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<td>テーマ</td>
<td>いくつかのオペレータの収束性についての研究。不可積分のオペレータの収束性を検討し、その性質を明らかにした。</td>
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<td>著者群</td>
<td>伊佐 誠史、伊藤 公智、亀井 榕三郎、遠山 宏明、渡邉 雅之</td>
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<tr>
<td>引用</td>
<td>数理解析研究所講究録 1996: 1-12</td>
</tr>
<tr>
<td>発行日</td>
<td>2016-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/224730">http://hdl.handle.net/2433/224730</a></td>
</tr>
<tr>
<td>種別</td>
<td>部門別論文</td>
</tr>
<tr>
<td>出版者</td>
<td>京都大学</td>
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Some operator divergences based on Petz-Bregman divergence

1. Introduction

This report is based on [12]. Throughout this report, a bounded linear operator $T$ on a Hilbert space $H$ is positive (denoted by $T \geq 0$) if $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$, and $T$ is strictly positive (denoted by $T > 0$) if $T$ is invertible and positive.

For strictly positive operators $A$ and $B$, $A \natural_{x} B$ is defined as follows ([3, 4, 13] etc.):

$$A \natural_{x} B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{x}A^{\frac{1}{2}}, \quad x \in \mathbb{R}.$$  

We call $A \natural_{x} B$ a path passing through $A = A \natural_{0} B$ and $B = A \natural_{1} B$. If $x \in [0, 1]$, the path $A \natural_{x} B$ coincides with the weighted geometric operator mean denoted by $A \#_{x} B$ (cf. [15]). We remark that $A \natural_{x} B = B \natural_{1-x} A$ holds for $x \in \mathbb{R}$.

Fujii and Kamei [2] defined the following relative operator entropy for strictly positive operators $A$ and $B$:

$$S(A|B) \equiv A^{\frac{1}{2}} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = \frac{d}{dx} A \natural_{x} B \big|_{x=0}.$$  

We can regard $S(A|B)$ as the gradient of the tangent line at $x = 0$ of the path $A \natural_{x} B$. Furuta [7] defined generalized relative operator entropy as follows:

$$S_{\alpha}(A|B) \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = \frac{d}{dx} A \natural_{x} B \big|_{x=\alpha}, \quad \alpha \in \mathbb{R}.$$  

We regard $S_{\alpha}(A|B)$ as the gradient of the tangent line at $x = \alpha$ of the path. We know immediately $S_{0}(A|B) = S(A|B)$. Yanagi, Kuriyama and Furutichi [18] introduced Tsallis relative operator entropy as follows:

$$T_{\alpha}(A|B) \equiv \frac{A \#_{\alpha} B - A}{\alpha}, \quad \alpha \in (0, 1].$$  

$T_{\alpha}(A|B)$ can be regarded as the average rate of change of $A \#_{x} B$ from $x = 0$ to $x = \alpha$. Since $\lim_{x \to 0} \frac{a^{x} - 1}{x} = \log a$ holds for $a > 0$, we have $T_{0}(A|B) \equiv \lim_{\alpha \to 0} T_{\alpha}(A|B) = S(A|B)$. Tsallis relative operator entropy can be extended as the notion for $\alpha \in \mathbb{R}$. In this case, we use $\natural_{\alpha}$ instead of $\#_{\alpha}$. In [8], we had given the following relations among these relative operator entropies:

$$S_{1}(A|B) \geq -T_{1-\alpha}(B|A) \geq S_{\alpha}(A|B) \geq T_{\alpha}(A|B) \geq S(A|B), \quad \alpha \in (0, 1).$$
Fujii [1] defined operator valued \(\alpha\)-divergence \(D_\alpha(A|B)\) for \(\alpha \in (0, 1)\) as follows:

\[
D_\alpha(A|B) \equiv \frac{A \triangledown_\alpha B - A \#_\alpha B}{\alpha(1-\alpha)},
\]

where \(A \triangledown_\alpha B \equiv (1-\alpha)A + \alpha B\) is the weighted arithmetic operator mean. The operator valued \(\alpha\)-divergence has the following relations at end points for interval \((0, 1)\).

**Theorem A** ([5, 6]). For strictly positive operators \(A\) and \(B\), the following hold:

\[
D_0(A|B) \equiv \lim_{\alpha \to +0} D_\alpha(A|B) = B - A - S(A|B),
\]

\[
D_1(A|B) \equiv \lim_{\alpha \to 1-0} D_\alpha(A|B) = A - B - S(B|A).
\]

Petz [17] introduced the right hand side in the first equation in Theorem A as an operator divergence, so we call \(D_0(A|B)\) Petz-Bregman divergence. We remark that \(D_1(A|B) = D_0(B|A)\) holds. Figure 1 shows our interpretation of \(D_0(A|B)\).

In [10], we showed the following relation between operator valued \(\alpha\)-divergence and Tsallis relative operator entropy:

**Theorem B** ([10]). For strictly positive operators \(A\) and \(B\), the following holds:

\[
D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B) \quad \text{for} \ \alpha \in (0, 1).
\]

Theorem B shows that \(D_\alpha(A|B)\) is a difference between two terms in \((*)\). From this fact, we regard the differences between the relative operator entropies in \((*)\) as operator divergences. In section 2, we represent these operator divergences by using Petz-Bregman divergence.

On the other hand, for an operator valued smooth function \(\Psi : C \to B(H)\) and \(X, Y \in C\), where \(C\) is a convex set in a Banach space, Petz [17] defined a divergence.
\[ D_\Psi(X, Y) \] as follows:

\[ D_\Psi(X, Y) \equiv \Psi(X) - \Psi(Y) - \lim_{\alpha \to +0} \frac{\Psi(Y + \alpha(X - Y)) - \Psi(Y)}{\alpha}. \]

We call \( D_\Psi(X, Y) \) a \( \Psi \)-Bregman divergence of \( Y \) and \( X \) in this report. Petz gave some examples for invertible density matrices \( X \) and \( Y \). If \( \Psi(X) = \eta(X) \equiv X \log X \) and \( X \) commutes with \( Y \), then \( D_\Psi(X, Y) = Y - X + X(\log X - \log Y) \), which is the usual quantum relative entropy.

In section 3, we let \( C = \mathbb{R} \) and show \( D_\Psi(x, y) = D_0(A \natural_v B \mid A \natural_x B) \) for \( \Psi(t) = A \natural_t B \) and \( x, y \in \mathbb{R} \). Then we have \( D_\Psi(1, 0) = D_0(A \mid B) \) in particular. Based on this interpretation, we discuss \( \Psi \)-Bregman divergences \( D_\Psi(1, 0) \) for several functions \( \Psi \) which relate to the operator divergences given in section 2.

In section 4, we show the results corresponding to those in section 2 on expanded relative operator entropies defined by operator power mean.

### 2. Divergences given by the differences among relative operator entropies

In this section, we regard the differences between the relative operator entropies in (*) as operator divergences. There are 10 such divergences. For convenience, we use symbols \( \Delta_i \) for them as follows:

- \( \Delta_1 = T_\alpha(A \mid B) - S(A \mid B) \)
- \( \Delta_2 = S_\alpha(A \mid B) - T_\alpha(A \mid B) \)
- \( \Delta_3 = -T_{1-\alpha}(B \mid A) - S_\alpha(A \mid B) \)
- \( \Delta_4 = S_1(A \mid B) + T_{1-\alpha}(B \mid A) \)
- \( \Delta_5 = S_\alpha(A \mid B) - S(A \mid B) \)
- \( \Delta_6 = -T_{1-\alpha}(B \mid A) - T_\alpha(A \mid B) = D_\alpha(A \mid B) \)
- \( \Delta_7 = S_1(A \mid B) - S_\alpha(A \mid B) \)
- \( \Delta_8 = -T_{1-\alpha}(B \mid A) - S(A \mid B) \)
- \( \Delta_9 = S_1(A \mid B) - T_\alpha(A \mid B) \)
- \( \Delta_{10} = S_1(A \mid B) - S(A \mid B) \)

We represent each of \( \Delta_1, \cdots, \Delta_{10} \) by using Petz-Bregman divergence. It is sufficient to consider \( \Delta_1, \Delta_2, \Delta_3 \) and \( \Delta_4 \) since the following relations hold:

\[ \Delta_5 = \Delta_1 + \Delta_2, \quad \Delta_6 = \Delta_2 + \Delta_3, \quad \Delta_7 = \Delta_3 + \Delta_4, \quad \Delta_8 = \Delta_1 + \Delta_2 + \Delta_3, \quad \Delta_9 = \Delta_2 + \Delta_3 + \Delta_4, \quad \Delta_{10} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4. \]

The relation of the differences among \( \Delta_1, \cdots, \Delta_{10} \) are given as in Table 1.

The next lemma is essential in our discussion.

**Lemma 2.1** ([8, 10]). For strictly positive operators \( A \) and \( B \), the following hold for \( s, t \in \mathbb{R} \):

1. \( S_t(A \mid A \natural_s B) = sS_{st}(A \mid B) \),
2. \( S_t(A \mid B) = -S_{1-t}(B \mid A) \).
The following are the results on $\Delta_1$ and $\Delta_2$.

**Theorem 2.2.** For strictly positive operators $A$ and $B$, the following hold:

(1) $\Delta_1 = T_\alpha(A|B) - S(A|B) = \frac{1}{\alpha}D_0(A|A \#_\alpha B)$ for $\alpha \in (0, 1]$,

(2) $\Delta_2 = S_\alpha(A|B) - T_\alpha(A|B) = \frac{1}{\alpha}D_0(A \#_\alpha B|A)$ for $\alpha \in (0, 1]$.

**Proof.** (1) By (1) in Lemma 2.1, we have

$$T_\alpha(A|B) - S(A|B) = \frac{A \#_\alpha B - A}{\alpha} - S(A|B) = \frac{1}{\alpha}(A \#_\alpha B - A - \alpha S(A|B)) = \frac{1}{\alpha}D_0(A|A \#_\alpha B).$$

(2) By Lemma 2.1, we have

$$S_\alpha(A|B) - T_\alpha(A|B) = \frac{A - A \#_\alpha B}{\alpha} + S_\alpha(A|B) = \frac{1}{\alpha}(A - A \#_\alpha B + \alpha S_\alpha(A|B)) = \frac{1}{\alpha}D_0(A \#_\alpha B|A).$$

**Remark 1.** By exchanging $A$ for $B$ and replacing $\alpha$ with $1 - \alpha$ for (1) and (2), we have the following relations:

(1) $\Delta_4 = S_1(A|B) + T_{1-\alpha}(B|A) = \frac{1}{1-\alpha}D_0(B|A \#_\alpha B)$ for $\alpha \in [0, 1)$,

(2) $\Delta_3 = -T_{1-\alpha}(B|A) - S_\alpha(A|B) = \frac{1}{1-\alpha}D_0(A \#_\alpha B|B)$ for $\alpha \in [0, 1)$. 

---

**Table 1**

| $S_1(A|B) - S(A|B)$ | $\geq S_1(A|B) - T_\alpha(A|B)$ | $\geq S_1(A|B) - S_\alpha(A|B)$ | $\geq S_1(A|B) + T_{1-\alpha}(B|A)$ | $\geq 0$ |
|---------------------|-------------------------------|-------------------------------|-------------------------------|---------|
| $\forall$           | $\forall$                      | $\forall$                      | $\forall$                      |         |
| $-T_{1-\alpha}(B|A) - S(A|B)$ | $\geq -T_{1-\alpha}(B|A) - T_\alpha(A|B)$ | $\geq -T_{1-\alpha}(B|A) - S_\alpha(A|B)$ | $\geq 0$ |         |
| $\forall$           | $\forall$                      | $\forall$                      | $\forall$                      |         |
| $S_\alpha(A|B) - S(A|B)$ | $\geq S_\alpha(A|B) - T_\alpha(A|B)$ | $\geq S_\alpha(A|B) - S_\alpha(A|B)$ | $\geq 0$ |         |
| $\forall$           | $\forall$                      | $\forall$                      | $\forall$                      |         |
| $T_\alpha(A|B) - S(A|B)$ | $\geq 0$                      |                                 |                                 |         |
| $\forall$           |                                 |                                 |                                 |         |
We give a geometrical interpretation for (1) in Theorem 2.2. Figure 2 and Figure 3 show $T_\alpha(A|B) - S(A|B)$ and $D_0(A|A \#_\alpha B)$ appeared in (1) in Theorem 2.2, respectively. Figure 4 is an image of (1) in Theorem 2.2.

Figure 2: An interpretation of $T_\alpha(A|B) - S(A|B)$.

Figure 3: An interpretation of $D_0(A|A \#_\alpha B) = A \#_\alpha B - A - S(A|A \#_\alpha B)$. 
Theorem 2.2 leads the next theorem.

**Theorem 2.3.** For strictly positive operators \( A \) and \( B \), the following holds:

\[
D_{\alpha}(A|B) = \frac{1}{1-\alpha}D_{0}(A \#_{\alpha} B|B) + \frac{1}{\alpha}D_{0}(A \#_{\alpha} B|A)
\]

for \( \alpha \in (0, 1) \).

**Proof.** By (2) in Theorem 2.2 and (2) in Remark 1, we have

\[
D_{\alpha}(A|B) = -T_{1-\alpha}(B|A) - T_{\alpha}(A|B)
\]

\[
= (-T_{1-\alpha}(B|A) - S_{\alpha}(A|B)) + (S_{\alpha}(A|B) - T_{\alpha}(A|B))
\]

\[
= \frac{1}{1-\alpha}D_{0}(A \#_{\alpha} B|B) + \frac{1}{\alpha}D_{0}(A \#_{\alpha} B|A).
\]

By Theorem 2.3, we have

\[
\alpha(1-\alpha)D_{\alpha}(A|B) = \alpha D_{0}(A \#_{\alpha} B|B) + (1-\alpha)D_{0}(A \#_{\alpha} B|A)
\]

\[
= \alpha(B - A \#_{\alpha} B - S(A \#_{\alpha} B|B)) + (1-\alpha)(A - A \#_{\alpha} B - S(A \#_{\alpha} B|A))
\]

\[
= A \nabla_{\alpha} B - A \#_{\alpha} B - ((1-\alpha)S(A \#_{\alpha} B|A) + \alpha S(A \#_{\alpha} B|B)),
\]

and then

\[
(1-\alpha)S(A \#_{\alpha} B|A) + \alpha S(A \#_{\alpha} B|B) = 0,
\]

since \( D_{\alpha}(A|B) = \frac{A \nabla_{\alpha} B - A \#_{\alpha} B}{\alpha(1-\alpha)} \). This means that \( A \#_{\alpha} B \) is a solution of

\[
(1-\alpha)S(X|A) + \alpha S(X|B) = 0
\]

which is the Karcher equation concerning two operators \( A \) and \( B \). In this case, we can rewrite the result of Lawson-Lim [16] as follows:
**Theorem 2.4** ([16]). For strictly positive operators $A$, $B$ and $X$, and for $\alpha \in [0,1]$, 

$$(1 - \alpha)S(X|A) + \alpha S(X|B) = 0 \text{ if and only if } X = A \#_{\alpha} B.$$ 

For readers’ convenience, we give a direct proof of this theorem.

**Proof.** It is obvious if $\alpha = 0$. Otherwise, we have

$$(1 - \alpha)S(X|A) + \alpha S(X|B) = 0 \iff \log(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^{1-\alpha} + \log(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{\alpha} = 0$$

$\iff (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{\alpha} = (X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{1-\alpha}$

$\iff X^{-\frac{1}{2}}BX^{-\frac{1}{2}} = (X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{\frac{1}{\alpha}-1}$

$\iff (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{\alpha} = (X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{\frac{1}{\alpha}}X^{-\frac{1}{2}}A$

$\iff X = A \#_{\alpha} B.$

**Remark 2.** This theorem holds even if $\alpha$ is any real number [14].

3. **$\Psi$-Bregman divergences on the differences of relative operator entropies**

In this section, we consider $\Psi$-Bregman divergence in the case $C = \mathbb{R}$ as follows: For an operator valued smooth function $\Psi : \mathbb{R} \to B(H)$ and $x, y \in \mathbb{R}$,

$$D_{\Psi}(x,y) \equiv \Psi(x) - \Psi(y) - \lim_{\alpha \to +0} \frac{\Psi(y + \alpha(x-y)) - \Psi(y)}{\alpha}.$$ 

From the following theorem, it is natural that we consider $D_{\Psi}(1,0)$ as a divergence of operators $A$ and $B$.

**Theorem 3.1.** Let $\Psi(t) = A \natural_{t} B$ for strictly positive operators $A$ and $B$. Then for $x, y \in \mathbb{R}$,

$$D_{\Psi}(x,y) = D_{0}(A \#_{y} B|A \#_{x} B).$$

In particular, $D_{\Psi}(1,0) = D_{0}(A|B)$.

**Proof.**

$$D_{\Psi}(x,y) = A \#_{x} B - A \#_{y} B - \lim_{\alpha \to +0} \frac{A \#_{y + \alpha(x-y)} B - A \#_{y} B}{\alpha}$$

$$= A \#_{x} B - A \#_{y} B - \lim_{\alpha \to +0} \frac{(A \#_{y} B) \#_{\alpha} (A \#_{x} B) - A \#_{y} B}{\alpha} \text{ by [11, Lemma 2.2]}$$

$$= A \#_{x} B - A \#_{y} B - S(A \#_{y} B|A \#_{x} B) = D_{0}(A \#_{y} B|A \#_{x} B).$$

In the rest of this section, we obtain $D_{\Psi}(1,0)$ for functions $\Psi$ which relate to the operator divergences $\Delta_{1}, \Delta_{2}, \Delta_{5}$ and $\Delta_{6}$ in section 2.
Theorem 3.2. For strictly positive operators $A$ and $B$, the following hold:

(1) If $\Psi(t) = T_t(A|B) - S(A|B)$, then

$$D_{\Psi}(1,0) = D_0(A|B) - \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

(2) If $\Psi(t) = S_t(A|B) - S(A|B)$, then

$$D_{\Psi}(1,0) = D_0(A|B) + D_0(B|A) - S(A|B)A^{-1}S(A|B).$$

(3) If $\Psi(t) = S_t(A|B) - T_t(A|B)$, then

$$D_{\Psi}(1,0) = D_0(B|A) - \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

(4) If $\Psi(t) = D_t(A|B)$ for $t \in [0,1]$, then

$$D_{\Psi}(1,0) = D_0(B|A) - 2D_0(A|B) + \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

Here, we give a proof of (1). The others are obtained similarly.

Proof. (1) For $a > 0$, we have

$$\lim_{\alpha \to +0} \frac{a^\alpha - 1 - \alpha \log a}{\alpha^2} = \frac{1}{2} (\log a)^2.$$

Replacing $a$ by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\lim_{\alpha \to +0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha} = \lim_{\alpha \to +0} \frac{A^{\frac{1}{2}}((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha - I - \alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))A^{\frac{1}{2}}}{\alpha^2}$$

$$= \frac{1}{2} A^{\frac{1}{2}}(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))^2 A^{\frac{1}{2}} = \frac{1}{2} S(A|B)A^{-1}S(A|B),$$

then

$$D_{\Psi}(1,0) = T_1(A|B) - S(A|B) - (T_0(A|B) - S(A|B))$$

$$- \lim_{\alpha \to +0} \frac{T_\alpha(A|B) - S(A|B) - (T_0(A|B) - S(A|B))}{\alpha}$$

$$= T_1(A|B) - S(A|B) - \lim_{\alpha \to +0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha}$$

$$= D_0(A|B) - \frac{1}{2} S(A|B)A^{-1}S(A|B).$$
4. Divergences given by the differences of expanded relative operator entropies

In this section, we try to generalize Theorems 2.2 and 2.3 in Section 2 for operator power mean. For $A, B > 0$, $x \in [0, 1]$ and $r \in [-1, 1]$, operator power mean $A \#_{x,r} B$ is defined as follows:

$$A \#_{x,r} B \equiv A^{\frac{1}{2}} \left( \left( 1 - x \right) I + x \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^{r} \right)^{\frac{1}{r}} A^{\frac{1}{2}} = A \#_{\frac{1}{r}} \left( A \nabla_{x} (A \#_{r} B) \right).$$

We remark that $A \#_{x,r} B = B \#_{1-x,r} A$ holds for $x \in [0, 1]$ and $r \in [-1, 1]$ (cf. [9], [11]). To preserve $(1-x)I + x (A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^{r} \geq 0$, we have to impose $x$ in $[0, 1]$.

The operator power mean is a path passing through $A = A \#_{0,r} B$ and $B = A \#_{1,r} B$, and combines arithmetic, geometric and harmonic means, that is, $A \#_{x,1} B = A \nabla_{x} B$, $A \#_{x,0} B \equiv \lim_{r \to 0} A \#_{x,r} B = A \#_{x} B$ and $A \#_{x,-1} B = A \Delta_{x} B = (A^{-1} \nabla_{x} B^{-1})^{-1}$.

For $\alpha \in [0, 1]$ and $r \in [-1, 1]$, expanded relative operator entropy $S_{\alpha,r}(A|B)$ and expanded Tsallis relative operator entropy $T_{\alpha,r}(A|B)$ are defined as follows (cf. [9]):

$$S_{\alpha,r}(A|B) \equiv A^{\frac{1}{2}} \left( \left( 1 - \alpha \right) I + \alpha \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^{r} \right)^{\frac{1}{r}} \frac{(A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^{r} - I}{r} A^{\frac{1}{2}}$$

$$= \frac{d}{dx} A \#_{x,r} B \bigg|_{x=\alpha} = \left( A \#_{\alpha,r} B \right) \left( A \nabla_{\alpha} (A \#_{r} B) \right)^{-1} S_{0,r}(A|B) \quad (r \neq 0),$$

$$S_{\alpha,0}(A|B) \equiv \lim_{r \to 0} S_{\alpha,r}(A|B) = S_{\alpha}(A|B),$$

$$T_{\alpha,r}(A|B) \equiv \frac{A \#_{\alpha,r} B - A}{\alpha} \quad (\alpha \neq 0),$$

$$T_{0,r}(A|B) \equiv \lim_{\alpha \to 0} T_{\alpha,r}(A|B) = T_{r}(A|B).$$

We remark that $S_{0,r}(A|B) = T_{r}(A|B)$ and $S_{1,r}(A|B) = -T_{r}(B|A)$ hold for $r \in [-1, 1]$. A similar inequality to $(*)$ also holds for these expanded relative operator entropies, which is given as follows [9]:

$$S_{0,r}(A|B) \leq T_{\alpha,r}(A|B) \leq S_{\alpha,r}(A|B) \leq -T_{1-\alpha,r}(B|A) \leq S_{1,r}(A|B), \quad \alpha \in [0, 1], \ r \in [-1, 1].$$

If $r = 0$, then this inequality becomes $(*)$.

We defined expanded operator valued $\alpha$-divergence $D_{\alpha,r}(A|B)$ as follows [11]:

$$D_{\alpha,r}(A|B) \equiv -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B), \quad \alpha \in [0, 1], \ r \in [-1, 1].$$

For $\alpha \in (0, 1)$, we can represent $D_{\alpha,r}(A|B)$ as follows:

$$D_{\alpha,r}(A|B) = \frac{A \nabla_{\alpha} B - A \#_{\alpha,r} B}{\alpha(1-\alpha)}. $$

We gave the following relations on expanded operator valued $\alpha$-divergence.

**Proposition 4.1** ([11], Proposition 4.4). *For strictly positive operators $A$ and $B$, the following hold:*

1. $D_{\alpha,0}(A|B) = D_{\alpha}(A|B)$ for $\alpha \in [0, 1]$,
2. $D_{\alpha,1}(A|B) = 0$ for $\alpha \in [0, 1]$,
3. $D_{0,r}(A|B) = B - A - S_{0,r}(A|B)$ for $r \in [-1, 1]$,
4. $D_{1,r}(A|B) = A - B - S_{0,r}(B|A) = D_{0,r}(B|A)$ for $r \in [-1, 1]$.
We call $D_{0,r}(A|B) = B - A - S_{0,r}(A|B)$ expanded Petz-Bregman divergence.

Similarly to section 2, we consider the differences between two expanded relative operator entropies in (**) as operator divergence. There are 10 such divergences.

$$
\begin{align*}
T_{\alpha,r}(A|B) - S_{0,r}(A|B), & \quad S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B), \\
-T_{1-\alpha,r}(B|A) - S_{\alpha,r}(A|B), & \quad S_{1,r}(A|B) + T_{1-\alpha,r}(B|A), \\
S_{\alpha,r}(A|B) - S_{0,r}(A|B), & \quad -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B) = D_{\alpha,r}(A|B), \\
S_{1,r}(A|B) - S_{\alpha,r}(A|B), & \quad -T_{1-\alpha,r}(B|A) - S_{0,r}(A|B), \\
S_{1,r}(A|B) - T_{\alpha,r}(A|B), & \quad S_{1,r}(A|B) - S_{0,r}(A|B). \\
\end{align*}
$$

The relations of these differences are given as in Table 2. If $r = 0$, then this table coincides with Table 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$S_{1,r}(A|B) - S_{0,r}(A|B) \geq S_{1,r}(A|B) - T_{\alpha,r}(A|B) \geq S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B) \geq S_{1,r}(A|B) + T_{1-\alpha,r}(B|A)$ & $\geq 0$ \\
\hline
$\nabla$ & $\nabla$ & $\nabla$ \\
\hline
$-T_{1-\alpha,r}(B|A) - S_{0,r}(A|B) \geq -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B) \geq -T_{1-\alpha,r}(B|A) - S_{\alpha,r}(A|B) \geq 0$ & $\nabla$ & $\nabla$ \\
\hline
$\nabla$ & $\nabla$ & $\nabla$ \\
\hline
$S_{\alpha,r}(A|B) - S_{1,r}(A|B) \geq S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B)$ & $\quad$ & $\nabla$ \\
\hline
$\nabla$ & $\nabla$ & $\nabla$ \\
\hline
$T_{\alpha,r}(A|B) - S_{0,r}(A|B)$ & $\quad$ & $0$ \\
\hline
$\nabla$ & $\nabla$ & $\nabla$ \\
\hline
$0$ & $\quad$ & $\nabla$ \\
\hline
\end{tabular}
\caption{Table 2}
\end{table}

We represent these operator divergences by using expanded Petz-Bregman divergence. Theorems 4.2 and 4.3 correspond to Theorems 2.2 and 2.3, respectively.

**Theorem 4.2.** For strictly positive operators $A$ and $B$ and $r \in [-1, 1]$, the following hold:

1. \[ T_{\alpha,r}(A|B) - S_{0,r}(A|B) = \frac{1}{\alpha}D_{0,r}(A|A \#_{\alpha,r} B) \quad \text{for } \alpha \in (0, 1), \]
2. \[ S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B) = \frac{1}{\alpha}D_{0,r}(A \#_{\alpha,r} B|A) \quad \text{for } \alpha \in (0, 1). \]

**Theorem 4.3.** For strictly positive operators $A$ and $B$ and $r \in [-1, 1]$, the following holds:

\[ D_{\alpha,r}(A|B) = \frac{1}{1 - \alpha}D_{0,r}(A \#_{\alpha,r} B|B) + \frac{1}{\alpha}D_{0,r}(A \#_{\alpha,r} B|A) \quad \text{for } \alpha \in (0, 1). \]
References


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