Some operator divergences based on Petz-Bregman divergence

1. Introduction

This report is based on [12]. Throughout this report, a bounded linear operator \( T \) on a Hilbert space \( H \) is positive (denoted by \( T \geq 0 \)) if \( \langle T\xi, \xi \rangle \geq 0 \) for all \( \xi \in H \), and \( T \) is strictly positive (denoted by \( T > 0 \)) if \( T \) is invertible and positive.

For strictly positive operators \( A \) and \( B \), \( A \triangleleft_{x} B \) is defined as follows ([3, 4, 13] etc.):

\[
A \triangleleft_{x} B \equiv A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{x} A^{\frac{1}{2}}, \quad x \in \mathbb{R}.
\]

We call \( A \triangleleft_{x} B \) a path passing through \( A = A \triangleleft_{0} B \) and \( B = A \triangleleft_{1} B \). If \( x \in [0, 1] \), the path \( A \triangleleft_{x} B \) coincides with the weighted geometric operator mean denoted by \( A \natural_{x} B \) (cf. [15]). We remark that \( A \triangleleft_{x} B = B \triangleleft_{1-x} A \) holds for \( x \in \mathbb{R} \).

Fujii and Kamei [2] defined the following relative operator entropy for strictly positive operators \( A \) and \( B \):

\[
S(A|B) \equiv A^{\frac{1}{2}} \log \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}
\]

\[
= \frac{d}{dx} A \triangleleft_{x} B \bigg|_{x=0}.
\]

We can regard \( S(A|B) \) as the gradient of the tangent line at \( x = 0 \) of the path \( A \triangleleft_{x} B \). Furuta [7] defined generalized relative operator entropy as follows:

\[
S_{\alpha}(A|B) \equiv A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}
\]

\[
= \frac{d}{dx} A \triangleleft_{x} B \bigg|_{x=\alpha}, \quad \alpha \in \mathbb{R}.
\]

We regard \( S_{\alpha}(A|B) \) as the gradient of the tangent line at \( x = \alpha \) of the path. We know immediately \( S_{0}(A|B) = S(A|B) \). Yanagi, Kuriyama and Furuiichi [18] introduced Tsallis relative operator entropy as follows:

\[
T_{\alpha}(A|B) \equiv \frac{A \triangleleft_{\alpha} B - A}{\alpha}, \quad \alpha \in (0, 1].
\]

\( T_{\alpha}(A|B) \) can be regarded as the average rate of change of \( A \triangleleft_{x} B \) from \( x = 0 \) to \( x = \alpha \). Since \( \lim_{x \to 0} \frac{a^{x} - 1}{x} = \log a \) holds for \( a > 0 \), we have \( T_{0}(A|B) \equiv \lim_{\alpha \to 0} T_{\alpha}(A|B) = S(A|B) \).

Tsallis relative operator entropy can be extended as the notion for \( \alpha \in \mathbb{R} \). In this case, we use \( \triangleleft_{\alpha} \) instead of \( \triangleleft_{x} \). In [8], we had given the following relations among these relative operator entropies:

\[
(*) \quad S_{1}(A|B) \geq -T_{1-\alpha}(B|A) \geq S_{\alpha}(A|B) \geq T_{\alpha}(A|B) \geq S(A|B), \quad \alpha \in (0, 1].
\]
Fujii [1] defined operator valued $\alpha$-divergence $D_\alpha(A|B)$ for $\alpha \in (0,1)$ as follows:

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \#_\alpha B}{\alpha(1-\alpha)},$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is the weighted arithmetic operator mean. The operator valued $\alpha$-divergence has the following relations at end points for interval $(0,1)$.

**Theorem A ([5, 6]).** For strictly positive operators $A$ and $B$, the following hold:

$$D_0(A|B) \equiv \lim_{\alpha \to +0} D_\alpha(A|B) = B - A - S(A|B),$$

$$D_1(A|B) \equiv \lim_{\alpha \to 1-0} D_\alpha(A|B) = A - B - S(B|A).$$

Petz [17] introduced the right hand side in the first equation in Theorem A as an operator divergence, so we call $D_0(A|B)$ Petz-Bregman divergence. We remark that $D_1(A|B) = D_0(B|A)$ holds. Figure 1 shows our interpretation of $D_0(A|B)$.

In [10], we showed the following relation between operator valued $\alpha$-divergence and Tsallis relative operator entropy:

**Theorem B ([10]).** For strictly positive operators $A$ and $B$, the following holds:

$$D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B) \quad \text{for } \alpha \in (0,1).$$

Theorem B shows that $D_\alpha(A|B)$ is a difference between two terms in $(*)$. From this fact, we regard the differences between the relative operator entropies in $(*)$ as operator divergences. In section 2, we represent these operator divergences by using Petz-Bregman divergence.

On the other hand, for an operator valued smooth function $\Psi : C \to B(H)$ and $X, Y \in C$, where $C$ is a convex set in a Banach space, Petz [17] defined a divergence
$D_{\Psi}(X, Y)$ as follows:

$$D_{\Psi}(X, Y) \equiv \Psi(X) - \Psi(Y) - \lim_{\alpha \to +0} \frac{\Psi(Y + \alpha(X - Y)) - \Psi(Y)}{\alpha}.$$ 

We call $D_{\Psi}(X, Y)$ \textit{\Psi-Bregman divergence} of $Y$ and $X$ in this report. Petz gave some examples for invertible density matrices $X$ and $Y$. If $\Psi(X) = \eta(X) \equiv X \log X$ and $X$ commutes with $Y$, then $D_{\Psi}(X, Y) = Y - X + X(\log X - \log Y)$, which is the usual quantum relative entropy.

In section 3, we let $C = \mathbb{R}$ and show $D_{\Psi}(x, y) = D_{0}(A \natural_{v} B | A \natural_{x} B)$ for $\Psi(t) = A \natural_{t} B$ and $x, y \in \mathbb{R}$. Then we have $D_{\Psi}(1, 0) = D_{0}(A|B)$ in particular.

The relation of the differences among $\Delta_{1}, \ldots, \Delta_{10}$ are given as in Table 1.

**Lemma 2.1** ([8, 10]). For strictly positive operators $A$ and $B$, the following hold for $s, t \in \mathbb{R}$:

(1) $S_{t}(A|A \natural_{s} B) = sS_{st}(A|B)$,

(2) $S_{t}(A|B) = -S_{1-t}(B|A)$. 

**In section 4**, we show the results corresponding to those in section 2 on expanded relative operator entropies defined by operator power mean.
The following are the results on $\Delta_1$ and $\Delta_2$.

**Theorem 2.2.** For strictly positive operators $A$ and $B$, the following hold:

(1) \[ \Delta_1 = T_\alpha(A|B) - S(A|B) = \frac{1}{\alpha}D_0(A|A \#_{\alpha} B) \] for $\alpha \in (0, 1]$,

(2) \[ \Delta_2 = S_\alpha(A|B) - T_\alpha(A|B) = \frac{1}{\alpha}D_0(A \#_{\alpha} B|A) \] for $\alpha \in (0, 1]$.

**Proof.** (1) By (1) in Lemma 2.1, we have

\[
T_\alpha(A|B) - S(A|B) = \frac{A \#_{\alpha} B - A}{\alpha} - S(A|B) = \frac{1}{\alpha} \left( A \#_{\alpha} B - A - \alpha S(A|B) \right) = \frac{1}{\alpha} \left( A \#_{\alpha} B - A - S(A|A \#_{\alpha} B) \right) = \frac{1}{\alpha} D_0(A|A \#_{\alpha} B).
\]

(2) By Lemma 2.1, we have

\[
S_\alpha(A|B) - T_\alpha(A|B) = \frac{A - A \#_{\alpha} B}{\alpha} + S_\alpha(A|B) = \frac{1}{\alpha} \left( A - A \#_{\alpha} B + \alpha S_\alpha(A|B) \right) = \frac{1}{\alpha} \left( A - A \#_{\alpha} B + S_1(A|A \#_{\alpha} B) \right) = \frac{1}{\alpha} D_0(A \#_{\alpha} B|A).
\]

**Remark 1.** By exchanging $A$ for $B$ and replacing $\alpha$ with $1 - \alpha$ for (1) and (2), we have the following relations:

(1) \[ \Delta_4 = S_1(A|B) + T_{1-\alpha}(B|A) = \frac{1}{1-\alpha}D_0(B|A \#_\alpha B) \] for $\alpha \in [0, 1)$,

(2) \[ \Delta_3 = -T_{1-\alpha}(B|A) - S_\alpha(A|B) = \frac{1}{1-\alpha}D_0(A \#_{\alpha} B|B) \] for $\alpha \in [0, 1)$. 

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**Table 1**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Condition</th>
</tr>
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<tbody>
<tr>
<td>$S_1(A</td>
<td>B) - S(A</td>
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<tr>
<td>$T_\alpha(A</td>
<td>B)$</td>
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<td>$S_\alpha(A</td>
<td>B) - S(A</td>
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<td>$T_\alpha(A</td>
<td>B) - S(A</td>
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<tr>
<td>$S_1(A</td>
<td>B) - S_\alpha(A</td>
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<tr>
<td>$S(A</td>
<td>B) - T_\alpha(A</td>
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<td>$S_\alpha(A</td>
<td>B) - S(A</td>
</tr>
<tr>
<td>$T_\alpha(A</td>
<td>B) - S(A</td>
</tr>
<tr>
<td>$0$</td>
<td>$\forall$</td>
</tr>
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</table>

\[
\begin{align*}
S_1(A|B) - S(A|B) & \geq S_1(A|B) - T_\alpha(A|B) \geq S_1(A|B) - S_\alpha(A|B) \geq S_1(A|B) + T_{1-\alpha}(B|A) \geq 0, \\
-T_\alpha(A|B) - S(A|B) & \geq -T_{1-\alpha}(B|A) - T_\alpha(A|B) \geq -T_{1-\alpha}(B|A) - S_\alpha(A|B) \geq 0, \\
S_\alpha(A|B) - S(A|B) & \geq S_\alpha(A|B) - T_\alpha(A|B), \\
T_\alpha(A|B) - S(A|B) & = 0, \\
0 & = 0.
\end{align*}
\]
We give a geometrical interpretation for (1) in Theorem 2.2. Figure 2 and Figure 3 show $T_\alpha(A|B) - S(A|B)$ and $D_0(A|A \#_\alpha B)$ appeared in (1) in Theorem 2.2, respectively. Figure 4 is an image of (1) in Theorem 2.2.

Figure 2: An interpretation of $T_\alpha(A|B) - S(A|B)$.

Figure 3: An interpretation of $D_0(A|A \#_\alpha B) = A \#_\alpha B - A - S(A|A \#_\alpha B)$. 
Figure 4: An image of $T_{\alpha}(A|B) - S(A|B) = \frac{1}{\alpha}D_{0}(A|A \#_{\alpha} B)$.

Theorem 2.2 leads to the next theorem.

**Theorem 2.3.** For strictly positive operators $A$ and $B$, the following holds:

$$D_{\alpha}(A|B) = \frac{1}{1-\alpha}D_{0}(A \#_{\alpha} B|B) + \frac{1}{\alpha}D_{0}(A \#_{\alpha} B|A) \quad \text{for } \alpha \in (0,1).$$

**Proof.** By (2) in Theorem 2.2 and (2) in Remark 1, we have

$$D_{\alpha}(A|B) = -T_{1-\alpha}(B|A) - T_{\alpha}(A|B)$$
$$= (-T_{1-\alpha}(B|A) - S_{\alpha}(A|B)) + (S_{\alpha}(A|B) - T_{\alpha}(A|B))$$
$$= \frac{1}{1-\alpha}D_{0}(A \#_{\alpha} B|B) + \frac{1}{\alpha}D_{0}(A \#_{\alpha} B|A).$$

\[\square\]

By Theorem 2.3, we have

$$\alpha(1-\alpha)D_{\alpha}(A|B) = \alpha D_{0}(A \#_{\alpha} B|B) + (1-\alpha)D_{0}(A \#_{\alpha} B|A)$$
$$= \alpha(B - A \#_{\alpha} B - S(A \#_{\alpha} B|B)) + (1-\alpha)(A - A \#_{\alpha} B - S(A \#_{\alpha} B|A))$$
$$= A \nabla_{\alpha} B - A \#_{\alpha} B - ((1-\alpha)S(A \#_{\alpha} B|A) + \alpha S(A \#_{\alpha} B|B)),$$

and then

$$(1-\alpha)S(A \#_{\alpha} B|A) + \alpha S(A \#_{\alpha} B|B) = 0,$$

since $D_{\alpha}(A|B) = \frac{A \nabla_{\alpha} B - A \#_{\alpha} B}{\alpha(1-\alpha)}$. This means that $A \#_{\alpha} B$ is a solution of $(1-\alpha)S(X|A) + \alpha S(X|B) = 0$ which is the Karcher equation concerning two operators $A$ and $B$. In this case, we can rewrite the result of Lawson-Lim [16] as follows:
Theorem 2.4 ([16]). For strictly positive operators $A$, $B$ and $X$, and for $\alpha \in [0,1],
(1 - \alpha)S(X|A) + \alpha S(X|B) = 0$ if and only if $X = A \natural_{\alpha} B$.

For readers’ convenience, we give a direct proof of this theorem.

Proof. It is obvious if $\alpha = 0$. Otherwise, we have
$$
(1 - \alpha)S(X|A) + \alpha S(X|B) = 0 $$
$$
\iff \log(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^{1-\alpha} + \log(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{\alpha} = 0 $$
$$
\iff (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{\alpha} = (X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^{1-\alpha} $$
$$
\iff X^{-\frac{1}{2}}BX^{-\frac{1}{2}} = (X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^{\alpha-1} $$
$$
\iff B = X^{\frac{1}{2}}(X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{\frac{1}{2}}X^{-\frac{1}{2}}A $$
$$
\iff A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{\frac{1}{2}} $$
$$
\iff (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha} = A^{-\frac{1}{2}}XA^{-\frac{1}{2}} $$
$$
\iff X = A \natural_{\alpha} B.
$$

Remark 2. This theorem holds even if $\alpha$ is any real number [14].

3. $\Psi$-Bregman divergences on the differences of relative operator entropies

In this section, we consider $\Psi$-Bregman divergence in the case $C = \mathbb{R}$ as follows: For an operator valued smooth function $\Psi : \mathbb{R} \to B(H)$ and $x, y \in \mathbb{R},$
$$
D_{\Psi}(x, y) \equiv \Psi(x) - \Psi(y) - \lim_{\alpha \to +0} \frac{\Psi(y + \alpha(x - y)) - \Psi(y)}{\alpha}.
$$
From the following theorem, it is natural that we consider $D_{\Psi}(1,0)$ as a divergence of operators $A$ and $B$.

Theorem 3.1. Let $\Psi(t) = A \natural_{t} B$ for strictly positive operators $A$ and $B$. Then for $x, y \in \mathbb{R},$
$$
D_{\Psi}(x, y) = D_{0}(A \mathfrak{h}_{y} B|A \natural_{x} B).
$$
In particular, $D_{\Psi}(1, 0) = D_{0}(A|B)$.

Proof. $D_{\Psi}(x, y) = A \mathfrak{h}_{x} B - A \mathfrak{h}_{y} B - \lim_{\alpha \to +0} \frac{A \mathfrak{h}_{y + \alpha(x-y)} B - A \mathfrak{h}_{y} B}{\alpha}$
$$
= A \mathfrak{h}_{x} B - A \mathfrak{h}_{y} B - \lim_{\alpha \to +0} \frac{(A \mathfrak{h}_{y} B) \mathfrak{h}_{\alpha} (A \mathfrak{h}_{x} B) - A \mathfrak{h}_{y} B}{\alpha} \quad \text{by [11, Lemma 2.2]}
$$
$$
= A \mathfrak{h}_{x} B - A \mathfrak{h}_{y} B - S(A \mathfrak{h}_{y} B|A \mathfrak{h}_{x} B) = D_{0}(A \mathfrak{h}_{y} B|A \mathfrak{h}_{x} B).
$$

In the rest of this section, we obtain $D_{\Psi}(1, 0)$ for functions $\Psi$ which relate to the operator divergences $\triangle_{1}, \triangle_{2}, \triangle_{5}$ and $\triangle_{6}$ in section 2.
Theorem 3.2. For strictly positive operators $A$ and $B$, the following hold:

1. If $\Psi(t) = T_t(A|B) - S(A|B)$, then
   \[ D_{\Psi}(1,0) = D_0(A|B) - \frac{1}{2}S(A|B)A^{-1}S(A|B). \]

2. If $\Psi(t) = S_t(A|B) - S(A|B)$, then
   \[ D_{\Psi}(1,0) = D_0(A|B) + D_0(B|A) - S(A|B)A^{-1}S(A|B). \]

3. If $\Psi(t) = S_t(A|B) - T_t(A|B)$, then
   \[ D_{\Psi}(1,0) = D_0(B|A) - \frac{1}{2}S(A|B)A^{-1}S(A|B). \]

4. If $\Psi(t) = D_t(A|B)$ for $t \in [0,1]$, then
   \[ D_{\Psi}(1,0) = D_0(B|A) - 2D_0(A|B) + \frac{1}{2}S(A|B)A^{-1}S(A|B). \]

Here, we give a proof of (1). The others are obtained similarly.

Proof. (1) For $a > 0$, we have
   \[ \lim_{\alpha \to 0} \frac{a^{\alpha} - 1 - \alpha \log a}{\alpha^2} = \frac{1}{2}(\log a)^2. \]
Replacing $a$ by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have
   \[ \lim_{\alpha \to 0} \frac{T_{\alpha}(A|B) - S(A|B)}{\alpha} = \frac{1}{2}A^{\frac{1}{2}}(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))^2 A^{\frac{1}{2}} = \frac{1}{2}S(A|B)A^{-1}S(A|B), \]
then
   \[ D_{\Psi}(1,0) = T_1(A|B) - S(A|B) - (T_0(A|B) - S(A|B)) - \lim_{\alpha \to 0} \frac{T_{\alpha}(A|B) - S(A|B) - (T_0(A|B) - S(A|B))}{\alpha} \]
   \[ = T_1(A|B) - S(A|B) - \lim_{\alpha \to 0} \frac{T_{\alpha}(A|B) - S(A|B)}{\alpha} \]
   \[ = D_0(A|B) - \frac{1}{2}S(A|B)A^{-1}S(A|B). \]
4. Divergences given by the differences of expanded relative operator entropies

In this section, we try to generalize Theorems 2.2 and 2.3 in section 2 for operator power mean. For \( A, B > 0, \ x \in [0, 1] \) and \( r \in [-1, 1] \), operator power mean \( A \#_{x,r} B \) is defined as follows:

\[
A \#_{x,r} B \equiv A^{\frac{1}{2}} \left\{ (1-x)I + x \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^r \right\}^{\frac{1}{r}} A^{\frac{1}{2}} = A \#_{\frac{r}{2}} \left\{ A \nabla_x (A \#_{r} B) \right\}.
\]

We remark that \( A \#_{x,r} B = B \#_{1-x,r} A \) holds for \( x \in [0, 1] \) and \( r \in [-1, 1] \) (cf. [9], [11]). To preserve \( (1-x)I + x \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^r \geq 0 \), we have to impose \( x \in [0, 1] \).

The operator power mean is a path passing through \( A = A \#_{0,r} B \) and \( B = A \#_{1,r} B \), and combines arithmetic, geometric and harmonic means, that is, \( A \#_{x,1} B = A \nabla_{x} B, A \#_{x,0} B \equiv \lim_{r \arrow 0} A \#_{x,r} B = A \#_{x} B \) and \( A \#_{x,-1} B = A \Delta_{x} B = (A^{-1} \nabla_{x} B^{-1})^{-1} \).

For \( \alpha \in [0, 1] \) and \( r \in [-1, 1] \), expanded relative operator entropy \( S_{\alpha,r}(A|B) \) and expanded Tsallis relative operator entropy \( T_{\alpha,r}(A|B) \) are defined as follows (cf. [9]):

\[
S_{\alpha,r}(A|B) \equiv A^{\frac{1}{2}} \left\{ (1-\alpha \lambda)I + \alpha \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^r \right\}^{\frac{1}{r}} \frac{(A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^r - I}{r} A^{\frac{1}{2}} = \frac{d}{dx} A \#_{x,r} B \bigg|_{x=\alpha} = (A \#_{\alpha,r} B) (A \nabla_{\alpha} (A \#_{r} B))^{-1} S_{0,r}(A|B) (r \neq 0),
\]

\[
S_{\alpha,0}(A|B) \equiv \lim_{r \arrow 0} S_{\alpha,r}(A|B) = S_{\alpha}(A|B),
\]

\[
T_{\alpha,r}(A|B) \equiv \frac{A \#_{\alpha,r} B - A}{\alpha} (\alpha \neq 0), \quad T_{0,r}(A|B) \equiv \lim_{\alpha \arrow 0} T_{\alpha,r}(A|B) = T_{r}(A|B).
\]

We remark that \( S_{0,r}(A|B) = T_{r}(A|B) \), \( S_{1,r}(A|B) = -T_{r}(B|A) \) and \( T_{1,r}(A|B) = B - A \) hold for \( r \in [-1, 1] \). A similar inequality to (\*) also holds for these expanded relative operator entropies, which is given as follows [9]:

\[
(**) \ S_{0,r}(A|B) \leq T_{\alpha,r}(A|B) \leq S_{\alpha,r}(A|B) \leq -T_{1-\alpha,r}(B|A) \leq S_{1,r}(A|B), \quad \alpha \in [0, 1], \ r \in [-1, 1].
\]

If \( r = 0 \), then this inequality becomes (\*). 

We defined expanded operator valued \( \alpha \)-divergence \( D_{\alpha,r}(A|B) \) as follows [11]:

\[
D_{\alpha,r}(A|B) \equiv -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B), \quad \alpha \in [0, 1], \ r \in [-1, 1].
\]

For \( \alpha \in (0, 1) \), we can represent \( D_{\alpha,r}(A|B) \) as follows:

\[
D_{\alpha,r}(A|B) = \frac{A \nabla_{\alpha} B - A \#_{\alpha,r} B}{\alpha(1-\alpha)}.
\]

We gave the following relations on expanded operator valued \( \alpha \)-divergence.

**Proposition 4.1** ([11], Proposition 4.4). For strictly positive operators \( A \) and \( B \), the following hold:

1. \( D_{\alpha,0}(A|B) = D_{\alpha}(A|B) \) for \( \alpha \in [0, 1] \),
2. \( D_{\alpha,1}(A|B) = 0 \) for \( \alpha \in [0, 1] \),
3. \( D_{0,r}(A|B) = B - A - S_{0,r}(A|B) \) for \( r \in [-1, 1] \),
4. \( D_{1,r}(A|B) = A - B - S_{0,r}(B|A) = D_{0,r}(B|A) \) for \( r \in [-1, 1] \).
We call $D_{0,r}(A|B) = B - A - S_{0,r}(A|B)$ expanded Petz-Bregman divergence.

Similarly to section 2, we consider the differences between two expanded relative operator entropies in $(**)$ as operator divergence. There are 10 such divergences.

\[ T_{\alpha,r}(A|B) - S_{0,r}(A|B), \quad S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B), \]
\[ -T_{1-\alpha,r}(B|A) - S_{\alpha,r}(A|B), \quad S_{1,r}(A|B) + T_{1-\alpha,r}(B|A), \]
\[ S_{\alpha,r}(A|B) - S_{0,r}(A|B), \quad -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B) = D_{\alpha,r}(A|B), \]
\[ S_{1,r}(A|B) - S_{\alpha,r}(A|B), \quad -T_{1-\alpha,r}(B|A) - S_{0,r}(A|B), \]
\[ S_{1,r}(A|B) - T_{\alpha,r}(A|B), \quad S_{1,r}(A|B) - S_{0,r}(A|B). \]

The relations of these differences are given as in Table 2. If $r = 0$, then this table coincides with Table 1.

| $S_{1,r}(A|B) - S_{0,r}(A|B) \geq S_{1,r}(A|B) - T_{\alpha,r}(A|B) \geq S_{\alpha,r}(A|B) - S_{0,r}(A|B) \geq S_{1,r}(A|B) + T_{1-\alpha,r}(B|A) \geq 0$ | $\forall$ | $\forall$ | $\forall$ |
| $-T_{1-\alpha,r}(B|A) - S_{0,r}(A|B) \geq -T_{1-\alpha,r}(B|A) - T_{\alpha,r}(A|B) \geq -T_{1-\alpha,r}(B|A) - S_{\alpha,r}(A|B) \geq 0$ | $\forall$ | $\forall$ |
| $S_{\alpha,r}(A|B) - S_{0,r}(A|B) \geq S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B)$ | $\forall$ | $\forall$ |
| $T_{\alpha,r}(A|B) - S_{0,r}(A|B)$ | $0$ |
| $\forall$ | $0$ |

We represent these operator divergences by using expanded Petz-Bregman divergence. Theorems 4.2 and 4.3 correspond to Theorems 2.2 and 2.3, respectively.

**Theorem 4.2.** For strictly positive operators $A$ and $B$ and $r \in [-1, 1]$, the following hold:

1. $T_{\alpha,r}(A|B) - S_{0,r}(A|B) = \frac{1}{\alpha} D_{0,r}(A|A \#_{0,r} B)$ for $\alpha \in (0, 1)$,
2. $S_{\alpha,r}(A|B) - T_{\alpha,r}(A|B) = \frac{1}{\alpha} D_{0,r}(A \#_{\alpha,r} B|A)$ for $\alpha \in (0, 1)$.

**Theorem 4.3.** For strictly positive operators $A$ and $B$ and $r \in [-1, 1]$, the following holds:

\[ D_{\alpha,r}(A|B) = \frac{1}{1-\alpha} D_{0,r}(A \#_{\alpha,r} B|B) + \frac{1}{\alpha} D_{0,r}(A \#_{\alpha,r} B|A) \] for $\alpha \in (0, 1)$. 

References


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