数理解析研究所講究録
第 1998 巻 2016 年 46-52

Quantum groups, quiver varieties, and Lusztig's symmetries

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Abstract

In this talk, I will give a geometric construction of the quantized enveloping algebras of type ADE and their bases via cyclic quiver varieties. The construction respects BGP reflections, which turns out to be Lusztigs symmetries acting on these algebras.

1 Introduction

1.1 Quantum group $U_t(g)$

We take the following notations:

- I is the set of vertices $\{1, 2, \ldots, n\}$.
- $C = (C_{ij})_{i,j \in I}$ is a symmetric generalized Cartan matrix .
- g is complex Kac-Moody Lie algebra associated with C.
- t is an indeterminate.

The quantum group $U_t(g)$ is the $\mathbb{Q}(t)$-algebra generated by the Chevalley generators $E_i, K_i^{\pm}, F_i$, $i \in I$, subject to the quantum Serre relations and other relations $\sim$:

$$U_t(g) = \langle E_i, K_i^{\pm}, F_i \rangle / \sim .$$

It has the triangular decomposition into the sub-algebras $U_t(n^+) = \langle E_i \rangle / \sim$, $U_t(h) = \langle K_i^{\pm} \rangle / (K_i K_i^{-1} = 1, K_i K_j = K_j K_i)$, $U_t(n^-) = \langle F_i \rangle / \sim$.

Now let us slightly enlarge the quantum group into $\tilde{U}_t(g)$, generated by E_i, K_i, K_i', F_i, subject to similar relations. Then this algebra has the triangular decomposition into the sub-algebras $U_t(n^+) = \langle E_i \rangle / \sim$, $\tilde{U}_t(h) = \langle K_i, K_i' \rangle / (K_i K_j' = K_j' K_i)$, $U_t(n^-) = \langle F_i \rangle / \sim$.

Taking the reduction of $\tilde{U}_t(g)$ by imposing the relation $K_i K_i' = 1$, we obtain the usual quantum group $U_t(g)$.

1.2 Categorification of $U_t(n^+)$

We let Γ denote the diagram of C, namely, it has the vertex set I, and $-C_{ij}$ edges between any two different vertices i, j.

By choosing an orientation Ω on the diagram Γ, we obtain an oriented graph (called quiver) $Q = (\Gamma, \Omega)$.

We work over the base field $k = \mathbb{C}$. Then we have the path algebra $\mathbb{C}Q$, whose category of left modules will be denoted by $\mathbb{C}Q$-mod.

Recall that, by naturally viewing Q as a category, its representations are the functors from the category Q to the category of $\mathbb{C}$-vector spaces. The category of the representations of Q, which we denote by $Rep(Q)$, is equivalent to the module category $\mathbb{C}Q$-mod. For any $d = (d_i) \in \mathbb{N}^I$, let $Rep(Q, d)$ denote the vector space of representations which sends i to $\mathbb{C}^{d_i}$.

Theorem 1.1 (Ringel [Rin90], Green [Gre95]). *Assume Q is acyclic, namely, it has no oriented cycles. Let the base field k be a finite field and take t to be $\sqrt{|k|}$. Let $H(Rep(Q))$ denote the* Hall algebra *of the abelian category $Rep(Q)$. Then we have an embedding of algebra*

$$U_t(n^+) \hookrightarrow H(Rep(Q)).$$

This embedding is an isomorphism when g is of type ADE.

Here, the Hall algebra $H(Rep(Q))$ has the natural basis $\{[M]\}$, where $[M]$ denote the isoclass of an object M in $Rep(Q)$. Its multiplication is determined by counting the short exact sequences.

Theorem 1.2 (Lusztig [Lus90] [Lus91]). *Let the base field k be $\mathbb{C}$.*

1. *There is an embedding from $U_t(n^+)$ to the Grothendieck ring of perverse sheaves over the vector spaces $Rep(Q, d)$, $d \in \mathbb{N}^I$. This embedding is an isomorphism if g is of type ADE.*

2. *Via this embedding, we obtain the canonical basis of $U_t(n^+)$ which consists of perverse sheaves and whose structure constants are in $\mathbb{N}[t^{\pm}]$.*

Theorem 1.3 (Hernandez-Leclerc [HL11]). *Let g be of type ADE. Then $U_t(n^+)$ is isomorphic to the dual of Grothendieck ring of perverse sheaves over graded quiver varieties $\mathcal{M}(w^d)$, where w^d are some dimension vectors associated with $d \in \mathbb{N}^I$.*

Proof. They prove that the graded quiver varieties $\mathcal{M}(w^d)$ are isomorphic to the vector space $Rep(Q, d)$. □

1.3 Categorification of $U_t(g)$

Let $\dot{U}_t(g)$ be the idempotended form of $U_t(g)$ introduced by Lusztig [Lus93]. It can be categorified by using quiver Hecke algebras (Khovanov-Lauda [KL09], Rouquier [Rou08]).

Let $C_2(Rep(Q))$ denote the abelian category of 2-periodic complexes of Q-representations $M^\bullet : M^0 \leftrightarrow M^1$. Let $\sim$ denote the quasi-isomorphisms.

Theorem 1.4 (Bridgeland[Bri13]). *Let k be a finite field and specialize t to $\sqrt{|k|}$. Assume Q to be acyclic. Then there is an algebra embedding from localized quantum algebra $\tilde{U}_t(g)[K_i^{-1}, K_i'^{-1}]$ to the localized Hall algebra $H(C_2(Rep(Q)))[[M^\bullet] : H^\bullet(M^\bullet) = 0]/\sim$, such that K_i and K_i' correspond to $[S_i \xrightarrow{1} S_i]$ and $[S_i \xleftarrow{1} S_i]$ respectively.*

This embedding is an isomorphism if g is of type ADE.

1.4 Main result

We take the base field $k = \mathbb{C}$. Let g be of type ADE. h the Coxeter number. Take the complex number $q = e^{\frac{2\pi i}{2h}}$. Let $\mathcal{M}_0(w)$ denote the cyclic quiver variety introduced by Nakajima, for any function w from the cyclic group $\langle q \rangle$ to $\mathbb{N}$. This variety depends on the orientation of the associated quiver Q, which we always take to be acyclic.

Theorem 1.5 (Main Theorem [Qin13]). *(1) After the field extension to $\mathbb{Q}(\sqrt{t})$, we have the isomorphism of algebras*

$$R_t(Q) \otimes \mathbb{Q}(\sqrt{t}) \xrightarrow{\kappa_Q} \tilde{U}_t(g) \otimes \mathbb{Q}(\sqrt{t})$$

where

$$R_t(Q) = \oplus_{\text{special } w} K_0^*(w),$$

$K_0(w)$ is the Grothendieck ring of some perverse sheaves over the cyclic quiver variety $\mathcal{M}_0(w)$ and $K_0^*(w)$ its dual.

As a consequence, the natural geometric basis $L(Q)$ in $R_t(Q)$ gives us a basis $\kappa_Q(L(Q))$ in $R_t(Q)$. Moreover, it has the following property by construction.

Theorem 1.6 ([HL11]). *$L(Q)$ contains the dual canonical basis of $U_t(Q)$ (dual to the canonical basis with respect to Lusztig's bilinear form).*

Corollary 1.7. *Let $\underline{R}_t(Q)$ be the reduction of $R_t(Q)$ by taking reduction $K_iK_i' = 1$. Then we obtain corresponding claims for the quantum group $U_t(g)$ and the Grothendieck ring $\underline{R}_t(Q)$.*

Remark 1.8.
- *Let Σ denote the shift functor on complexes. Then $\Sigma^2 = 1$ in $C_2(Rep(Q))$. This gives the indication of our choice of q such that $q^{2h} = 1$.*
- *Our choice of special w is inspired from the choice of Hernandez-Leclerc. We generalize their result from $U_t(n^+)$ to $U_t(g)$.*
- *In Bridgeland's work, the Cartan part $U_t(h)$ is realized by contractible complexes, which are redundant information in the study of triangulated categories. In our approach, the Cartan part are associated with some strata of $\mathcal{M}_0(w)$. Their counterparts for generic $q \in \mathbb{C}^*$ choice are redundant information, by Nakajima, in the study of finite dimensional representations of quantum affine algebras.*

2 Construction

2.1 Cyclic quiver variety $\mathcal{M}_0(w)$

We use the language of Keller-Scherotzke [KS13] to define quiver varieties [Nak01].

Let $D^b(Q)$ denote the bounded derived category of $Rep(Q)$, $\Sigma = [1]$ its shift functor, τ the Auslander-Reiten translation.

We choose a representative for each isoclass of an indecomposable object. Let $IndD^b(Q)$ denote the corresponding full subcategory.

Example 2.1. *We take the quiver Q to be the graph $2 \to 1$. S_i and P_i its i-th simple and injective respectively. Then $IndD^b(Q)$ is drawn in Figure 1 where each arrow denotes an irreducible (minimal non-isomorphic) morphism. The functor τ is the horizontal one-step shift to the left.*

$$\begin{tikzcd}
\cdots & & P_2 \arrow[dr] & & \Sigma S_1 \arrow[dr] & & \Sigma S_2 \arrow[dr] & & \Sigma^2 P_2 & \cdots \\
& S_1 \arrow[ur] & & S_2 \arrow[ur] & & \Sigma P_2 \arrow[ur] & & \Sigma^2 S_1 \arrow[ur] & &
\end{tikzcd}$$

Figure 1: $IndD^b(Q)$ for a type A_2 quiver Q.

We deform $(IndD^b(Q))^{op}$ into $R(Q)$ the regular Nakajima category by:

1. inserting a vertex σx between τx and x, $\forall x \in IndD^b(Q)$,
2. adding horizontal arrows from x to σx and from σx to τx,

3. imposing the mesh relations on this category (namely, sums of triangles vanish).

see Figure 2 for an example.

$$\begin{tikzcd}
\cdots & \sigma P_2 & P_2 \arrow[l] \arrow[dl] & \sigma\Sigma S_1 \arrow[l] & \Sigma S_1 \arrow[l] \arrow[dl] & \sigma\Sigma S_2 \arrow[l] & \Sigma S_2 \arrow[l] \arrow[dl] & \sigma\Sigma^2 P_2 \arrow[l] & \Sigma^2 P_2 \arrow[l] \arrow[dl] \\
\sigma S_1 & S_1 \arrow[l] & \sigma S_2 \arrow[l] & S_2 \arrow[l] \arrow[ul] & \sigma\Sigma P_2 \arrow[l] & \Sigma P_2 \arrow[l] \arrow[ul] & \sigma\Sigma^2 S_1 \arrow[l] & \Sigma^2 S_1 \arrow[l] \arrow[ul] & \cdots
\end{tikzcd}$$

Figure 2: $R(Q)$ for a type A_2 quiver Q.

Define the operator σ such that $\sigma^2 = \tau$.

Define the singular Nakajima category $S(Q)$ to be the full subcategory of $R(Q)$ generated on σx, $x \in IndD^b(Q)$

Fold the categories $R(Q)$ and $S(Q)$ to $R(Q)/\Sigma^2$ and $S(Q)/\Sigma^2$ respectively.

We assign the an element in $\langle q \rangle$ (called the q-degree) to each object u in $R(Q)/\Sigma^2$ such that the arrows decrease the degrees by q.

We take dimension vectors $v \in \mathbb{N}^{IndD^b(Q)/\Sigma^2}$, $w \in \mathbb{N}^{\sigma S(Q)/\Sigma^2}$. By standard argument in Nakajima's work, we obtain cyclic quiver varieties

$$\mathcal{M}(v,w) = Rep(R(Q)/\Sigma^2, v, w)//GL(v), \text{ (GITquotient)}$$

and

$$\mathcal{M}_0(w) = Rep(S(Q)/\Sigma^2, w).$$

There is a natural proper map π from $\mathcal{M}(v,w)$ to $\mathcal{M}_0(w)$. The derived push forward $\pi_!$ on the constant perverse sheaf gives us the decomposition into perverse sheaves

$$\pi_! 1_{\mathcal{M}(v,w)} = \oplus_{v' \leq v : (v',w) l-\text{dominant}} IC(\mathcal{M}_0(v',w)), \tag{1}$$

where *l-dominant* is some combinatorial condition on the pair (v,w), $\mathcal{M}_0(v',w)$ is a closed subvariety in $\mathcal{M}_0(w)$, $IC(\ )$ denote the intersection cohomology sheaf.

The ring $\mathbb{Z}[t^{\pm}]$ acts on the Grothendieck group of derived categories of sheafs over $\mathcal{M}_0(w)$ such that t acts as the shift functor. Let $K_0(w)$ denote the submodule spanned by the IC sheaves appearing in Equation (1).

2.2 Special w

We define $W^S = \mathbb{N}^{\{\sigma S_i\}}$, $W^{\Sigma S} = \mathbb{N}^{\{\sigma \Sigma S_i\}}$.

Define $R_t(Q)$ as $\oplus_{w \in W^S \oplus W^{\Sigma S}} K_0^*(w)$. Its multiplication is defined geometrically via restriction functors on quiver varieties.

Define

$$
\begin{aligned}
R_t^+(Q) &= \oplus_{w \in W^S} K_0^*(w) \\
R_t^-(Q) &= \oplus_{w \in W^{\Sigma S}} K_0^*(w) \\
R_t^0(Q) &= \langle K_i, K_i' \rangle,
\end{aligned}
$$

where K_i, K_i' are special central element in $R_t(Q)$.

Proof of Theorem 1.5. We show the triangular decomposition $R_t(Q) = R_t^+(Q) \otimes R_t^0(Q) \otimes R_t^-(Q)$. It is then easy to verify the quantum Serre relations which implies $R_t^+(Q) \otimes \mathbb{Q}(\sqrt{t})$, $R_t^0(Q) \otimes \mathbb{Q}(\sqrt{t})$, $R_t^-(Q) \otimes \mathbb{Q}(\sqrt{t})$ are isomorphic to $U_t(n^+) \otimes \mathbb{Q}(\sqrt{t})$, $\tilde{U}_t(h) \otimes \mathbb{Q}(\sqrt{t})$, $U_t(n^-) \otimes \mathbb{Q}(\sqrt{t})$ respectively. $\square$

3 Reflection

Theorem 3.1. *Let j be a sink point (no outgoing arrow) in Q. Let Q' denote the quiver obtained from Q by a reflection at j and $T''_{j,-1}$ and $T'_{j,1}$ Lusztig's symmetries. Then we can construct an isomorphism θ such that the diagram in Figure 3 is commutative.*

$$
\begin{tikzcd}
\mathbb{Q}(\sqrt{t}) \otimes U_t(g) \arrow[r, shift left, "T''_{j,-1}"] & U_t(g) \otimes \mathbb{Q}(\sqrt{t}) \arrow[l, shift left, "T'_{j,1}"] \\
\mathbb{Q}(\sqrt{t}) \otimes \underline{R}_t(Q) \arrow[u, "\kappa_Q"] \arrow[r, "\text{iso. } \theta"] & \underline{R}_t(Q') \otimes \mathbb{Q}(\sqrt{t}) \arrow[u, "\kappa_{Q'}"'] \\
L(Q) \arrow[r, "\text{iso. } \theta"] & L(Q')
\end{tikzcd}
$$

Figure 3: Changing quiver orientations.

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