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Mathematical analysis for a Warren–Kobayashi–Lobkovsky–Carter type system

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Introduction

Let $1 < N \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$, and let $\nu_{\partial\Omega}$ be the unit outer normal on $\partial\Omega$. Besides, let us set $Q := (0,\infty) \times \Omega$ and $\Sigma := (0,\infty) \times \partial\Omega$.

Let $\nu > 0$ be a fixed constant. In this paper, the following parabolic type system, denoted by (S), is considered.

(S):

$$\begin{cases} \partial_t(u - L\eta) - \Delta u = f(t,x), \quad (t,x) \in Q, \\ u(t,x) = 0, \quad (t,x) \in \Sigma, \\ u(0,x) = u_0(x), \quad x \in \Omega; \end{cases} \tag{0.1}$$

$$\begin{cases} \partial_t\eta - \Delta\eta + \partial I_{[0,1]}(\eta) - \left(\eta - u - \frac{1}{2}\right) + \alpha'(\eta)|\nabla\theta| + \nu\beta'(\eta)|\nabla\theta|^2 \ni 0 \ \text{ in } Q, \\ \nabla\eta \cdot \nu_{\partial\Omega} = 0 \ \text{ on } \Sigma, \\ \eta(0,x) = \eta_0(x), \quad x \in \Omega; \end{cases} \tag{0.2}$$

$$\begin{cases} \alpha_0(\eta)\,\partial_t\theta - \operatorname{div}\left(\alpha(\eta)\dfrac{\nabla\theta}{|\nabla\theta|} + 2\nu\beta(\eta)\nabla\theta\right) = 0 \ \text{ in } Q, \\ \left(\alpha(\eta)\dfrac{\nabla\theta}{|\nabla\theta|} + 2\nu\beta(\eta)\nabla\theta\right) \cdot \nu_{\partial\Omega} = 0 \ \text{ on } \Sigma, \\ \theta(0,x) = \theta_0(x), \quad x \in \Omega. \end{cases} \tag{0.3}$$

The system (S) is based on a non-isothermal model of grain boundary motion, proposed by Warren–Kobayashi–Lobkovsky–Carter [25]. In the context, $u = u(t,x)$ is the relative temperature with the zero-critical degree, $\eta = \eta(t,x)$ is an order parameter which indicates the solidification degree of grains in a polycrystal, and $\theta = \theta(t, x)$ is an order parameter which indicates the orientation angle of grain. In particular, the value of η is supposed to be constrained on the closed interval $[0, 1]$. Then, the cases when "$\eta = 1$" and "$\eta = 0$" are assigned to "completely solidifying phase" and "completely melting phase", respectively, and also, the solidification degree is supposed to link to the orientation degree of grain, directly. The term $\partial I_{[0,1]}$ as in (0.1) is the subdifferential of the indicator function $I_{[0,1]}$ on the closed interval $[0, 1]$, i.e.:

$$r \in \mathbb{R} \mapsto I_{[0,1]}(r) := \begin{cases} 0, & \text{if } r \in [0, 1], \\ \infty, & \text{otherwise;} \end{cases}$$

and one of roles of this term is to constrain the value of η onto the required range $[0, 1]$. $L > 0$ is a constant of the latent heat. $f = f(t, x)$ is a given heat source. $0 < \alpha_0 = \alpha_0(\eta)$, $0 \leq \alpha = \alpha(\eta)$ and $0 < \beta = \beta(\eta)$ are given mobility functions, and α' and β' denote the differentials $\frac{d\alpha}{d\eta}$ and $\frac{d\beta}{d\eta}$ of α and β, respectively.

The initial-boundary value problem (0.1) is to reproduce the process of heat exchanges, and the term $u - L\eta$ denotes the *enthalpy,* as in the weak formulation of the Stefan problem (cf. [24]).

On the other hand, the remaining coupling system $\{(0.2),(0.3)\}$ is derived as a gradient flow of the following function, called *free-energy:*

$$\begin{aligned} [\eta, \theta] \in H^1(\Omega)^2 \mapsto \mathscr{F}_u(\eta, \theta) := \frac{1}{2} \int_\Omega |\nabla \eta|^2 \, dx + \int_\Omega I_{[0,1]}(\eta) \, dx \\ -\frac{1}{2} \int_\Omega \left(\eta - u - \tfrac{1}{2}\right)^2 dx + \int_\Omega \alpha(\eta) |\nabla \theta| \, dx + \nu \int_\Omega \beta(\eta) |\nabla \theta|^2 \, dx, \end{aligned} \qquad (0.4)$$

with given $u \in L^2(\Omega)$.

More precisely, (0.2) is an initial-boundary value problem of an Allen–Cahn type equation, which is governed by the following double-well function (cf. [24]):

$$\eta \in \mathbb{R} \mapsto I_{[0,1]}(\eta) - \frac{1}{2}\left(\eta - u - \tfrac{1}{2}\right)^2 \in (-\infty, \infty], \text{ with } u \in \mathbb{R};$$

and the role of (0.3) is to reproduce the crystalline orientation process by means of the singular type diffusion $-\mathrm{div}\left(\alpha(\eta)\frac{\nabla\theta}{|\nabla\theta|} + 2\nu\beta(\eta)\nabla\theta\right)$ (cf. [12, 14, 15, 25]). Besides, the term $\alpha'(\eta)|\nabla\theta| + \nu\beta'(\eta)|\nabla\theta|^2$ in (0.2) is an additional perturbation to reproduce the interactions between solidifications and crystalline orientations.

Under the isothermal settings, i.e. the constant settings of temperature u, there are a number of relevant studies, e.g. [4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 26, 27], which worked on mathematical analysis for some simplified versions of the system (S). The line of the previous results can be summarized as follows.

(Ref.1) [4, 12, 13, 14, 15]: the modellings and auxiliary studies.

(Ref.2) [5, 6, 7, 16, 20, 21, 22, 27]: the existence of solutions to isothermal systems.

(Ref.3) [11, 19, 26]: the energy-dissipations and asymptotic behavior for solutions to isothermal systems.

Now, the objective of this paper is to expand the applicable scope of the mathematical methods developed in (Ref.1)–(Ref.3), to the non-isothermal system (S). For this purpose, we set the goal in this paper to prove the following two Main Theorems.

Main Theorem 1: to show the existence of solution $[u, \eta, \theta]$ to (S), which reproduce the energy-dissipation, appropriately.

Main Theorem 2: to show the association between the steady-state problem for (S), and the asymptotic behavior of the orbit $[u(t), \eta(t), \theta(t)]$ as $t \to \infty$.

The statements of Main Theorems are presented in Section 2, on the basis of the preliminaries outlined in Section 1. These two Main Theorems are proved in the following Sections 3 and 4, respectively.

1 Preliminaries

In this Section, we outline some basic notations and known facts, as preliminaries of the study.

Notation 1 (Notations in real analysis) Let $d \in \mathbb{N}$ be any fixed number. Then, we simply denote by $|x|$ and $x \cdot y$ the Euclidean norm of $x \in \mathbb{R}^d$ and the standard scalar product of $x, y \in \mathbb{R}^d$, respectively, i.e.:

$$|x| := \sqrt{x_1^2 + \cdots + x_d^2} \text{ and } x \cdot y := x_1 y_1 + \cdots + x_d y_d,$$
$$\text{for all } x = [x_1, \cdots, x_d], y = [y_1, \cdots, y_d] \in \mathbb{R}^d.$$

The d-dimensional Lebesgue measure is denoted by $\mathscr{L}^d$. Also, unless otherwise specified, the measure theoretical phrases, such as "a.e.", "dt" and "dx", and so on, are with respect to the Lebesgue measure in each corresponding dimension.

Notation 2 (Notations of functional analysis) For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ its inner product.

Let $\mathrm{Id} : L^2(\Omega) \longrightarrow L^2(\Omega)$ be the identity map on $L^2(\Omega)$. Let $F_0 : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ and $F : H^1(\Omega) \longrightarrow H^1(\Omega)^*$ be the duality maps, defined as:

$$\langle F_0 z, w \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} := \int_\Omega \nabla z \cdot \nabla w \, dx, \quad \text{for } [z, w] \in H_0^1(\Omega)^2,$$

and

$$\langle F z, w \rangle_{H^1(\Omega)^*, H^1(\Omega)} := \int_\Omega \nabla z \cdot \nabla w \, dx + \int_\Omega zw \, dx, \quad \text{for } [z, w] \in H^1(\Omega)^2,$$

respectively.

In this paper, we simply put $V_0 := H_0^1(\Omega)$, and we prescribe the dual space $V_0^* = H^{-1}(\Omega)$ as a Hilbert space endowed with the following inner product:

$$(z, w)_{V_0^*} := \langle z, F_0^{-1} w \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \text{for all } [z, w] \in [V_0^*]^2 \ (= V_0^* \times V_0^*).$$

Besides, we denote by $P_0 > 0$ the constant of Poincaré's inequality. More precisely, P_0 is the constant of continuous embedding $V_0 \subset L^2(\Omega)$, such that:

$$|v|_{L^2(\Omega)} \leq P_0 |v|_{V_0}, \quad \text{for any } v \in V_0. \tag{1.1}$$

We define the operator of Laplacian Δ_0, subject to the Dirichlet-zero boundary condition, by letting:

$$\Delta_0 : v \in W_0 := V_0 \cap H^2(\Omega) \subset L^2(\Omega) \mapsto \Delta v \in L^2(\Omega).$$

Also, we define the operator of Laplacian Δ_{N}, subject to the Neumann-zero boundary condition, by letting:

$$\Delta_{\mathrm{N}} : v \in W_{\mathrm{N}} := \left\{ \, z \in H^2(\Omega) \, \middle| \, \nabla z \cdot \nu_{\partial\Omega} = 0 \text{ in } H^{\frac{1}{2}}(\partial\Omega) \, \right\} \subset L^2(\Omega) \mapsto \Delta v \in L^2(\Omega).$$

By the definitions, it is easily checked that:

$$-\Delta_0 = F_0|_{H^2(\Omega)} \quad \text{and} \quad -\Delta_{\mathrm{N}} = (F - \mathrm{Id})|_{W_{\mathrm{N}}}. \tag{1.2}$$

Notation 3 (Notations in convex analysis) For any proper lower semi-continuous (l.s.c. from now on) and convex function Ψ defined on a Hilbert space X, we denote by $D(\Psi)$ its effective domain, and denote by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of Ψ, and it has a maximal monotone graph in the product space X^2. More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as a set of all elements $z_0^* \in X$ which satisfy the following variational inequality:

$$(z_0^*, z - z_0)_X \leq \Psi(z) - \Psi(z_0) \quad \text{for any } z \in D(\Psi).$$

The set $D(\partial\Psi) := \{ z \in X \,|\, \partial\Psi(z) \neq \emptyset \}$ is called the domain of $\partial\Psi$. We often use the notation "$[z_0, z_0^*] \in \partial\Psi$ in X^2", to mean that "$z_0^* \in \partial\Psi(z_0)$ in X with $z_0 \in D(\partial\Psi)$", by identifying the operator $\partial\Psi$ with its graph in X^2.

Remark 1.1 Let $X_0 \subset H^1(\Omega)$ be a closed linear subspace in $H^1(\Omega)$, and let Ψ_0 be a proper l.s.c. and convex function on $L^2(\Omega)$, defined as:

$$z \in L^2(\Omega) \mapsto \Psi_0(z) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla z|^2 \, dx, & \text{if } z \in X_0, \\ \infty, & \text{otherwise.} \end{cases}$$

Then, the subdifferential $\partial\Psi_0$ of this convex function is directly associated with the operator of Laplacian.

For instance (cf. [1, 2]), if $X_0 = V_0$, then:

$$\partial\Psi_0(z) = \{-\Delta_0 z\} = \{F_0 z\}, \quad \text{for } z \in W_0.$$

As well as, if $X_0 = H^1(\Omega)$, then:

$$\partial\Psi_0(z) = \{-\Delta_{\mathrm{N}} z\} = \{Fz - z\}, \quad \text{for } z \in W_{\mathrm{N}}.$$

As another example, we mention about the subdifferential $\partial\Psi_{[0,1]} \subset L^2(\Omega)^2$ of a proper l.s.c. and convex function $\Psi_{[0,1]} : L^2(\Omega) \longrightarrow [0,\infty]$, defined as:

$$z \in L^2(\Omega) \mapsto \Psi_{[0,1]}(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2\,dx + \int_\Omega I_{[0,1]}(z)\,dx, \\ \qquad \text{if } z \in H^1(\Omega), \\ \infty, \quad \text{otherwise.} \end{cases}$$

In this example, it is known that (cf. [2, 10]):

$$\begin{cases} D(\Psi_{[0,1]}) = \{\ z \in H^1(\Omega) \mid 0 \le z \le 1 \text{ a.e. in } \Omega\ \}, \\ D(\partial\Psi_{[0,1]}) = D(\Psi_{[0,1]}) \cap W_{\mathrm{N}}, \end{cases}$$

and for any $z \in D(\partial\Psi_{[0,1]})$,

$$\begin{aligned} \partial\Psi_{[0,1]}(z) &= -\Delta_{\mathrm{N}} z + \{\ \xi \in L^2(\Omega) \mid \xi(x) \in \partial I_{[0,1]}(z(x)) \text{ a.e. } x \in \Omega\ \} \\ &= \left\{ w + \xi \;\middle|\; \begin{array}{l} w = -\Delta_{\mathrm{N}} z \text{ in } L^2(\Omega), \text{ and } \xi(x)(\sigma - z(x)) \le 0 \\ \text{for a.e. } x \in \Omega \text{ and any } \sigma \in [0,1] \end{array} \right\}. \end{aligned}$$

Remark 1.2 (Time-dependent subdifferentials) It is often useful to consider the subdifferentials under time-dependent settings of convex functions. With regard to this topic, certain general theories were established by a number of researchers (e.g. Kenmochi [8] and Ôtani [18]). So, referring to some of these (e.g. [8, Chapter 2]), we can see the following fact.

(Fact 0) Let E_0 be a convex subset in a Hilbert space X, let $I \subset [0,\infty)$ be a time-interval, and for any $t \in I$, let $\Psi^t : X \longrightarrow (-\infty,\infty]$ be a proper l.s.c. and convex function, such that $D(\Psi^t) = E_0$ for all $t \in I$. Based on this, let us define a convex function $\hat{\Psi}^I : L^2(I;X) \longrightarrow (-\infty,\infty]$, by putting:

$$\zeta \in L^2(I;X) \mapsto \hat{\Psi}^I(\zeta) := \begin{cases} \displaystyle\int_I \Psi^t(\zeta(t))\,dt, & \text{if } \Psi^{(\cdot)}(\zeta) \in L^1(I), \\ \infty, & \text{otherwise.} \end{cases}$$

Here, if $E_0 \subset D(\hat{\Psi}^I)$, i.e. if the function $t \in I \mapsto \Psi^t(z)$ is integrable for any $z \in E_0$, then it holds that:

$$[\zeta,\zeta^*] \in \partial\hat{\Psi}^I \text{ in } L^2(I;X)^2, \text{ iff.}$$
$$\zeta \in D(\hat{\Psi}^I) \text{ and } [\zeta(t),\zeta^*(t)] \in \partial\Psi^t \text{ in } X^2, \text{ a.e. } t \in I.$$

Finally, we mention about the Mosco convergence, that is known as a representative notion of the functional-convergence.

Definition 1.1 (Mosco convergence: cf. [17]) Let X be an abstract Hilbert space. Let $\Psi : X \longrightarrow (-\infty,\infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \longrightarrow (-\infty,\infty]$, $n \in \mathbb{N}$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco [17], as $n \to \infty$, iff. the following two conditions are fulfilled.

1° **The condition of lower-bound:** $\liminf_{n\to\infty} \Psi_n(z_n^\dagger) \geq \Psi(z^\dagger)$, if $z^\dagger \in X$, $\{z_n^\dagger\}_{n=1}^\infty \subset X$, and $z_n^\dagger \to z^\dagger$ weakly in X as $n \to \infty$.

2° **The condition of optimality:** for any $z^\ddagger \in D(\Psi)$, there exists a sequence $\{z_n^\ddagger\}_{n=1}^\infty \subset X$ such that $z_n^\ddagger \to z^\ddagger$ in X and $\Psi_n(z_n^\ddagger) \to \Psi(z^\ddagger)$, as $n \to \infty$.

Remark 1.3 As a basic matter of the Mosco-convergence, we can see the following fact (see [8, Chapter 2], for example).

(Fact 1) Let X, Ψ and $\{\Psi_n\}_{n=1}^\infty$ be as in Definition 1.1. Besides, let us assume that:

$$\Psi_n \to \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \to \infty,$$

and

$$\begin{cases} [z, z^*] \in X^2, \quad [z_n, z_n^*] \in \partial\Psi_n \text{ in } X^2,\ n \in \mathbb{N}, \\ z_n \to z \text{ in } X \text{ and } z_n^* \to z^* \text{ weakly in } X, \text{ as } n \to \infty. \end{cases}$$

Then, it holds that:

$$[z, z^*] \in \partial\Psi \text{ in } X^2, \text{ and } \Psi_n(z_n) \to \Psi(z), \text{ as } n \to \infty.$$

2 Statements of Main Theorems

We begin with prescribing the assumptions in our study.

(A0) $\nu > 0$ and $L > 0$ are given positive constants, and $f \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$ is a given function.

(A1) $0 < \alpha_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$, $0 \leq \alpha \in C^2(\mathbb{R})$ and $0 < \beta \in C^2(\mathbb{R})$ are given functions, such that α and β are convex functions, $\alpha'(0) = \beta'(0) = 0$, and

$$\delta_* := \inf\big(\alpha_0(\mathbb{R}) \cup \beta(\mathbb{R})\big) > 0.$$

(A2) $[u_0, \eta_0, \theta_0]$ is a triplet of initial data, and this is taken from a class $D_* \subset L^2(\Omega)^3$, prescribed as:

$$D_* := V_0 \times D(\Psi_{[0,1]}) \times \big(H^1(\Omega) \cap L^\infty(\Omega)\big).$$

Note that D_* is a subset of the domain of free-energy $\mathscr{F}_u$, given in (0.4).

Under these assumptions, we define the solution to (S) as follows.

Definition 2.1 (Definition of solution) A triplet of functions $[u, \eta, \theta] \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)^3)$ is called a solution to (S), iff. $[u, \eta, \theta]$ fulfills the following conditions.

(S0) $u \in W^{1,2}_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,\infty); V_0)$ with $u(0) = u_0$ in $L^2(\Omega)$;
$\eta \in W^{1,2}_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,\infty); H^1(\Omega))$ with $\eta(0) = \eta_0$ in $L^2(\Omega)$;
$\theta \in W^{1,2}_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,\infty); H^1(\Omega))$ with $\theta(0) = \theta_0$ in $L^2(\Omega)$;

$$[u(t), \eta(t), \theta(t)] \in D_*(\theta_0) := \left\{ [\tilde{u}, \tilde{\eta}, \tilde{\theta}] \in D_* \;\middle|\; |\tilde{\theta}|_{L^\infty(\Omega)} \leq |\theta_0|_{L^\infty(\Omega)} \right\},$$
for a.e. $t > 0$.

(S1) u solves (0.1) in the following variational sense:

$$\int_\Omega \partial_t(u - L\eta)(t)z\,dx + \int_\Omega \nabla u(t)\cdot\nabla z\,dx = \int_\Omega f(t)z\,dx, \qquad (2.1)$$
$$\text{for any } z \in V_0, \text{ a.e. } t > 0.$$

(S2) η solves (0.2) in the following variational sense:

$$\begin{aligned}&\int_\Omega \partial_t\eta(t)(\eta(t)-\varphi)\,dx + \int_\Omega \nabla\eta(t)\cdot\nabla(\eta(t)-\varphi)\,dx \\ &\quad - \int_\Omega \left(\eta(t) - u(t) - \tfrac{1}{2}\right)(\eta(t)-\varphi)\,dx \\ &+ \int_\Omega (\eta(t)-\varphi)\alpha'(\eta(t))|\nabla\theta(t)|\,dx + \nu\int_\Omega (\eta(t)-\varphi)\beta'(\eta(t))|\nabla\theta(t)|^2\,dx \le 0,\end{aligned} \qquad (2.2)$$
$$\text{for any } \varphi \in D(\Psi_{[0,1]}), \text{ and a.e. } t > 0.$$

(S3) θ solves (0.3) in the following variational sense:

$$\begin{aligned}&\int_\Omega \alpha_0(\eta(t))\partial_t\theta(t)(\theta(t)-\omega)\,dx + 2\nu\int_\Omega \beta(\eta(t))\nabla\theta(t)\cdot\nabla(\theta(t)-\omega)\,dx \\ &\quad + \int_\Omega \alpha(\eta(t))|\nabla\theta(t)|\,dx \le \int_\Omega \alpha(\eta(t))|\nabla\omega|\,dx,\end{aligned} \qquad (2.3)$$
$$\text{for any } \omega \in H^1(\Omega), \text{ a.e. } t > 0.$$

Next, for simplicities of descriptions, we add some specific notations in our study.

Notation 4 (Specific notations) For any function $\tilde{\eta} \in L^\infty(\Omega)$, we denote by $\Phi(\tilde{\eta};\cdot)$ a proper l.s.c. and convex function on $L^2(\Omega)$, defined as:

$$z \in L^2(\Omega) \mapsto \Phi(\tilde{\eta}; z) := \begin{cases} \displaystyle\int_\Omega \alpha(\tilde{\eta})|\nabla z|\,dx + \nu\int_\Omega \beta(\tilde{\eta})|\nabla z|^2\,dx, \\ \qquad \text{if } z \in H^1(\Omega), \\ \infty, \quad \text{otherwise,} \end{cases}$$

and we denote by $\partial\Phi(\tilde{\eta};\cdot)$ the subdifferential of $\Phi(\tilde{\eta};\cdot)$ in the topology of $L^2(\Omega)$.

Next, we define a functional $\mathscr{F}_0$ on $L^2(\Omega)^2$, by letting:

$$[\eta,\theta] \in L^2(\Omega)^2 \mapsto \mathscr{F}_0(\eta,\theta) := \Psi_{[0,1]}(\eta) - \frac{1}{2}\int_\Omega \left(\eta - \tfrac{1}{2}\right)^2 dx + \Phi(\eta;\theta) \in (0,\infty]. \qquad (2.4)$$

Note that $\mathscr{F}_0$ corresponds to the free-energy $\mathscr{F}_u$, given in (0.4), in the case of $u \equiv 0$.

Finally, we set the following two key-constants:

$$A_0 := \frac{1}{2(1+L^2)} \quad \text{and} \quad B_0 := A_0 + \frac{P_0^2}{L}, \qquad (2.5)$$

by using the constant $P_0 > 0$ as in (1.1), and for any $\varpi \in V_0$, we define a functional $\mathscr{G}_\varpi$ on $L^2(\Omega)^3$, by letting:

$$\begin{aligned}[u,\eta,\theta] \in L^2(\Omega)^3 \quad \mapsto \quad \mathscr{G}_\varpi(u,\eta,\theta) &:= \frac{1}{2L}|u-\varpi|^2_{L^2(\Omega)} + (\eta,\varpi)_{L^2(\Omega)} \\ &+\frac{A_0}{2}|u-\varpi|^2_{V_0} + \mathscr{F}_0(\eta,\theta) \in (-\infty,\infty].\end{aligned} \tag{2.6}$$

Remark 2.1 By using the notations in Notation 4, the variational inequalities (2.1) and (2.3) can be reformulated to the following forms of evolution equations:

$$\partial_t(u-L\eta)(t) - \Delta_0 u(t) = f(t) \text{ in } L^2(\Omega), \text{ a.e. } t>0,$$

and

$$\alpha_0(\eta(t))\partial_t\theta(t) + \partial\Phi(\eta(t);\theta(t)) \ni 0 \text{ in } L^2(\Omega), \text{ a.e. } t>0,$$

respectively. However, it must be noted that the reformulation by L^2-subdifferential is not available for (2.2), due to the L^1-perturbation term $\nu\beta'(\eta)|\nabla\theta|^2$ $(\in L^\infty_{\rm loc}([0,\infty);L^1(\Omega)))$.

Now, our two Main Theorems are stated as follows.

Main Theorem 1 (Existence of solution with energy-dissipation) *Under the assumptions (A0)–(A2), let A_0 and B_0 be the constants given in (2.5). Then, the system (S) admits at least one solution $[u,\eta,\theta]$ which fulfills the following condition.*

(S4) (Energy-dissipation) For any $\varrho \in L^2(\Omega)$ with $\varpi := F_0^{-1}\varrho \in W_0$, the function

$$\begin{aligned} t\in[0,\infty) \mapsto \mathscr{J}_\varrho(t) &:= \frac{A_0}{2}\int_0^t |\partial_t u(\tau)|^2_{L^2(\Omega)}\,d\tau + \frac{1}{4}\int_0^t |\partial_t\eta(\tau)|^2_{L^2(\Omega)}\,d\tau \\ &+\int_0^t |(\sqrt{\alpha_0(\eta)}\partial_t\theta)(\tau)|^2_{L^2(\Omega)}\,d\tau + \frac{1}{2L}\int_0^t |u(\tau)-\varpi|^2_{V_0}\,d\tau \\ &+\mathscr{G}_\varpi(u(t),\eta(t),\theta(t)) - \frac{B_0}{2}\int_0^t |f(\tau)-\varrho|^2_{L^2(\Omega)}\,d\tau \in \mathbb{R},\end{aligned}$$

satisfies the following dissipation property:

$$\mathscr{J}_\varrho(t) \le \mathscr{J}_\varrho(s) \quad \textit{for a.e. } s>0 \textit{ and any } t\ge s,$$

and in particular,

$$\mathscr{J}_\varrho(t) \le \mathscr{J}_\varrho(0) \quad \textit{for any } t\ge 0. \tag{2.7}$$

Main Theorem 2 (Asymptotic behavior) *In addition to (A0)–(A2), let us assume the following condition.*

(A3) There exists a function $f_\infty \in L^2(\Omega)$ such that $f - f_\infty \in L^2(0,\infty;L^2(\Omega))$.

Also, let $[u,\eta,\theta]$ be the solution to (S) obtained in Main Theorem 1, and let $\omega_\infty(u,\eta,\theta) \subset L^2(\Omega)^3$ be the ω-limit set of the orbit $[u(t),\eta(t),\theta(t)]$, $t>0$, i.e.:

$$\omega_\infty(u,\eta,\theta) := \left\{ [u_\infty,\eta_\infty,\theta_\infty] \in L^2(\Omega)^3 \;\middle|\; \begin{array}{l} \text{there exists a sequence of time } 0< \\ t_1<t_2<t_3<\cdots<t_n\uparrow\infty \text{ and} \\ [u(t_n),\eta(t_n),\theta(t_n)] \to [u_\infty,\eta_\infty,\theta_\infty] \\ \text{in } L^2(\Omega)^3 \text{ as } n\to\infty. \end{array} \right\}.$$

Then, the following three items hold.

(O) $\omega_\infty(u,\eta,\theta)$ *is nonempty and compact in* $L^2(\Omega)^3$.

(I) Any ω*-limit point* $[u_\infty,\eta_\infty,\theta_\infty] \in \omega_\infty(u,\eta,\theta)$ *fulfills that:*

(i-a) $u_\infty = F_0^{-1} f_\infty$ *in* V_0, *i.e.* $u_\infty \in W_0$ *and* $-\Delta_0 u_\infty = f_\infty$ *in* $L^2(\Omega)$.

(i-b) η_∞ *is a solution to*

$$\begin{gathered}\big(\nabla\eta_\infty, \nabla(\eta_\infty - \varphi)\big)_{L^2(\Omega)^N} - \big(\eta_\infty - u_\infty - \tfrac{1}{2}, \eta_\infty - \varphi\big)_{L^2(\Omega)} \le 0, \\ \text{for any } \varphi \in D(\Psi_{[0,1]}),\end{gathered} \tag{2.8}$$

i.e. $\eta_\infty \in D(\partial\Psi_{[0,1]})$ *and* $\partial\Psi_{[0,1]}(\eta_\infty) - \big(\eta_\infty - u_\infty - \tfrac{1}{2}\big) \ni 0$ *in* $L^2(\Omega)$.

(i-c) θ_∞ *is a constant on* Ω, *i.e.* $\partial\Phi(\eta_\infty;\theta_\infty) \ni 0$ *in* $L^2(\Omega)$, *and moreover, the constant* θ_∞ *fulfills that* $|\theta_\infty| \le |\theta_0|_{L^\infty(\Omega)}$.

3 Proof of Main Theorem 1

The Main Theorem 1 is proved by means of the time-discretization method. To this end, we denote by $h \in (0,1]$ the argument of time-step, and we set the following time-discretization scheme, denoted by $(\mathrm{AP})_h$, as the approximating problem for (S).

$(\mathrm{AP})_h$:

$$\frac{u_i^h - u_{i-1}^h}{h} - L\frac{\eta_i^h - \eta_{i-1}^h}{h} - \Delta_0 u_i^h = f_i^h \text{ in } L^2(\Omega),$$

$$\frac{\eta_i^h - \eta_{i-1}^h}{h} - \Delta_{\mathrm{N}}\eta_i^h + \partial I_{[0,1]}(\eta_i^h) - \big(\eta_i^h - u_i^h - \tfrac{1}{2}\big)$$
$$+\alpha'(\eta_i^h)|\nabla\theta_{i-1}^h| + \nu\beta'(\eta_i^h)|\nabla\theta_{i-1}^h|^2 \ni 0 \text{ in a weak variational sense,}$$

$$\alpha_0(\eta_i^h)\frac{\theta_i^h - \theta_{i-1}^h}{h} + \partial\Phi(\eta_i^h;\theta_i^h) \ni 0 \text{ in } L^2(\Omega),$$

where

$$f_i^h := \frac{1}{h}\int_{(i-1)h}^{ih} f(\tau)\,d\tau \text{ in } L^2(\Omega), \text{ for } i = 1,2,3,\ldots. \tag{3.1}$$

Definition 3.1 For any $h \in (0,1]$, a sequence $\{[u_i^h,\eta_i^h,\theta_i^h]\}_{i=0}^\infty \subset L^2(\Omega)^3$ is called a solution to $(\mathrm{AP})_h$, iff. $\{[u_i^h,\eta_i^h,\theta_i^h]\}_{i=1}^\infty \subset D_*$, $[u_0^h,\eta_0^h,\theta_0^h] = [u_0,\eta_0,\theta_0]$ in $L^2(\Omega)^3$, and for any $i \in \mathbb{N}$, the following variational inequalities are fulfilled,

$$\begin{gathered}\frac{1}{h}\int_\Omega (u_i^h - u_{i-1}^h)z\,dx - \frac{L}{h}\int_\Omega(\eta_i^h - \eta_{i-1}^h)z\,dx + \int_\Omega \nabla u_i^h\cdot\nabla z\,dx \\ = \int_\Omega f_i^h z\,dx, \quad \text{for any } z \in V_0,\end{gathered} \tag{3.2}$$

$$\begin{gathered}\frac{1}{h}\int_\Omega(\eta_i^h - \eta_{i-1}^h)(\eta_i^h - \varphi)\,dx + \int_\Omega \nabla\eta_i^h\cdot\nabla(\eta_i^h - \varphi)\,dx \\ - \int_\Omega \big(\eta_i^h - u_i^h - \tfrac{1}{2}\big)(\eta_i^h - \varphi)\,dx \\ + \int_\Omega (\eta_i^h - \varphi)\alpha'(\eta_i^h)|\nabla\theta_{i-1}^h|\,dx + \nu\int_\Omega(\eta_i^h - \varphi)\beta'(\eta_i^h)|\nabla\theta_{i-1}^h|^2\,dx \le 0, \\ \text{for any } \varphi \in D(\Psi_{[0,1]}) \text{ a.e. in } \Omega,\end{gathered} \tag{3.3}$$

and

$$\frac{1}{h}\int_\Omega \alpha_0(\eta_i^h)(\theta_i^h-\theta_{i-1}^h)(\theta_i^h-\omega)\,dx+\Phi(\eta_i^h;\theta_i^h)\le\Phi(\eta_i^h;\omega),\qquad\qquad (3.4)$$
$$\text{for any } \omega\in H^1(\Omega).$$

Now, let us set our first task to prove the following Proposition.

Proposition 3.1 *There exists a positive constant* $h_0 \in (0,1]$, *such that for any* $h \in (0,h_0]$, *the problem* $(AP)_h$ *admits a unique solution* $\{[u_i^h,\eta_i^h,\theta_i^h]\}_{i=0}^\infty \subset L^2(\Omega)^3$, *such that*

$$[u_i^h,\eta_i^h,\theta_i^h]\in D_*(\theta_0),\quad \textit{for } i=0,1,2,\ldots.\qquad\qquad (3.5)$$

For the proof of the above Proposition, we prepare some additional notations. In the problem $(\mathrm{AP})_h$, we simply put

$$e_i^h := u_i^h - L\eta_i^h \ \text{ for } i=0,1,2,3,\ldots.$$

Then, the system $\{(3.2),(3.3)\}$ can be reformulated to a minimization problem for the following proper l.s.c . (however possibly non-convex) functional:

$$[e,\eta]\ \in\ V_0^*\times L^2(\Omega)\mapsto \Upsilon_h(e,\eta)$$
$$:=\begin{cases} \dfrac{1}{2Lh}|e-e_{i-1}^h|_{V_0^*}^2+\dfrac{1}{2h}\displaystyle\int_\Omega|\eta-\eta_{i-1}^h|^2\,dx+\dfrac{1}{2L}\int_\Omega|e+L\eta|^2\,dx \\ \qquad +\Psi_{[0,1]}(\eta)+\displaystyle\int_\Omega\alpha(\eta)|\nabla\theta_{i-1}^h|\,dx+\nu\int_\Omega\beta(\eta)|\nabla\theta_{i-1}^h|^2\,dx \\ \qquad -\dfrac{1}{2}\displaystyle\int_\Omega\left(\eta-\tfrac{1}{2}\right)^2dx-\dfrac{1}{L}(f_i^h,e)_{V_0^*}, \\ \qquad \text{if } [e,\eta]\in L^2(\Omega)\times H^1(\Omega), \\ \infty, \quad \text{otherwise,}\end{cases}$$

with $i\in\mathbb{N}$ and the given data $f_i^h\in L^2(\Omega)$ $(\subset V_0^*)$ and $[e_{i-1}^h,\eta_{i-1}^h,\theta_{i-1}^h]\in D_*$.

The following lemma is to verify the validity of this reformulation.

Lemma 3.1 *Let us assume* $i\in\mathbb{N}$ *and* $[u_{i-1}^h,\eta_{i-1}^h,\theta_{i-1}^h]\in D_*$. *Then, there exists a positive constant* $h_0\in(0,1]$, *such that for any* $h\in(0,h_0]$, *the solving pair* $[u_i^h,\eta_i^h]\in V_0\times D(\Psi_{[0,1]})$ *to* (3.2)–(3.3) *coincides with the unique minimizer of* Υ_h.

Proof. The non-convex part in Υ_h:

$$[e,\eta]\in L^2(\Omega)^2\mapsto -\frac{1}{2}\int_\Omega\left(\eta-\tfrac{1}{2}\right)^2dx\in\mathbb{R},$$

is independent of the variable $e\in V_0^*$, and has a quadratic growth order for the variable $\eta\in L^2(\Omega)$. So, there will be a small constant $h_0\in(0,1]$ such that for any $h\in(0,h_0]$, Υ_h forms a proper l.s.c and strictly convex function on $V_0^*\times L^2(\Omega)$, and hence Υ_h has a unique minimizer in $D(\Upsilon_h)=L^2(\Omega)\times D(\Psi_{[0,1]})$.

Now, let us suppose $h \in (0, h_0]$. Then, applying the standard methods of convex analysis (cf. [3, 9, 24]), it is inferred that $[e_*, \eta_*] \in L^2(\Omega) \times D(\Psi_{[0,1]})$ is the minimizer of Υ_h iff.:

$$F_0^{-1}\left(\frac{e_* - e_{i-1}^h}{h} - f_i^h\right) = e_* + L\eta_* \text{ in } V_0,$$

and

$$\frac{1}{h}\int_\Omega (\eta_* - \eta_{i-1}^h)(\eta_* - \varphi)\,dx + \int_\Omega \nabla\eta_* \cdot \nabla(\eta_* - \varphi)\,dx$$
$$+ \int_\Omega \left((e_* + L\eta_*) - (\eta_* - \tfrac{1}{2})\right)(\eta_* - \varphi)\,dx$$
$$+ \int_\Omega (\eta_* - \varphi)\alpha'(\eta_*)|\nabla\theta_{i-1}^h|\,dx + \nu\int_\Omega (\eta_* - \varphi)\beta'(\eta_*)|\nabla\theta_{i-1}^h|^2\,dx \le 0,$$
$$\text{for any } \varphi \in D(\Psi_{[0,1]}).$$

Thus, we can take the minimizer $[u_*, \eta_*] := [e_* + L\eta_*, \eta_*] \in V_0 \times D(\Psi_{[0,1]})$ as the approximation data $[u_i^h, \eta_i^h]$ at the i-th step. ■

On the other hand, for the inclusion (3.4), we can see the following lemma, by referring to the previous studies [21, 22].

Lemma 3.2 *Let $i \in \mathbb{N}$, and let $h_0 \in (0,1]$ be the constant as in Lemma 3.1. Let us assume $h \in (0, h_0]$, $\theta_{i-1}^h \in H^1(\Omega) \cap L^\infty(\Omega)$ is given, and $[u_i^h, \eta_i^h] \in V_0 \times D(\Psi_{[0,1]})$ is the solution pair to $\{(3.2),(3.3)\}$ obtained in Lemma 3.1. Then, the variational inequality* (3.4) *admits a unique solution $\theta_i^h \in H^1(\Omega)$, such that:*

$$|\theta_i^h|_{L^\infty(\Omega)} \le |\theta_{i-1}^h|_{L^\infty(\Omega)}. \tag{3.6}$$

Proof. This lemma can be proved by applying similar analytic methods as in [21, 22].
In fact, the inclusion (3.4) is equivalent to the minimization problem for the following proper l.s.c. and strictly convex function:

$$\theta \in L^2(\Omega) \mapsto \frac{1}{2h}|\sqrt{\alpha_0(\eta_i^h)}(\theta - \theta_{i-1}^h)|_{L^2(\Omega)}^2 + \Phi(\eta_i^h; \theta), \tag{3.7}$$

namely the variational inequality (3.4) corresponds to the Euler-Lagrange equation for this convex function.

Therefore, the existence and uniqueness for (3.4) will be straightforward consequence of the coercivity and strictly convexity of the functional given in (3.7). Also, the inequality (3.6) is obtained by applying the result of comparison principle for (3.4), discussed in [21, Lemma 3.5] and [22, Lemma 4.4]. ■

Proof of Proposition 3.1. On the basis of the previous lemmas 3.1–3.2, we can prove Proposition 3.1 through the following iteration steps.

(Step 0) Put $i = 1$, and set $[u_0, \eta_0, \theta_0] \in D_*(\theta_0)$ as the initial value $[u_0^h, \eta_0^h, \theta_0^h]$.

(Step 1) Obtain the data $[u_i^h, \eta_i^h] \in V_0 \times D(\Psi_{[0,1]})$ by applying Lemma 3.1.

(Step 2) Obtain the data $\theta_i^h \in H^1(\Omega)$ with (3.6) by applying Lemma 3.2.

(Step 3) Advance the value of i, i.e. $i \leftarrow i + 1$, and return to (Step 1).

Here, note that in the above iterations, the property (3.5) can be obtained, inductively, through the process of (Step 1)–(Step 2). ■

Next, we verify the following lemma, concerned with the energy-estimate for approximating solutions.

Lemma 3.3 *Let $h_0 \in (0,1]$ be the constant as in Lemma 3.1, and for any $h \in (0, h_0]$, let $\{[u_i^h, \eta_i^h, \theta_i^h]\}_{i=0}^{\infty} \subset D_*(\theta_0)$ be the solution to $(AP)_h$ obtained in Proposition 3.1. Let us take any $\varrho \in L^2(\Omega)$, and let us put $\varpi = F_0^{-1}\varrho \in W_0$. Then, there exists a small constant $h_* \in (0, h_0]$, such that for any $h \in (0, h_*]$, the solution $\{[u_i^h, \eta_i^h, \theta_i^h]\}_{i=0}^{\infty}$ satisfies the following energy-inequality:*

$$\begin{aligned}&\frac{A_0}{2h}|u_i^h - u_{i-1}^h|_{L^2(\Omega)}^2 + \frac{1}{4h}|\eta_i^h - \eta_{i-1}^h|_{L^2(\Omega)}^2 + \frac{1}{h}|\sqrt{\alpha_0(\eta_i^h)}(\theta_i^h - \theta_{i-1}^h)|_{L^2(\Omega)}^2 \\ &\quad + \frac{h}{2L}|u_i^h - \varpi|_{V_0}^2 - \frac{B_0 h}{2}|f_i^h - \varrho|_{L^2(\Omega)}^2 \\ &\quad + \mathscr{G}_\varpi(u_i^h, \eta_i^h, \theta_i^h) \le \mathscr{G}_\varpi(u_{i-1}^h, \eta_{i-1}^h, \theta_{i-1}^h), \ i = 1, 2, 3, \ldots.\end{aligned} \tag{3.8}$$

Proof. With (1.2) and the relation $\varrho = F_0\varpi = -\Delta_0\varpi$ in $L^2(\Omega)$ in mind, we can see from (3.2) that:

$$\frac{1}{h}(u_i^h - u_{i-1}^h) - \Delta_0(u_i^h - \varpi) = (f_i^h - \varrho) + \frac{L}{h}(\eta_i^h - \eta_{i-1}^h) \ \text{ in } L^2(\Omega), \ i = 1, 2, 3, \ldots. \tag{3.9}$$

Here, let us multiply the both sides of (3.9) by $u_i^h - \varpi$. Then, by using (1.2) and Schwarz's inequality, we have:

$$\begin{aligned}&\frac{1}{h}|u_i^h - \varpi|_{L^2(\Omega)}^2 + |u_i^h - \varpi|_{V_0}^2 = \frac{1}{h}(u_{i-1}^h - \varpi, u_i^h - \varpi)_{L^2(\Omega)} \\ &\quad + (f_i^h - \varrho, u_i^h - \varpi)_{L^2(\Omega)} + \frac{L}{h}(\eta_i^h - \eta_{i-1}^h, u_i^h - \varpi)_{L^2(\Omega)} \\ \le\ & \frac{1}{2h}|u_{i-1}^h - \varpi|_{L^2(\Omega)}^2 + \frac{1}{2h}|u_i^h - \varpi|_{L^2(\Omega)}^2 + P_0|f_i^h - \varrho|_{L^2(\Omega)}|u_i^h - \varpi|_{V_0} \\ &\quad + \frac{L}{h}(u_i, \eta_i^h - \eta_{i-1}^h)_{L^2(\Omega)} - \frac{L}{h}(\eta_i^h, \varpi)_{L^2(\Omega)} + \frac{L}{h}(\eta_{i-1}^h, \varpi)_{L^2(\Omega)}, \ i = 1, 2, 3, \ldots,\end{aligned}$$

so that:

$$\begin{aligned}&\frac{1}{2L}|u_i^h - \varpi|_{L^2(\Omega)}^2 + (\eta_i^h, \varpi)_{L^2(\Omega)} + \frac{h}{2L}|u_i^h - \varpi|_{V_0}^2 \\ \le\ & \frac{1}{2L}|u_{i-1}^h - \varpi|_{L^2(\Omega)}^2 + (\eta_{i-1}^h, \varpi)_{L^2(\Omega)} + \frac{P_0^2 h}{2L}|f_i^h - \varrho|_{L^2(\Omega)}^2 \\ &\quad + (u_i^h, \eta_i^h - \eta_{i-1}^h)_{L^2(\Omega)}, \ i = 1, 2, 3, \ldots.\end{aligned} \tag{3.10}$$

Also, we multiply the both sides of (3.9) by $u_i^h - u_{i-1}^h$. Then, by using (1.2) and Schwarz's inequality, it is computed that:

$$\begin{aligned}&\frac{1}{h}|u_i^h - u_{i-1}^h|_{L^2(\Omega)}^2 + \frac{1}{2}|u_i^h - \varpi|_{V_0}^2 - \frac{1}{2}|u_{i-1}^h - \varpi|_{V_0}^2 \\ \le\ & \frac{1}{2h}|u_i^h - u_{i-1}^h|_{L^2(\Omega)}^2 + h|f_i^h - \varrho|_{L^2(\Omega)}^2 + \frac{L^2}{h}|\eta_i^h - \eta_{i-1}^h|_{L^2(\Omega)}^2, \ i = 1, 2, 3, \ldots.\end{aligned}$$

So, multiplying the both sides by $A_0 h$, and applying (2.5), the above inequality can be reduced to:

$$\begin{aligned}&\frac{A_0}{2h}|u_i^h - u_{i-1}^h|_{L^2(\Omega)}^2 - \frac{1}{2h}|\eta_i^h - \eta_{i-1}^h|_{L^2(\Omega)}^2 - A_0 h|f_i^h - \varrho|_{L^2(\Omega)}^2 \\ &\quad + \frac{A_0}{2}|u_i^h - \varpi|_{V_0}^2 \leq \frac{A_0}{2}|u_{i-1}^h - \varpi|_{V_0}^2, \ i = 1, 2, 3, \ldots.\end{aligned} \tag{3.11}$$

Next, let us take η_{i-1}^h as the test function in (3.3). Besides, with the convexities of α and β in mind, we apply Schwarz's inequality and Taylor's theorem to derive that:

$$\begin{aligned}&\left(\frac{1}{h} - \frac{1}{2}\right)|\eta_i - \eta_{i-1}|_{L^2(\Omega)}^2 + \Psi_{[0,1]}(\eta_i^h) - \Psi_{[0,1]}(\eta_{i-1}^h) \\ &\quad -\frac{1}{2}\int_\Omega \left(\eta_i^h - \tfrac{1}{2}\right)^2 dx + \frac{1}{2}\int_\Omega \left(\eta_{i-1}^h - \tfrac{1}{2}\right)^2 dx \\ &\quad + \int_\Omega \alpha(\eta_i^h)|\nabla \theta_{i-1}^h|\,dx - \int_\Omega \alpha(\eta_{i-1}^h)|\nabla \theta_{i-1}^h|\,dx \\ &\quad + \nu \int_\Omega \beta(\eta_i^h)|\nabla \theta_{i-1}^h|^2\,dx - \nu\int_\Omega \beta(\eta_{i-1}^h)|\nabla \theta_{i-1}^h|^2\,dx \\ &\quad + (u_i^h, \eta_i^h - \eta_{i-1}^h)_{L^2(\Omega)} \leq 0, \ i = 1, 2, 3, \ldots.\end{aligned} \tag{3.12}$$

Finally, we put $\omega = \theta_{i-1}^h$ in (3.4). Then, by using basic theory of convex analysis, it is inferred that:

$$\begin{aligned}&\frac{1}{h}|\sqrt{\alpha_0(\eta_i^h)}(\theta_i^h - \theta_{i-1}^h)|_{L^2(\Omega)}^2 + \int_\Omega \alpha(\eta_i^h)|\nabla\theta_i^h|\,dx - \int_\Omega \alpha(\eta_i^h)|\nabla \theta_{i-1}^h|\,dx \\ &\quad + \nu\int_\Omega \beta(\eta_i^h)|\nabla\theta_i^h|^2\,dx - \nu\int_\Omega \beta(\eta_i^h)|\nabla\theta_{i-1}^h|^2\,dx \leq 0, \ i = 1, 2, 3, \ldots.\end{aligned} \tag{3.13}$$

Now, let us set $h_* := \min\{h_0, \frac{1}{2}\}$. Then, for any $h \in (0, h_*]$, we can see that:

$$\frac{1}{2h} - \frac{1}{2} = \frac{1}{2h}(1 - h) \geq \frac{1}{4h},$$

and hence, we can obtain the energy-inequality (3.8) by taking the sum of (3.10)–(3.13). ■

Hereafter, let h_* be the constants as in Lemma 3.3, and for any $h \in (0, h_*]$ ($\subset (0, h_0]$), let $\{[u_i^h, \eta_i^h, \theta_i^h]\}_{i=0}^\infty$ be the solution to $(\mathrm{AP})_h$ obtained in Proposition 3.1. On this basis, we define three different kinds of time-interpolations $[\overline{u}_h, \overline{\eta}_h, \overline{\theta}_h] \in L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega)^3)$, $[\underline{u}_h, \underline{\eta}_h, \underline{\theta}_h] \in L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega)^3)$ and $[\widehat{u}_h, \widehat{\eta}_h, \widehat{\theta}_h] \in W^{1,\infty}_{\mathrm{loc}}([0,\infty); L^2(\Omega)^3)$, by letting

$$\begin{cases} [\overline{u}_h(t), \overline{\eta}_h(t), \overline{\theta}_h(t)] := [u_i^h, \eta_i^h, \theta_i^h], \\ \qquad \text{for any } t \geq 0 \text{ and any } 0 \leq i \in \mathbb{Z} \text{ satisfying } t \in ((i-1)h, ih], \\ [\underline{u}_h(t), \underline{\eta}_h(t), \underline{\theta}_h(t)] := [u_{i-1}^h, \eta_{i-1}^h, \theta_{i-1}^h], \\ \qquad \text{for any } t \geq 0 \text{ and any } i \in \mathbb{N} \text{ satisfying } t \in [(i-1)h, ih), \\ [\widehat{u}_h(t), \widehat{\eta}_h(t), \widehat{\theta}_h(t)] := \dfrac{ih - t}{h}[u_{i-1}^h, \eta_{i-1}^h, \theta_{i-1}^h] + \dfrac{t - (i-1)h}{h}[u_i^h, \eta_i^h, \theta_i^h], \\ \qquad \text{for any } t \geq 0 \text{ and any } i \in \mathbb{N} \text{ satisfying } t \in [(i-1)h, ih). \end{cases}$$

Then, from Proposition 3.1, it immediately follows that:

$$\left.\begin{array}{l}[\overline{u}_h(t),\overline{\eta}_h(t),\overline{\theta}_h(t)]\in D_*(\theta_0),\\ [\underline{u}_h(t),\underline{\eta}_h(t),\underline{\theta}_h(t)]\in D_*(\theta_0),\\ [\widehat{u}_h(t),\widehat{\eta}_h(t),\widehat{\theta}_h(t)]\in D_*(\theta_0),\end{array}\right\}\text{for all } t\geq 0 \text{ and } h\in(0,h_*]. \tag{3.14}$$

Also, from (3.8) in Lemma 3.3, we can see that:

$$\begin{aligned}\frac{A_0}{2}\int_s^t|\partial_t\widehat{u}_h(\tau)|^2_{L^2(\Omega)}\,d\tau+\frac{1}{4}\int_s^t|\partial_t\widehat{\eta}_h(\tau)|^2_{L^2(\Omega)}\,d\tau+\int_s^t|(\sqrt{\alpha_0(\overline{\eta}_h)}\partial_t\widehat{\theta}_h)(\tau)|^2_{L^2(\Omega)}\,d\tau\\ +\frac{1}{2L}\int_s^t|\overline{u}_h(\tau)-\varpi|^2_{V_0}\,d\tau-\frac{B_0}{2}\int_s^t|f_h(\tau)-\varrho|^2_{L^2(\Omega)}\,d\tau\\ +\mathscr{G}_\varpi(\overline{u}_h(t),\overline{\eta}_h(t),\overline{\theta}_h(t))\leq\mathscr{G}_\varpi(\underline{u}_h(s),\underline{\eta}_h(s),\underline{\theta}_h(s)),\\ \text{for all } 0\leq s\leq t<\infty, \text{ and any } [\varrho,\varpi]=[\varrho,F_0^{-1}\varrho]\in L^2(\Omega)\times W_0.\end{aligned} \tag{3.15}$$

where

$$f_h(t):=f_i^h \text{ in } L^2(\Omega), \text{ for any } t\geq 0 \text{ and any } i\in\mathbb{N} \text{ satisfying } t\in[(i-1)h,ih).$$

Note that the assumption (A0) and (3.1) imply:

$$f_h\to f \text{ in } L^2_{\rm loc}([0,\infty);L^2(\Omega)), \text{ as } h\downarrow 0. \tag{3.16}$$

The above (3.14)–(3.16) enable us to say that:

(♯1-a) $\{[\widehat{u}_h,\widehat{\eta}_h,\widehat{\theta}_h]\}_{h\in(0,h_*]}$ is a bounded sequence in the space $W^{1,2}_{\rm loc}([0,\infty);L^2(\Omega)^3)\cap L^\infty_{\rm loc}([0,\infty);V_0\times H^1(\Omega)^2)$;

(♯1-b) $\{[\overline{u}_h,\overline{\eta}_h,\overline{\theta}_h]\}_{h\in(0,h_*]}$ and $\{[\underline{u}_h,\underline{\eta}_h,\underline{\theta}_h]\}_{h\in(0,h_*]}$ are bounded sequences in the space $L^\infty_{\rm loc}([0,\infty);V_0\times H^1(\Omega)^2)$.

Therefore, by applying the compactness theory of Aubin's type [23], we find a sequence $h_*>h_1>h_2>h_3>\cdots>h_n\downarrow 0$ as $n\to\infty$, and a triplet of functions $[u,\eta,\theta]\in L^2_{\rm loc}([0,\infty);L^2(\Omega)^3)$, such that the sequences:

$$\left\{\begin{array}{l}\{[\overline{u}_n,\overline{\eta}_n,\overline{\theta}_n]\}_{n=1}^\infty:=\{[\overline{u}_{h_n},\overline{\eta}_{h_n},\overline{\theta}_{h_n}]\}_{n=1}^\infty,\\ \{[\underline{u}_n,\underline{\eta}_n,\underline{\theta}_n]\}_{n=1}^\infty:=\{[\underline{u}_{h_n},\underline{\eta}_{h_n},\underline{\theta}_{h_n}]\}_{n=1}^\infty,\\ \{[\widehat{u}_n,\widehat{\eta}_n,\widehat{\theta}_n]\}_{n=1}^\infty:=\{[\widehat{u}_{h_n},\widehat{\eta}_{h_n},\widehat{\theta}_{h_n}]\}_{n=1}^\infty,\end{array}\right.$$

fulfill the following properties:

$$\left\{\begin{array}{l}\bullet\ [u,\eta,\theta]\in W^{1,2}_{\rm loc}([0,\infty);L^2(\Omega)^3)\cap L^\infty_{\rm loc}([0,\infty);V_0\times H^1(\Omega)^2),\\ \bullet\ [u(t),\eta(t),\theta(t)]\in D_*(\theta_0),\ \text{ for any } t\geq 0,\\ \bullet\ [u(0),\eta(0),\theta(0)]=[\widehat{u}_n(0),\widehat{\eta}_n(0),\widehat{\theta}_n(0)]=[u_0,\eta_0,\theta_0] \text{ in } L^2(\Omega)^3,\\ \quad\text{for any } n\in\mathbb{N};\end{array}\right. \tag{3.17}$$

$$\begin{aligned}[\widehat{u}_n,\widehat{\eta}_n,\widehat{\theta}_n]\to[u,\eta,\theta] \text{ in } C_{\rm loc}([0,\infty);L^2(\Omega)^2), \text{ weakly in } W^{1,2}_{\rm loc}([0,\infty);L^2(\Omega)^3)\\ \text{and weakly-}* \text{ in } L^\infty_{\rm loc}([0,\infty);V_0\times H^1(\Omega)^2), \text{ as } n\to\infty;\end{aligned} \tag{3.18}$$

$$[\overline{u}_n,\overline{\eta}_n,\overline{\theta}_n]\to[u,\eta,\theta] \text{ and } [\underline{u}_n,\underline{\eta}_n,\underline{\theta}_n]\to[u,\eta,\theta] \text{ in } L^\infty_{\rm loc}([0,\infty);L^2(\Omega)^2) \\ \text{and weakly-}* \text{ in } L^\infty_{\rm loc}([0,\infty);V_0\times H^1(\Omega)^2), \text{ as } n\to\infty; \tag{3.19}$$

$$\begin{cases} \overline{u}_n(t)\to u(t),\ \underline{u}_n(t)\to u(t) \text{ and } \widehat{u}_n(t)\to u(t) \text{ weakly in } V_0, \\ \overline{\eta}_n(t)\to \eta(t),\ \underline{\eta}_n(t)\to \eta(t) \text{ and } \widehat{\eta}_n(t)\to \eta(t) \text{ weakly in } H^1(\Omega), \\ \overline{\theta}_n(t)\to \theta(t),\ \underline{\theta}_n(t)\to \theta(t) \text{ and } \widehat{\theta}_n(t)\to \theta(t) \text{ weakly in } H^1(\Omega), \end{cases} \\ \text{as } n\to\infty, \text{ for any } t\in I; \tag{3.20}$$

$$\big(\alpha(\underline{\eta}_n)\nabla\underline{\theta}_n\big)(t)\to\big(\alpha(\eta)\nabla\theta\big)(t),\ \big(\sqrt{\beta(\underline{\eta}_n)}\nabla\underline{\theta}_n\big)(t)\to\big(\sqrt{\beta(\eta)}\nabla\theta\big)(t), \\ \text{weakly in } L^2(\Omega)^N, \text{ as } n\to\infty, \text{ for any } t\in I; \tag{3.21}$$

and in particular,

$$\begin{cases} 0\le\overline{\eta}_n(t)\le 1,\ 0\le\underline{\eta}_n(t)\le 1,\ 0\le\widehat{\eta}_n(t)\le 1,\ 0\le\eta(t)\le 1, \\ |\overline{\theta}_n(t)|\le|\theta_0|_{L^\infty(\Omega)},\ |\underline{\theta}_n(t)|\le|\theta_0|_{L^\infty(\Omega)},\ |\widehat{\theta}_n(t)|\le|\theta_0|_{L^\infty(\Omega)},\ |\theta(t)|\le|\theta_0|_{L^\infty(\Omega)}, \end{cases} \\ \text{a.e. in } \Omega, \text{ for any } t\ge 0 \text{ and any } n\in\mathbb{N}. \tag{3.22}$$

Based on these, we can refer to the previous study [21], to check the following lemma.

Lemma 3.4 *Let $I\subset(0,T)$ be any open interval. Let $\hat{\Phi}^I : L^2(I;L^2(\Omega))\longrightarrow[0,\infty]$ and $\hat{\Phi}^I_n : L^2(I;L^2(\Omega))\longrightarrow[0,\infty]$, $n\in\mathbb{N}$, be functionals, defined as:*

$$\begin{cases} \zeta\in L^2(I;L^2(\Omega))\mapsto\hat{\Phi}^I(\zeta):=\displaystyle\int_I\Phi(\eta(t);\zeta(t))\,dt, \\ \zeta\in L^2(I;L^2(\Omega))\mapsto\hat{\Phi}^I_n(\zeta):=\displaystyle\int_I\Phi(\overline{\eta}_n(t);\zeta(t))\,dt, \text{ for } n\in\mathbb{N}, \end{cases}$$

by using $\eta\in L^2(0,T;L^2(\Omega))$ and $\overline{\eta}_n\in L^2(0,T;L^2(\Omega))$, $n\in\mathbb{N}$, as in (3.17)–(3.22). Then, the following items hold.

(A) $\hat{\Phi}^I$ and $\hat{\Phi}^I_n$, $n\in\mathbb{N}$, are proper l.s.c and convex functions on $L^2(I;L^2(\Omega))$, such that $D(\hat{\Phi}^I)=D(\hat{\Phi}^I_n)=L^2(I;H^1(\Omega))$, for all $n\in\mathbb{N}$.

(B) $\hat{\Phi}^I_n\to\hat{\Phi}^I$ on $L^2(I;L^2(\Omega))$, in the sense of Mosco, as $n\to\infty$.

(C) If $\theta^\dagger\in L^2(I;H^1(\Omega))$, $\{\theta^\dagger_n\}_{n=1}^\infty\subset L^2(I;H^1(\Omega))$, $\theta^\dagger_n\to\theta^\dagger$ in $L^2(I;L^2(\Omega))$ and $\hat{\Phi}^I_n(\theta^\dagger_n)\to\hat{\Phi}^I(\theta^\dagger)$, as $n\to\infty$, then $\theta^\dagger_n\to\theta^\dagger$ in $L^2(I;H^1(\Omega))$ as $n\to\infty$.

Proof. We omit to show the detailed proof, because the demonstration scenario is just a slight modifications of those as in [21, Lemmas 4.1–4.2]. ∎

Now, the Main Theorem 1 is proved as follows.

Proof of Main Theorem 1. First, the condition (S0) can be obtained, immediately, as a straightforward consequence of (3.17).

Next, we verify conditions (S1)–(S3). Let us fix any bounded open interval $I\subset(0,\infty)$. Then, due to (3.2)–(3.4) and Remark 1.2 (Fact 0), the functions $[\overline{u}_n,\overline{\eta}_n,\overline{\theta}_n]$, $[\underline{u}_h,\underline{\eta}_n,\underline{\theta}_n]$ and $[\widehat{u}_n,\widehat{\eta}_n,\widehat{\theta}_n]$, for $n\in\mathbb{N}$, must satisfy

$$\int_I\big(\partial_t(\widehat{u}_n-L\widehat{\eta}_n)(t),z\big)_{L^2(\Omega)}\,dt+\int_I(\overline{u}_n(t),z)_{V_0}\,dt=\int_I(f_{h_n}(t),z)_{L^2(\Omega)}\,dt \\ \text{for any } z\in V_0 \text{ and any } n\in\mathbb{N}, \tag{3.23}$$

$$\begin{aligned}\int_I \big(\partial_t \widehat{\eta}_n(t), (\overline{\eta}_n - \psi)(t)\big)_{L^2(\Omega)}\,dt + \int_I \big(\nabla \overline{\eta}_n(t), \nabla(\overline{\eta}_n - \psi)(t)\big)_{L^2(\Omega)^N}\,dt \\ - \int_I \big((\overline{\eta}_n - \overline{u}_n - \tfrac{1}{2})(t), (\overline{\eta}_n - \psi)(t)\big)_{L^2(\Omega)}\,dt \\ + \int_I\int_\Omega (\overline{\eta}_n - \psi)(t)\alpha'(\overline{\eta}_n(t))|\nabla \underline{\theta}_n(t)|\,dxdt \\ + \nu \int_I\int_\Omega (\overline{\eta}_n - \psi)(t)\beta'(\overline{\eta}_n(t))|\nabla \underline{\theta}_n(t)|^2\,dxdt \le 0\end{aligned} \qquad (3.24)$$

for any $\psi \in L^2(I; H^1(\Omega))$ with $\psi(t) \in D(\Psi_{[0,1]})$ a.e. $t \in I$, and any $n \in \mathbb{N}$,

and

$$\big[\overline{\theta}_n,\ -\alpha_0(\overline{\eta}_n)\partial_t\widehat{\theta}_n\big] \in \partial\Phi_n^I \ \text{ in } L^2(I; L^2(\Omega))^2, \text{ for any } n \in \mathbb{N}. \qquad (3.25)$$

By virtue of (3.17)–(3.19), (3.25), Lemma 3.4 (A)–(B) and Remark 1.3 (Fact 1), it is deduced that:

$$\big[\theta,\ -\alpha_0(\eta)\partial_t\theta\big] \in \partial\Phi^I \ \text{ in } L^2(I; L^2(\Omega))^2, \qquad (3.26)$$

and

$$\Phi_n^I(\overline{\theta}_n) \to \Phi^I(\theta) \text{ as } n \to \infty. \qquad (3.27)$$

Here, on account of (3.26), Lemma 3.4 (A) and Remark 1.2 (Fact 0), we can show the compatibility of the pair $[\eta, \theta]$ with (S3).

In the meantime, from (3.17)–(3.19), (3.27) and Lemma 3.4 (C), we infer that

$$\begin{cases} \overline{\theta}_n \to \theta \text{ in } L^2(I; H^1(\Omega)) \text{ as } n \to \infty, \text{ and hence,} \\ \underline{\theta}_n \to \theta \text{ and } \widehat{\theta}_n \to \theta \text{ in } L^2(I; H^1(\Omega)) \text{ as } n \to \infty. \end{cases}$$

So, invoking (A1) and (3.22), we further have:

$$\begin{aligned}\alpha'(\overline{\eta}_n)\nabla\underline{\theta}_n \to \alpha'(\eta)\nabla\theta \text{ and } \sqrt{\beta'(\overline{\eta}_n)}\nabla\underline{\theta}_n \to \sqrt{\beta'(\eta)}\nabla\theta \\ \text{in } L^2(I; L^2(\Omega)^N) \text{ as } n \to \infty.\end{aligned} \qquad (3.28)$$

Besides, taking a subsequence if necessary, it will be seen that:

$$\begin{aligned}\overline{\theta}_n(s) \to \theta(s),\ \underline{\theta}_n(s) \to \theta(s) \text{ and } \widehat{\theta}_n(s) \to \theta(s) \\ \text{in } H^1(\Omega) \text{ as } n \to \infty, \text{ for a.e. } s \in I,\end{aligned}$$

and furthermore,

$$\begin{aligned}\big(\alpha(\underline{\eta}_n)\nabla\underline{\theta}_n\big)(s) \to \big(\alpha(\eta)\nabla\theta\big)(s),\ \big(\sqrt{\beta(\underline{\eta}_n)}\nabla\underline{\theta}_n\big)(s) \to \big(\sqrt{\beta(\eta)}\nabla\theta\big)(s), \\ \text{in } L^2(\Omega)^N, \text{ as } n \to \infty, \text{ for a.e. } s \in I.\end{aligned} \qquad (3.29)$$

By virtue of (3.16)–(3.19), (3.22) and (3.28), letting $n \to \infty$ in (3.23) and (3.24) yield that:

$$\int_I \big(\partial_t(u - L\eta)(t), z\big)_{L^2(\Omega)}\,dt + \int_I (u(t), z)_{V_0}\,dt = \int_I (f(t), z)_{L^2(\Omega)}\,dt$$
$$\text{for any } z \in V_0,$$

and

$$\int_I \big(\partial_t\eta(t), (\eta-\psi)(t)\big)_{L^2(\Omega)}\,dt + \int_I \big(\nabla\eta(t), \nabla(\eta-\psi)(t)\big)_{L^2(\Omega)^N}\,dt$$
$$- \int_I \big((\eta - u - \tfrac{1}{2})(t), (\eta-\psi)(t)\big)_{L^2(\Omega)}\,dt$$
$$+ \int_I\int_\Omega (\eta-\psi)(t)\alpha'(\eta(t))|\nabla\theta(t)|\,dxdt + \nu \int_I\int_\Omega (\eta-\psi)(t)\beta'(\eta(t))|\nabla\theta(t)|^2\,dxdt \le 0$$
$$\text{for any } \psi \in L^2(I; H^1(\Omega)) \text{ with } \psi(t) \in D(\Psi_{[0,1]}) \text{ a.e. } t \in I,$$

respectively.

We thus obtain the compatibility of $[u, \eta, \theta]$ with (S1)–(S2), because the choice of the bounded open interval $I \subset (0, \infty)$ is arbitrary.

Finally, we verify the condition (S4). For this purpose, let us put $\psi = \eta$ in (3.24) to see the limiting situation as $n \to \infty$. Then, having in mind (3.16)–(3.19), (3.22) and (3.28), we observe from (3.23) and (3.24) that:

$$\begin{aligned}\limsup_{n\to\infty} \int_I |\overline{u}_n(t)|_{V_0}^2\,dt &\le \lim_{n\to\infty} \int_I \big(\overline{u}_n(t), u(t)\big)_{V_0}\,dt \\ &\quad + \lim_{n\to\infty} \int_I \big(-\partial_t(\widehat{u}_n - L\widehat{\eta}_n)(t) + f_n(t), (\overline{u}_n - u)(t)\big)_{L^2(\Omega)}\,dt \\ &= \int_I |u(t)|_{V_0}^2\,dt, \end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \int_I |\nabla\overline{\eta}_n(t)|_{L^2(\Omega)^N}^2\,dt &\le \lim_{n\to\infty} \int_I \big(\nabla\overline{\eta}_n(t), \nabla\eta(t)\big)_{L^2(\Omega)^N}\,dt \\ &\quad + \lim_{n\to\infty} \int_I \big(-\partial_t\widehat{\eta}_n(t) + (\overline{\eta}_n - \overline{u}_n - \tfrac{1}{2})(t), (\overline{\eta}_n - \eta)(t)\big)_{L^2(\Omega)}\,dt \\ &\quad + \lim_{n\to\infty} \int_I\int_\Omega (\overline{\eta}_n - \eta)(t)\alpha'(\overline{\eta}_n(t))|\nabla\underline{\theta}_n(t)|\,dxdt \\ &\quad + \nu \lim_{n\to\infty} \int_I\int_\Omega (\overline{\eta}_n - \eta)(t)\beta'(\overline{\eta}_n(t))|\nabla\underline{\theta}_n(t)|^2\,dxdt \\ &= \int_I |\nabla\eta(t)|_{L^2(\Omega)^N}^2\,dt, \end{aligned} \tag{3.31}$$

respectively.

From (3.19), (3.30)–(3.31) and the uniform convexities of L^2-base topologies, it follows that:

$$\overline{u}_n \to u \text{ in } L^2(I; V_0) \text{ and } \overline{\eta}_n \to \eta \text{ in } L^2(I; H^1(\Omega)), \text{ as } n \to \infty,$$

and hence,

$$\begin{cases} \underline{u}_n \to u \text{ and } \widehat{u}_n \to u \text{ in } L^2(I; V_0), \\ \underline{\eta}_n \to \eta \text{ and } \widehat{\eta}_n \to \eta \text{ in } L^2(I; H^1(\Omega)), \end{cases} \quad \text{as } n \to \infty.$$

Besides, taking a subsequence if necessary, it is further seen that:

$$\begin{cases} \overline{u}_n(s) \to u(s),\ \underline{u}(s) \to u(s) \text{ and } \widehat{u}_n(s) \to u(s) \text{ in } V_0, \\ \overline{\eta}_n(s) \to \eta(s),\ \underline{\eta}(s) \to \eta(s) \text{ and } \widehat{\eta}_n(s) \to \eta(s) \text{ in } H^1(\Omega), \end{cases} \tag{3.32}$$
$$\text{as } n \to \infty, \text{ for a.e. } s \in I.$$

Now, the condition (S4) can be verified by putting $h = h_n$ in (3.15), letting $n \to \infty$, and taking into account (3.17)–(3.22), (3.29) and (3.32). ■

4 Proof of Main Theorem 2

In the proof of Main Theorem 2, the key-point is in the energy-inequality (2.7) obtained in the previous Main Theorem 1.

Let $[u, \eta, \theta] \in W^{1,2}_{\mathrm{loc}}([0, \infty); L^2(\Omega)^3) \cap L^\infty_{\mathrm{loc}}([0, \infty); V_0 \times H^1(\Omega)^2)$ be the solution to (S) obtained in Main Theorem 1, and let $f_\infty \in L^2(\Omega)$ be the function as in (A3). Besides, in the energy-inequality (2.7), let us put:

$$\varrho = f_\infty \text{ and } \varpi = w_\infty := F_0^{-1} f_\infty \text{ in } V_0 \ (w_\infty \in W_0).$$

Then, with (2.4) and (2.6) in mind, we can see from (2.7) that:

$$\begin{aligned} &\frac{A_0}{2}|u(t) - w_\infty|^2_{V_0} + \mathscr{F}_0(\eta(t), \theta(t)) \\ &\quad + \frac{A_0}{2}\int_0^t |\partial_t u(\tau)|^2_{L^2(\Omega)}\,d\tau + \frac{1}{4}\int_0^t |\partial_t \eta(\tau)|^2_{L^2(\Omega)}\,d\tau \\ &\quad + \int_0^t |(\sqrt{\alpha_0(\eta)}\partial_t\theta)(\tau)|^2_{L^2(\Omega)}\,d\tau + \frac{1}{2L}\int_0^t |u(\tau) - w_\infty|^2_{V_0}\,d\tau \\ \le\ & \frac{1}{2L}|u_0 - w_\infty|^2_{L^2(\Omega)} + \frac{A_0}{2}|u_0 - w_\infty|^2_{V_0} + \mathscr{F}_0(\eta_0, \theta_0) \\ &\quad + (\eta_0 - \eta(t), w_\infty)_{L^2(\Omega)} + \frac{B_0}{2}\int_0^t |f(\tau) - f_\infty|^2_{L^2(\Omega)}\,d\tau \\ \le\ & \frac{B_0}{2}\left(|u_0 - w_\infty|^2_{L^2(\Omega)} + |f - f_\infty|^2_{L^2(0,\infty;L^2(\Omega))}\right) \\ &\quad + \mathscr{F}_0(\eta_0, \theta_0) + \mathscr{L}^N(\Omega)^{\frac{1}{2}}|w_\infty|_{L^2(\Omega)} \\ =:\ & \mathscr{J}_\infty < \infty, \text{ for any } t \ge 0. \end{aligned} \tag{4.1}$$

By using the above estimate, the Main Theorem 2 is proved as follows.

Proof of Main Theorem 2. First, we verify (O). From (4.1), it is observed that:

(♯2-a) $[\partial_t u, \partial_t \eta, \partial_t \theta] \in L^2(0, \infty; L^2(\Omega)^3)$, and $u - w_\infty \in L^2(0, \infty; V_0) \cap L^\infty(0, \infty; V_0)$;

(♯2-b) the orbit $\{[u(t), \eta(t), \theta(t)] \,|\, t \ge 0\}$ is contained in a class $\mathcal{K}_\infty$, defined as

$$\mathcal{K}_\infty := \left\{ [\tilde{u}, \tilde{\eta}, \tilde{\theta}] \in D_*(\theta_0) \ \middle|\ \frac{A_0}{2}|\tilde{u} - w_\infty|^2_{L^2(\Omega)} + \mathscr{F}_0(\tilde{\eta}, \tilde{\theta}) \le \mathscr{J}_\infty \right\};$$

(♯2-c) the class $\mathcal{K}_\infty$ is a compact set in $L^2(\Omega)^3$.

Therefore, we find a triplet $[u_\infty, \eta_\infty, \theta_\infty] \in L^2(\Omega)^3$ and a sequence of times $0 < t_1 < t_2 < t_3 < \cdots < t_n \uparrow \infty$ as $n \to \infty$, such that:

$$[u(t_n), \eta(t_n), \theta(t_n)] \to [u_\infty, \eta_\infty, \theta_\infty] \text{ in } L^2(\Omega)^3, \text{ as } n \to \infty. \tag{4.2}$$

This implies that $\omega(u, \eta, \theta) \neq \emptyset$. Also, the compactness of $\omega(u, \eta, \theta)$ is obtained by taking into account (♯2-b) and the fact that:

$$\omega(u, \eta, \theta) = \bigcap_{s \geq 0} \overline{\{ [u(t), \eta(t), \theta(t)] \mid t \geq s \}} \subset \overline{\mathcal{K}_\infty} = \mathcal{K}_\infty.$$

Thus, the item (O) holds.

Next, we verify (I). Let us take any $[u_\infty, \eta_\infty, \theta_\infty] \in \omega(u, \eta, \theta)$ with a sequence $0 < t_1 < t_2 < t_3 < \cdots < t_n \uparrow \infty$ as $n \to \infty$, such that (4.2) holds. In this situation, we can see from (♯2-a)–(♯2-c) that:

(♯2-d) $\{u_n\}_{n=1}^\infty := \{u(\cdot + t_n)\}_{n=1}^\infty$ is a bounded sequence in $W^{1,2}(0,1; L^2(\Omega)) \cap L^\infty(0,1; V_0)$;

(♯2-e) $\{\eta_n\}_{n=1}^\infty := \{\eta(\cdot + t_n)\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty := \{\theta(\cdot + t_n)\}_{n=1}^\infty$ are bounded sequences in $W^{1,2}(0,1; L^2(\Omega)) \cap L^\infty(0,1; H^1(\Omega))$;

(♯2-f) $\{[u_n(t), \eta_n(t), \theta_n(t)] \mid t \in [0,1],\ n \in \mathbb{N}\} \subset \mathcal{K}_\infty$, and in particular, $0 \leq \eta_n(t) \leq 1$ and $|\theta_n(t)| \leq |\theta_0|_{L^\infty(\Omega)}$ a.e. in Ω, for any $t \geq 0$ and any $n \in \mathbb{N}$.

Owing to (A1), (♯2-a)–(♯2-f) and the compactness theory of Aubin's type [23], we infer that:

$$\begin{cases} \partial_t u_n \to 0 & \text{in } L^2(0,1; L^2(\Omega)), \\ u_n \to w_\infty & \text{in } L^2(0,1; V_0), \\ & \text{in } C([0,1]; L^2(\Omega)) \text{ and} \\ & \text{weakly-}* \text{ in } L^\infty(0,1; V_0), \end{cases} \quad \text{as } n \to \infty, \tag{4.3}$$

$$\begin{cases} \partial_t \eta_n \to 0 \text{ and } \partial_t \theta_n \to 0 & \text{in } L^2(0,1; L^2(\Omega)), \\ \eta_n \to \eta_\infty \text{ and } \theta_n \to \theta_\infty & \text{in } C([0,1]; L^2(\Omega)) \text{ and} \\ & \text{weakly-}* \text{ in } L^\infty(0,1; H^1(\Omega)), \end{cases} \quad \text{as } n \to \infty, \tag{4.4}$$

$$\begin{gathered} \alpha(\eta_n) \nabla \theta_n \to \alpha(\eta_\infty) \nabla \theta_\infty \text{ and } \sqrt{\beta(\eta_n)} \nabla \theta_n \to \sqrt{\beta(\eta_\infty)} \nabla \theta_\infty, \\ \text{weakly in } L^2(0,1; L^2(\Omega)^N), \text{ as } n \to \infty, \end{gathered} \tag{4.5}$$

and

$$0 \leq \eta_\infty \leq 1 \text{ and } |\theta_\infty| \leq |\theta_0|_{L^\infty(\Omega)} \text{ a.e. in } \Omega. \tag{4.6}$$

Now, by the uniqueness of limit, the convergences (4.2)–(4.3) lead to:

$$u_\infty = w_\infty = F_0^{-1} f_\infty \text{ in } V_0. \tag{4.7}$$

It implies the validity of (i-a).

In the meantime, due to Definition 2.1 (S2)–(S3), the sequence $\{[u_n, \eta_n, \theta_n]\}_{n=1}^{\infty}$ must satisfy that:

$$\begin{aligned}
&\int_0^1 (\partial_t \eta_n(t), \eta_n(t) - \varphi)_{L^2(\Omega)}\,dt + \int_0^1 \big(\nabla \eta_n(t), \nabla(\eta_n(t) - \varphi)\big)_{L^2(\Omega)^N}\,dt \\
&\qquad - \int_0^1 \big((\eta_n - u_n - \tfrac{1}{2})(t), \eta_n(t) - \varphi\big)_{L^2(\Omega)}\,dt \\
&\qquad + \int_0^1 \int_\Omega (\eta_n(t) - \varphi)\alpha'(\eta_n(t))|\nabla \theta_n(t)|\,dxdt \\
&\qquad + \nu \int_0^1 \int_\Omega (\eta_n(t) - \varphi)\beta'(\eta_n(t))|\nabla \theta_n(t)|^2\,dxdt \leq 0, \\
&\qquad \text{for any } \varphi \in D(\Psi_{[0,1]}) \text{ and any } n \in \mathbb{N},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
&\int_0^1 \big((\alpha_0(\eta_n)\partial_t \theta)(t), \theta_n(t)\big)_{L^2(\Omega)}\,dt + \int_0^1 \Phi(\eta_n(t); \theta_n(t))\,dt \\
&\qquad \leq \int_0^1 \Phi(\eta(t); 0)\,dt = 0, \quad \text{for any } n \in \mathbb{N}.
\end{aligned} \tag{4.9}$$

From (4.4) and (4.9), it is observed that:

$$\begin{aligned}
0 &\leq \nu\delta_* \int_0^1 |\nabla \theta_\infty|_{L^2(\Omega)^N}^2\,dt \leq \nu\delta_* \limsup_{n\to\infty} \int_0^1 |\nabla \theta_n|_{L^2(\Omega)^N}^2\,dt \\
&\leq \nu \limsup_{n\to\infty} \int_0^1 \int_\Omega \beta(\eta_n(t))|\nabla \theta_n(t)|^2\,dxdt \\
&\leq \limsup_{n\to\infty} \int_0^1 \Phi(\eta_n(t); \theta_n(t))\,dt \\
&\leq -\lim_{n\to\infty} \int_0^1 \big((\alpha_0(\eta_n)\partial_t \theta_n)(t), \theta_n(t)\big)_{L^2(\Omega)}\,dt = 0,
\end{aligned}$$

and it implies that:

$$|\nabla \theta_\infty|_{L^2(\Omega)^N}^2 = \lim_{n\to\infty} |\nabla \theta_n|_{L^2(\Omega)^N}^2 = 0. \tag{4.10}$$

The item (i-c) will be obtained as a consequence of (4.6) and (4.10).

Furthermore, by virtue of (A1), (4.10) and (♯2-f), we can compute that:

$$\begin{aligned}
&\bigg| \int_0^1 \int_\Omega (\eta_n(t) - \varphi)\alpha'(\eta_n(t))|\nabla \theta_n(t)|\,dxdt \\
&\qquad + \nu \int_0^1 \int_\Omega (\eta_n(t) - \varphi)\beta'(\eta_n(t))|\nabla \theta_n(t)|^2\,dxdt \bigg| \\
&\leq |\alpha'|_{C([0,1])} |\nabla \theta_n|_{L^2(0,1;L^1(\Omega)^N)} + \nu |\beta'|_{C([0,1])} |\nabla \theta_n|_{L^2(0,1;L^2(\Omega)^N)}^2 \\
&\to 0 \text{ as } n \to \infty, \text{ for any } \varphi \in D(\Psi_{[0,1]}).
\end{aligned} \tag{4.11}$$

With (4.3)–(4.7) and (4.11) in mind, letting $n \to \infty$ in (4.8) yields the variational inequality (2.8) asserted in (i-b). ∎

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