数理解析研究所講究録
第 1996 巻 2016 年 1-12

Some operator divergences based on Petz-Bregman divergence

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1. Introduction

This report is based on [12]. Throughout this report, a bounded linear operator T on a Hilbert space H is positive (denoted by $T\geq 0$) if $(T\xi,\xi)\geq 0$ for all $\xi\in H$, and T is strictly positive (denoted by $T>0$) if T is invertible and positive.

For strictly positive operators A and B, $A\natural_{x}B$ is defined as follows ([3, 4, 13] etc.):

$$A\natural_{x}B\equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{x}A^{\frac{1}{2}},\ x\in\mathbb{R}.$$

We call $A\natural_{x}B$ a path passing through $A=A\natural_{0}B$ and $B=A\natural_{1}B$. If $x\in[0,1]$, the path $A\natural_{x}B$ coincides with the weighted geometric operator mean denoted by $A\sharp_{x}B$ (cf. [15]). We remark that $A\natural_{x}B=B\natural_{1-x}A$ holds for $x\in\mathbb{R}$.

Fujii and Kamei [2] defined the following relative operator entropy for strictly positive operators A and B:

$$\begin{aligned}S(A|B)&\equiv A^{\frac{1}{2}}\log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}\\&=\left.\frac{d}{dx}A\natural_{x}B\right|_{x=0}.\end{aligned}$$

We can regard $S(A|B)$ as the gradient of the tangent line at $x=0$ of the path $A\natural_{x}B$. Furuta [7] defined generalized relative operator entropy as follows:

$$\begin{aligned}S_{\alpha}(A|B)&\equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}\\&=\left.\frac{d}{dx}A\natural_{x}B\right|_{x=\alpha},\ \alpha\in\mathbb{R}.\end{aligned}$$

We regard $S_{\alpha}(A|B)$ as the gradient of the tangent line at $x=\alpha$ of the path. We know immediately $S_{0}(A|B)=S(A|B)$. Yanagi, Kuriyama and Furuichi [18] introduced Tsallis relative operator entropy as follows:

$$T_{\alpha}(A|B)\equiv\frac{A\sharp_{\alpha}B-A}{\alpha},\ \alpha\in(0,1].$$

$T_{\alpha}(A|B)$ can be regarded as the average rate of change of $A\sharp_{x}B$ from $x=0$ to $x=\alpha$. Since $\lim_{x\to 0}\frac{a^{x}-1}{x}=\log a$ holds for $a>0$, we have $T_{0}(A|B)\equiv\lim_{\alpha\to 0}T_{\alpha}(A|B)=S(A|B)$. Tsallis relative operator entropy can be extended as the notion for $\alpha\in\mathbb{R}$. In this case, we use $\natural_{\alpha}$ instead of $\sharp_{\alpha}$. In [8], we had given the following relations among these relative operator entropies:

$$(*)\qquad S_{1}(A|B)\geq-T_{1-\alpha}(B|A)\geq S_{\alpha}(A|B)\geq T_{\alpha}(A|B)\geq S(A|B),\ \alpha\in(0,1).$$

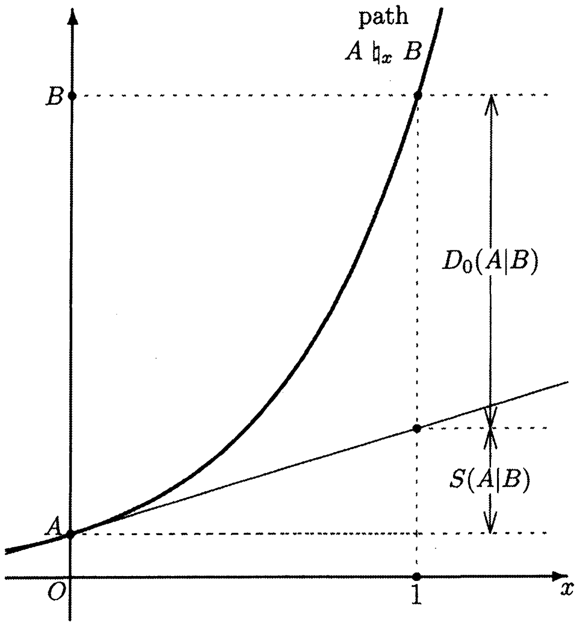

Figure 1: An interpretation of $D_0(A|B)$.

Fujii [1] defined operator valued α-divergence $D_\alpha(A|B)$ for $\alpha \in (0, 1)$ as follows:

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \natural_\alpha B}{\alpha(1-\alpha)},$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is the weighted arithmetic operator mean. The operator valued α-divergence has the following relations at end points for interval $(0, 1)$.

Theorem A ([5, 6]). *For strictly positive operators A and B, the following hold:*

$$\begin{aligned} D_0(A|B) &\equiv \lim_{\alpha \to +0} D_\alpha(A|B) &= B - A - S(A|B), \\ D_1(A|B) &\equiv \lim_{\alpha \to 1-0} D_\alpha(A|B) &= A - B - S(B|A). \end{aligned}$$

Petz [17] introduced the right hand side in the first equation in Theorem A as an operator divergence, so we call $D_0(A|B)$ *Petz-Bregman divergence.* We remark that $D_1(A|B) = D_0(B|A)$ holds. Figure 1 shows our interpretation of $D_0(A|B)$.

In [10], we showed the following relation between operator valued α-divergence and Tsallis relative operator entropy:

Theorem B ([10]). *For strictly positive operators A and B, the following holds:*

$$D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B) \quad \textit{for } \alpha \in (0, 1).$$

Theorem B shows that $D_\alpha(A|B)$ is a difference between two terms in $(*)$. From this fact, we regard the differences between the relative operator entropies in $(*)$ as operator divergences. In section 2, we represent these operator divergences by using Petz-Bregman divergence.

On the other hand, for an operator valued smooth function $\Psi : \mathbf{C} \to B(H)$ and $X, Y \in \mathbf{C}$, where $\mathbf{C}$ is a convex set in a Banach space, Petz [17] defined a divergence

$D_\Psi(X,Y)$ as follows:

$$D_\Psi(X,Y) \equiv \Psi(X) - \Psi(Y) - \lim_{\alpha \to +0} \frac{\Psi(Y + \alpha(X-Y)) - \Psi(Y)}{\alpha}.$$

We call $D_\Psi(X,Y)$ *Ψ-Bregman divergence* of Y and X in this report. Petz gave some examples for invertible density matrices X and Y. If $\Psi(X) = \eta(X) \equiv X \log X$ and X commutes with Y, then $D_\Psi(X,Y) = Y - X + X(\log X - \log Y)$, and if $\Psi(X) = tr\ \eta(X)$, then $D_\Psi(X,Y) = tr\ X(\log X - \log Y)$, which is the usual quantum relative entropy.

In section 3, we let $\mathbf{C} = \mathbb{R}$ and show $D_\Psi(x,y) = D_0(A \natural_y B | A \natural_x B)$ for $\Psi(t) = A \natural_t B$ and $x, y \in \mathbb{R}$. Then we have $D_\Psi(1,0) = D_0(A|B)$ in particular. Based on this interpretation, we discuss Ψ-Bregman divergences $D_\Psi(1,0)$ for several functions Ψ which relate to the operator divergences given in section 2.

In section 4, we show the results corresponding to those in section 2 on expanded relative operator entropies defined by operator power mean.

2. Divergences given by the differences among relative operator entropies

In this section, we regard the differences between the relative operator entropies in $(*)$ as operator divergences. There are 10 such divergences. For convenience, we use symbols Δ_i for them as follows:

$$\begin{array}{ll}
\Delta_1 = T_\alpha(A|B) - S(A|B), & \Delta_2 = S_\alpha(A|B) - T_\alpha(A|B), \\
\Delta_3 = -T_{1-\alpha}(B|A) - S_\alpha(A|B), & \Delta_4 = S_1(A|B) + T_{1-\alpha}(B|A), \\
\Delta_5 = S_\alpha(A|B) - S(A|B), & \Delta_6 = -T_{1-\alpha}(B|A) - T_\alpha(A|B) = D_\alpha(A|B), \\
\Delta_7 = S_1(A|B) - S_\alpha(A|B), & \Delta_8 = -T_{1-\alpha}(B|A) - S(A|B), \\
\Delta_9 = S_1(A|B) - T_\alpha(A|B), & \Delta_{10} = S_1(A|B) - S(A|B).
\end{array}$$

We represent each of $\Delta_1, \cdots, \Delta_{10}$ by using Petz-Bregman divergence. It is sufficient to consider Δ_1, Δ_2, Δ_3 and Δ_4 since the following relations hold:

$$\begin{array}{lll}
\Delta_5 = \Delta_1 + \Delta_2, & \Delta_6 = \Delta_2 + \Delta_3, & \Delta_7 = \Delta_3 + \Delta_4, \\
\Delta_8 = \Delta_1 + \Delta_2 + \Delta_3, & \Delta_9 = \Delta_2 + \Delta_3 + \Delta_4, & \Delta_{10} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4.
\end{array}$$

The relation of the differences among $\Delta_1, \cdots, \Delta_{10}$ are given as in Table 1.

The next lemma is essential in our discussion.

Lemma 2.1 ([8, 10]). *For strictly positive operators A and B, the following hold for $s, t \in \mathbb{R}$:*

(1) $$S_t(A|A \natural_s B) = sS_{st}(A|B),$$

(2) $$S_t(A|B) = -S_{1-t}(B|A).$$

Table 1

$$
\begin{array}{lllllll}
S_1(A|B)-S(A|B) & \geq & S_1(A|B)-T_\alpha(A|B) & \geq & S_1(A|B)-S_\alpha(A|B) & \geq & S_1(A|B)+T_{1-\alpha}(B|A) \geq 0 \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
-T_{1-\alpha}(B|A)-S(A|B) & \geq & -T_{1-\alpha}(B|A)-T_\alpha(A|B) & \geq & -T_{1-\alpha}(B|A)-S_\alpha(A|B) \geq 0 & & \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
S_\alpha(A|B)-S(A|B) & \geq & S_\alpha(A|B)-T_\alpha(A|B) & & & & \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
T_\alpha(A|B)-S(A|B) & & 0 & & & & \\
\rotatebox{90}{$\leq$} & & & & & & \\
0 & & & & & &
\end{array}
$$

The following are the results on Δ_1 and Δ_2.

Theorem 2.2. *For strictly positive operators A and B, the following hold:*

$$
\begin{array}{llllll}
(1) & \Delta_1 & = & T_\alpha(A|B)-S(A|B) & = & \dfrac{1}{\alpha}D_0(A|A \natural_\alpha B) \quad \textit{for } \alpha \in (0,1], \\
(2) & \Delta_2 & = & S_\alpha(A|B)-T_\alpha(A|B) & = & \dfrac{1}{\alpha}D_0(A \natural_\alpha B|A) \quad \textit{for } \alpha \in (0,1].
\end{array}
$$

Proof. (1) By (1) in Lemma 2.1, we have

$$
\begin{aligned}
T_\alpha(A|B)-S(A|B) &= \frac{A \natural_\alpha B - A}{\alpha} - S(A|B) = \frac{1}{\alpha}(A \natural_\alpha B - A - \alpha S(A|B)) \\
&= \frac{1}{\alpha}(A \natural_\alpha B - A - S(A|A \natural_\alpha B)) = \frac{1}{\alpha}D_0(A|A \natural_\alpha B).
\end{aligned}
$$

(2) By Lemma 2.1, we have

$$
\begin{aligned}
S_\alpha(A|B)-T_\alpha(A|B) &= \frac{A - A \natural_\alpha B}{\alpha} + S_\alpha(A|B) = \frac{1}{\alpha}(A - A \natural_\alpha B + \alpha S_\alpha(A|B)) \\
&= \frac{1}{\alpha}(A - A \natural_\alpha B + S_1(A|A \natural_\alpha B)) \\
&= \frac{1}{\alpha}(A - A \natural_\alpha B - S(A \natural_\alpha B|A)) = \frac{1}{\alpha}D_0(A \natural_\alpha B|A).
\end{aligned}
$$

□

Remark 1. *By exchanging A for B and replacing α with $1-\alpha$ for (1) and (2), we have the following relations:*

$$
\begin{array}{llllll}
(1) & \Delta_4 & = & S_1(A|B)+T_{1-\alpha}(B|A) & = & \dfrac{1}{1-\alpha}D_0(B|A \natural_\alpha B) \quad \textit{for } \alpha \in [0,1), \\
(2) & \Delta_3 & = & -T_{1-\alpha}(B|A)-S_\alpha(A|B) & = & \dfrac{1}{1-\alpha}D_0(A \natural_\alpha B|B) \quad \textit{for } \alpha \in [0,1).
\end{array}
$$

We give a geometrical interpretation for (1) in Theorem 2.2. Figure 2 and Figure 3 show $T_\alpha(A|B)-S(A|B)$ and $D_0(A|A \sharp_\alpha B)$ appeared in (1) in Theorem 2.2, respectively. Figure 4 is an image of (1) in Theorem 2.2.

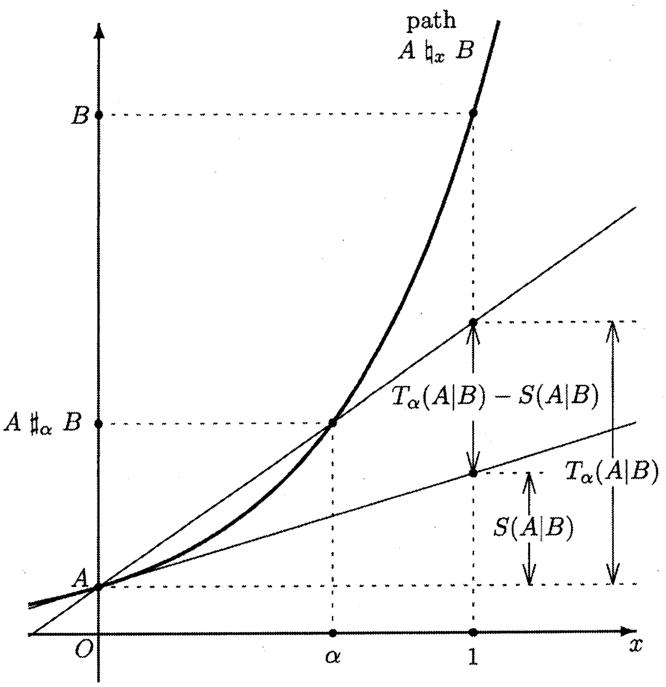

Figure 2: An interpretation of $T_\alpha(A|B) - S(A|B)$.

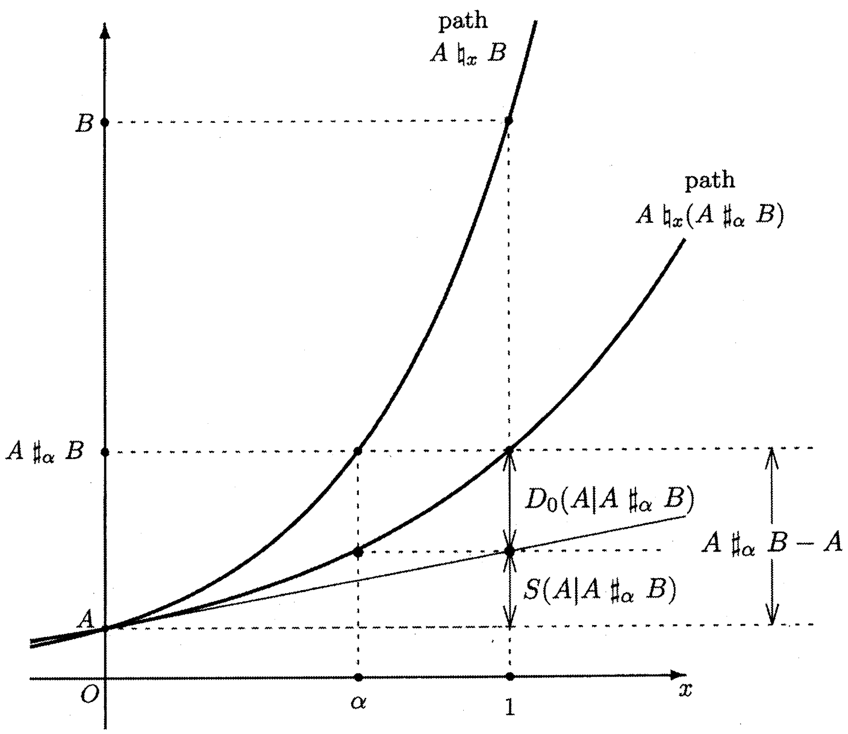

Figure 3: An interpretation of $D_0(A|A \sharp_\alpha B) = A \sharp_\alpha B - A - S(A|A \sharp_\alpha B)$.

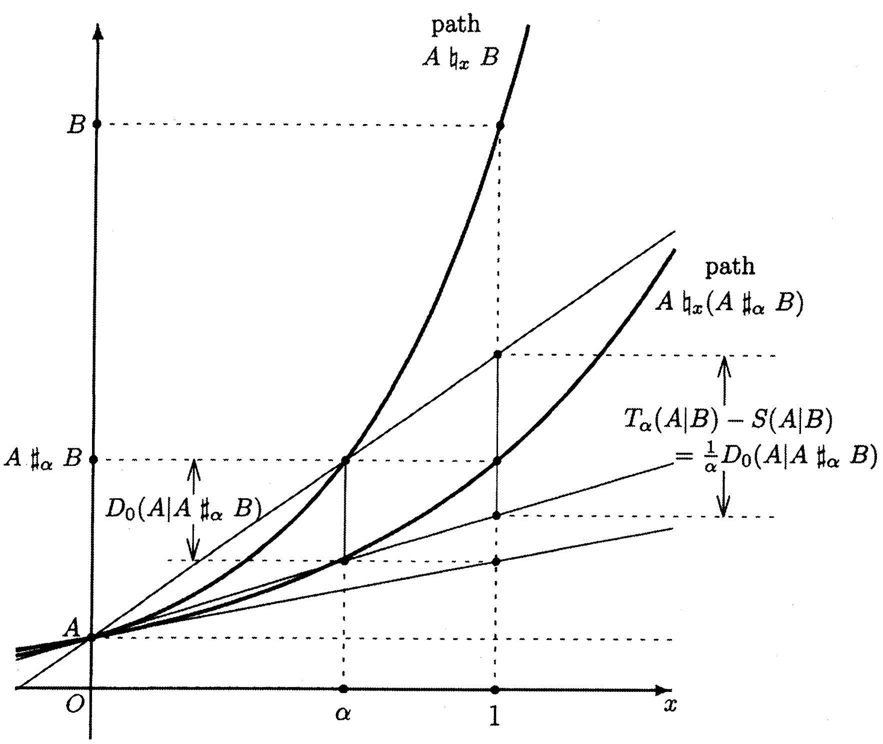

Figure 4: An image of $T_\alpha(A|B) - S(A|B) = \frac{1}{\alpha} D_0(A|A \natural_\alpha B)$.

Theorem 2.2 leads the next theorem.

Theorem 2.3. *For strictly positive operators A and B, the following holds:*

$$D_\alpha(A|B) = \frac{1}{1-\alpha} D_0(A \sharp_\alpha B|B) + \frac{1}{\alpha} D_0(A \sharp_\alpha B|A) \quad \textit{for } \alpha \in (0,1).$$

Proof. By (2) in Theorem 2.2 and (2) in Remark 1, we have

$$\begin{aligned} D_\alpha(A|B) &= -T_{1-\alpha}(B|A) - T_\alpha(A|B) \\ &= \big(-T_{1-\alpha}(B|A) - S_\alpha(A|B)\big) + \big(S_\alpha(A|B) - T_\alpha(A|B)\big) \\ &= \frac{1}{1-\alpha} D_0(A \sharp_\alpha B|B) + \frac{1}{\alpha} D_0(A \sharp_\alpha B|A). \end{aligned}$$

□

By Theorem 2.3, we have

$$\begin{aligned} \alpha(1-\alpha)D_\alpha(A|B) &= \alpha D_0(A \sharp_\alpha B|B) + (1-\alpha) D_0(A \sharp_\alpha B|A) \\ &= \alpha\big(B - A \sharp_\alpha B - S(A \sharp_\alpha B|B)\big) + (1-\alpha)\big(A - A \sharp_\alpha B - S(A \sharp_\alpha B|A)\big) \\ &= A \nabla_\alpha B - A \sharp_\alpha B - \big((1-\alpha)S(A \sharp_\alpha B|A) + \alpha S(A \sharp_\alpha B|B)\big), \end{aligned}$$

and then

$$(1-\alpha)S(A \sharp_\alpha B|A) + \alpha S(A \sharp_\alpha B|B) = 0,$$

since $D_\alpha(A|B) = \dfrac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)}$. This means that $A \sharp_\alpha B$ is a solution of $(1-\alpha)S(X|A) + \alpha S(X|B) = 0$ which is the Karcher equation concerning two operators A and B. In this case, we can rewrite the result of Lawson-Lim [16] as follows:

Theorem 2.4 ([16]). *For strictly positive operators A, B and X, and for $\alpha \in [0,1]$,*

$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0 \quad \textit{if and only if } X = A \sharp_\alpha B.$$

For readers' convenience, we give a direct proof of this theorem.

Proof. It is obvious if $\alpha = 0$. Otherwise, we have

$$\begin{aligned}
& (1-\alpha)S(X|A) + \alpha S(X|B) = 0 \\
\Longleftrightarrow \quad & \log(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^{1-\alpha} + \log(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{\alpha} = 0 \\
\Longleftrightarrow \quad & (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{\alpha} = (X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{1-\alpha} \\
\Longleftrightarrow \quad & X^{-\frac{1}{2}}BX^{-\frac{1}{2}} = (X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{\frac{1}{\alpha}-1} \\
\Longleftrightarrow \quad & B = X^{\frac{1}{2}}(X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{\frac{1}{\alpha}}X^{-\frac{1}{2}}A \\
\Longleftrightarrow \quad & A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = A^{-\frac{1}{2}}X^{\frac{1}{2}}(X^{\frac{1}{2}}A^{-\frac{1}{2}}A^{-\frac{1}{2}}X^{\frac{1}{2}})^{\frac{1}{\alpha}}X^{-\frac{1}{2}}A^{\frac{1}{2}} \\
\Longleftrightarrow \quad & A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{\frac{1}{\alpha}} \\
\Longleftrightarrow \quad & (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha} = A^{-\frac{1}{2}}XA^{-\frac{1}{2}} \\
\Longleftrightarrow \quad & X = A \sharp_\alpha B.
\end{aligned}$$

□

Remark 2. *This theorem holds even if α is any real number* [14].

3. Ψ-Bregman divergences on the differences of relative operator entropies

In this section, we consider Ψ-Bregman divergence in the case $\mathbf{C} = \mathbb{R}$ as follows: For an operator valued smooth function $\Psi : \mathbb{R} \to B(H)$ and $x, y \in \mathbb{R}$,

$$D_\Psi(x,y) \equiv \Psi(x) - \Psi(y) - \lim_{\alpha \to +0} \frac{\Psi(y + \alpha(x-y)) - \Psi(y)}{\alpha}.$$

From the following theorem, it is natural that we consider $D_\Psi(1,0)$ as a divergence of operators A and B.

Theorem 3.1. *Let $\Psi(t) = A \natural_t B$ for strictly positive operators A and B. Then for $x, y \in \mathbb{R}$,*

$$D_\Psi(x,y) = D_0(A \natural_y B | A \natural_x B).$$

In particular, $D_\Psi(1,0) = D_0(A|B)$.

Proof.

$$\begin{aligned}
D_\Psi(x,y) &= A \natural_x B - A \natural_y B - \lim_{\alpha \to +0} \frac{A \natural_{y+\alpha(x-y)} B - A \natural_y B}{\alpha} \\
&= A \natural_x B - A \natural_y B - \lim_{\alpha \to +0} \frac{(A \natural_y B) \natural_\alpha (A \natural_x B) - A \natural_y B}{\alpha} \quad \text{by [11, Lemma 2.2]} \\
&= A \natural_x B - A \natural_y B - S(A \natural_y B | A \natural_x B) = D_0(A \natural_y B | A \natural_x B).
\end{aligned}$$

□

In the rest of this section, we obtain $D_\Psi(1,0)$ for functions Ψ which relate to the operator divergences Δ_1, Δ_2, Δ_5 and Δ_6 in section 2.

Theorem 3.2. *For strictly positive operators A and B, the following hold:*

(1) *If* $\Psi(t) = T_t(A|B) - S(A|B)$, *then*

$$D_\Psi(1,0) = D_0(A|B) - \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

(2) *If* $\Psi(t) = S_t(A|B) - S(A|B)$, *then*

$$D_\Psi(1,0) = D_0(A|B) + D_0(B|A) - S(A|B)A^{-1}S(A|B).$$

(3) *If* $\Psi(t) = S_t(A|B) - T_t(A|B)$, *then*

$$D_\Psi(1,0) = D_0(B|A) - \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

(4) *If* $\Psi(t) = D_t(A|B)$ *for* $t \in [0,1]$, *then*

$$D_\Psi(1,0) = D_0(B|A) - 2D_0(A|B) + \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

Here, we give a proof of (1). The others are obtained similarly.

Proof. (1) For $a > 0$, we have

$$\lim_{\alpha \to +0} \frac{a^\alpha - 1 - \alpha \log a}{\alpha^2} = \frac{1}{2}(\log a)^2.$$

Replacing a by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\begin{aligned}\lim_{\alpha \to +0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha} &= \lim_{\alpha \to +0} \frac{A^{\frac{1}{2}}\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha - I - \alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)A^{\frac{1}{2}}}{\alpha^2} \\ &= \frac{1}{2}A^{\frac{1}{2}}\left(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)^2 A^{\frac{1}{2}} = \frac{1}{2}S(A|B)A^{-1}S(A|B),\end{aligned}$$

then

$$\begin{aligned}D_\Psi(1,0) &= T_1(A|B) - S(A|B) - \left(T_0(A|B) - S(A|B)\right) \\ &\quad - \lim_{\alpha \to +0} \frac{T_\alpha(A|B) - S(A|B) - \left(T_0(A|B) - S(A|B)\right)}{\alpha} \\ &= T_1(A|B) - S(A|B) - \lim_{\alpha \to +0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha} \\ &= D_0(A|B) - \frac{1}{2}S(A|B)A^{-1}S(A|B).\end{aligned}$$

□

4. Divergences given by the differences of expanded relative operator entropies

In this section, we try to generalize Theorems 2.2 and 2.3 in section 2 for operator power mean. For $A, B > 0$, $x \in [0,1]$ and $r \in [-1,1]$, operator power mean $A \,\sharp_{x,r}\, B$ is defined as follows:

$$A \,\sharp_{x,r}\, B \equiv A^{\frac{1}{2}} \left\{(1-x)I + x\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}} A^{\frac{1}{2}} \;=\; A \,\natural_{\frac{1}{r}}\, \{A \,\nabla_x\, (A \,\natural_r\, B)\}.$$

We remark that $A \,\sharp_{x,r}\, B = B \,\sharp_{1-x,r}\, A$ holds for $x \in [0,1]$ and $r \in [-1,1]$ (cf. [9], [11]). To preserve $(1-x)I + x\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r \geq 0$, we have to impose x in $[0,1]$. The operator power mean is a path passing through $A = A \,\sharp_{0,r}\, B$ and $B = A \,\sharp_{1,r}\, B$, and combines arithmetic, geometric and harmonic means, that is, $A \,\sharp_{x,1}\, B = A \,\nabla_x\, B$, $A \,\sharp_{x,0}\, B \equiv \lim_{r\to 0} A \,\sharp_{x,r}\, B = A \,\sharp_x\, B$ and $A \,\sharp_{x,-1}\, B = A \,\Delta_x\, B = (A^{-1} \,\nabla_x\, B^{-1})^{-1}$. For $\alpha \in [0,1]$ and $r \in [-1,1]$, expanded relative operator entropy $S_{\alpha,r}(A|B)$ and expanded Tsallis relative operator entropy $T_{\alpha,r}(A|B)$ are defined as follows (cf. [9]):

$$\begin{aligned}
S_{\alpha,r}(A|B) &\equiv A^{\frac{1}{2}}\left(\left\{(1-\alpha)I + \alpha\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}-1} \frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}\right)A^{\frac{1}{2}} \\
&= \frac{d}{dx}\, A \,\sharp_{x,r}\, B\Big|_{x=\alpha} = (A \,\sharp_{\alpha,r}\, B)\big(A \,\nabla_\alpha (A \,\natural_r\, B)\big)^{-1} S_{0,r}(A|B) \ (r \neq 0), \\
S_{\alpha,0}(A|B) &\equiv \lim_{r\to 0} S_{\alpha,r}(A|B) = S_\alpha(A|B), \\
T_{\alpha,r}(A|B) &\equiv \frac{A \,\sharp_{\alpha,r}\, B - A}{\alpha}\ (\alpha \neq 0),\ T_{0,r}(A|B) \equiv \lim_{\alpha\to 0} T_{\alpha,r}(A|B) = T_r(A|B).
\end{aligned}$$

We remark that $S_{0,r}(A|B) = T_r(A|B)$, $S_{1,r}(A|B) = -T_r(B|A)$ and $T_{1,r}(A|B) = B - A$ hold for $r \in [-1,1]$. A similar inequality to $(*)$ also holds for these expanded relative operator entropies, which is given as follows [9]:

$$(**) \quad S_{0,r}(A|B) \leq T_{\alpha,r}(A|B) \leq S_{\alpha,r}(A|B) \leq -T_{1-\alpha,r}(B|A) \leq S_{1,r}(A|B), \qquad \alpha \in [0,1],\ r \in [-1,1].$$

If $r = 0$, then this inequality becomes $(*)$.

We defined expanded operator valued α-divergence $D_{\alpha,r}(A|B)$ as follows [11]:

$$D_{\alpha,r}(A|B) \equiv -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B), \quad \alpha \in [0,1],\ r \in [-1,1].$$

For $\alpha \in (0,1)$, we can represent $D_{\alpha,r}(A|B)$ as follows:

$$D_{\alpha,r}(A|B) = \frac{A \,\nabla_\alpha\, B - A \,\sharp_{\alpha,r}\, B}{\alpha(1-\alpha)}.$$

We gave the following relations on expanded operator valued α-divergence.

Proposition 4.1 ([11], Proposition 4.4). *For strictly positive operators A and B, the following hold:*

$$\begin{aligned}
&(1) \quad D_{\alpha,0}(A|B) = D_\alpha(A|B) \ \textit{for}\ \alpha \in [0,1], \\
&(2) \quad D_{\alpha,1}(A|B) = 0 \ \textit{for}\ \alpha \in [0,1], \\
&(3) \quad D_{0,r}(A|B) = B - A - S_{0,r}(A|B) \ \textit{for}\ r \in [-1,1], \\
&(4) \quad D_{1,r}(A|B) = A - B - S_{0,r}(B|A) = D_{0,r}(B|A) \ \textit{for}\ r \in [-1,1].
\end{aligned}$$

We call $D_{0,r}(A|B) = B - A - S_{0,r}(A|B)$ expanded Petz-Bregman divergence.

Similarly to section 2, we consider the differences between two expanded relative operator entropies in $(**)$ as operator divergence. There are 10 such divergences.

$$
\begin{array}{ll}
T_{\alpha,r}(A|B) - S_{0,r}(A|B), & S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B), \\
-T_{1-\alpha,r}(B|A) - S_{\alpha,r}(A|B), & S_{1,r}(A|B) + T_{1-\alpha,r}(B|A), \\
S_{\alpha,r}(A|B) - S_{0,r}(A|B), & -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B) = D_{\alpha,r}(A|B), \\
S_{1,r}(A|B) - S_{\alpha,r}(A|B), & -T_{1-\alpha,r}(B|A) - S_{0,r}(A|B), \\
S_{1,r}(A|B) - T_{\alpha,r}(A|B), & S_{1,r}(A|B) - S_{0,r}(A|B).
\end{array}
$$

The relations of these differences are given as in Table 2. If $r = 0$, then this table coincides with Table 1.

Table 2

$$
\begin{array}{ccccccc}
S_{1,r}(A|B) - S_{0,r}(A|B) & \geq & S_{1,r}(A|B) - T_{\alpha,r}(A|B) & \geq & S_{1,r}(A|B) - S_{\alpha,r}(A|B) & \geq & S_{1,r}(A|B) + T_{1-\alpha,r}(B|A) \geq 0 \\
\mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} & & \\
-T_{1-\alpha,r}(B|A) - S_{0,r}(A|B) & \geq & -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B) & \geq & -T_{1-\alpha,r}(B|A) - S_{\alpha,r}(A|B) \geq 0 & & \\
\mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} & & & & \\
S_{\alpha,r}(A|B) - S_{0,r}(A|B) & \geq & S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B) & & & & \\
\mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} & & & & \\
T_{\alpha,r}(A|B) - S_{0,r}(A|B) & & 0 & & & & \\
\mathrel{\rotatebox{90}{$\geq$}} & & & & & & \\
0 & & & & & &
\end{array}
$$

We represent these operator divergences by using expanded Petz-Bregman divergence. Theorems 4.2 and 4.3 correspond to Theorems 2.2 and 2.3, respectively.

Theorem 4.2. *For strictly positive operators A and B and $r \in [-1, 1]$, the following hold:*

$$
\begin{array}{llcl}
(1) & T_{\alpha,r}(A|B) - S_{0,r}(A|B) & = & \frac{1}{\alpha} D_{0,r}(A|A \natural_{\alpha,r} B) \quad \text{for } \alpha \in (0, 1], \\
(2) & S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B) & = & \frac{1}{\alpha} D_{0,r}(A \natural_{\alpha,r} B|A) \quad \text{for } \alpha \in (0, 1].
\end{array}
$$

Theorem 4.3. *For strictly positive operators A and B and $r \in [-1, 1]$, the following holds:*

$$
D_{\alpha,r}(A|B) = \frac{1}{1-\alpha} D_{0,r}(A \natural_{\alpha,r} B|B) + \frac{1}{\alpha} D_{0,r}(A \natural_{\alpha,r} B|A) \quad \text{for } \alpha \in (0, 1).
$$

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