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Existence of weak solution for volume preserving mean curvature flow via phase field method

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1 Introduction

Let $U_t \subset \mathbb{R}^d$ be a open set with smooth boundary M_t for any $t \in [0,T)$. A family of hypersurfaces $\{M_t\}_{t\in[0,T)}$ is called the volume preserving mean curvature flow if the velocity vector v of M_t is given by

$$v = h - \langle h \cdot \nu \rangle \nu \quad \text{on } M_t, \quad t \in (0,T), \tag{1.1}$$

where h and ν are the mean curvature vector and the inner unit normal vector for M_t respectively, and

$$\langle h \cdot \nu \rangle := \frac{1}{\mathcal{H}^{d-1}(M_t)} \int_{M_t} h \cdot \nu \, d\mathcal{H}^{d-1}.$$

Here $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. By (1.1), M_t satisfies the volume preserving property, that is

$$\frac{d}{dt}\mathcal{L}^d(U_t) = -\int_{M_t} v \cdot \nu \, d\mathcal{H}^{d-1} = 0. \tag{1.2}$$

Here $\mathcal{L}^d$ is the d-dimensional Lebesgue measure. By (1.2) we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{H}^{d-1}(M_t) &= -\int_{M_t} h \cdot v \, d\mathcal{H}^{d-1} = -\int_{M_t} (v + \langle h \cdot \nu \rangle \nu) \cdot v \, d\mathcal{H}^{d-1} \\ &= -\int_{M_t} |v|^2 \, d\mathcal{H}^{d-1} - \langle h \cdot \nu \rangle \int_{M_t} v \cdot \nu \, d\mathcal{H}^{d-1} = -\int_{M_t} |v|^2 \, d\mathcal{H}^{d-1}. \end{aligned} \tag{1.3}$$

The time global existence of the classical solution to (1.1) with a convex initial data U_0 is proved by Gage [11] ($d = 2$) and Huisken [13] ($d \geq 2$). Escher and Simonett [8] proved that if M_0 is sufficiently close to a Euclidean sphere, then there exists a time global solution for (1.1). Recently, Mugnai, Seis and Spadaro [19] showed the time global existence of the distributional solution for (1.1) by using a variational approach.

In this article we study the phase field method for the volume preserving mean curvature flow and show the global existence theorem which is obtained in [23].

2 L^2-flow

The L^2-flow is a kind of an weak solution for the surface evolution equations [17, 18]. In this section, we derive the concept of the L^2-flow. The precise definition is given in Section 4.

Let M_t be a closed smooth hyper surface in $\mathbb{R}^d$ and v be the normal velocity vector. Then for $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$ we have

$$\frac{d}{dt}\int_{M_t} \eta \, d\mathcal{H}^{d-1} = \int_{M_t} (-h\eta + \nabla\eta)\cdot v + \eta_t \, d\mathcal{H}^{d-1}. \tag{2.1}$$

Here, h is the mean curvature vector of M_t. Then, by (2.1), for any $T>0$, there exists $C_1>0$ such that

$$\left|\int_0^T \int_{M_t} \eta_t + \nabla\eta \cdot v \, d\mathcal{H}^{d-1} dt\right| \leq C_1 \|\eta\|_{C^0(\mathbb{R}^d\times(0,T))} \tag{2.2}$$

for any $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$. Note that $C_1>0$ is given by

$$C_1 = \Big(\int_0^T\int_{M_t} |v|^2 \, d\mathcal{H}^{d-1}dt\Big)^{\frac{1}{2}} \Big(\int_0^T\int_{M_t} |h|^2 \, d\mathcal{H}^{d-1}dt\Big)^{\frac{1}{2}} \tag{2.3}$$

and

$$\int_{M_t} \eta \, d\mathcal{H}^{d-1}\Big|_{t=0} = \int_{M_t} \eta \, d\mathcal{H}^{d-1}\Big|_{t=T} = 0.$$

By the following proposition, we can regard v of (2.2) as a normal velocity vector of M_t and this is the concept of the L^2-flow.

Proposition 2.1. Assume that $\{U_t\}_{t\in[0,T]}$ is a family of open sets. Set $M_t := \partial U_t$. Assume that $\cup_{t\in[0,T]} M_t \times \{t\} \subset \mathbb{R}^{d+1}$ is smooth. Then the normal velocity vector of M_t is v if and only if there exists $C_1>0$ such that (2.2) holds for any $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$.

Proof. It is clear that if v is the normal velocity of M_t then there exists $C_1>0$ such that (2.2) holds for any $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$. Assume that (2.2) holds for any $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$ and w is a normal velocity vector of M_t. We only need to prove that $v=w$. By the assumption there exists $C_2>0$ such that

$$\begin{aligned} \left|\int_0^T \int_{M_t} \eta_t + \nabla\eta \cdot v \, d\mathcal{H}^{d-1} dt\right| &\leq C_1 \|\eta\|_{C^0(\mathbb{R}^d\times(0,T))}, \\ \left|\int_0^T \int_{M_t} \eta_t + \nabla\eta \cdot w \, d\mathcal{H}^{d-1} dt\right| &\leq C_2 \|\eta\|_{C^0(\mathbb{R}^d\times(0,T))} \end{aligned} \tag{2.4}$$

for any $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$. Thus for any $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$ we have

$$\begin{aligned} &\left|\int_0^T \int_{M_t} \nabla\eta \cdot (v-w) \, d\mathcal{H}^{d-1} dt\right| \\ \leq &\left|\int_0^T \int_{M_t} \eta_t + \nabla\eta \cdot v \, d\mathcal{H}^{d-1} dt\right| + \left|\int_0^T \int_{M_t} \eta_t + \nabla\eta \cdot w \, d\mathcal{H}^{d-1} dt\right| \\ \leq &(C_1+C_2)\|\eta\|_{C^0(\mathbb{R}^d\times(0,T))}. \end{aligned} \tag{2.5}$$

Define f by $v - w = f\nu$, where ν is the inner unit normal vector of M_t. Then by (2.5) we obtain

$$\left| \int_0^T \int_{M_t} (\nabla \eta \cdot \nu) f \, d\mathcal{H}^{d-1} dt \right| \leq (C_1 + C_2) \|\eta\|_{C^0(\mathbb{R}^d \times (0,T))}. \tag{2.6}$$

Assume that there exist $\delta > 0$, (x_0, t_0) and $R > 0$ such that $f > \delta > 0$ or $-f > \delta > 0$ on $B_R(x_0, t_0)$. Define

$$r(x,t) = \begin{cases} \operatorname{dist}(x, M_t), & x \in U_t, \\ -\operatorname{dist}(x, M_t), & x \notin U_t \end{cases} \tag{2.7}$$

and $\phi_1^\varepsilon(x,t) := \tanh(r(x,t)/\varepsilon)$. Choose ϕ_2 such that

$$\phi_2(x,t) = \begin{cases} 1, & x \in B_{R/2}(x_0, t_0), \\ 0, & x \notin B_R(x_0, t_0) \end{cases} \tag{2.8}$$

and $0 \leq \phi_2 \leq 1$. Set $\eta^\varepsilon := \phi_1^\varepsilon \phi_2$. Then $0 \leq \eta \leq 1$ and

$$\nabla \eta^\varepsilon = \nabla \phi_1^\varepsilon \phi_2 + \nabla \phi_2 \phi_1^\varepsilon = \varepsilon^{-1} \phi_2 \nu + \nabla \phi_2 \phi_1^\varepsilon.$$

Note that there exists $C > 0$ which does not depend on ε such that $\|\nabla \phi_2 \phi_1^\varepsilon\|_\infty \leq C$. Thus we have

$$\left| \int_0^T \int_{M_t} \nabla \eta^\varepsilon \cdot \nu f \, d\mathcal{H}^{d-1} dt \right| \to \infty \quad \text{as} \quad \varepsilon \to 0.$$

But this is contradiction to (2.6). Hence we obtain $f = 0$ and $v = w$. □

Remark 2.2 (Brakke's mean curvature flow [4]). The definition of the L^2-flow is similar to the formulation of Brakke's mean curvature flow. Let M_t be a closed smooth hyper surface in $\mathbb{R}^d$ and v be the normal velocity vector again. Then by (2.1) clearly we have

$$\frac{d}{dt} \int_{M_t} \eta \, d\mathcal{H}^{d-1} \leq \int_{M_t} (-h\eta + \nabla \eta) \cdot v + \eta_t \, d\mathcal{H}^{d-1} \tag{2.9}$$

for $\eta \in C_c^1(\mathbb{R}^d \times (0,T); \mathbb{R}^+)$. (2.9) is called Brakke's inequality. Formally, the definition of Brakke's mean curvature flow is defined by (2.9) with $v = h$ [4]. In fact, if M_t is smooth, then the normal velocity of M_t is v if and only if (2.9) holds for any $\eta \in C_c^1(\mathbb{R}^d \times (0,T); \mathbb{R}^+)$ (see [24, Section 2]).

3 Phase field method for volume preserving MCF

In this section we consider the three types of the phase field method for (1.1). Let $\varepsilon \in (0,1)$, $W(s) := \dfrac{(1-s^2)^2}{2}$ and $\Omega := \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. We also use Ω to a set $[0,1)^d \subset \mathbb{R}^d$. We consider the following Allen-Cahn equation :

$$\begin{cases} \varepsilon \varphi_t^\varepsilon = \varepsilon \Delta \varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon}, & (x,t) \in \Omega \times (0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega. \end{cases} \tag{3.1}$$

Note that Ω is divided into $\{\varphi^\varepsilon(\cdot,t)\approx 1\}$ and $\{\varphi^\varepsilon(\cdot,t)\approx -1\}$ for the solution φ^ε to (3.1). Letting $\varepsilon\to 0$, the zero level set of (3.1) converges to the mean curvature flow, that is, for the mean curvature flow N_t, if $\{\varphi^\varepsilon(\cdot,0)=0\}=N_0$ for any $\varepsilon>0$, then $\{\varphi^\varepsilon(\cdot,t)=0\}\approx N_t$ for sufficiently small $\varepsilon>0$ under the suitable conditions [9, 15].

For the volume preserving mean curvature flow, Rubinstein and Sternberg [21] studied the following Allen-Cahn equation with non-local term:

$$\begin{cases} \varepsilon\varphi_t^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda_1^\varepsilon, & (x,t)\in\Omega\times(0,\infty), \\ \varphi^\varepsilon(x,0)=\varphi_0^\varepsilon(x), & x\in\Omega, \end{cases} \tag{3.2}$$

where $\lambda_1(t) := \fint_\Omega \frac{W'(\varphi^\varepsilon)}{\varepsilon}\,dx = \frac{1}{\mathcal{L}^d(\Omega)}\int_\Omega \frac{W'(\varphi^\varepsilon)}{\varepsilon}\,dx$.

Using the divergence theorem, we obtain the volume preserving property for (3.2):

$$\frac{d}{dt}\int_\Omega \varphi^\varepsilon\,dx = \int_\Omega \varphi_t^\varepsilon\,dx = 0, \quad t>0. \tag{3.3}$$

(3.3) means that the ratio of the volume of $\{\varphi^\varepsilon(\cdot,t)\approx 1\}$ and $\{\varphi^\varepsilon(\cdot,t)\approx -1\}$ is constant with respect to $t\geq 0$.

We assume that $\{\varphi^\varepsilon(\cdot,t)=0\}\approx M_t$ for sufficiently small $\varepsilon>0$, where φ^ε and M_t are the solution for (3.2) and the solution for (1.1), respectively. Then we have

$$v\cdot\nu\approx\frac{-\varphi_t^\varepsilon}{|\nabla\varphi^\varepsilon|},\quad h\cdot\nu\approx\frac{-\Delta\varphi^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon}}{|\nabla\varphi^\varepsilon|},\quad \langle h\cdot\nu\rangle\approx\frac{\lambda_1^\varepsilon}{\varepsilon|\nabla\varphi^\varepsilon|},\quad \nu\approx\frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}. \tag{3.4}$$

Note that the velocity and the unit normal vector for the zero level set $\{\varphi^\varepsilon(\cdot,t)=0\}$ are given by $\frac{-\varphi_t^\varepsilon}{|\nabla\varphi^\varepsilon|}$ and $\frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}$, respectively. Using (3.3) and the integration by parts we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}+\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx = \int_\Omega\Big(\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\varphi_t^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon}\varphi_t^\varepsilon\Big)\,dx \\ &= \int_\Omega\Big(-\varepsilon\Delta\varphi^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)\varphi_t^\varepsilon\,dx = \int_\Omega(-\varepsilon\varphi_t^\varepsilon+\lambda_1^\varepsilon)\varphi_t^\varepsilon\,dx \\ &= -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx + \lambda_1^\varepsilon\int_\Omega\varphi_t^\varepsilon\,dx = -\int_\Omega\varepsilon(\varphi_t^\varepsilon)^2\,dx. \end{aligned} \tag{3.5}$$

Thus we obtain

$$\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}+\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx\Big|_{t=0}^{T} + \int_0^T\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dxdt = 0 \tag{3.6}$$

and $\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}+\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx$ is a monotone decreasing function with respect to t. The formula (3.5) corresponds to (1.3) by using the approximations [15]

$$\mathcal{H}^{d-1}(M_t)\approx\frac{1}{\sigma}\int_\Omega\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}+\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx \quad\text{and}\quad \int_\Omega|v|^2\,d\mathcal{H}^{d-1}\approx\frac{1}{\sigma}\int_\Omega\varepsilon(\varphi_t^\varepsilon)^2\,dx, \tag{3.7}$$

where $\sigma:=\int_{-1}^1\sqrt{2W(s)}\,ds$. Chen, Hilhorst and Logak [7] proved that the zero level set of the solution to (3.2) converges to the volume preserving mean curvature flow under the suitable conditions.

Remark 3.1. Whether the solution for (3.2) converges to the time global weak solution of the volume preserving mean curvature flow or not is an open problem, due to the difficulty of estimates of the Lagrange multiplier. From

$$\frac{1}{\sigma}\int_0^T\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dxdt \approx \int_0^T\int_{M_t} |v|^2\,d\mathcal{H}^{n-1}dt,$$

$$\frac{1}{\sigma}\int_0^T\int_\Omega \varepsilon^{-1}\Big(\varepsilon\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)^2\,dxdt \approx \int_0^T\int_{M_t} |h|^2\,d\mathcal{H}^{n-1}dt$$

and the estimate of (2.2), to obtain the existence of the L^2-flow for (1.1), we need the boundedness of

$$\frac{1}{\sigma}\sup_{\varepsilon\in(0,1)}\int_0^T\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dxdt \tag{3.8}$$

and

$$\frac{1}{\sigma}\sup_{\varepsilon\in(0,1)}\int_0^T\int_\Omega \varepsilon^{-1}\Big(\varepsilon\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)^2\,dxdt. \tag{3.9}$$

Assume that

$$\frac{1}{\sigma}\sup_{\varepsilon\in(0,T)}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx\Big|_{t=0} < \infty.$$

Then the boundedness of (3.8) is clear from (3.6). In [3], they proved that there exists $C > 0$ such that

$$\sup_{\varepsilon\in(0,1)}\int_0^T (\lambda_1^\varepsilon)^2\,dt \le C \tag{3.10}$$

for (3.2). But to obtain the boundedness of (3.9), it is clear that we need the boundedness of

$$\sup_{\varepsilon\in(0,1)}\int_0^T \varepsilon^{-1}(\lambda_1^\varepsilon)^2\,dt.$$

In 2011, Brassel and Bretin [6] studied the following reaction diffusion equation:

$$\begin{cases} \varepsilon\varphi_t^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda_2^\varepsilon\sqrt{2W(\varphi^\varepsilon)}, & (x,t)\in\Omega\times(0,\infty),\\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x\in\Omega, \end{cases} \tag{3.11}$$

where

$$\lambda_2 = \lambda_2(t) := \frac{\int_\Omega W'(\varphi^\varepsilon)/\varepsilon\,dx}{\int_\Omega \sqrt{2W(\varphi^\varepsilon)}\,dx}.$$

The solution for (3.11) also has the volume preserving property (3.3). Alfaro and Alifrangis [1] proved that the zero level set of the solution to (3.2) also converges to the volume preserving mean curvature flow under the suitable conditions and [6] proved that the numerical experiments via (3.11) is better than (3.2).

Assume that $\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}=\frac{W(\varphi^\varepsilon)}{\varepsilon}$ and the zero level set of φ^ε approximates the volume preserving mean curvature flow, for the solution φ^ε to (3.11). Then we have

$$v\cdot\nu\approx\frac{-\varphi_t^\varepsilon}{|\nabla\varphi^\varepsilon|},\quad h\cdot\nu\approx\frac{-\Delta\varphi^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon}}{|\nabla\varphi^\varepsilon|},\quad \langle h\cdot\nu\rangle\approx\lambda_2^\varepsilon\frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon|\nabla\varphi^\varepsilon|}=\lambda_2^\varepsilon,\quad \nu\approx\frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}. \tag{3.12}$$

By the maximum principle, we have

$$\sup_{\Omega\times[0,\infty)}|\varphi^\varepsilon|\leq 1$$

for the solution φ^ε to (3.11) with $\sup_\Omega|\varphi^\varepsilon(\cdot,0)|\leq 1$ (moreover the solution to (3.14) also has the property). Note that $\sqrt{2W(s)}=0$ if and only if $s=\pm1$.

We assume that there exists $C>0$ such that

$$\sup_{\varepsilon\in(0,1)}\left\{\sup_{t\in(0,T)}\int_\Omega\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}+\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx,\ \int_0^T\int_\Omega\varepsilon(\varphi_t^\varepsilon)^2\,dxdt,\ \int_0^T(\lambda_2^\varepsilon)^2\,dt\right\}\leq C \tag{3.13}$$

for (3.11). Then we have

$$\begin{aligned}&\int_0^T\int_\Omega\varepsilon\Big(\Delta\varphi^\varepsilon-\frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)^2\,dxdt\\ \leq&\int_0^T\int_\Omega\varepsilon(\varphi_t^\varepsilon)^2\,dxdt+\int_0^T\int_\Omega\varepsilon\Big(\lambda_2^\varepsilon\frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\Big)^2\,dxdt\\ \leq&C+\int_0^T(\lambda^\varepsilon)^2\int_\Omega\frac{2W(\varphi^\varepsilon)}{\varepsilon}\,dxdt\leq C+2C\int_0^T(\lambda_2^\varepsilon)^2dt\leq C.\end{aligned}$$

Hence, from the arguments of Remark 3.1 we may obtain the L^2-flow for (1.1) via (3.11). But whether the solution for (3.11) has estimates (3.13) or not is unknown. One of the reasons is that the solution for (3.11) has not the property such as (3.5).

In this article, we consider the following reaction diffusion equation studied by Golovaty [12]:

$$\begin{cases}\varepsilon\varphi_t^\varepsilon=\varepsilon\Delta\varphi^\varepsilon-\dfrac{W'(\varphi^\varepsilon)}{\varepsilon}+\lambda^\varepsilon\sqrt{2W(\varphi^\varepsilon)}, & (x,t)\in\Omega\times(0,\infty),\\ \varphi^\varepsilon(x,0)=\varphi_0^\varepsilon(x), & x\in\Omega,\end{cases} \tag{3.14}$$

where

$$\lambda^\varepsilon=\lambda^\varepsilon(t)=\frac{-\int_\Omega\sqrt{2W(\varphi^\varepsilon)}\Big(\varepsilon\Delta\varphi^\varepsilon-\frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx}{2\int_\Omega W(\varphi^\varepsilon)\,dx}. \tag{3.15}$$

Using the integration by parts, we have

$$\lambda^\varepsilon=\frac{-2\int_\Omega\varphi^\varepsilon\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}+\frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx}{\int_\Omega W(\varphi^\varepsilon)\,dx}.$$

In [12], Golovaty studied the singular limit of radially symmetric solutions to (3.14). Define $k(s) := \int_0^s \sqrt{2W(\tau)}\,d\tau = s - \frac{1}{3}s^3$. By the definition of λ^ε we have

$$\frac{d}{dt}\int_\Omega k(\varphi^\varepsilon)\,dx = \int_\Omega \varphi_t^\varepsilon \sqrt{2W(\varphi^\varepsilon)}\,dx = 0. \tag{3.16}$$

Using (3.16) we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx = \int_\Omega \Big(\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\varphi_t^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon}\varphi_t^\varepsilon\Big)\,dx \\ &= \int_\Omega \varepsilon\Big(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)\varphi_t^\varepsilon\,dx = \int_\Omega \varepsilon\Big(-\varphi_t^\varepsilon + \lambda^\varepsilon\frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\Big)\varphi_t^\varepsilon\,dx \\ &= -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx + \lambda^\varepsilon\int_\Omega \varphi_t^\varepsilon\sqrt{2W(\varphi^\varepsilon)}\,dx = -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx. \end{aligned} \tag{3.17}$$

Thus by using (3.7), (3.17) also corresponds to (1.3).

Assume that $\lim_{\varepsilon\to 0}\varphi^\varepsilon = \pm 1$ a.e. Then by $\lim_{\varepsilon\to 0} k(\varphi^\varepsilon) = \pm\frac{\sigma}{2}$ a.e. we have

$$\lim_{\varepsilon\to 0}\int_\Omega k(\varphi^\varepsilon)\,dx = \frac{\sigma}{2}\lim_{\varepsilon\to 0}\int_\Omega \varphi^\varepsilon\,dx. \tag{3.18}$$

Hence we can regard (3.16) as the volume preserving property.

We also assume that there exists $D_1 > 0$ such that

$$\frac{1}{\sigma}\sup_{\varepsilon\in(0,T)}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx\Big|_{t=0} \le D_1. \tag{3.19}$$

Then by (3.17) we have

$$\sup_{\varepsilon\in(0,1)}\Big\{\sup_{t\in(0,T)}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx,\ \int_0^T\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dxdt\Big\} \le D_1 \tag{3.20}$$

for the solution to (3.14) and $T > 0$. So if there exists $C > 0$ such that

$$\sup_{\varepsilon\in(0,1)}\int_0^T (\lambda^\varepsilon)^2\,dt \le C,$$

we may obtain the L^2-flow for (1.1). We show the estimate in Section 5 (see (5.1)).

Remark 3.2. From a different perspective, we can regard (3.16) as the volume preserving property. We show

$$-\int_{M_t} v\cdot\nu\,d\mathcal{H}^{d-1} \approx \frac{1}{\sigma}\int_\Omega \varphi_t^\varepsilon\sqrt{2W(\varphi^\varepsilon)}\,dx$$

under the suitable conditions. Assume that $\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} = \frac{W(\varphi^\varepsilon)}{\varepsilon}$. Using (3.7) and $v\cdot\nu \approx \frac{-\varphi_t^\varepsilon}{|\nabla\varphi^\varepsilon|}$ we have

$$\begin{aligned} \int_\Omega \varphi_t^\varepsilon\sqrt{2W(\varphi^\varepsilon)}\,dx &\approx -\int_\Omega v\cdot\nu|\nabla\varphi^\varepsilon|\sqrt{2W(\varphi^\varepsilon)}\,dx = -\int_\Omega v\cdot\nu\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx \\ &\approx -\sigma\int_{M_t} v\cdot\nu\,d\mathcal{H}^{d-1}. \end{aligned}$$

Remark 3.3. The property $\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} \approx \frac{W(\varphi^\varepsilon)}{\varepsilon}$ is known as the key point of the proof for the existence of the mean curvature flow via the phase field method [15]. For the solution φ^ε of (3.1), we define the signed measure ξ_t^ε by

$$\xi_t^\varepsilon(A) := \int_A \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}\, dx, \quad A \subset \mathbb{R}^d. \tag{3.21}$$

The measure ξ_t^ε is called the discrepancy measure. In [15], Ilmanen proved that $\xi_t^\varepsilon \le 0$ for any $t \ge 0$ if $\xi_0^\varepsilon \le 0$, and $|\xi_t^\varepsilon| \to 0$ as $\varepsilon \to 0$ a.e. $t \ge 0$ for the solution φ^ε of (3.1), under the suitable conditions.

4 Global existence of L^2-flow

In this section we define the weak solution (L^2-flow) and explain the time global existence theorem of [23]. We recall some notations and definitions from geometric measure theory and refer to [2, 4, 5, 10, 22] for more details.

Let $d \ge k+1$ and $G_k(\mathbb{R}^d)$ be a Grassman manifold of unoriented k-dimensional subspaces in $\mathbb{R}^d$.

Definition 4.1. A set $M \subset \mathbb{R}^d$ is called a countably k-rectifiable set if M is $\mathcal{H}^k$-measurable and there exists a family of C^1 k-dimensional embedded submanifolds $\{M_i\}_{i=1}^\infty$ such that $\mathcal{H}^k(M \setminus \cup_{i=1}^\infty M_i) = 0$.

Definition 4.2. Let M be an $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$ and $\theta \in L^1_{loc}(\mathcal{H}^k(M))$ is a positive function. We say M has an approximate tangent plane $P \in G_k(\mathbb{R}^d)$ at $x_0 \in M$ with respect to θ if

$$\lim_{\lambda\downarrow 0} \int_{\eta_{x_0,\lambda}(M)} f(y)\theta(x_0+\lambda y)\, d\mathcal{H}^k(y) = \theta(x_0)\int_P f(y)\, d\mathcal{H}^k(y)$$

holds for any $f \in C_c(\mathbb{R}^d)$. Here $\eta_{x_0,\lambda}(x) := \frac{1}{\lambda}(x-x_0)$.

Remark 4.3. If $M \subset \mathbb{R}^d$ is a k-rectifiable set and $\mathcal{H}^k(M) < \infty$, then there exists an approximate tangent plane $\mathcal{H}^k$-a.e. on M.

Definition 4.4. A Radon measure μ is called k-rectifiable if there exists a countable k-rectifiable set M and a function $\theta : M \to [0,\infty)$ such that $\theta \in L^1_{loc}(\mathcal{H}^k\lfloor_M)$ and $\mu = \theta\mathcal{H}^k\lfloor_M$, that is, $\mu(A) = \int_{A\cap M}\theta\, d\mathcal{H}^k$ for any measurable set $A \subset \mathbb{R}^d$. Moreover if $\theta \in \mathbb{N}$ $\mathcal{H}^k$-a.e. on M, μ is called k-integral.

Definition 4.5. Let M be an $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$ and $\theta \in L^1_{loc}(\mathcal{H}^k(M))$ is a positive function. For a $(d-1)$-rectifiable Radon measure $\mu = \theta\mathcal{H}^k\lfloor_M$, h is called a generalized mean curvature vector if

$$\int_{\mathbb{R}^d} \mathrm{div}_M\, g\, d\mu = -\int_{\mathbb{R}^d} h\cdot g\, d\mu$$

holds for any $g \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$. Here, $\mathrm{div}_M\, g = \sum_{k,l=1}^d \partial_{x_k} g_l(\delta_{kl} - \nu_k\nu_l)$, $\nu = (\nu_1,\dots,\nu_d)$ is the unit normal vector of the approximate tangent plane of M and $g = (g_1,\dots,g_d)$.

Remark 4.6. If $M \subset \mathbb{R}^d$ is a smooth hypersurface, then by the divergence theorem for manifolds, we have

$$\int_M \mathrm{div}_M\, g\, d\mathcal{H}^{d-1} = -\int_M h \cdot g\, d\mathcal{H}^{d-1} + \int_{\partial M} \gamma \cdot g\, d\mathcal{H}^{d-2}$$

for any $g \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$, where h and γ are the mean curvature vector of M and the outer unit normal vector of M on ∂M, respectively.

Definition 4.7 (L^2-flow [17]). Let $\{\mu_t\}_{t\in(0,T)}$ be a family of Radon measures on $\mathbb{R}^d$ and define $d\mu := d\mu_t dt$. We say $\{\mu_t\}_{t\in(0,T)}$ is an L^2-flow if

1. μ_t is $(d-1)$-rectifiable and has a generalized mean curvature vector $h \in (L^2(\mu_t))^d$ for a.e. $t \in (0,T)$,
2. and there exist $C > 0$ and $v \in L^2(0,T;(L^2(\mu_t))^d)$ such that

$$v(x,t) \perp T_x\mu_t \quad \text{for } \mu\text{-a.e. } (x,t) \in \mathbb{R}^d \times (0,T) \tag{4.1}$$

and

$$\left| \int_0^T \int_{\mathbb{R}^d} (\eta_t + \nabla\eta \cdot v)\, d\mu_t dt \right| \le C\|\eta\|_{C^0(\mathbb{R}^d\times(0,T))} \tag{4.2}$$

hold for any $\eta \in C^1_c(\mathbb{R}^d \times (0,T))$ with $\mathrm{diam}\,(\mathrm{spt}\,\eta) \le 1$. Here $T_x\mu_t$ is the approximate tangent plane of μ_t at x.

A function $v \in L^2(0,T;(L^2(\mu_t))^d)$ with (4.1) and (4.2) is called a generalized velocity vector.

We define

$$\mu_t^\varepsilon(\phi) := \frac{1}{\sigma}\int_{\mathbb{R}^d} \phi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)dx \tag{4.3}$$

and

$$\mu^\varepsilon(\psi) := \frac{1}{\sigma}\int_0^\infty\int_{\mathbb{R}^d} \psi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)dxdt \tag{4.4}$$

for any $\phi \in C_c(\mathbb{R}^d)$ and $\psi \in C_c(\mathbb{R}^d \times [0,\infty))$. Denote

$$v^\varepsilon = \begin{cases} \frac{-\varphi_t^\varepsilon}{|\nabla\varphi^\varepsilon|}\frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|} & \text{if } |\nabla\varphi^\varepsilon| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.8 ([23]). Let $d = 2,3$ and $U_0 \subset \Omega$ be a bounded open set with C^1 boundary M_0. Then the following hold:

(a) There exist a family of solutions $\{\varphi^{\varepsilon_i}\}_{i=1}^\infty$ for (3.14) and a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ on $\mathbb{R}^d$ such that

(a1) $\mu_0 = \mathcal{H}^{d-1}\lfloor_{M_0}$.

(a2) $\mu_t^\varepsilon \to \mu_t$ as Radon measures for any $t \in [0,\infty)$.

(b) There exists $\psi \in BV_{loc}(\Omega \times [0,\infty)) \cap C^{\frac{1}{2}}_{loc}([0,\infty); L^1(\Omega))$ such that

(b1) $\varphi^\varepsilon \to 2\psi - 1$ in $L^1_{loc}(\Omega \times [0,\infty))$ and a.e. pointwise.

(b2) $\psi(\cdot,0) = \chi_{U_0}$ a.e. on Ω and $\psi = 0$ or $+1$ a.e. in $\Omega \times (0,\infty)$.

(b3) (Volume preserving property 1) $\psi(\cdot,t)$ satisfies

$$\int_\Omega \psi(x,t)\,dx = \mathcal{L}^d(U_0)$$

for any $t \in [0,\infty)$.

(b4) For any $t \in [0,\infty)$ and for any $\phi \in C_c(\mathbb{R}^d;\mathbb{R}^+)$ we have $\|\nabla\psi(\cdot,t)\|(\phi) \leq \mu_t(\phi)$ and $\operatorname{spt}\|\nabla\psi(\cdot,t)\| \subset \operatorname{spt}\mu_t$. Here $\|\nabla\psi(\cdot,t)\|$ is the total variation measure of the distributional derivative $\nabla\psi(\cdot,t)$.

(c) There exists $\lambda \in L^2_{loc}(0,\infty)$ such that $\lambda^\varepsilon \rightharpoonup \lambda$ in $L^2(0,T)$ for any $T > 0$.

(d) There exists $g \in L^2_{loc}(0,\infty;(L^2(\mu_t))^d)$ such that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\times(0,\infty)} -\lambda^\varepsilon\sqrt{2W(\varphi^\varepsilon)}\nabla\varphi^\varepsilon\cdot\Phi\,d\mu^\varepsilon = \int_{\mathbb{R}^d\times(0,\infty)} g\cdot\Phi\,d\mu \tag{4.5}$$

for any $\Phi \in C_c(\mathbb{R}^d \times [0,\infty);\mathbb{R}^d)$, where $d\mu := d\mu_t dt$.

(e) $\{\mu_t\}_{t\in(0,\infty)}$ is a L^2-flow with the generalized velocity vector $v = h + g$, and $\{\mu_t\}_{t\in(0,\infty)}$ satisfies

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\times(0,\infty)} v^\varepsilon\cdot\Phi\,d\mu^\varepsilon = \int_{\mathbb{R}^d\times(0,\infty)} v\cdot\Phi\,d\mu \tag{4.6}$$

for any $\Phi \in C_c(\mathbb{R}^d \times [0,\infty);\mathbb{R}^d)$. Moreover there exists $\partial^*\{\psi = 1\} \to \mathbb{N}$ such that

$$v = h - \frac{\lambda}{\theta}\nu \qquad \mathcal{H}^d\text{-a.e. on } \partial^*\{\psi(\cdot,t) = 1\}, \tag{4.7}$$

where $\partial^*\{\psi(\cdot,t) = 1\}$ is the reduced boundary of $\{\psi(\cdot,t) = 1\}$ and ν is the inner unit normal vector of $\{\psi(\cdot,t) = 1\}$ on $\partial^*\{\psi(\cdot,t) = 1\}$.

(f) (Volume preserving property 2)

$$\int_\Omega v\cdot\nu\,d\|\nabla\psi(\cdot,t)\| = 0 \tag{4.8}$$

for a.e. $t \in (0,\infty)$.

5 Lagrange multiplier λ^ε

Assume that $U_0 \subset \Omega$ is a open set with a smooth boundary M_0, and the solution φ^ε of (3.14) satisfies (3.19), $\|\varphi^\varepsilon\|_\infty \le 1$ and $M_0 = \{\varphi_0^\varepsilon = 0\}$. Then we may suppose that there exists $\omega > 0$ such that

$$\Big|\int_\Omega k(\varphi_0^\varepsilon)\,dx\Big| \le \frac{2}{3}|\Omega| - \omega.$$

Note that $\Big|\int_\Omega k(\varphi_0^\varepsilon)\,dx\Big| = \frac{2}{3}|\Omega|$ if and only if $\varphi_0^\varepsilon \equiv 1$ or $\varphi_0^\varepsilon \equiv -1$. By an argument similar to [3], we obtain the following estimate:

Proposition 5.1. There exists $C_3 = C_3(D_1,\omega) > 0$ and $\epsilon_1 = \epsilon_1(D_1,\omega) > 0$ such that

$$\sup_{\varepsilon\in(0,\epsilon_1)} \int_0^T |\lambda^\varepsilon(t)|^2\,dt \le C_3(1+T). \tag{5.1}$$

6 Outline of the proof of Theorem 4.8

In this section we only show the estimate (4.2) and the volume preserving property (4.8) under the suitable conditions.

Proposition 6.1. Let φ^ε be a solution to (3.14) with (3.19) and (5.1). Let a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ satisfy $\mu_t^\varepsilon \to \mu_t$ as Radon measures for any $t \in [0,\infty)$. Then there exist a subsequence $\varepsilon \to 0$ and $v \in L^2(0,T;(L^2(\mu_t))^d)$ such that (4.2) holds.

Proof. Fix $T > 0$. By (3.17) and (3.19), we have (3.20). Note that by (3.20) and (5.1) there exists $C > 0$ such that

$$\sup_{\varepsilon\in(0,1)} \int_0^T\int_\Omega \varepsilon^{-1}\Big(\varepsilon\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)^2\,dxdt \le C. \tag{6.1}$$

For any $\eta \in C_c^1(\Omega\times(0,T))$ with $\operatorname{diam}(\operatorname{spt}\eta) \le 1$ we compute that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \eta\,d\mu_t^\varepsilon =& \int_\Omega \eta_t\,d\mu_t^\varepsilon + \frac{1}{\sigma}\int_\Omega \eta\Big(\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\varphi_t^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon}\varphi_t^\varepsilon\Big)\,dx\\
=&\int_\Omega \eta_t\,d\mu_t^\varepsilon + \frac{1}{\sigma}\int_\Omega \varepsilon\eta\Big(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)\varphi_t^\varepsilon\,dx - \frac{1}{\sigma}\int_\Omega \varepsilon(\nabla\eta\cdot\nabla\varphi^\varepsilon)\varphi_t^\varepsilon\,dx\\
=&\int_\Omega \eta_t\,d\mu_t^\varepsilon + \frac{1}{\sigma}\int_\Omega \varepsilon\eta\Big(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)\varphi_t^\varepsilon\,dx + \int_\Omega \nabla\eta\cdot v^\varepsilon\,d\tilde\mu_t^\varepsilon,
\end{aligned} \tag{6.2}$$

where $d\tilde\mu_t^\varepsilon := \frac{\varepsilon}{\sigma}|\nabla\varphi^\varepsilon|^2dx$. By (3.20), (6.1) and (6.2), there exists $C > 0$ such that

$$\begin{aligned}
&\Big|\int_0^T\Big(\int_\Omega \eta_t\,d\mu_t^\varepsilon + \int_\Omega \nabla\eta\cdot v^\varepsilon\,d\tilde\mu_t^\varepsilon\Big)dt\Big|\\
\le&\|\eta\|_{C^0(\Omega\times(0,T))}\\
&\times\Big\{\frac{1}{\sigma}\Big(\int_0^T\int_{\operatorname{spt}\eta}\varepsilon^{-1}\Big(\varepsilon\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)^2\,dxdt\Big)^{\frac12}\Big(\int_0^T\int_{\operatorname{spt}\eta}\varepsilon(\varphi^\varepsilon)^2\,dxdt\Big)^{\frac12}\Big\}\\
\le&C\|\eta\|_{C^0(\Omega\times(0,T))}.
\end{aligned} \tag{6.3}$$

Note that by (3.20), (6.1) and [20, Proposition 4.9] there exists a subsequence $\varepsilon \to 0$ such that $\xi_t^\varepsilon \to 0$ as Radon measures for a.e. $t \in [0,T)$. Hence $\tilde{\mu}_t^\varepsilon \to \mu_t$ as Radon measures for a.e. $t \in [0,T)$. By (3.20), we have

$$\sup_{\varepsilon\in(0,\epsilon_1)} \int_0^T \int_\Omega |v^\varepsilon|^2 \, d\tilde{\mu}_t^\varepsilon dt = \sup_{\varepsilon\in(0,\epsilon_1)} \int_0^T \int_\Omega \varepsilon|\varphi_t^\varepsilon|^2 \, dxdt \le \sigma D_1.$$

Hence there exist $v \in L^2(0,T;(L^2(\mu_t))^d)$ and a subsequence $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Phi \cdot v^\varepsilon \, d\tilde{\mu}_t^\varepsilon dt = \int_0^T \int_\Omega \Phi \cdot v \, d\mu \tag{6.4}$$

for any $\Phi \in C_c(\Omega \times (0,\infty);\mathbb{R}^d)$ (see [14, Theorem 4.4.2]). Hence by (6.3) and (6.4) we obtain (2.9). □

Proposition 6.2. Let all the assumptions of Proposition 6.1 and (b) of Theorem 4.8 hold. Then we have (4.8) for a.e. $t \in (0,\infty)$.

Proof. Let $\varphi := \lim_{\varepsilon\to 0} \varphi^\varepsilon$. By [17, Proposition 4.5] we have

$$\int_0^T \int_{\mathbb{R}^d} v \cdot \nu\eta \, d\|\nabla\varphi(\cdot,t)\| dt = -\int_0^T \int_{\mathbb{R}^d} \varphi\eta_t \, dxdt \tag{6.5}$$

for any $T > 0$ and $\eta \in C_c^1(\mathbb{R}^d \times (0,T))$. Here ν is the inner unit normal vector of $\{\varphi(\cdot,t) = 1\}$ on $\partial^*\{\varphi(\cdot,t) = 1\}$. By (6.5), (b3) and the periodic boundary condition, we have

$$\int_0^T \zeta \int_\Omega v \cdot \nu \, d\|\nabla\varphi(\cdot,t)\| dt = -\int_0^T \zeta_t \int_\Omega \varphi \, dxdt = -(2\mathcal{L}^d(U_0) - 1) \int_0^T \zeta_t \, dt = 0 \tag{6.6}$$

for any $\zeta \in C_c^1((0,T))$. Using (6.6) and $\|\nabla\psi(\cdot,t)\| = \frac{1}{2}\|\nabla\varphi(\cdot,t)\|$ for any $t \ge 0$,

$$\int_0^T \zeta \int_\Omega v \cdot \nu \, d\|\nabla\psi(\cdot,t)\| dt = \frac{1}{2} \int_0^T \zeta \int_\Omega v \cdot \nu \, d\|\nabla\varphi(\cdot,t)\| dt = 0$$

holds for any $\zeta \in C_c^1((0,T))$. Hence we have (4.8) for a.e. $t \in (0,\infty)$. □

7 Monotonicity formula

Finally, we show the negativity of the discrepancy measure ξ_t^ε and the monotonicity formula for (3.14).

Set $\xi_\varepsilon = \xi_\varepsilon(x,t) := \frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon}$ and define ξ_t^ε by (3.21) for a solution φ^ε for (3.14). In this section, we assume $\xi_\varepsilon(x,0) \le 0$ for any $x \in \Omega$. By the maximum principle, we have

Proposition 7.1. $\xi_\varepsilon(x,t) \le 0$ for any $(x,t) \in \Omega\times[0,\infty)$. Moreover ξ_t^ε is a non-positive measure for $t \in [0,\infty)$.

Define the backward heat kernel ρ by

$$\rho = \rho_{y,s}(x,t) := \frac{1}{(4\pi(s-t))^{\frac{d-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}}, \qquad t < s,\ x, y \in \mathbb{R}^d.$$

To localize the computations, choose a radially symmetric cut-off function

$$\eta(x) \in C_c^\infty(B_{\frac{1}{2}}(0)) \quad \text{with} \quad \eta = 1 \text{ on } B_{\frac{1}{4}}(0) \quad \text{and} \quad 0 \le \eta \le 1.$$

Define

$$\tilde\rho_{(y,s)}(x,t) := \rho_{(y,s)}(x,t)\eta(x-y) = \frac{1}{(4\pi(s-t))^{\frac{d-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}} \eta(x-y), \qquad t < s,\ x, y \in \mathbb{R}^d.$$

Proposition 7.2. There exists $C_4 = C_4(d) > 0$ such that

$$\int_\Omega \tilde\rho \, d\mu_t^\varepsilon(x)\Big|_{t=t_2} \le \Big(\int_\Omega \tilde\rho \, d\mu_t^\varepsilon(x)\Big|_{t=t_1} + C_4 \int_{t_1}^{t_2} e^{-\frac{1}{128(s-t)}} \mu_t^\varepsilon(B_{\frac{1}{2}}(y))\, dt \Big) e^{C_3(1+(t_2-t_1))} \tag{7.1}$$

for any $y \in \Omega$ and $0 \le t_1 < t_2$.

Proof. By an argument similar to [16], there exists $C > 0$ such that

$$\frac{d}{dt}\int_\Omega \tilde\rho \, d\mu_t^\varepsilon \le \frac{1}{2(s-t)} \int_\Omega \tilde\rho \, d\xi_t^\varepsilon + \frac{1}{2}(\lambda^\varepsilon)^2 \int_\Omega \tilde\rho \, d\mu_t^\varepsilon + C e^{-\frac{1}{128(s-t)}} \mu_t^\varepsilon(B_{\frac{1}{2}}(y)) \tag{7.2}$$

for any $y \in \Omega$ and $0 \le t < s$. By Proposition 7.1, (5.1) and (7.2) we obtain (7.1). □

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