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An approach from the Yosida approximation to a quasilinear degenerate parabolic-elliptic chemotaxis system with growth term

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1. Introduction

In this report we consider the solvability of the parabolic-elliptic chemotaxis system

$$(\mathrm{P})\qquad \begin{cases} \dfrac{\partial b}{\partial t}-\Delta D(b)+\nabla\cdot(K(b,c)b\nabla c)=f(b,c) & \text{in } Q:=(0,\infty)\times\Omega,\\ -\Delta c+c=b & \text{in } (0,\infty)\times\Omega,\\ (-\nabla D(b)+K(b,c)b\nabla c)\cdot\nu=0,\quad \dfrac{\partial c}{\partial\nu}=0 & \text{on } (0,\infty)\times\partial\Omega,\\ b(0,x)=b_0(x), & x\in\Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ $(n\leq 3)$ with C^2-boundary, $b:Q\to\mathbb{R}$, $c:Q\to\mathbb{R}$ are unknown functions and $D\in C(\mathbb{R})$, $K,f\in C(\mathbb{R}^2)$ are given functions. In 1970 Keller and Segel proposed the fully parabolic version of (P) with $D(b)=b$, $K(b,c)=1$ and $f(b,c)=0$. This system describes a part of the life cycle of cellular slime molds with chemotaxis. In more detail, slime molds move towards higher concentrations of the chemical substance. Here $b(t,x)$ represents the density of the cell population and $c(t,x)$ shows the concentration of the signal substance at place x and time t.

As introduced by Bellomo, Bellouquid, Tao and Winkler [**2**], a number of variations of the original Keller–Segel system are proposed and studied. In those studies, the proof of existence of local solutions is based on the theory by Ladyzhenskaya, Solonnikova and Uraltseva [**7**] or the theory by Amann [**1**]. These are based on linear theory, which need linearlization, and thus the proof is indirect; note that these studies need the smoothness or boundedness for initial data to prove existence of local solutions. As to the problem (P), Marinoschi [**9**] established existence of local solutions to (P) by an operator theoretic approach under the Lipschitz condition for D,K,f. This approach for existence of solutions to (P) by Marinoschi was new, however, it is insufficient in terms of imposing the smallness of $\|b_0\|_{L^2(\Omega)}$. Concerning this problem, in [**12**] the smallness assumption was removed in the case with Lipschitz and nondegenerate diffusion and with superlinear growth term $f(b,c)$. However these results cannot be applied to the more general case such as porous medium-type diffusion $D(r)=r^m$, which is studied in many papers (see e.g., Chung, Kang and Kim [**4**]). More precisely, porous medium-type diffusion is motivated from a biological point of view (see Szymanska, Morales-Rodrigo, Lachowicz and Chaplain [**11**]), furthermore, more many studies with quasilinear diffusion are found in [**6**]. Therefore it is important to extend the result by Marinoschi to the case with more general diffusion. Recently, we obtained existence results in the case of non-Lipschitz and degenerate diffusion in [**13**]. These results are obtained as an extension of [**12**], which is proved by the approximation of diffusion and it is effective that the smallness assumption for $\|b_0\|_{L^2(\Omega)}$ was removed. However the assumption in these results were strong, because of the way of approximation. In [**13**] we consider the linear approximation of D as follows:

$$D_\varepsilon(r) := D(r+\varepsilon),$$
$$D_\varepsilon^R(r) := \begin{cases} D_\varepsilon(r), & r \le R, \\ D_\varepsilon(R) + D_\varepsilon'(R)(r-R), & r \ge R, \end{cases} \qquad 0<\varepsilon<1<R.$$

This approximation loses some condition for D and thus to prove local existence of solutions we need a technical condition. Also we note that the result in [**13**] did not assert the case with growth term.

The purpose of this report is to improve this problem and obtain existence results in more general case. To overcome this problem, instead of linear approximation, we consider the Yosida approximation of D_ε as

$$D_{\varepsilon,\lambda}(r) := D_\varepsilon(J_{\varepsilon,\lambda}(r)),$$
$$J_{\varepsilon,\lambda}(r) := (I+\lambda D_\varepsilon)^{-1}(r), \quad 0<\varepsilon,\lambda<1,$$

where D_ε is the function defined as above. Note that the Yosida approximation preserves a growth property:

$$D(r) \ge d_1 r^m \Rightarrow D_{\varepsilon,\lambda}(r) \ge d_1 J_{\varepsilon,\lambda}(r)^m.$$

This is one of advantages of the Yosida approximation, whereas linear approximation in [**13**] loses such property.

In this report we make the following assumption on D, K and f:

(A1) $\quad D \in C^1(\mathbb{R}),\ \ D'(r) > 0\ (r>0),\ \ D(r) \ge d_1 r^m \quad (\exists m > n-1,\ \exists d_1 > 0),$

(A2) $\quad D'^{\frac12}(r) \le d_2 \left(\int_0^r D'^{\frac12}(s)\,ds + 1 \right), \quad rD'(r) \le d_3 D(r) \quad (\exists d_2, d_3 > 0),$

(A3) $\quad (r_1,r_2) \mapsto K(r_1,r_2)r_1 \in C^1(\mathbb{R}^2),$

$$\left| \frac{\partial}{\partial r_1}(K(r_1,r_2)r_1) \right| \le k_1 \left(D'^{\frac12}(r_1) + 1 \right) \quad (\exists k_1 > 0),$$
$$\left| \frac{\partial}{\partial r_2}(K(r_1,r_2)r_1) \right| \le k_2 \quad (\exists k_2 > 0),$$

(A4) $\quad |K(r_1,r_2)r_1| \le k_3 \left(r_1^\beta D'^{\frac12}(r_1) + 1 \right) \quad (\exists \beta \in [0, 1-\frac{2}{2^*}],\ \exists k_3 > 0),$

where 2^* denotes the Sobolev embedding exponent with $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$,

(A5) $\quad$ (i) f is Lipschitz continuous on $\mathbb{R}^2$ or

(ii) $f(b,c) = |b|^{\alpha-1}b$, where $2 < \alpha + 1 < 2m + (m+1)\frac{2}{n}$.

We also define weak solutions of (P) as follows.

Definition 1.1. Let $T > 0$. A pair (b, c) is said to be a *weak solution* of (P) on $[0, T]$ if

(a) $0 \le b \in C([0,T]; L^2(\Omega)) \cap H^1(0,T; (H^1(\Omega))')$, $D(b) \in L^2(0,T; H^1(\Omega))$,

(b) $0 \le c \in C([0,T]; H^2(\Omega))$,

(c) $b(0) = b_0$ and for any $\psi \in H^1(\Omega)$,

$$\left\langle \frac{db}{dt}(t), \psi \right\rangle_{(H^1(\Omega))', H^1(\Omega)} + \int_\Omega \nabla D(b) \cdot \nabla\psi - \int_\Omega K(b,c) b \nabla c \cdot \nabla \psi = \int_\Omega f(b,c)\psi,$$

$$\int_\Omega \nabla c \cdot \nabla \psi + \int_\Omega c\psi = \int_\Omega b\psi.$$

In particular, if $T > 0$ can be taken arbitrarily, then (b, c) is called a *global weak solution* of (P).

Then our main results read as follows.

Theorem 1.1. *Let $n \le 3$. Assume that the conditions* (A1)–(A5) *are satisfied. Let $0 \le b_0 \in L^2(\Omega)$ and $\int_0^{b_0} D(r)\,dr \in L^1(\Omega)$. Then there exists $T > 0$ such that* (P) *possesses a weak solution (b, c) on $[0, T]$. Moreover, the following estimates hold:*

$$\|b(t)\|_{L^2(\Omega)} \le C, \quad t \in [0,T],$$

$$\left\| \int_0^{b(t)} D(r)\,dr \right\|_{L^1(\Omega)} \le C, \quad t \in [0,T],$$

$$\|\nabla c(t)\|_{L^\infty(\Omega)} \le C, \quad t \in [0,T],$$

where C is a constant which depends on $\|b_0\|_{L^2(\Omega)}$ and $\left\| \int_0^{b_0} D(r)\,dr \right\|_{L^1(\Omega)}$.

Under an additional condition, global existence of solutions is established.

Theorem 1.2. *Under the assumption of* Theorem 1.1 *suppose further that $\beta = 0$ in the condition* (A4), *that $D'(r) \le d_4 r^{m-1}$ for some $d_4 > 0$ and that $\alpha \le m$ in the condition* (A5). *Then there exists a global weak solution of* (P).

This report is organized as follows. In Section 2 we introduce an approximate problem and give an existence result for approximate solutions. Section 3 gives estimates for the approximate solutions. Section 4 is devoted to convergence of approximate solutions and gives the proof of Theorem 1.1. Finally we deal with global existence of solutions in Section 5.

2. Approximate Problem

In what follows, we assume the same hypothesis as in Theorem 1.1 and assume (ii) in (A5); we can also prove the case (i) in (A5) by a similar way. We define the real Hilbert spaces V and H as

$$V := H^1(\Omega) \quad \text{and} \quad H := L^2(\Omega)$$

equipped with standard inner products. We shall denote by $\|\cdot\|_V$ and $\|\cdot\|_H$ the norms in V and H, respectively. Then we have $V \subset H \subset V'$ with dense and continuous injections. Introducing the operator $A_\Delta : D(A_\Delta) \subset H \to H$ as

$$A_\Delta := -\Delta \quad \text{with } D(A_\Delta) = \left\{ u \in H^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\},$$

we define the inner product and norm on V' as

$$(v, \overline{v})_{V'} := \langle v, (I + A_\Delta)^{-1}\overline{v}\rangle_{V',V} \quad \text{for } v, \overline{v} \in V',$$
$$\|v\|_{V'} := \left\|(I + A_\Delta)^{-1}v\right\|_V \quad \text{for } v \in V'.$$

To show existence of solutions to (P) we introduce the approximate system

$$\text{(2.1)} \quad \begin{cases} \dfrac{\partial b}{\partial t} - \Delta D_{\varepsilon,\lambda}(b) + \nabla \cdot (K_{\varepsilon,\lambda}(b,c)b\nabla c) = f_{\varepsilon,\lambda}(b,c) & \text{in } (0,\infty)\times\Omega, \\ -\Delta c + c = J_{\varepsilon,\lambda}(b) & \text{in } (0,\infty)\times\Omega, \\ (-\nabla D_{\varepsilon,\lambda}(b) + K_{\varepsilon,\lambda}(b,c)b\nabla c)\cdot\nu = 0, \quad \dfrac{\partial c}{\partial \nu} = 0 & \text{on } (0,\infty)\times\partial\Omega. \\ b(0,x) = b_0(x), & x \in \Omega, \end{cases}$$

where $0 < \varepsilon, \lambda < 1$ and

$$D_\varepsilon(r) := D(r+\varepsilon), \quad D_{\varepsilon,\lambda}(r) := D_\varepsilon(J_{\varepsilon,\lambda}(r)), \quad J_{\varepsilon,\lambda}(r) := (I + \lambda D_\varepsilon)^{-1}(r),$$
$$K_\varepsilon(r_1, r_2) := K(r_1+\varepsilon, r_2), \quad K_{\varepsilon,\lambda}(r_1,r_2) := \frac{K_\varepsilon(J_{\varepsilon,\lambda}(r_1), r_2)J_{\varepsilon,\lambda}(r_1)}{r_1}$$

and $f_{\varepsilon,\lambda}$ is the approximation which varies depending on the form of f: if f is Lipschitz continuous then $f_{\varepsilon,\lambda}(r_1,r_2) := f(J_{\varepsilon,\lambda}(r_1), r_2)$, else if $f(r_1,r_2) == |r_1|^{\alpha-1}r_1$ then

$$f_{\varepsilon,\lambda}(r_1,r_2) = f_{\varepsilon,\lambda}(r_1) := f((I+\lambda f)^{-1}(J_{\varepsilon,\lambda}(r_1)), r_2).$$

Lemma 2.1. *Let $0 < \varepsilon, \lambda < 1$. Let $D_{\varepsilon,\lambda}$ and $K_{\varepsilon,\lambda}$ be as above. Then*

$$\text{(2.2)} \quad 0 < \frac{D'(\varepsilon)}{1+\lambda D'(\varepsilon)} \le D'_{\varepsilon,\lambda}(r) \le \frac{1}{\lambda} < \infty \quad (r \ge 0)$$

and $(r_1,r_2) \mapsto K_{\varepsilon,\lambda}(r_1,r_2)r_1$ is Lipschitz continuous on $\mathbb{R}^2$. Moreover,

$(\mathrm{A1})_{\varepsilon,\lambda}$ $D'_{\varepsilon,\lambda}(r) > 0, \quad D_{\varepsilon,\lambda}(r) \ge d_1 J_{\varepsilon,\lambda}(r)^m,$

$(\mathrm{A2})_{\varepsilon,\lambda}$ $D'^{\frac{1}{2}}_{\varepsilon,\lambda}(r) \le d_2\left(\int_0^r D'^{\frac{1}{2}}_{\varepsilon,\lambda}(s)\,ds + 1\right), \quad J_{\varepsilon,\lambda}(r)D'_{\varepsilon,\lambda}(r) \le d_3 D_{\varepsilon,\lambda}(r),$

$(\mathrm{A3})_{\varepsilon,\lambda}$ $(r_1,r_2) \mapsto K_{\varepsilon,\lambda}(r_1,r_2)r_1 \in C^1(\mathbb{R}^2),$

$$\left|\frac{\partial}{\partial r_1}(K_{\varepsilon,\lambda}(r_1,r_2)r_1)\right| \le k_1\left(D'^{\frac{1}{2}}_\varepsilon(J_{\varepsilon,\lambda}(r_1)) + 1\right), \quad \left|\frac{\partial}{\partial r_2}(K_{\varepsilon,\lambda}(r_1,r_2)r_1)\right| \le k_2,$$

$(\mathrm{A4})_{\varepsilon,\lambda}$ $|K_{\varepsilon,\lambda}(r_1,r_2)r_1| \le k_3\left(J_{\varepsilon,\lambda}(r_1)^\beta D'^{\frac{1}{2}}_\varepsilon(J_{\varepsilon,\lambda}(r_1)) + 1\right),$

where m, d_i, k_i and β are the same constants as in the conditions (A1)–(A4).

Proof. See [**14**]. □

We now state existence of solutions to the approximate problem.

Proposition 2.2 (Existence of Approximate Solutions). *Let $n \le 3$ and $0 < \varepsilon, \lambda < 1$. Then there exists $T_{\varepsilon,\lambda} > 0$ such that (2.1) has a unique weak solution $(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})$ satisfying*

$$0 \le b_{\varepsilon,\lambda} \in C([0, T_{\varepsilon,\lambda}];\ H) \cap L^2(0, T_{\varepsilon,\lambda};\ V) \cap H^1(0, T_{\varepsilon,\lambda};\ V'),$$
$$0 \le c_{\varepsilon,\lambda} \in C([0, T_{\varepsilon,\lambda}];\ D(A_\Delta)).$$

Proof. Let $0 < \varepsilon, \lambda < 1$. In the same way as in [**9, 12**], we rewrite (2.1) as the abstract Cauchy problem

$$\begin{cases} \dfrac{db}{dt}(t) + A_{\varepsilon,\lambda} b(t) = 0 \quad \text{a.a. } t \in (0, T), \\ b(0) = b_0, \end{cases} \tag{2.3}$$

where $A_{\varepsilon,\lambda} : D(A_{\varepsilon,\lambda}) := \{ b \in V;\ D_{\varepsilon,\lambda}(b) \in V \} = V \subset V' \to V'$ is the nonlinear operator defined as

$$\langle A_{\varepsilon,\lambda} b, \psi \rangle_{V',V} := \int_\Omega \nabla D_{\varepsilon,\lambda}(b) \cdot \nabla \psi - \int_\Omega K_{\varepsilon,\lambda}(b, c_b) b \nabla c_b \cdot \nabla \psi - \int_\Omega f_{\varepsilon,\lambda}(b, c_b) \psi$$

for any $\psi \in V$, where we have denoted

$$c_b := (I + A_\Delta)^{-1} J_{\varepsilon,\lambda}(b).$$

Then (b, c) is the weak solution of (2.1) if and only if b is the solution of (2.3). In the previous papers [**9, 12**], we prove existence of solutions by considering the approximate abstract Cauchy problem of (2.3), by proving the quasi-m-accretivity for an approximate operator of $A_{\varepsilon,\lambda}$, and by discussing convergence. We note that, as to the estimate for c_b, it was sufficient to have

$$\|c_b\|_{H^2(\Omega)} \le C_{\mathrm{R}} \|b\|_{L^2(\Omega)} \quad (\exists\, C_{\mathrm{R}} > 0).$$

Though the second equation in the approximate problem in present report seems to be different from one in [**9, 12**], we can derive the same estimate for c_b as

$$\|c_b\|_{H^2(\Omega)} \le C_{\mathrm{R}} \|J_{\varepsilon,\lambda}(b)\|_{L^2(\Omega)} \le C_{\mathrm{R}} \|b\|_{L^2(\Omega)},$$

and hence we can prove existence of solutions to (2.1) by a similar way. □

We conclude this section by a useful inequality for D, which will be used in estimates for the approximate solutions.

Lemma 2.3. *For each $b \in H^1(\Omega)$ it holds that*

$$\left\| \int_0^{J_{\varepsilon,\lambda}(b)} D_\varepsilon'^{\frac{1}{2}}(s)\, ds \right\|_{H^1(\Omega)}^2 \le \left\| \nabla \int_0^{J_{\varepsilon,\lambda}(b)} D_\varepsilon'^{\frac{1}{2}}(s)\, ds \right\|_{L^2(\Omega)}^2 + (1 + d_3) \left\| \int_0^{J_{\varepsilon,\lambda}(b)} D_\varepsilon(s)\, ds \right\|_{L^1(\Omega)}.$$

Proof. We note that the assumption (A2) gives

$$rD_\varepsilon(r) = \int_0^r (D_\varepsilon(s) + sD'_\varepsilon(s))\, ds \le (1+d_3)\int_0^r D_\varepsilon(s)\, ds, \quad r > 0. \tag{2.4}$$

In light of Schwarz's inequality and (2.4), we have

$$\left(\int_0^{J_{\varepsilon,\lambda}(b)} D_\varepsilon'^{\frac{1}{2}}(s)\, ds\right)^2 \le J_{\varepsilon,\lambda}(b)\int_0^{J_{\varepsilon,\lambda}(b)} D'_\varepsilon(s)\, ds \le (1+d_3)\int_0^{J_{\varepsilon,\lambda}(b)} D_\varepsilon(s)\, ds.$$

Hence the assertion follows. □

3. Estimates for Approximate Solutions

In this section we derive some estimates for approximate solutions independent of ε, λ. We give alower estimate for $T^{\max}_{\varepsilon,\lambda}$, where $T^{\max}_{\varepsilon,\lambda}$ is the maximal existence time of the weak solutions to (2.1) in Proposition 2.2.

Lemma 3.1 (Lower Bound for the Existence Time). *There exists a constant $T > 0$ such that for all $0 < \varepsilon, \lambda < 1$,*

$$T^{\max}_{\varepsilon,\lambda} \ge T.$$

Next we give estimates for the approximate solutions.

Lemma 3.2 (Estimates for Approximate Solutions). *Let T be as in* Lemma 3.1. *Then for all $0 < \varepsilon, \lambda < 1$,*

$$\|J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}(t))\|_{L^2(\Omega)} \le \mu_0 = \sqrt{\|b_0\|^2_{L^2(\Omega)} + 1}, \quad t \in [0,T], \tag{3.1}$$

$$\left\|\int_0^{J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda})} D_\varepsilon'^{\frac{1}{2}}(s)\, ds\right\|^2_{L^2(0,T;V)} \le M_1, \tag{3.2}$$

$$\left\|\int_0^{J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}(t))} D_\varepsilon(s)\, ds\right\|_{L^1(\Omega)} \le M_2, \quad t \in [0,T], \tag{3.3}$$

$$\|\nabla c_{\varepsilon,\lambda}(t)\|_{L^\infty(\Omega)} \le M'_2, \quad t \in [0,T], \tag{3.4}$$

$$\|D_{\varepsilon,\lambda}(b_{\varepsilon,\lambda})\|^2_{L^2(0,T;V)} \le 2M_2, \tag{3.5}$$

$$\|f_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})\|^2_{L^2(0,T;V')} \le 2M_3, \tag{3.6}$$

$$\left\|\frac{db_{\varepsilon,\lambda}}{dt}\right\|^2_{L^2(0,T;V')} \le M_4 \tag{3.7}$$

where M_1, M_2, M'_2, M_3 and M_4 are positive constants which do not depend on ε, λ. Moreover there exists $T' \in (0,T)$ such that for each $\delta \in (0,T')$,

$$\left\|\frac{d}{dt}\int_0^{J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda})} D_\varepsilon'^{\frac{1}{2}}(s)\, ds\right\|^2_{L^2(\delta,T';V')} \le M_5, \tag{3.8}$$

where M_5 is a positive constant.

4. Passage to the Limit as $\varepsilon, \lambda \to 0$ (Local Existence)

Letting $\varepsilon, \lambda \to 0$ in (2.1), we can obtain a pair (b, c) which solves (P). To discuss convergence we note the following lemma (see [**10**, p. 51, Lemma 3.9]).

Lemma 4.1. *Put* $1 \le p < \infty$, $u \in L^p(\Omega)$ *and* $(u_\alpha)_{\alpha>0}$ *satisfies*

$$\begin{aligned} u_\alpha &\to u \quad \text{weakly in } L^p(\Omega), \\ u_\alpha &\to v \quad \text{a.e. on } \Omega, \end{aligned}$$

where v *is a measurable function on* Ω. *Then* $u = v$.

Proof of Theorem 1.1. Put $T := T'$. From (3.1) there exists $b \in L^2(0,T;L^2(\Omega))$ such that the following convergence holds:

$$J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) \to b \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \tag{4.1}$$

as $\varepsilon, \lambda \to 0$. Hereafter, we denote a suitable subnet of $(J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}))_{0<\varepsilon,\lambda<1}$ again by the same notation $(J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}))_{0<\varepsilon,\lambda<1}$. Moreover, in light of (3.2) and (3.8), the Lions–Aubin theorem (see [**8**, p. 57]) says that for each $\delta \in (0,T)$ there exists $\zeta_\delta \in L^2(\delta,T;L^2(\Omega)$ such that

$$\tilde{D}_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) = \int_0^{J_{\varepsilon,\lambda}(b)} D_\varepsilon'^{\frac{1}{2}}(s)\,ds \to \zeta_\delta \quad \text{in } L^2(\delta,T;L^2(\Omega)) \text{ and a.e. on } (\delta,T)\times\Omega$$

as $\varepsilon, \lambda \to 0$. Since $\tilde{D}_{\varepsilon,\lambda}^{-1} \searrow \tilde{D}^{-1}$ as $\varepsilon, \lambda \to 0$, where $\tilde{D}(r) := \int_0^r D'^{\frac{1}{2}}(s)\,ds$, we observe

$$J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) = \tilde{D}_{\varepsilon,\lambda}^{-1}(\tilde{D}_{\varepsilon,\lambda}(J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}))) \to \tilde{D}^{-1}(\zeta_\delta) \quad \text{a.e. on } (\delta,T)\times\Omega \tag{4.2}$$

as $\varepsilon, \lambda \to 0$. We can thus apply Lemma 4.1 for (4.1) and (4.2) to conclude that $b = \tilde{D}^{-1}(\zeta_\delta)$ a.e. on $(\delta,T)\times\Omega$. Since δ is arbitrarily, it follows from (4.2) that

$$J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) \to b \quad \text{a.e. on } (0,T)\times\Omega$$

as $\varepsilon, \lambda \to 0$. Moreover, by (3.5), there exists a function $\zeta \in L^2(0,T;V)$ such that

$$D_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) \to \zeta \quad \text{weakly in } L^2(0,T;V)$$

as $\varepsilon, \lambda \to 0$. In particular, $D_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) \to \zeta$ weakly in $L^2((0,T)\times\Omega)$ as $\varepsilon, \lambda \to 0$. Noting that $D_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) \to D(b)$ a.e. on $(0,T)\times\Omega$, we observe from Lemma 4.1 that $\zeta = D(b)$. Thus we have

$$D_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}) \to D(b) \quad \text{weakly in } L^2(0,T;V)$$

as $\varepsilon, \lambda \to 0$. Moreover, (3.1) and (3.6) imply that

$$b \in L^2(0,T;V) \cap H^1(0,T;V') \tag{4.3}$$

and

$$\frac{db_{\varepsilon,\lambda}}{dt} \to \frac{db}{dt} \quad \text{weakly in } L^2(0,T;V')$$

as $\varepsilon, \lambda \to 0$. On the other hand, using (3.1) together with the regularity result and the Sobolev embedding yields that $(c_{\varepsilon,\lambda}(t))_{0<\varepsilon,\lambda<1}$ and $(\nabla c_{\varepsilon,\lambda}(t))_{0<\varepsilon,\lambda<1}$ are bounded in $H^1(\Omega)$ for each $t \in (0,T)$, and hence we see that

$$c_{\varepsilon,\lambda} \to c := (I + A_\Delta)^{-1} b \quad \text{in } L^2(0,T;H^2(\Omega)) \text{ and a.e. on } (0,T)\times\Omega,$$
$$\nabla c_{\varepsilon,\lambda} \to \nabla c \quad \text{in } L^2(0,T;L^{2^*}(\Omega)) \text{ and a.e. on } (0,T)\times\Omega$$

as $\varepsilon, \lambda \to 0$. Moreover, the condition $(\mathrm{A3})_{\varepsilon,\lambda}$ and the Sobolev embedding yield

$$\begin{aligned}\|K_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})b_{\varepsilon,\lambda}\|^2_{L^{\left(1-\frac{1}{2^*}\right)^{-1}}(\Omega)} &\le k_1^2 C_{\mathrm{GN}}'^2 \left\| \int_0^{J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda})} \left(D_\varepsilon'^{\frac12}(s) + 1\right) ds \right\|^2_{H^1(\Omega)} \\ &\le 2k_1^2 C_{\mathrm{GN}}'^2 \left(\left\| \int_0^{J_{\varepsilon,\lambda}(b)} D_\varepsilon'^{\frac12}(s)\, ds \right\|^2_{H^1(\Omega)} + \|J_{\varepsilon,\lambda}(b_{\varepsilon,\lambda})\|^2_{H^1(\Omega)} \right).\end{aligned}$$

Therefore we see that $(K_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})b_{\varepsilon,\lambda})_{0<\varepsilon,\lambda<1}$ is bounded in $L^2(0,T;L^{\left(1-\frac{1}{2^*}\right)^{-1}}(\Omega))$ by the results produced in Lemma 3.2. So there exists a function $\xi \in L^2(0,T;L^{\left(1-\frac{1}{2^*}\right)^{-1}}(\Omega))$ such that

$$K_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})b_{\varepsilon,\lambda} \to \xi \quad \text{weakly in } L^2(0,T;L^{\left(1-\frac{1}{2^*}\right)^{-1}}(\Omega))$$

as $\varepsilon, \lambda \to 0$. In particular, $K_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})b_{\varepsilon,\lambda} \to K(b,c)b$ weakly in $L^2((0,T)\times\Omega)$ as $\varepsilon, \lambda \to 0$. In the same argument as above, we deduce from Lemma 4.1 that $\xi = K(b,c)b$ and hence

$$K_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})b_{\varepsilon,\lambda} \to K(b,c)b \quad \text{weakly in } L^2(0,T;L^{\left(1-\frac{1}{2^*}\right)^{-1}}(\Omega))$$

as $\varepsilon, \lambda \to 0$. Therefore for any $\psi \in V$, we have

$$\int_\Omega K_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda})b_{\varepsilon,\lambda}\nabla c_{\varepsilon,\lambda} \cdot \nabla\psi \to \int_\Omega K(b,c)b\nabla c \cdot \nabla\psi$$

as $\varepsilon, \lambda \to 0$. Moreover the property of the Yosida approximation and (3.6) imply that

$$f_{\varepsilon,\lambda}(b_{\varepsilon,\lambda}, c_{\varepsilon,\lambda}) \to f(b,c) \quad \text{weakly in } L^2(0,T;V')$$

as $\varepsilon, \lambda \to 0$. Thus we conclude that (b,c) solves (P) in V'; note that $b \in C([0,T];L^2(\Omega))$ by (4.3) so that $c \in C([0,T];H^2(\Omega))$. Finally we prove that $b \in C([0,T];L^2(\Omega))$. We first show the weak continuity in $L^2(\Omega)$:

$$\lim_{t\to t_0} (b(t),\psi)_{L^2(\Omega)} = (b(t_0),\psi)_{L^2(\Omega)} \quad (t_0 \in [0,T],\ \psi \in L^2(\Omega)). \tag{4.4}$$

If $\psi \in V$, then we deduce

$$\begin{aligned}|(b(t)-b(t_0),\psi)_{L^2(\Omega)}| = \left| \left\langle \int_{t_0}^t \frac{db}{dt}(s)\,ds, \psi \right\rangle_{V',V} \right| &\le \left| \int_{t_0}^t \left\| \frac{db}{dt}(s) \right\|_{V'} ds \right| \|\psi\|_V \\ &\le |t-t_0|^{\frac12} \left\| \frac{db}{dt} \right\|_{L^2(0,T;V')} \|\psi\|_V \to 0\end{aligned}$$

as $t \to t_0$. If $\psi \in H$, then for all $\varepsilon > 0$ we choose $\psi_\varepsilon \in V$ satisfying $\|\psi - \psi_\varepsilon\|_{L^2(\Omega)} \le \varepsilon$, so that

$$\begin{aligned}\left| (b(t) - b(t_0), \psi)_{L^2(\Omega)} \right| &\le \|b(t) - b(t_0)\|_{L^2(\Omega)} \|\psi - \psi_\varepsilon\|_{L^2(\Omega)} + \left| (b(t) - b(t_0), \psi_\varepsilon)_{L^2(\Omega)} \right| \\ &\le 2\mu_0\varepsilon + \left| (b(t) - b(t_0), \psi_\varepsilon)_{L^2(\Omega)} \right|\end{aligned}$$

and hence

$$\limsup_{t\to t_0} \left| (b(t) - b(t_0), \psi)_{L^2(\Omega)} \right| \le 2\mu_0\varepsilon,$$

which implies (4.4). Next, we can show that

$$\left| \|b(t)\|^2_{L^2(\Omega)} - \|b(t_0)\|^2_{L^2(\Omega)} \right| \le M_0|t - t_0| \to 0 \quad \text{as } t \to t_0,$$

that is,

$$\lim_{t\to t_0} \|b(t)\|_{L^2(\Omega)} = \|b(t_0)\|_{L^2(\Omega)}.$$

This fact and (4.4) imply that $b(t) \to b(t_0)$ in $L^2(\Omega)$ as $t \to t_0$ (see [**3**, Proposition 3.32]). Therefore it turns out that $b \in C([0,T]; L^2(\Omega))$. Thus we conclude that (b, c) is a weak solution of (P). This completes the proof. □

5. Proof of Theorem 1.2 (Global Existence)

The goal of this last section is to prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to show that for all $T > 0$ there exists a constant $C_T > 0$ such that

$$\sup_{t\in[0,T)} \left(\|b(t)\|_{L^2(\Omega)} + \left\| \int_0^{b(t)} D(s)\, ds \right\|_{L^1(\Omega)} \right) \le C_T,$$

where (b, c) is a weak solution of (P) on $[0, T)$. Indeed, we can show that

$$\begin{aligned}&\frac{1}{2} \|b(t)\|^2_{L^2(\Omega)} + \frac{1}{2} \left\| \int_0^{b(t)} D(s)\, ds \right\|_{L^1(\Omega)} \\ &\le e^{L_4 T} \left(\frac{1}{2} \|b_0\|^2_{L^2(\Omega)} + \frac{1}{2} \left\| \int_0^{b_0} D(s)\, ds \right\|_{L^1(\Omega)} \right) + \left(e^{L_4 T} - 1\right), \quad t \in [0, T).\end{aligned}$$

This completes the proof of Theorem 1.2. □

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