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Periodic solutions of double-diffusive convection system in the whole space [*1]

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1 Introduction

We consider the time periodic problem of the following system (DCBF), which describes double-diffusive convection phenomena of incompressible viscous fluid contained in some porous medium.

$$
\text{(DCBF)}\begin{cases}
\partial_t \boldsymbol{u} = \nu\Delta\boldsymbol{u} - a\boldsymbol{u} - \nabla p + \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1 & (x,t)\in\mathbb{R}^N\times[0,S],\\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t)\in\mathbb{R}^N\times[0,S],\\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t)\in\mathbb{R}^N\times[0,S],\\
\nabla\cdot\boldsymbol{u} = 0 & (x,t)\in\mathbb{R}^N\times[0,S],\\
\boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S),\ T(\cdot,0) = T(\cdot,S),\ C(\cdot,0) = C(\cdot,S),
\end{cases}
$$

where $\mathbb{R}^N$ denotes N-dimension Euclidean space. Unknown functions of (DCBF) are

$$
\begin{aligned}
&\boldsymbol{u} = \boldsymbol{u}(x,t) = (u^1(x,t), u^2(x,t), \cdots, u^N(x,t)) && : \text{Fluid velocity},\\
&T = T(x,t) && : \text{Temperature of fluid},\\
&C = C(x,t) && : \text{Concentration of solute},\\
&p = p(x,t) && : \text{Pressure of fluid}.
\end{aligned}
$$

Given positive constants ν, a and ρ are called the viscosity coefficient, Darcy's coefficient and Soret's coefficient respectively. Constant vectors $\boldsymbol{g} = (g^1, g^2, \cdots, g^N)$ and $\boldsymbol{h} = (h^1, h^2, \cdots, h^N)$ describe the effects of gravity. Moreover $\boldsymbol{f}_1 = \boldsymbol{f}_1(x,t) = (f_1^1(x,t), f_1^2(x,t), \cdots, f_1^N(x,t))$, $f_2 = f_2(x,t)$ and $f_3 = f_3(x,t)$ are given external forces.

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When there exist two different diffusion processes with different diffusion speeds (e.g., heat and solute) in the fluid and when the distributions of these diffusion processes are heterogeneous, the behavior of fluid becomes more complicated than those of simplified diffusion models. Such complex fluid phenomena, the so-called double-diffusive convection, can be observed in various fields, for instance, oceanography, geology and astrophysics. Particularly, the double-diffusive convection phenomenon in porous media is regarded as one of the important subjects, since it has large area of application, for example, models of the soil pollution, the storage of heat-generating materials (e.g., grain and coal) and the chemical reaction in catalysts. When we deal with double-diffusive convection phenomena in porous media, the so-called Brinkman-Forchheimer equation, which is derived from a modified Darcy's law, is applied in order to describe the behavior of fluid velocity. Although the original Brinkman-Forchheimer equation has some nonlinear terms and a function which stands for the porosity (the rate of void space of the medium), we adopt a linearized Brinkman-Forchheimer equation as the first equation of (DCBF) on the basis of the fact that some recent researches suggest the smallness of these nonlinear terms and the assumption that the porous medium is homogeneous. Moreover, based on Oberbeck-Boussinesq approximation, the terms $\boldsymbol{g}T$ and $\boldsymbol{h}C$ are added to the first equation of (DCBF) in order to describe the effects of buoyancy. The second and third equations, derived from the results of non-equilibrium statistical physics, possess convection terms $\boldsymbol{u} \cdot \nabla T$ and $\boldsymbol{u} \cdot \nabla C$, which make (DCBF) difficult to deal with as non-monotone perturbations. Here, $\rho \Delta T$ in the third equation designates Soret's effect, one of the interactions between the temperature and the concentration. To be precise, we have to add the term $\rho' \Delta C$, called Dufour's effect, in the second equation of (DCBF). However, since Dufour's effect is much smaller than Soret's effect, we neglect $\rho' \Delta C$ in our model (for more details and examples, see, e.g., Brandt-Fernando [2], Nield-Bejan [10] and Radko [17]).

As for previous studies for (DCBF), the solvability of the initial boundary value problem and the time periodic problem in bounded domains is investigated in Terasawa–Ôtani [20] and Ôtani–U. [14] respectively. In spite of the presence of convection terms which are quite similar to $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in the Navier-Stokes equations, the global solvability of (DCBF) for $N \leqslant 3$ with large data (initial data and external forces without smallness assumptions) is shown in [20] and [14]. In these papers, the existence of solution is assured by the application of abstract results given in Ôtani [11] and [12], where evolution equations governed by subdifferential operators with non-monotone perturbations are considered. Since Rellich-Kondrachov's theorem plays a significant role in order to apply the abstract theory, the boundedness of domains is an essential condition in [20] and [14]. However, in our recent study [15], the global solvability of the initial boundary value problem in general domains for $N \leqslant 4$ with large data is assured via Banach's contraction mapping principle. Motivated by these results, we aim to extend the solvability of the time periodic problem to those for unbounded domains. In particular, since we obtained the existence of solution with large data in [20], [14] and [15], we construct a periodic solution of (DCBF) without smallness conditions of external forces.

However, to the best of our knowledge, there are very few studies for the solvability of time periodic problem in unbounded domains with large data, especially, for parabolic type equations with non-monotone perturbations, where the uniqueness of solution is not assured.

Time periodic problems in unbounded domains have been studied in, e.g., Maremonti [8], Kozono–Nakao [7] for the Navier-Stokes equations and Villamizar-Roa–Rodríguez-Bellido–Rojas-Medar [21] for Boussinesq system (coupling of the Navier-Stokes equations and the second equation of (DCBF)). In their arguments, the smallness of given data seems to be essential in order to assure the convergence of iterations. On the other hand, as for the solvability of time periodic problem with large data, abstract evolution equations associated with subdifferential operators in Hilbert space have been investigated so far, e.g., in Bénilan–Brézis [1], Nagai [9], Yamada [22] and Ôtani [12]. Moreover, in Inoue–Ôtani [6], the solvability of periodic problem for Boussinesq system in non-cylindrical domains (moving bounded domains) is shown by the application of result given in Ôtani [12]. In these abstract theories, the coercivity of subdifferential operators seems to be one of essential conditions. Particularly, in Ôtani [12], φ-level set compactness is assumed so that Schauder-Tychonoff-type fixed point theorem can be available. These conditions assumed in previous studies for abstract problems are usually guaranteed by the boundedness of space domains when we apply them to concrete partial differential equations.

The main purpose of this paper is to construct of a time periodic solution for (DCBF) in $\mathbb{R}^N$ with large data via the convergence of solutions for approximate equations in bounded domains. In the next section, we define some notations and state our main result. In Section 3, we give an outline of our proof. Our argument follows the basic strategy given in Ôtani [13], namely, relies on local strong convergence and diagonal argument. Our proof is roughly divide into three steps. In Sections 4-6, we give some details of each step.

2 Notation and Main Result

Let Ω stand for either a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary or $\mathbb{R}^N$ itself. We define $\mathbb{L}^q(\Omega) := (L^q(\Omega))^N$, $\mathbb{W}^{k,q}(\Omega) := (W^{k,q}(\Omega))^N$ and $\mathbb{H}^k(\Omega) := (H^k(\Omega))^N$, where $L^q(\Omega)$, $W^{k,q}(\Omega)$ and $H^k(\Omega) := W^{k,2}(\Omega)$ designate the standard Lebesgue and Sobolev spaces ($1 \leqslant q \leqslant \infty$, $k \in \mathbb{N}$).

We here recall that the Helmholtz decomposition holds for $\mathbb{L}^q(\Omega)$ with $q \in (1, \infty)$ (see, e.g., Fujiwara–Morimoto [4] and Galdi [5]). That is to say, for any $\boldsymbol{v} \in \mathbb{L}^q(\Omega)$, the following decomposition is uniquely determined.

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2, \quad \boldsymbol{w}_1 \in \mathbb{L}^q_\sigma(\Omega) \text{ and } \boldsymbol{w}_2 \in G_q(\Omega),$$

where each functional space is defined by

$$\begin{aligned}
&\mathbb{C}^\infty_\sigma(\Omega) := \{\boldsymbol{w} \in \mathbb{C}^\infty_0(\Omega) = (C^\infty_0(\Omega))^N;\ \nabla \cdot \boldsymbol{w}(x) = 0 \ \ \forall x \in \Omega\},\\
&\mathbb{L}^q_\sigma(\Omega) : \text{ the closure of } \mathbb{C}^\infty_\sigma(\Omega) \text{ in } \mathbb{L}^q(\Omega),\\
&G_q(\Omega) := \{\boldsymbol{w} \in \mathbb{L}^q(\Omega);\ \exists p \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}), \ \text{ s.t., } \boldsymbol{w} = \nabla p\}.
\end{aligned}$$

Let $\mathcal{P}_\Omega$ stand for the orthogonal projection from $\mathbb{L}^2(\Omega)$ onto $\mathbb{L}^2_\sigma(\Omega)$. Then we define the Stokes operator by $\mathcal{A}_\Omega := -\mathcal{P}_\Omega\Delta$ with domain $D(\mathcal{A}_\Omega) = \mathbb{H}^2(\Omega) \cap \mathbb{H}^1_\sigma(\Omega)$, where $\mathbb{H}^1_\sigma(\Omega)$ denotes the closure of $\mathbb{C}^\infty_\sigma(\Omega)$ in $\mathbb{H}^1(\Omega)$. We here remark that $\mathcal{A}_{\mathbb{R}^N}\boldsymbol{v} = -\Delta\boldsymbol{v}$ holds for any $\boldsymbol{v} \in D(\mathcal{A}_{\mathbb{R}^N})$, i.e., $\boldsymbol{v} \in D(\mathcal{A}_{\mathbb{R}^N})$ satisfies $\Delta\boldsymbol{v} \in \mathbb{L}^2_\sigma(\mathbb{R}^N)$ (see Constantin–Foais [3], Sohr [18] and Temam [19]).

Henceforth, q^* and q' stand for the critical Sobolev exponent and the conjugate Hölder exponent associated with $q \in [1, \infty]$, namely, $q^* := qN/(N-q)$ for $N > q$ and $q' := q/(q-1)$. Moreover, we define $C_\pi([0,S]; X) := \{U \in C([0,S]; X); U(0) = U(S)\}$ (the set of continuous periodic functions with value in Banach space X).

We deal with the periodic solution of (DCBF) in the following sense:

Definition 2.1 (Periodic solution of (DCBF)). *Let $N = 3$ or 4. Then $(\boldsymbol{u}, T, C)$ is* *called a (periodic) solution of* (DCBF), *if* $(\boldsymbol{u}, T, C)$ *satisfies the following conditions:*

1. $(\boldsymbol{u}, T, C)$ *satisfies the following regularities:*

$$\begin{aligned}
&\boldsymbol{u} \in C_\pi([0,S]; \mathbb{L}^{2^*}_\sigma(\mathbb{R}^N)), && T, C \in C_\pi([0,S]; L^{2^*}(\mathbb{R}^N)),\\
&\partial_{x_\mu}\boldsymbol{u} \in C_\pi([0,S]; \mathbb{L}^2(\mathbb{R}^N)), && \partial_{x_\mu}T,\ \partial_{x_\mu}C \in C_\pi([0,S]; L^2(\mathbb{R}^N)),\\
&\partial_t\boldsymbol{u} \in L^2(0,S; \mathbb{L}^2_\sigma(\mathbb{R}^N)), && \partial_t T,\ \partial_t C \in L^2(0,S; L^2(\mathbb{R}^N)),\\
&\partial_{x_\iota}\partial_{x_\mu}\boldsymbol{u} \in L^2(0,S; \mathbb{L}^2(\mathbb{R}^N)), && \partial_{x_\iota}\partial_{x_\mu}T,\ \partial_{x_\iota}\partial_{x_\mu}C \in L^2(0,S; L^2(\mathbb{R}^N)),\\
&\Delta\boldsymbol{u} \in L^2(0,S; \mathbb{L}^2_\sigma(\mathbb{R}^N)),
\end{aligned}$$

   where $\iota, \mu = 1, 2, \cdots, N$.

2. $(\boldsymbol{u}, T, C)$ *satisfies the second and third equations of* (DCBF) *in* $L^2(0,S; L^2(\mathbb{R}^N))$.
3. *For any* $\boldsymbol{\phi} \in L^2(0,S; \mathbb{L}^2_\sigma(\mathbb{R}^N)) \cap L^2(0,S; \mathbb{L}^{(2^*)'}_\sigma(\mathbb{R}^N))$, $(\boldsymbol{u}, T, C)$ *satisfies the following identity:*

$$\int_0^S \int_{\mathbb{R}^N} (\partial_t\boldsymbol{u} - \Delta\boldsymbol{u} + a\boldsymbol{u} - \boldsymbol{g}T - \boldsymbol{h}C - \boldsymbol{f}_1) \cdot \boldsymbol{\phi}\, dxdt = 0. \tag{2.1}$$

Then our main result can be stated as follows:

Theorem 2.2. *Let* $N = 3$ *or* 4 *and let* $a > 0$. *Moreover, assume that*

$$\begin{aligned}
&\boldsymbol{f}_1 \in W^{1,2}(0,S; \mathbb{L}^2(\mathbb{R}^N)), \qquad \boldsymbol{f}_1(0) = \boldsymbol{f}_1(S),\\
&f_2, f_3 \in L^2(0,S; L^2(\mathbb{R}^N)) \cap L^2(0,S; L^{(2^*)'}(\mathbb{R}^N)).
\end{aligned}$$

Then (DCBF) *possesses at least one periodic solution* $(\boldsymbol{u}, T, C)$.

Remark. We can show that the identity (2.1) in the condition *3* leads to the first equation of (DCBF). Indeed, recalling the basic property of the Helmholtz decomposition and the fact that the dual space of $L^2(0,S; \mathbb{L}^2_\sigma(\mathbb{R}^N)) \cap L^2(0,S; \mathbb{L}^{(2^*)'}_\sigma(\mathbb{R}^N))$ coincides with $L^2(0,S; \mathbb{L}^2_\sigma(\mathbb{R}^N)) + L^2(0,S; \mathbb{L}^{2^*}_\sigma(\mathbb{R}^N))$, we can show that the identity (2.1) yields the first equation of (DCBF) with $p = p_1 + p_2$, where

$$\begin{aligned}
&p_1(\cdot, t) \in W^{1,2^*}_{\text{loc}}(\mathbb{R}^N),\ \ p_2(\cdot, t) \in W^{1,2}_{\text{loc}}(\mathbb{R}^N) \text{ for any } t \in [0,S],\\
&\nabla p_1 \in C_\pi([0,S]; \mathbb{L}^{2^*}(\mathbb{R}^N)),\ \ \nabla p_2 \in C_\pi([0,S]; \mathbb{L}^2(\mathbb{R}^N)).
\end{aligned}$$

3 Strategy

Our proof consists of the following three steps:

Step 1: We consider the following problem with two approximation parameters $n \in \mathbb{N}$ and $\lambda > 0$:

$$(\mathrm{DCBF})_{n,\lambda} \begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}_{\Omega_n} \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_{\Omega_n} \boldsymbol{g} T + \mathcal{P}_{\Omega_n} \boldsymbol{h} C + \mathcal{P}_{\Omega_n} \boldsymbol{f}_1|_{\Omega_n} & (x,t) \in \Omega_n \times [0,S], \\ \partial_t T + \boldsymbol{u} \cdot \nabla T + \lambda T = \Delta T + f_2|_{\Omega_n} & (x,t) \in \Omega_n \times [0,S], \\ \partial_t C + \boldsymbol{u} \cdot \nabla C + \lambda C = \Delta C + \rho \Delta T + f_3|_{\Omega_n} & (x,t) \in \Omega_n \times [0,S], \\ \boldsymbol{u} = 0, \quad T = 0, \quad C = 0 & (x,t) \in \partial\Omega_n \times [0,S], \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S), \ T(\cdot,0) = T(\cdot,S), \ C(\cdot,0) = C(\cdot,S). \end{cases}$$

Here and henceforth, Ω_R stands for the open ball in $\mathbb{R}^N$ centered at the origin with radius $R > 0$, i.e., $\Omega_R := \{x \in \mathbb{R}^N; |x| < R\}$ and $F|_{\Omega_R}$ denotes the restriction of F onto Ω_R.

Step 2: Let $(\boldsymbol{u}_n, T_n, C_n)$ be a periodic solution of $(\mathrm{DCBF})_{n,\lambda}$ obtained in Step 1. Taking the limits of the solution $(\boldsymbol{u}_n, T_n, C_n)$ and the system $(\mathrm{DCBF})_{n,\lambda}$ as $n \to \infty$, we show that the following problem $(\mathrm{DCBF})_\lambda$ admits a periodic solution for each parameter $\lambda > 0$.

$$(\mathrm{DCBF})_\lambda \begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}_{\mathbb{R}^N} \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_{\mathbb{R}^N} \boldsymbol{g} T + \mathcal{P}_{\mathbb{R}^N} \boldsymbol{h} C + \mathcal{P}_{\mathbb{R}^N} \boldsymbol{f}_1 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \partial_t T + \boldsymbol{u} \cdot \nabla T + \lambda T = \Delta T + f_2 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \partial_t C + \boldsymbol{u} \cdot \nabla C + \lambda C = \Delta C + \rho \Delta T + f_3 & (x,t) \in \mathbb{R}^N \times [0,S], \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S), \ T(\cdot,0) = T(\cdot,S), \ C(\cdot,0) = C(\cdot,S). \end{cases}$$

Step 3: Let $(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)$ be a periodic solution of $(\mathrm{DCBF})_\lambda$ derived in Step 2. Taking the limits of the solution $(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)$ and the system $(\mathrm{DCBF})_\lambda$ as $\lambda \to 0$, we assure the existence of periodic solution for the original system (DCBF).

4 Step 1: Approximate Equation in Bounded Domain

Solvability of the time periodic problem for (DCBF) in bounded domains with large data has been already shown in [14]. To be precise, we have to consider the case where $N = 4$ additionally. However, we can easily show that arguments in [14] also can be carried out for $N = 4$, if the domain has sufficiently smooth boundary (see also [16], where another proof via Schauder's fixed point theorem is given).

Therefore, we can assure the following solvability for equations defined in bounded domains with large data.

Lemma 4.1. *Let $N \leqslant 4$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Moreover, assume that $\boldsymbol{F}_1 \in L^2(0,S;\mathbb{L}^2(\Omega))$ and $F_2, F_3 \in$*

$L^2(0,S;L^2(\Omega))$. *Then for any non-negative constants a and λ, the following* (4.1) *admits at least one periodic solution $(\boldsymbol{u},T,C)$.*

$$
(4.1)\quad \begin{cases} \partial_t \boldsymbol{u} + \nu A_\Omega \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_\Omega \boldsymbol{g}T + \mathcal{P}_\Omega \boldsymbol{h}C + \mathcal{P}_\Omega \boldsymbol{F}_1 & (x,t)\in\Omega\times[0,S],\\ \partial_t T + \boldsymbol{u}\cdot\nabla T + \lambda T = \Delta T + F_2 & (x,t)\in\Omega\times[0,S],\\ \partial_t C + \boldsymbol{u}\cdot\nabla C + \lambda C = \Delta C + \rho\Delta T + F_3 & (x,t)\in\Omega\times[0,S],\\ \boldsymbol{u}=0,\quad T=0,\quad C=0 & (x,t)\in\partial\Omega\times[0,S],\\ \boldsymbol{u}(\cdot,0)=\boldsymbol{u}(\cdot,S),\ T(\cdot,0)=T(\cdot,S),\ C(\cdot,0)=C(\cdot,S). \end{cases}
$$

Here $(\boldsymbol{u},T,C)$ is said to be a periodic solution of (4.1), *if*

1. *$(\boldsymbol{u},T,C)$ satisfies the following regularities:*

$$
\begin{aligned} &\boldsymbol{u}\in C_\pi([0,S];\mathbb{H}^1_\sigma(\Omega))\cap L^2(0,S;\mathbb{H}^2(\Omega))\cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega)),\\ &T,C\in C_\pi([0,S];H^1_0(\Omega))\cap L^2(0,S;H^2(\Omega))\cap W^{1,2}(0,S;L^2(\Omega)). \end{aligned}
$$

2. *$(\boldsymbol{u},T,C)$ satisfies the first equation of* (4.1) *in $L^2(0,S;\mathbb{L}^2_\sigma(\Omega))$ and the second and third equations in $L^2(0,S;L^2(\Omega))$.*

5 Step 2: Enlargement of the Domain ($n\to\infty$)

According to Lemma 4.1, we can assure that $(\mathrm{DCBF})_{n,\lambda}$ possesses a periodic solution $(\boldsymbol{u}_n,T_n,C_n)$ such that

$$
\begin{aligned} &\boldsymbol{u}_n\in C_\pi([0,S];\mathbb{H}^1_\sigma(\Omega_n))\cap L^2(0,S;\mathbb{H}^2(\Omega_n))\cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\Omega_n)),\\ &T_n,C_n\in C_\pi([0,S];H^1_0(\Omega_n))\cap L^2(0,S;H^2(\Omega_n))\cap W^{1,2}(0,S;L^2(\Omega_n)) \end{aligned}
$$

for each parameter $n\in\mathbb{R}^N$. In this section, we consider Step 2 of our proof, namely, we demonstrate the following Lemma 5.1 by discussing the convergence of solutions $(\boldsymbol{u}_n,T_n,C_n)$ as $n\to\infty$.

Lemma 5.1. *Let $N=3,4$ and let $\boldsymbol{f}_1\in L^2(0,S;\mathbb{L}^2(\mathbb{R}^N))$, $f_2,f_3\in L^2(0,S;L^2(\mathbb{R}^N))$. Then for any positive constants a and λ, the following problem $(\mathrm{DCBF})_\lambda$ possesses at least one periodic solution $(\boldsymbol{u},T,C)$.*

$$
(\mathrm{DCBF})_\lambda\quad \begin{cases} \partial_t \boldsymbol{u} + \nu A_{\mathbb{R}^N} \boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_{\mathbb{R}^N} \boldsymbol{g}T + \mathcal{P}_{\mathbb{R}^N} \boldsymbol{h}C + \mathcal{P}_{\mathbb{R}^N} \boldsymbol{f}_1 & (x,t)\in\mathbb{R}^N\times[0,S],\\ \partial_t T + \boldsymbol{u}\cdot\nabla T + \lambda T = \Delta T + f_2 & (x,t)\in\mathbb{R}^N\times[0,S],\\ \partial_t C + \boldsymbol{u}\cdot\nabla C + \lambda C = \Delta C + \rho\Delta T + f_3 & (x,t)\in\mathbb{R}^N\times[0,S],\\ \boldsymbol{u}(\cdot,0)=\boldsymbol{u}(\cdot,S),\ T(\cdot,0)=T(\cdot,S),\ C(\cdot,0)=C(\cdot,S). \end{cases}
$$

Here $(\boldsymbol{u},T,C)$ is said to be a periodic solution of $(\mathrm{DCBF})_\lambda$, if

1. $(\boldsymbol{u}, T, C)$ *satisfies the following regularities:*

$$\boldsymbol{u} \in C_\pi([0,S]; \mathbb{H}^1_\sigma(\mathbb{R}^N)) \cap L^2(0,S; \mathbb{H}^2(\mathbb{R}^N)) \cap W^{1,2}(0,S; \mathbb{L}^2_\sigma(\mathbb{R}^N)),$$
$$T, C \in C_\pi([0,S]; H^1(\mathbb{R}^N)) \cap L^2(0,S; H^2(\mathbb{R}^N)) \cap W^{1,2}(0,S; L^2(\mathbb{R}^N)).$$

2. $(\boldsymbol{u}, T, C)$ *satisfies the first equation of* $(\mathrm{DCBF})_\lambda$ *in* $L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N))$ *and the second and third equations in* $L^2(0,S;L^2(\mathbb{R}^N))$.

Proof. To begin with, we prepare the uniform boundedness of $(\boldsymbol{u}_n, T_n, C_n)$ independent of the parameter n by establishing some a priori estimates. Multiplying the second equation of $(\mathrm{DCBF})_{n,\lambda}$ by T_n, we have

$$(5.1)\quad \frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} + 2|\nabla T_n|^2_{L^2(\Omega_n)} + \lambda |T_n|^2_{L^2(\Omega_n)} \leqslant \frac{1}{\lambda}|f_2|_{\Omega_n}|^2_{L^2(\Omega_n)} \leqslant \frac{1}{\lambda}|f_2|^2_{L^2(\mathbb{R}^N)}.$$

Since T_n belongs to $C_\pi([0,S]; H^1_0(\Omega_n))$, $|T_n(0)|^2_{L^2(\Omega_n)} = |T_n(S)|^2_{L^2(\Omega_n)}$ holds. Then integration of (5.1) over $[0,S]$ gives

$$(5.2)\qquad 2\int_0^S |\nabla T_n(s)|^2_{L^2(\Omega_n)}ds + \lambda \int_0^S |T_n(s)|^2_{L^2(\Omega_n)}ds \leqslant \frac{1}{\lambda}|f_2|^2_{L^2(0,S;L^2(\mathbb{R}^N))}.$$

Here, from the continuity of T_n, there exist $t^n_1 \in [0,S]$ where $|T(\cdot)|_{H^1(\Omega_n)}$ attains its minimum, i.e.,

$$|T(t^n_1)|_{H^1(\Omega_n)} = \min_{t\in[0,S]} |T(t)|_{H^1(\Omega_n)}$$

holds. By using (5.2), we obtain

$$(5.3)\qquad \begin{aligned} |T_n(t^n_1)|^2_{H^1(\Omega_n)} &\leqslant \frac{1}{S}\int_0^S |T_n(s)|^2_{H^1(\Omega_n)}ds \leqslant \frac{1}{\lambda S}\left(\frac{1}{2} + \frac{1}{\lambda}\right)|f_2|^2_{L^2(0,S;L^2(\mathbb{R}^N))} \\ &\leqslant \gamma_1. \end{aligned}$$

Here and henceforth, γ_1 denotes a general constant independent of n. Therefore, integrating (5.1) over $[t^n_1, t]$ with $t \in [t^n_1, t^n_1 + S]$ and recalling that T_n is a S-periodic function, we obtain

$$(5.4)\qquad \sup_{0\leqslant t\leqslant S} |T_n(t)|^2_{L^2(\Omega_n)} \leqslant \gamma_1.$$

Similarly, multiplying the third equation of $(\mathrm{DCBF})_{n,\lambda}$ by C_n, we get

$$\frac{d}{dt}|C_n|^2_{L^2(\Omega_n)} + |\nabla C_n|^2_{L^2(\Omega_n)} + \lambda |C_n|^2_{L^2(\Omega_n)} \leqslant \rho^2 |\nabla T_n|^2_{L^2(\Omega_n)} + \frac{1}{\lambda}|f_3|^2_{L^2(\mathbb{R}^N)},$$

which, together with (5.2), yields,

$$\int_0^S |C_n(s)|^2_{H^1(\Omega_n)} ds \leqslant \gamma_1 \tag{5.5}$$

and

$$\sup_{0\leqslant t\leqslant S} |C_n(t)|^2_{L^2(\Omega_n)} \leqslant \gamma_1. \tag{5.6}$$

Moreover, multiplying the first equation of $(\mathrm{DCBF})_{n,\lambda}$ by $\boldsymbol{u}_n$, $\mathcal{A}_{\Omega_n}\boldsymbol{u}_n$ and $\partial_t \boldsymbol{u}_n$, we have

$$\begin{aligned}
&\frac{d}{dt}|\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + 2\nu|\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + a|\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} \\
&\qquad\qquad \leqslant \frac{3|\boldsymbol{g}|^2}{a}|T_n|^2_{L^2(\Omega_n)} + \frac{3|\boldsymbol{h}|^2}{a}|C_n|^2_{L^2(\Omega_n)} + \frac{3}{a}|\boldsymbol{f}_1|^2_{\mathbb{L}^2(\mathbb{R}^N)}, \\
&\frac{d}{dt}|\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + \nu|\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} \\
&\qquad\qquad \leqslant \frac{3|\boldsymbol{g}|^2}{\nu}|T_n|^2_{L^2(\Omega_n)} + \frac{3|\boldsymbol{h}|^2}{\nu}|C_n|^2_{L^2(\Omega_n)} + \frac{3}{\nu}|\boldsymbol{f}_1|^2_{\mathbb{L}^2(\mathbb{R}^N)}, \\
&|\partial_t \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + \nu\frac{d}{dt}|\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} + a\frac{d}{dt}|\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} \\
&\qquad\qquad \leqslant 3|\boldsymbol{g}|^2|T_n|^2_{L^2(\Omega_n)} + 3|\boldsymbol{h}|^2|C_n|^2_{L^2(\Omega_n)} + 3|\boldsymbol{f}_1|^2_{\mathbb{L}^2(\mathbb{R}^N)}.
\end{aligned}$$

From (5.4) and (5.6), we can derive

$$\int_0^S |\boldsymbol{u}_n(s)|^2_{\mathbb{H}^1(\Omega_n)} ds + \int_0^S |\mathcal{A}_{\Omega_n}\boldsymbol{u}_n(s)|^2_{\mathbb{L}^2(\Omega_n)} ds + \int_0^S |\partial_t \boldsymbol{u}_n(s)|^2_{\mathbb{L}^2(\Omega_n)} ds \leqslant \gamma_1 \tag{5.7}$$

and

$$\sup_{0\leqslant t\leqslant S} |\boldsymbol{u}_n(t)|^2_{\mathbb{H}^1(\Omega_n)} \leqslant \gamma_1. \tag{5.8}$$

We here prepare the following inequalities so that we can accomplish the second energy estimates for T_n and C_n.

Lemma 5.2. *Let $R > 0$ and let $\boldsymbol{w} \in \mathbb{H}^2(\Omega_R) \cap \mathbb{H}^1_\sigma(\Omega_R)$ and $U \in H^2(\Omega_R) \cap H^1_0(\Omega_R)$. Then there exist some constant β which is independent of R such that the following inequalities hold:*

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\Omega_R)} \leqslant \beta|\nabla \boldsymbol{w}|^2_{\mathbb{L}^2(\Omega_R)}|\nabla U|_{L^2(\Omega_R)}|\Delta U|_{L^2(\Omega_R)} \tag{5.9}$$

for $N = 3$,

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\Omega_R)} \leqslant \beta|\nabla\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}|\mathcal{A}_{\Omega_R}\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}|\nabla U|_{L^2(\Omega_R)}|\Delta U|_{L^2(\Omega_R)} \tag{5.10}$$

for $N = 4$ *and*

$$\left|\partial_{x_\iota}\partial_{x_\mu}U\right|_{L^2(\Omega_R)} \leqslant \beta\,|\Delta U|_{L^2(\Omega_R)}, \quad \left|\partial_{x_\iota}\partial_{x_\mu}\boldsymbol{w}\right|_{\mathbb{L}^2(\Omega_R)} \leqslant \beta\,|\mathcal{A}_{\Omega_R}\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)} \tag{5.11}$$

for $N = 3, 4$, *where* $\iota, \mu = 1, 2, \cdots, N$.

Proof of Lemma 5.2. We here only prove (5.10), i.e., an estimate of convection term for $N = 4$ ((5.9) and (5.11) can be demonstrated by almost the same argument as that stated below).

From Hölder's inequality, we get

$$|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\Omega_R)} \leqslant |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega_R)}|\nabla U|_{L^2(\Omega_R)}|\nabla U|_{L^4(\Omega_R)}. \tag{5.12}$$

Moreover, by applying Sobolev's inequality, elliptic estimates and Poincaré's inequality,

$$|\nabla U|_{L^4(\Omega_R)} \leqslant \beta_{\Omega_R}|U|_{H^2(\Omega_R)} \leqslant \beta_{\Omega_R}|\Delta U|_{L^2(\Omega_R)} \tag{5.13}$$

and

$$\begin{aligned} |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega_R)} &\leqslant \beta'_{\Omega_R}|\boldsymbol{w}|^2_{\mathbb{W}^{1,8/3}(\Omega_R)} \leqslant \beta'_{\Omega_R}|\boldsymbol{w}|_{\mathbb{W}^{1,2}(\Omega_R)}|\boldsymbol{w}|_{\mathbb{W}^{1,4}(\Omega_R)} \\ &\leqslant \beta'_{\Omega_R}|\boldsymbol{w}|_{\mathbb{H}^1(\Omega_R)}|\boldsymbol{w}|_{\mathbb{H}^2(\Omega_R)} \leqslant \beta'_{\Omega_R}|\nabla\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}|\mathcal{A}_{\Omega_R}\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)} \end{aligned} \tag{5.14}$$

can be obtained, where β_{Ω_R} and β'_{Ω_R} are some general constants which may depend on R.

Here we define $U_R \in H^2(\Omega_1) \cap H^1_0(\Omega_1)$ and $\boldsymbol{w}_R \in \mathbb{H}^2(\Omega_1) \cap \mathbb{H}^1_\sigma(\Omega_1)$ by $U_R(y) := U(Ry)$ and $\boldsymbol{w}_R(y) := \boldsymbol{w}(Ry)$, where $y \in \Omega_1$. Then, under the scale conversion $y = x/R$, the following identities hold:

$$\begin{aligned} &|\nabla_x U|^4_{L^4(\Omega_R)} = |\nabla_y U_R|^4_{L^4(\Omega_1)}, && |\Delta_x U|^2_{L^2(\Omega_R)} = |\Delta_y U_R|^2_{L^2(\Omega_1)}, \\ &|\boldsymbol{w}|^8_{\mathbb{L}^8(\Omega_R)} = R^4|\boldsymbol{w}_R|^8_{\mathbb{L}^8(\Omega_1)}, && |\nabla_x \boldsymbol{w}|^2_{\mathbb{L}^2(\Omega_R)} = R^2|\nabla_y \boldsymbol{w}_R|^2_{\mathbb{L}^2(\Omega_1)}, \end{aligned}$$

where we use the fact that $\nabla_x = (\partial_{x_1}, \cdots, \partial_{x_N}) = \frac{1}{R}(\partial_{y_1}, \cdots, \partial_{y_N}) = \frac{1}{R}\nabla_y$ and $\Delta_x = \nabla_x^2 = \frac{1}{R^2}\nabla_y^2 = \frac{1}{R^2}\Delta_y$. Moreover, under the change of variable $y = x/R$, we can derive

$$\mathcal{P}_{\Omega_R}\Delta_x \boldsymbol{w}(x) = \frac{1}{R^2}\mathcal{P}_{\Omega_1}\Delta_y \boldsymbol{w}_R(y). \tag{5.15}$$

Indeed, since $\boldsymbol{w} \in \mathbb{H}^2(\Omega_R)$, the decomposition $\Delta_x \boldsymbol{w} = \boldsymbol{v}^1 + \boldsymbol{v}^2$ is valid with some $\boldsymbol{v}^1 \in \mathbb{L}^2_\sigma(\Omega_R)$ and $\boldsymbol{v}^2 \in G_2(\Omega_R)$. By the definition of $G_2(\Omega_R)$, there exists $P \in W^{1,2}(\Omega_R)$ such that $\boldsymbol{v}^2 = \nabla_x P$. Here we define $\boldsymbol{v}^1_R(y) := \boldsymbol{v}^1(Ry)$ and $P_R(y) := P(Ry)$, where $y \in \Omega_1$. Obviously, $\boldsymbol{v}^1_R \in \mathbb{L}^2_\sigma(\Omega_1)$ and $P_R \in W^{1,2}(\Omega_n)$ can be verified. Hence, converting the variables under the relationship $y = x/R$, we obtain $\frac{1}{R^2}\Delta_y \boldsymbol{w}_R = \boldsymbol{v}^1_R + \frac{1}{R}\nabla_y P_R$. Therefore, since the Helmholtz decomposition is uniquely determined, we can assure the identity (5.15). Then, from (5.15), we can derive

$$|\mathcal{A}_{\Omega_R}\boldsymbol{w}|^2_{\mathbb{L}^2(\Omega_R)} = \int_{\Omega_R} |\mathcal{P}_{\Omega_R}\Delta_x \boldsymbol{w}(x)|^2 dx = \int_{\Omega_1} |\mathcal{P}_{\Omega_1}\Delta_y \boldsymbol{w}_R(y)|^2\, dy = |\mathcal{A}_{\Omega_1}\boldsymbol{w}_R|^2_{\mathbb{L}^2(\Omega_1)}.$$

Therefore, using these identities under $y = x/R$ and recalling (5.12), (5.13), (5.14), we can deduce

$$\begin{aligned}
|\boldsymbol{w}\cdot\nabla U|^2_{L^2(\Omega_R)} &\leqslant |\boldsymbol{w}|^2_{\mathbb{L}^8(\Omega_R)}|\nabla_x U|_{L^2(\Omega_R)}|\nabla_x U|_{L^4(\Omega_R)} \\
&\leqslant R|\boldsymbol{w}_R|^2_{\mathbb{L}^8(\Omega_1)}|\nabla_x U|_{L^2(\Omega_R)}|\nabla_y U_R|_{L^4(\Omega_1)} \\
&\leqslant R\beta'_{\Omega_1}|\nabla_y \boldsymbol{w}_R|_{\mathbb{L}^2(\Omega_1)}|\mathcal{A}_{\Omega_1}\boldsymbol{w}_R|_{\mathbb{L}^2(\Omega_1)}|\nabla_x U|_{L^2(\Omega_R)}\beta_{\Omega_1}|\Delta_y U_R|_{L^2(\Omega_1)} \\
&\leqslant R\beta'_{\Omega_1}\beta_{\Omega_1}R^{-1}|\nabla_x \boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}|\mathcal{A}_{\Omega_R}\boldsymbol{w}|_{\mathbb{L}^2(\Omega_R)}|\nabla_x U|_{L^2(\Omega_R)}|\Delta_x U|_{L^2(\Omega_R)},
\end{aligned}$$

which implies that (5.10) holds for any $R > 0$ with the coefficient $\beta = \beta'_{\Omega_1}\beta_{\Omega_1}$. □

Proof of Lemma 5.1 (continued). Multiplying the second equation of $(\mathrm{DCBF})_{n,\lambda}$ by $-\Delta T_n$ and using (5.9) and (5.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \\
&\qquad\leqslant |\boldsymbol{u}_n\cdot\nabla T_n|_{L^2(\Omega_n)}|\Delta T_n|_{L^2(\Omega_n)} + |f_2|_{L^2(\mathbb{R}^N)}|\Delta T_n|_{L^2(\Omega_n)} \\
&\qquad\leqslant \gamma_1|\nabla \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^{1/2}_{L^2(\Omega_n)}|\Delta T_n|^{3/2}_{L^2(\Omega_n)} + |f_2|_{L^2(\mathbb{R}^N)}|\Delta T_n|_{L^2(\Omega_n)} \\
&\qquad\leqslant \frac{1}{2}|\Delta T_n|^2_{L^2(\Omega_n)} + \gamma_1|\nabla \boldsymbol{u}_n|^4_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + |f_2|^2_{L^2(\mathbb{R}^N)} \\
&\Rightarrow\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \leqslant \gamma_1|\nabla \boldsymbol{u}_n|^4_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.16}$$

for $N = 3$ and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \\
&\qquad\leqslant \frac{1}{2}|\Delta T_n|^2_{L^2(\Omega_n)} + \gamma_1|\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + |f_2|^2_{L^2(\mathbb{R}^N)} \\
&\Rightarrow\frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + |\Delta T_n|^2_{L^2(\Omega_n)} \\
&\qquad\leqslant \gamma_1|\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)}|\nabla T_n|^2_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.17}$$

for $N=4$. We here recall (5.3), i.e., $|\nabla T_n(t_1^n)|^2_{L^2(\Omega_n)} \leqslant \gamma_1$ holds for some $t_1^n \in [0,S]$. Then applying Gronwall's inequality to (5.16) and (5.17) over $[t_1^n, t]$ with $t \in [t_1^n, t_1^n+S]$, and using (5.7), (5.8) (uniform boundedness of $\boldsymbol{u}_n$), we obtain

$$\sup_{0\leqslant t\leqslant S} |\nabla T_n(t)|^2_{L^2(\Omega_n)} \leqslant \gamma_1. \tag{5.18}$$

Furthermore, integrations of (5.16) and (5.17) over $[0,S]$ yield

$$\int_0^S |\Delta T_n(s)|^2_{L^2(\Omega_n)} ds \leqslant \gamma_1. \tag{5.19}$$

Similarly, multiplying the second equation of $(\mathrm{DCBF})_{n,\lambda}$ by $\partial_t T_n$ and using (5.9) and (5.10), we get

$$\begin{aligned}
&|\partial_t T_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + \lambda\frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} \\
&\qquad \leqslant \gamma_1 |\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\nabla T_n|_{L^2(\Omega_n)} |\Delta T_n|_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.20}$$

for $N=3$ and

$$\begin{aligned}
&|\partial_t T_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla T_n|^2_{L^2(\Omega_n)} + \lambda\frac{d}{dt}|T_n|^2_{L^2(\Omega_n)} \\
&\qquad \leqslant \gamma_1 |\nabla \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\mathcal{A}_{\Omega_n} \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\nabla T_n|_{L^2(\Omega_n)} |\Delta T_n|_{L^2(\Omega_n)} + 2|f_2|^2_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{5.21}$$

for $N=4$. Integrating (5.20) and (5.21) over $[0,S]$, we have

$$\int_0^S |\partial_t T_n(s)|^2_{L^2(\Omega_n)} ds \leqslant \gamma_1. \tag{5.22}$$

By almost the same procedure as above, multiplications of the third equation by $-\Delta C_n$ and $\partial_t C_n$ yield

$$\begin{aligned}
&\frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + |\Delta C_n|^2_{L^2(\Omega_n)} \\
&\qquad \leqslant \gamma_1 |\nabla \boldsymbol{u}_n|^4_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|^2_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)}, \\
&|\partial_t C_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + \lambda\frac{d}{dt}|C_n|^2_{L^2(\Omega_n)} \\
&\qquad \leqslant \gamma_1 |\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|_{L^2(\Omega_n)} |\Delta C_n|_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)}
\end{aligned}$$

for $N=3$ and

$$\begin{aligned}
&\frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + |\Delta C_n|^2_{L^2(\Omega_n)} \\
&\leqslant \gamma_1 |\nabla \boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|^2_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|^2_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)}, \\
&|\partial_t C_n|^2_{L^2(\Omega_n)} + \frac{d}{dt}|\nabla C_n|^2_{L^2(\Omega_n)} + \lambda\frac{d}{dt}|C_n|^2_{L^2(\Omega_n)} \\
&\leqslant \gamma_1 |\nabla \boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\mathcal{A}_{\Omega_n}\boldsymbol{u}_n|_{\mathbb{L}^2(\Omega_n)} |\nabla C_n|_{L^2(\Omega_n)} |\Delta C_n|_{L^2(\Omega_n)} + 3\rho^2 |\Delta T_n|^2_{L^2(\Omega_n)} + 3|f_3|^2_{L^2(\mathbb{R}^N)}
\end{aligned}$$

for $N = 4$. From these inequalities, we can derive

$$(5.23)\qquad \sup_{0\leqslant t\leqslant S}|\nabla C_n|^2_{L^2(\Omega_n)}+\int_0^S|\Delta C_n(s)|^2_{L^2(\Omega_n)}ds+\int_0^S|\partial_t C_n(s)|^2_{L^2(\Omega_n)}ds\leqslant\gamma_1.$$

Hence, in view of (5.4), (5.6), (5.7), (5.8), (5.18), (5.19), (5.22) and (5.23), we get the followings:

$$(5.24)\qquad \sup_{0\leqslant t\leqslant S}\left|\widehat{T_n}(t)\right|^2_{H^1(\mathbb{R}^N)}+\sup_{0\leqslant t\leqslant S}\left|\widehat{C_n}(t)\right|^2_{H^1(\mathbb{R}^N)}+\sup_{0\leqslant t\leqslant S}|\widehat{\boldsymbol{u}_n}(t)|^2_{\mathbb{H}^1(\mathbb{R}^N)}\leqslant\gamma_1,$$

$$(5.25)\qquad \int_0^S\left(|[\Delta T_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}+|[\Delta C_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}+|[\mathcal{A}_{\Omega_n}\boldsymbol{u}_n]^\wedge(s)|^2_{\mathbb{L}^2(\mathbb{R}^N)}\right)ds\leqslant\gamma_1,$$

$$(5.26)\qquad \int_0^S\left(\left|\partial_t\widehat{T_n}(s)\right|^2_{L^2(\mathbb{R}^N)}+\left|\partial_t\widehat{C_n}(s)\right|^2_{L^2(\mathbb{R}^N)}+|\partial_t\widehat{\boldsymbol{u}_n}(s)|^2_{\mathbb{L}^2(\mathbb{R}^N)}\right)ds\leqslant\gamma_1,$$

where $\widehat{\cdot}$ and $[\cdot]^\wedge$ designate the zero-extension of function to the whole space $\mathbb{R}^N$, i.e., for example,

$$\widehat{T_n}(x,t)=[T_n]^\wedge(x,t):=\begin{cases}T_n(x,t) & (\text{ if } x\in\Omega_n),\\ 0 & (\text{ otherwise })\end{cases}$$

(remark that

$$\nabla[\boldsymbol{u}_n]^\wedge=[\nabla\boldsymbol{u}_n]^\wedge,\quad \nabla[T_n]^\wedge=[\nabla T_n]^\wedge,\quad \nabla[C_n]^\wedge=[\nabla C_n]^\wedge$$

are valid since $\boldsymbol{u}_n\in C([0,S];\mathbb{H}^1_\sigma(\Omega_n))$ and $T_n, C_n\in C([0,S];H^1_0(\Omega_n))$). Moreover, (5.11) and (5.25) yield

$$(5.27)\qquad \begin{aligned}&\int_0^S|[\partial_{x_\iota}\partial_{x_\mu}T_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}ds\leqslant\gamma_1,\quad \int_0^S|[\partial_{x_\iota}\partial_{x_\mu}C_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}ds\leqslant\gamma_1,\\ &\int_0^S|[\partial_{x_\iota}\partial_{x_\mu}\boldsymbol{u}_n]^\wedge(s)|^2_{\mathbb{L}^2(\mathbb{R}^N)}ds\leqslant\gamma_1\end{aligned}$$

for all $\iota,\mu=1,2,\cdots,N$. Using (5.9) and (5.10), we have

$$(5.28)\qquad \int_0^S|[\boldsymbol{u}_n\cdot\nabla T_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}ds+\int_0^S|[\boldsymbol{u}_n\cdot\nabla C_n]^\wedge(s)|^2_{L^2(\mathbb{R}^N)}ds\leqslant\gamma_1.$$

By (5.24), we can extract a subsequence $\{(\widehat{\boldsymbol{u}_{n_i}},\ \widehat{T_{n_i}},\ \widehat{C_{n_i}})\}_{i\in\mathbb{N}}$ of $\{(\widehat{\boldsymbol{u}_n},\ \widehat{T_n},\ \widehat{C_n})\}_{n\in\mathbb{N}}$ (simply denoted by $\{U_i\}_{i\in\mathbb{N}}:=\{(\widehat{\boldsymbol{u}_i},\ \widehat{T}_i,\ \widehat{C}_i)\}_{i\in\mathbb{N}}$ henceforth) which $*$-weakly converges in $L^\infty(0,S;\mathbb{H}^1_\sigma(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N))$. That is to say, there exist some

$U_* := (\boldsymbol{u}_*, T_*, C_*)$ such that

$$\begin{aligned}
&\widehat{\boldsymbol{u}_i} \rightharpoonup \boldsymbol{u}_* && *\text{-weakly in } L^\infty(0,S;\mathbb{H}^1_\sigma(\mathbb{R}^N)),\\
&\widehat{T_i} \rightharpoonup T_* && *\text{-weakly in } L^\infty(0,S;H^1(\mathbb{R}^N)),\\
&\widehat{C_i} \rightharpoonup C_* && *\text{-weakly in } L^\infty(0,S;H^1(\mathbb{R}^N)).
\end{aligned}$$

Furthermore, by (5.26) and (5.27), we can assure that U_* satisfies all the required regularities except the periodicity, i.e.,

$$\begin{aligned}
&\boldsymbol{u}_* \in C([0,S];\mathbb{H}^1_\sigma(\mathbb{R}^N)) \cap L^2(0,S;\mathbb{H}^2(\mathbb{R}^N)) \cap W^{1,2}(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)),\\
&T_*, C_* \in C([0,S];H^1(\mathbb{R}^N)) \cap L^2(0,S;H^2(\mathbb{R}^N)) \cap W^{1,2}(0,S;L^2(\mathbb{R}^N))
\end{aligned}$$

hold. Then (5.25) implies the following convergences:

$$\begin{aligned}
&[\mathcal{A}_{\Omega_i}\boldsymbol{u}_i]^\wedge \rightharpoonup \mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_* && \text{weakly in } L^2(0,S;\mathbb{L}^2_\sigma(\mathbb{R}^N)),\\
&[\Delta T_i]^\wedge \rightharpoonup \Delta T_* && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)),\\
&[\Delta C_i]^\wedge \rightharpoonup \Delta C_* && \text{weakly in } L^2(0,S;L^2(\mathbb{R}^N)),
\end{aligned}$$

namely, we can assure that all linear terms in the system $(\mathrm{DCBF})_{n,\lambda}$ to the corresponding terms in the system $(\mathrm{DCBF})_\lambda$.

In order to deduce the periodicity of U_* and assure the convergence of nonlinear terms $\{[\boldsymbol{u}_i \cdot \nabla T_i]^\wedge\}_{i\in\mathbb{N}}$, $\{[\boldsymbol{u}_i \cdot \nabla C_i]^\wedge\}_{i\in\mathbb{N}}$, we employ the following space-local strong convergence arguments. Recalling (5.24) and (5.26) , we get

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant S}\left|\widehat{T_i}\Big|_{\Omega_n}(t)\right|^2_{H^1(\Omega_n)} + \int_0^S \left|\partial_t\widehat{T_i}\Big|_{\Omega_n}(s)\right|^2_{L^2(\Omega_n)} ds \leqslant \gamma_1,\\
&\sup_{0\leqslant t\leqslant S}\left|\widehat{C_i}\Big|_{\Omega_n}(t)\right|^2_{H^1(\Omega_n)} + \int_0^S \left|\partial_t\widehat{C_i}\Big|_{\Omega_n}(s)\right|^2_{L^2(\Omega_n)} ds \leqslant \gamma_1,\\
&\sup_{0\leqslant t\leqslant S}|\widehat{\boldsymbol{u}_i}|_{\Omega_n}(t)|^2_{\mathbb{H}^1(\Omega_n)} + \int_0^S |\partial_t\widehat{\boldsymbol{u}_i}|_{\Omega_n}(s)|^2_{\mathbb{L}^2(\Omega_n)} ds \leqslant \gamma_1
\end{aligned}$$

for any $i \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $n_i \geqslant n$. These inequalities imply that we can apply Ascoli's theorem on Ω_n to the sequence $\{U_i\}_{i\in\mathbb{N}}$ and its subsequences for any $n \in \mathbb{N}$.

Therefore, applying Ascoli's theorem to $\{U_i\}_{i\in\mathbb{N}}$ with $n = 1$, we can extract a subsequence of $\{U_i\}_{i\in\mathbb{N}}$, which is simply denoted by $\{U_{i^1_j}\}_{j\in\mathbb{N}} := \{(\widehat{\boldsymbol{u}_{i^1_j}}, \widehat{T_{i^1_j}}, \widehat{C_{i^1_j}})\}_{j\in\mathbb{N}}$, such that

$$\widehat{T_{i^1_j}}|_{\Omega_1} \to T^1 \qquad \text{strongly in } C_\pi([0,S];L^2(\Omega_1)),$$
$$\widehat{C_{i^1_j}}|_{\Omega_1} \to C^1 \qquad \text{strongly in } C_\pi([0,S];L^2(\Omega_1)),$$
$$\widehat{\boldsymbol{u}_{i^1_j}}|_{\Omega_1} \to \boldsymbol{u}^1 \qquad \text{strongly in } C_\pi([0,S];\mathbb{L}^2(\Omega_1)).$$

Here we can easily deduce the periodicity of the limit $U^1 := (\boldsymbol{u}^1, T^1, C^1)$ from the periodicity of U_i for each $i \in \mathbb{N}$. Next, applying Ascoli's theorem to $\{U_{i^1_j}\}_{j\in\mathbb{N}}$ with $n = 2$, we can assure that there exists a subsequence $\{U_{i^2_j}\}_{j\in\mathbb{N}} := \{(\widehat{\boldsymbol{u}_{i^2_j}}, \widehat{T_{i^2_j}}, \widehat{C_{i^2_j}})\}_{j\in\mathbb{N}}$ which satisfies

$$\widehat{T_{i^2_j}}|_{\Omega_2} \to T^2 \qquad \text{strongly in } C_\pi([0,S];L^2(\Omega_2)),$$
$$\widehat{C_{i^2_j}}|_{\Omega_2} \to C^2 \qquad \text{strongly in } C_\pi([0,S];L^2(\Omega_2)),$$
$$\widehat{\boldsymbol{u}_{i^2_j}}|_{\Omega_2} \to \boldsymbol{u}^2 \qquad \text{strongly in } C_\pi([0,S];\mathbb{L}^2(\Omega_2)).$$

As for the relationship between U^1 and U^2, we can easily show that

$$U^1(x,t) = U^2(x,t) \quad \forall t \in [0,S], \quad \text{for a.e. } x \in \Omega_1.$$

Repeating these procedures inductively for each $n \in \mathbb{N}$, we can extract a subsequence $\{U_{i^n_j}\}_{j\in\mathbb{N}}$ of $\{U_{i^{(n-1)}_j}\}_{j\in\mathbb{N}}$ such that

$$\widehat{T_{i^n_j}}|_{\Omega_n} \to T^n \qquad \text{strongly in } C_\pi([0,S];L^2(\Omega_n)),$$
$$\widehat{C_{i^n_j}}|_{\Omega_n} \to C^n \qquad \text{strongly in } C_\pi([0,S];L^2(\Omega_n)),$$
$$\widehat{\boldsymbol{u}_{i^n_j}}|_{\Omega_2} \to \boldsymbol{u}^n \qquad \text{strongly in } C_\pi([0,S];\mathbb{L}^2(\Omega_n)),$$

where the limit $U^n := (\boldsymbol{u}^n, T^n, C^n)$ satisfies

$$U^{n_1}(x,t) = U^{n_2}(x,t) \quad \forall t \in [0,S], \quad \text{for a.e. } x \in \Omega_{n_1} \tag{5.29}$$

for $n_2 \geqslant n_1$. Moreover, extracting a subsequence along the diagonal part $\{U_{i^l_l}\}_{l\in\mathbb{N}}$, simply denoted by $\{U_l\}_{l\in\mathbb{N}}$, we can show that this subsequence satisfies the following convergences for all $n \in \mathbb{N}$:

$$\begin{aligned} &\widehat{T_l}|_{\Omega_n} \to T^n && \text{strongly in } C([0,S];L^2(\Omega_n)), \\ &\widehat{C_l}|_{\Omega_n} \to C^n && \text{strongly in } C([0,S];L^2(\Omega_n)), \\ &\widehat{\boldsymbol{u}_l}|_{\Omega_n} \to \boldsymbol{u}^n && \text{strongly in } C([0,S];\mathbb{L}^2(\Omega_n)). \end{aligned} \tag{5.30}$$

On the bases of (5.29), we can define

$$U(x,t) := U^n(x,t) \qquad\qquad \text{if } x \in \Omega_n.$$

Then, from the space-local strong convergence (5.30), it is easy to see that U coincides with the $*$-weak limit U_*, which implies that U_* is S-periodic.

Finally, we check the convergence of $\{[\boldsymbol{u}_l \cdot \nabla T_l]^\wedge\}_{l\in\mathbb{N}}$ and $\{[\boldsymbol{u}_l \cdot \nabla C_l]^\wedge\}_{l\in\mathbb{N}}$. From (5.28), $\{[\boldsymbol{u}_l \cdot \nabla T_l]^\wedge\}_{l\in\mathbb{N}}$ has a subsequence (still denoted by $\{[\boldsymbol{u}_l \cdot \nabla T_l]^\wedge\}_{l\in\mathbb{N}}$) which weakly converges in $L^2(0,S;L^2(\mathbb{R}^N))$. Let χ_1 be its limit. Here, we fix $\phi_1 \in C_0^\infty(\mathbb{R}^N \times (0,S))$ arbitrary and we assume that $M \in \mathbb{N}$ satisfies $supp\phi_1 \subset \Omega_M \times [0,S]$. Then, using the integration by parts, we have

$$\begin{aligned}\int_0^S\int_{\mathbb{R}^N} \phi_1[\boldsymbol{u}_l \cdot \nabla T_l]^\wedge dxdt &= \int_0^S\int_{\Omega_l} \phi_1|_{\Omega_l}\boldsymbol{u}_l \cdot \nabla T_l dxdt = -\int_0^S\int_{\Omega_l} \boldsymbol{u}_l T_l \cdot \nabla\phi_1|_{\Omega_l} dxdt \\ &= -\int_0^S\int_{\Omega_M} \boldsymbol{u}_l|_{\Omega_M} T_l|_{\Omega_M} \cdot \nabla\phi_1|_{\Omega_M} dxdt\end{aligned}$$

for any $l \in \mathbb{N}$ such that $n_{i_l^l} \geqslant M$. Therefore, taking the limit as $l \to \infty$, we obtain

$$\int_0^S\int_{\mathbb{R}^N} \phi_1\chi_1 dxdt = -\int_0^S\int_{\Omega_M} \boldsymbol{u}^M T^M \cdot \nabla\phi_1|_{\Omega_M} dxdt = -\int_0^S\int_{\mathbb{R}^N} \boldsymbol{u}T \cdot \nabla\phi_1 dxdt.$$

Moreover, by using the integration by parts again and recalling $\boldsymbol{u} = \boldsymbol{u}_*$, $T = T_*$, we can deduce

$$\int_0^S\int_{\mathbb{R}^N} \phi_1\chi_1 dxdt = -\int_0^S\int_{\mathbb{R}^N} \boldsymbol{u}_* T_* \cdot \nabla\phi_1 dxdt = \int_0^S\int_{\mathbb{R}^N} \phi_1\boldsymbol{u}_* \cdot \nabla T_* dxdt$$

for any $C_0^\infty(\mathbb{R}^N \times (0,S))$, which implies that χ_1 coincides with $\boldsymbol{u}_* \cdot \nabla T_*$. By exactly the same procedure, we can assure that $\{[\boldsymbol{u}_l \cdot \nabla C_l]^\wedge\}_{l\in\mathbb{N}}$ weakly converges to $\boldsymbol{u}_* \cdot \nabla C_*$ in $L^2(0,S;L^2(\mathbb{R}^N))$.

Consequently, we can assure that $(\boldsymbol{u}_*, T_*, C_*)$ becomes a periodic solution of $(\text{DCBF})_\lambda$. □

6 Step 3: Convergence as $\lambda \to 0$

In this section, we consider Step 3, namely, we show that the time periodic solution $(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)$ of $(\text{DCBF})_\lambda$, derived in Lemma 5.1, converges to a periodic solution of the original system (DCBF). Basic strategy in Step 3 is the same as those in Step 2, i.e., we first show some uniform boundedness of $(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)$ by establishing appropriate a priori estimates and we discuss weak-convergences and space-local strong convergence as $\lambda \to 0$ by using uniform a priori bounds. In this section, we only show a priori estimates. Henceforth, γ_2 designates a general constant independent of the parameter λ. Moreover, we write simply $|\cdot|_{L^p}$ and $|\cdot|_{H^k}$ in order to designate the norm in $L^p(\mathbb{R}^N)$ and $H^k(\mathbb{R}^N)$ respectively in this section, if there is no confusion.

Multiplying the second equation of $(\mathrm{DCBF})_\lambda$ by T_λ and applying Hölder's inequality, Sobolev's inequality, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|T_\lambda|^2_{L^2} + |\nabla T_\lambda|^2_{L^2} + \lambda|T_\lambda|^2_{L^2} &= \int_{\mathbb{R}^N} f_2 T_\lambda dx \\ &\leqslant |f_2|_{L^{(2^*)'}}|T_\lambda|_{L^{2^*}} \leqslant \gamma_2 |f_2|_{L^{(2^*)'}}|\nabla T_\lambda|_{L^2},\end{aligned}$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}|T_\lambda|^2_{L^2} + \frac{1}{2}|\nabla T_\lambda|^2_{L^2} + \lambda|T_\lambda|^2_{L^2} \leqslant \gamma_2 |f_2|^2_{L^{(2^*)'}}. \tag{6.1}$$

Under the assumption that f_2 belongs to $L^2(0,S;L^{(2^*)'}(\mathbb{R}^N))$, (6.1) yields

$$\int_0^S |\nabla T_\lambda(s)|^2_{L^2} ds + \lambda \int_0^S |T_\lambda(s)|^2_{L^2} ds \leqslant \gamma_2. \tag{6.2}$$

Similarly, multiplying the third equation of $(\mathrm{DCBF})_\lambda$ by C_λ, we have

$$\frac{1}{2}\frac{d}{dt}|C_\lambda|^2_{L^2} + \frac{1}{2}|\nabla C_\lambda|^2_{L^2} + \lambda|C_\lambda|^2_{L^2} \leqslant \rho^2|\nabla T_\lambda|^2_{L^2} + \gamma_2 |f_3|^2_{L^{(2^*)'}}.$$

Integrating this inequality over $[0,S]$ and using (6.2), we obtain

$$\int_0^S |\nabla C_\lambda(s)|^2_{L^2} ds + \lambda \int_0^S |C_\lambda(s)|^2_{L^2} ds \leqslant \gamma_2, \tag{6.3}$$

since $f_3 \in L^2(0,S;L^{(2^*)'}(\mathbb{R}^N))$.

Here we remark that the multiplications of the first equation by $\boldsymbol{u}_\lambda$ and $\partial_t \boldsymbol{u}_\lambda$ do not yield useful estimates, since we do not obtain L^2-estimates for $\boldsymbol{g}T_\lambda$ and $\boldsymbol{h}C_\lambda$ in (6.2) and (6.3). However, multiplying the first equation of $(\mathrm{DCBF})_\lambda$ by $\mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_\lambda$, we can obtain the following useful estimate:

$$\frac{d}{dt}|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \nu|\Delta \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + a|\nabla \boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} \leqslant \frac{2|\boldsymbol{g}|^2}{a}|\nabla T_\lambda|^2_{L^2} + \frac{2|\boldsymbol{h}|^2}{a}|\nabla C_\lambda|^2_{L^2} + \frac{1}{\nu}|\boldsymbol{f}_1|^2_{\mathbb{L}^2}. \tag{6.4}$$

Indeed, recalling the regularity of $\boldsymbol{u}_\lambda$, in particular, the fact that $\boldsymbol{u}_\lambda(t) \in D(\mathcal{A}_{\mathbb{R}^N})$ holds for almost all $t \in [0, S]$, we can assure that

$$\mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_\lambda(t) = -\Delta\boldsymbol{u}_\lambda(t) \qquad \text{for a.e. } t \in [0, S]$$

can be verified. Hence, the integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N}\boldsymbol{g}T_\lambda dx &= -\int_{\mathbb{R}^N} \Delta\boldsymbol{u}_\lambda \cdot \boldsymbol{g}T_\lambda dx = \int_{\mathbb{R}^N} \nabla\boldsymbol{u}_\lambda \cdot \nabla\boldsymbol{g}T_\lambda dx \\ &\leqslant |\nabla\boldsymbol{u}_\lambda|_{\mathbb{L}^2}|\boldsymbol{g}||\nabla T_\lambda|_{L^2} \leqslant \frac{a}{4}|\nabla\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \frac{|\boldsymbol{g}|^2}{a}|\nabla T_\lambda|^2_{L^2}\end{aligned}$$

and

$$\int_{\mathbb{R}^N} \Delta\boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N}\boldsymbol{h}C_\lambda dx \leqslant \frac{a}{4}|\nabla\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \frac{|\boldsymbol{h}|^2}{a}|\nabla C_\lambda|^2_{L^2}.$$

Therefore, multiplying the first equation of $(\mathrm{DCBF})_\lambda$ by $\mathcal{A}_{\mathbb{R}^N}\boldsymbol{u}_\lambda = -\Delta\boldsymbol{u}_\lambda$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\nabla\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \nu|\Delta\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + a|\nabla\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} \\ &\quad = -\int_{\mathbb{R}^N} \Delta\boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N}\boldsymbol{g}T_\lambda dx - \int_{\mathbb{R}^N} \Delta\boldsymbol{u}_\lambda \cdot \mathcal{P}_{\mathbb{R}^N}\boldsymbol{h}C_\lambda dx - \int_{\mathbb{R}^N} \boldsymbol{f}_1 \cdot \Delta\boldsymbol{u}_\lambda dx \\ &\quad \leqslant \frac{a}{2}|\nabla\boldsymbol{u}_\lambda|^2_{\mathbb{L}^2} + \frac{|\boldsymbol{g}|^2}{a}|\nabla T_\lambda|^2_{L^2} + \frac{|\boldsymbol{h}|^2}{a}|\nabla C_\lambda|^2_{L^2} + |\Delta\boldsymbol{u}_\lambda|_{\mathbb{L}^2}|\boldsymbol{f}_1|_{\mathbb{L}^2},\end{aligned}$$

which yields (6.4). Integrating (6.4) over $[0, S]$ and using (6.2) and (6.3), we have

$$\int_0^S |\Delta\boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2}ds + \int_0^S |\nabla\boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2}ds \leqslant \gamma_2. \tag{6.5}$$

Since $\boldsymbol{u}_\lambda \in C_\pi([0, S]; \mathbb{H}^1_\sigma(\mathbb{R}^N))$, there exists $t_2^\lambda \in [0, S]$ where $|\nabla\boldsymbol{u}(\cdot)|^2_{\mathbb{L}^2}$ attains its minimum. From (6.5), we can derive $|\nabla\boldsymbol{u}(t_2^\lambda)|_{\mathbb{L}^2} \leqslant \gamma_2$. Therefore integrating (6.4) over $[t_2^\lambda, t]$ $(t \in [t_2^\lambda, t_2^\lambda + S])$, we obtain

$$\sup_{0\leqslant t\leqslant S} |\nabla\boldsymbol{u}_\lambda(t)|_{\mathbb{L}^2}ds \leqslant \gamma_2. \tag{6.6}$$

Moreover, since $\boldsymbol{u}_\lambda \in C([0, S]; \mathbb{H}^1_\sigma(\mathbb{R}^N))$, Sobolev's inequality and (6.6) lead to $\boldsymbol{u}_\lambda \in C([0, S]; \mathbb{L}^{2^*}_\sigma(\mathbb{R}^N))$ and

$$\sup_{0\leqslant t\leqslant S} |\boldsymbol{u}_\lambda(t)|_{\mathbb{L}^{2^*}}ds \leqslant \gamma_2. \tag{6.7}$$

Here, by using almost the same argument as that in our proof for Lemma 5.2 and the fact that $C^\infty(\mathbb{R}^N)$ (resp. $\mathbb{C}^\infty(\mathbb{R}^N)$) is dense in $H^2(\mathbb{R}^N)$ (resp. $\mathbb{H}^2(\mathbb{R}^N)$), we can obtain the following inequalities: for any $w \in \mathbb{H}^2(\mathbb{R}^N)$ and $U \in H^2(\mathbb{R}^N)$, there exist a constant β such that

$$(6.8) \qquad |w \cdot \nabla U|^2_{L^2(\mathbb{R}^N)} \leqslant \beta |\nabla w|^2_{\mathbb{L}^2(\mathbb{R}^N)} |\nabla U|_{L^2(\mathbb{R}^N)} |\Delta U|_{L^2(\mathbb{R}^N)}$$

for $N = 3$,

$$(6.9) \qquad |w \cdot \nabla U|^2_{L^2(\mathbb{R}^N)} \leqslant \beta |\nabla w|_{\mathbb{L}^2(\mathbb{R}^N)} |\Delta w|_{\mathbb{L}^2(\mathbb{R}^N)} |\nabla U|_{L^2(\mathbb{R}^N)} |\Delta U|_{L^2(\mathbb{R}^N)}$$

for $N = 4$ and

$$(6.10) \qquad \left|\partial_{x_\iota}\partial_{x_\mu} U\right|_{L^2(\mathbb{R}^N)} \leqslant \beta \left|\Delta U\right|_{L^2(\mathbb{R}^N)}, \qquad \left|\partial_{x_\iota}\partial_{x_\mu} w\right|_{\mathbb{L}^2(\mathbb{R}^N)} \leqslant \beta \left|\Delta w\right|_{\mathbb{L}^2(\mathbb{R}^N)}$$

for $N = 3, 4$, where $\iota, \mu = 1, 2, \cdots, N$. Multiplying the second equation of $(\mathrm{DCBF})_\lambda$ by $-\Delta T_\lambda$ and $\partial_t T_\lambda$, using (6.8), (6.9) and repeating exactly the same calculations as those for (5.16), (5.17), (5.20), (5.21), we obtain

$$(6.11) \qquad \begin{aligned} &\frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + |\Delta T_\lambda|^2_{L^2} \leqslant \gamma_2 |\nabla u_\lambda|^4_{\mathbb{L}^2} |\nabla T_\lambda|^2_{L^2} + 2|f_2|^2_{L^2}, \\ &|\partial_t T_\lambda|^2_{L^2} + \frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + \lambda \frac{d}{dt}|T_\lambda|^2_{L^2} \leqslant \gamma_2 |\nabla u_\lambda|^2_{\mathbb{L}^2} |\nabla T_\lambda|_{L^2} |\Delta T_\lambda|_{L^2} + 2|f_2|^2_{L^2} \end{aligned}$$

for $N = 3$ and

$$(6.12) \qquad \begin{aligned} &\frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + |\Delta T_\lambda|^2_{L^2} \leqslant \gamma_2 |\nabla u_\lambda|^2_{\mathbb{L}^2} |\Delta u_\lambda|^2_{\mathbb{L}^2} |\nabla T_\lambda|^2_{L^2} + 2|f_2|^2_{L^2}, \\ &|\partial_t T_\lambda|^2_{L^2} + \frac{d}{dt}|\nabla T_\lambda|^2_{L^2} + \lambda \frac{d}{dt}|T_\lambda|^2_{L^2} \\ &\qquad \leqslant \gamma_2 |\nabla u_\lambda|_{\mathbb{L}^2} |\Delta u_\lambda|_{\mathbb{L}^2} |\nabla T_n|_{L^2} |\Delta T_\lambda|_{L^2} + 2|f_2|^2_{L^2} \end{aligned}$$

for $N = 4$. From the fact that $T_\lambda \in C([0, S]; H^1(\mathbb{R}^N))$ and (6.2) holds, there exists $t_3^\lambda \in [0, S]$ such that

$$|\nabla T_\lambda(t_3^\lambda)|^2_{L^2} + \lambda |T_\lambda(t_3^\lambda)|^2_{L^2} = \min_{0 \leqslant t \leqslant S} \left(|\nabla T_\lambda(t)|^2_{L^2} + \lambda |T_\lambda(t)|^2_{L^2}\right) \leqslant \gamma_2.$$

Then applying Gronwall's inequality to (6.11) and (6.12) over $[t_3^\lambda, t]$ ($t \in [t_3^\lambda, t_3^\lambda + S]$), we have

$$(6.13) \qquad \sup_{0 \leqslant t \leqslant S} |\nabla T_\lambda(t)|^2_{L^2} + \int_0^S |\Delta T_\lambda(s)|^2_{L^2} ds + \int_0^S |\partial_t T_\lambda(s)|^2_{L^2} ds \leqslant \gamma_2.$$

Similarly, the third equation of $(\mathrm{DCBF})_\lambda$ gives

$$\frac{d}{dt}|\nabla C_\lambda|_{L^2}^2 + |\Delta C_\lambda|_{L^2}^2 \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^4 |\nabla C_\lambda|_{L^2}^2 + 3\rho^2 |\Delta T_\lambda|_{L^2}^2 + 3|f_3|_{L^2}^2,$$

$$|\partial_t C_\lambda|_{L^2}^2 + \frac{d}{dt}|\nabla C_\lambda|_{L^2}^2 + \lambda \frac{d}{dt}|C_\lambda|_{L^2}^2 \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 |\nabla C_\lambda|_{L^2} |\Delta C_\lambda|_{L^2} + 3\rho^2 |\Delta T_\lambda|_{L^2}^2 + 3|f_3|_{L^2}^2$$

for $N = 3$ and

$$\frac{d}{dt}|\nabla C_\lambda|_{L^2}^2 + |\Delta C_\lambda|_{L^2}^2 \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 |\Delta \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 |\nabla C_\lambda|_{L^2}^2 + 3\rho^2 |\Delta T_\lambda|_{L^2}^2 + 3|f_3|_{L^2}^2,$$

$$|\partial_t C_\lambda|_{L^2}^2 + \frac{d}{dt}|\nabla C_\lambda|_{L^2}^2 + \lambda \frac{d}{dt}|C_\lambda|_{L^2}^2 \leqslant \gamma_3 |\nabla \boldsymbol{u}_\lambda|_{\mathbb{L}^2} |\Delta \boldsymbol{u}_\lambda|_{\mathbb{L}^2} |\nabla C_\lambda|_{L^2} |\Delta C_\lambda|_{L^2} + 3\rho^2 |\Delta T_\lambda|_{L^2}^2 + 3|f_2|_{L^2}^2$$

for $N = 4$, which yields

$$\sup_{0 \leqslant t \leqslant S} |\nabla C_\lambda(t)|_{L^2}^2 + \int_0^S |\Delta C_\lambda(s)|_{L^2}^2 ds + \int_0^S |\partial_t C_\lambda(s)|_{L^2}^2 ds \leqslant \gamma_2. \tag{6.14}$$

In order to deduce L^2-estimate for $\partial_t \boldsymbol{u}_\lambda$, we consider the time subtractions of $\boldsymbol{u}_\lambda$, which is denoted by $D_h \boldsymbol{u}_\lambda(t) := \boldsymbol{u}_\lambda(t+h) - \boldsymbol{u}_\lambda(t)$ for $h > 0$. From the first equation of $(\mathrm{DCBF})_\lambda$, $D_h \boldsymbol{u}_\lambda(t)$, $D_h T_\lambda(t) := T_\lambda(t+h) - T_\lambda(t)$, $D_h C_\lambda(t) := C_\lambda(t+h) - C_\lambda(t)$ and $D_h \boldsymbol{f}_1(t) := \boldsymbol{f}_1(t+h) - \boldsymbol{f}_1(t)$ satisfy

$$\partial_t D_h \boldsymbol{u}_\lambda - \nu \mathcal{A}_{\mathbb{R}^N} D_h \boldsymbol{u}_\lambda + a D_h \boldsymbol{u}_\lambda = \mathcal{P}_{\mathbb{R}^N} \boldsymbol{g} D_h T_\lambda + \mathcal{P}_{\mathbb{R}^N} \boldsymbol{h} D_h C_\lambda + \mathcal{P}_{\mathbb{R}^N} D_h \boldsymbol{f}_1. \tag{6.15}$$

Multiplying (6.15) by $D_h \boldsymbol{u}_\lambda$, we get

$$\frac{d}{dt}|D_h \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 + a|D_h \boldsymbol{u}_\lambda|_{\mathbb{L}^2}^2 \leqslant \frac{3|\boldsymbol{g}|^2}{a}|D_h T_\lambda|_{L^2}^2 + \frac{3|\boldsymbol{h}|^2}{a}|D_h C_\lambda|_{L^2}^2 + \frac{3}{a}|D_h \boldsymbol{f}_1|_{\mathbb{L}^2}^2.$$

Since $D_h \boldsymbol{u}_\lambda \in C_\pi([0,S]; \mathbb{L}^2_\sigma(\mathbb{R}^N))$, $\boldsymbol{f}_1 \in W^{1,2}(0,S; \mathbb{L}^2(\mathbb{R}^N))$ and we already have estimates for $\partial_t T_\lambda$ and $\partial_t C_\lambda$ in (6.13) and (6.14), we obtain

$$\int_0^S |D_h \boldsymbol{u}_\lambda(s)|_{\mathbb{L}^2}^2 ds \leqslant \gamma_2 h^2$$

for any $h > 0$, which immediately yields

$$\int_0^S |\partial_t \boldsymbol{u}_\lambda(s)|_{\mathbb{L}^2}^2 ds \leqslant \gamma_2. \tag{6.16}$$

Consequently, from (6.2), (6.3), (6.5), (6.6), (6.13), (6.14) and (6.16), we can derive

$$\sup_{0\leqslant t\leqslant S} |\nabla T_\lambda(t)|_{L^2(\mathbb{R}^N)} + \sup_{0\leqslant t\leqslant S} |\nabla C_\lambda(t)|_{L^2(\mathbb{R}^N)} + \sup_{0\leqslant t\leqslant S} |\nabla \boldsymbol{u}_\lambda(t)|_{\mathbb{L}^2(\mathbb{R}^N)} \leqslant \gamma_2, \tag{6.17}$$

$$\int_0^S \left(|\Delta T_\lambda(s)|^2_{L^2(\mathbb{R}^N)} + |\Delta C_\lambda(s)|^2_{L^2(\mathbb{R}^N)} + |A_{\mathbb{R}^N} \boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2(\mathbb{R}^N)} \right) ds \leqslant \gamma_2, \tag{6.18}$$

$$\int_0^S \left(|\partial_t T_\lambda(s)|^2_{L^2(\mathbb{R}^N)} + |\partial_t C_\lambda(s)|^2_{L^2(\mathbb{R}^N)} + |\partial_t \boldsymbol{u}_\lambda(s)|^2_{\mathbb{L}^2(\mathbb{R}^N)} \right) ds \leqslant \gamma_2. \tag{6.19}$$

Moreover, (6.10) and (6.18) give

$$\begin{aligned} &\int_0^S \left|\partial_{x_\iota}\partial_{x_\mu} T_\lambda(s)\right|^2_{L^2(\mathbb{R}^N)} ds \leqslant \gamma_2, \quad \int_0^S \left|\partial_{x_\iota}\partial_{x_\mu} C_\lambda(s)\right|^2_{L^2(\mathbb{R}^N)} ds \leqslant \gamma_2, \\ &\int_0^S \left|\partial_{x_\iota}\partial_{x_\mu} \boldsymbol{u}_\lambda(s)\right|^2_{\mathbb{L}^2(\mathbb{R}^N)} ds \leqslant \gamma_2 \end{aligned} \tag{6.20}$$

for all $\iota, \mu = 1, 2, \cdots, N$. Using (6.8) and (6.9), we have

$$\int_0^S |\boldsymbol{u}_\lambda \cdot \nabla T_\lambda(s)|^2_{L^2(\mathbb{R}^N)} ds + \int_0^S |\boldsymbol{u}_\lambda \cdot \nabla C_\lambda(s)|^2_{L^2(\mathbb{R}^N)} ds \leqslant \gamma_2. \tag{6.21}$$

Furthermore, from Sobolev's inequality and (6.17), we can derive

$$\sup_{0\leqslant t\leqslant S} |T_\lambda(t)|_{L^{2^*}(\mathbb{R}^N)} + \sup_{0\leqslant t\leqslant S} |C_\lambda(t)|_{L^{2^*}(\mathbb{R}^N)} + \sup_{0\leqslant t\leqslant S} |\boldsymbol{u}_\lambda(t)|_{\mathbb{L}^{2^*}(\mathbb{R}^N)} \leqslant \gamma_2. \tag{6.22}$$

Hence, (6.17), (6.18), (6.19), (6.20), (6.21) and (6.22) allow us to repeat exactly the same convergence argument as that in Step 2. We also remark that λT_λ and λC_λ strongly converge to zero in $L^2(0, S; L^2(\mathbb{R}^N))$ as $\lambda \to 0$, since

$$\int_0^S |\lambda T_\lambda|^2_{L^2} dt = \lambda \int_0^S \lambda |T_\lambda|^2_{L^2} dt \leqslant \lambda\gamma_2, \qquad \int_0^S |\lambda C_\lambda|^2_{L^2} dt \leqslant \lambda\gamma_2$$

hold from (6.2) and (6.3). Thus, letting λ tend to 0 and following our procedure for convergence stated in Section 5, $\{(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)\}_{\lambda>0}$ (to be precise, some suitable subsequence of $\{(\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)\}_{\lambda>0}$) converges to a time periodic solution of the original system (DCBF), whence follows our result. □

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