
It's enough to think about the case $[a,b] \subset [a,d]$ and $[c,d] \subset [a,d]$. $\square$

Corollary 4. *For $t \in [0,1]$. let $m_t : (0,\infty)^2 \to \mathbb{R}$ be a real-valued continuous function on each variables satisfying the following condition*

$$[(1-t)a^{-1} + tb^{-1}]^{-1} \le m_t(a,b) \le (1-t)a + tb \tag{4.3}$$

for all $a, b \in (0,\infty)$ and $t \in [0,1]$. Assume that $\{m_t\}_{t\in[0,1]}$ is point wise continuous on $t \in [0,1]$. Then the following assertions are equivalent:

(1) *$\{m_t\}_{t\in[0,1]}$ is an interpolational mean,*
(2) *there exists a real-valued function f such that*

$$m_t(a,b) = f^{-1}[(1-t)f(a) + tf(b)] \quad \text{for all } a,b \in (0,\infty) \text{ and } t \in [0,1].$$

Proof. It is enough to show that the condition (4.3) satisfies the conditions (i) and (ii) of Theorem 3. Since (i) is easy, here we only show that (4.3) implies the condition (ii) of Theorem 3. If $m_{\frac{1}{2}}(x,y) = x$ satisfies, then

$$\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \le x \le \frac{x+y}{2}$$

by (4.3). By the first inequality of the above, we get $y \le x$, and also we obtain $x \le y$ from the second inequality of the above. Therefore $x = y$ holds and condition (ii) is satisfied. $\square$

Lastly we derive a characterization of operator interpolational means from the above results. The characterization gives us the fact that the weighted power mean is the only operator interplational mean.

Theorem 5. *For $\alpha \in [0,1]$, let $\mathfrak{M}_\alpha$ be a weighted operator mean with the representing functions $\mathfrak{m}_\alpha(x)$. If $\{\mathfrak{m}_\alpha(x)\}_{\alpha\in[0,1]}$ is point wise continuous on $\alpha \in [0,1]$ and*

$$[(1-\alpha) + \alpha x^{-1}]^{-1} \le \mathfrak{m}_\alpha(x) \le (1-\alpha) + \alpha x$$

holds for all $\alpha \in [0,1]$ and $x > 0$, then they are mutually equivalent:

(1) *$\{\mathfrak{M}_\alpha\}_{\alpha\in[0,1]}$ is an operator interpolational mean,*
(2) *there exists $r \in [-1,1]$, $\mathfrak{m}_\alpha(x) = [(1-\alpha) + \alpha x^r]^{\frac{1}{r}}$.*

In (2), we consider the case $r = 0$ as x^α.

To prove Theorem 5, we prepare the next lemma;

Lemma E ([4, Theorem 84]). *For a real-valued continuous function f such that its inverse function exists, let $m_\alpha(a,b) = f^{-1}[(1-\alpha)f(a) + \alpha f(b)]$ for $a,b > 0$ and $\alpha \in [0,1]$. If*

$$m_\alpha(ka,kb) = km_\alpha(a,b)$$

holds for all $k > 0$ and $a,b \in (0,\infty)$, then there exists $r \in \mathbb{R}$ such that $m_\alpha(a,b)$ can be determined as

$$m_\alpha(a,b) = [(1-\alpha)a^r + \alpha b^r]^{\frac{1}{r}}.$$

Proof of Theorem 5. (2) $\Longrightarrow$ (1) is clear. We only show (1) $\Longrightarrow$ (2). For $a, b > 0$ and $\alpha \in [0,1]$, $\mathfrak{M}_\alpha(aI, bI) = a\mathfrak{m}_\alpha(\frac{b}{a})I$ holds from Theorem A and

$$a\left[(1-\alpha)+\alpha\left(\frac{b}{a}\right)^{-1}\right]^{-1} \leq a\mathfrak{m}_\alpha\left(\frac{b}{a}\right) \leq a\left[(1-\alpha)+\alpha\frac{b}{a}\right]$$

follows from the assumption of Theorem 5. This relation is equivalent to the following inequality;

$$[(1-\alpha)a^{-1}+\alpha b^{-1}]^{-1} \leq \mathfrak{M}_\alpha(a,b) \leq (1-\alpha)a+\alpha b,$$

here we identify $\mathfrak{M}_\alpha(a,b)$ and c by $\mathfrak{M}_\alpha(aI,bI)$ and cI, respectively. Hence by the assumption and Corollary 4, there exists a real-valued function f such that

$$\mathfrak{M}_\alpha(a,b) = f^{-1}[(1-\alpha)f(a)+\alpha f(b)].$$

Moreover, $\mathfrak{M}_\alpha$ satisfies the transformer equality

$$\mathfrak{M}_\alpha(cA,cB) = c\mathfrak{M}_\alpha(A,B) \quad \text{for } c>0$$

because $\mathfrak{M}_\alpha$ is an operator mean. These facts and Lemma E implies

$$\mathfrak{M}_\alpha(a,b) = [(1-\alpha)a^r+\alpha b^r]^{\frac{1}{r}}, \quad r \in \mathbb{R}.$$

Moreover since it is increasing on $r \in \mathbb{R}$, we have $r \in [-1,1]$ by the assumption. Therefore we obtain

$$\mathfrak{m}_\alpha(xI) = \mathfrak{M}_\alpha(I,xI) = [(1-\alpha)+\alpha x^r]^{\frac{1}{r}}I.$$

□

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