Entropy of random chaotic interval map with noise which causes coarse-graining

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Abstract

A random chaotic interval map with noise which causes coarse-graining induces a finite-state Markov chain. For a map topologically conjugate to a piecewise-linear map with the Lebesgue measure being ergodic, we prove that the Shannon entropy for the induced Markov chain possesses a finite limit as the noise level tends to zero. In most cases, the limit turns out to be strictly greater than the Lyapunov exponent of the original map without noise.

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1. Introduction

For the study of random mapping dynamics, Lyapunov exponents play key roles. Furstenberg–Kesten [4] proved convergence of upper Lyapunov exponent for products of independent random matrices (see also Bougerol–Lacroix [2]). Diaconis–Freedman [3] proved almost sure convergence of the backward iteration if the random mapping is contracting on the average. Steinsaltz [12] proved almost sure convergence of the backward iteration for random logistic maps under the assumption that the averaged Lyapunov exponent is negative.

Matsumoto–Tsuda [8] observed that the numerical KS entropy for a modified BZ map with noise may fall below that for the original map without

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noise, and called this phenomenon the *noise-induced order*. For mathematical results, Sumi [13] proved that the chaos disappears for most of random complex dynamical systems for rational chaotic maps.

In order to study how a (non-random) mapping dynamics is affected by a noise, it may be useful to study how the Lyapunov exponent is related to some entropies for random chaotic maps. Araújo–Tahzibi [1] proved that the metric entropy of a random mapping dynamics, which was introduced by Kifer [6] via its skew product realization, falls below the KS entropy of the noise zero limit of the random mapping dynamics. Kozlov–Treshchev [7] and Piftankin–Treschev [11] proved that the coarse-graining Gibbs entropy converges to the KS entropy in the noise zero limit.

In this paper, we study the noise zero limit of the entropy of random chaotic maps through an approach which is different from all the above results.

Let f be a chaotic map on the interval $[0,1]$ with invariant probability measure μ and consider a device which is designed to return $f(x)$ as output if input is x and if there is no noise. Suppose there is a noise which affects the device in such a way as coarse-graining the states; more precisely, the states are clustered into the set of subintervals $\Delta = \{A^{(1)}, \ldots, A^{(N)}\}$ equivolume with respect to μ, and, if input is n taken from $\{1, \ldots, N\}$, the device picks a point U from the subinterval $A^{(n)}$ at random with respect to μ conditional on $A^{(n)}$ and returns n' such that $f(U) \in A^{(n')}$ as output. To iterate this procedure independently induces a Markov chain taking values in $\{1, \ldots, N\}$.

The purpose of this paper is to study the fine-graining limit as the noise level $1/N$ tends to zero of the Shannon entropy $H_\Delta(f)$ for the induced Markov chain. We shall prove that $\limsup H_\Delta(f)$ and $\liminf H_\Delta(f)$ are invariants with respect to topological conjugate. We shall also prove that, for piecewise-linear map with the Lebesgue measure being ergodic, the fine-graining limit does exist and is obtained explicitly. It is remarkable that the limit is always no less, and, in most cases strictly greater, than the Lyapunov exponent $\lambda(f)$ of the original (non-random) dynamical system (f, μ).

Let us give a small remark. Misiurewicz [9] and [10] studied continuity and discontinuity of topological entropies for piecewise monotone interval maps under perturbations preserving the number of pieces of monotonicity. He proved that the topological entropy for the skew tent maps is continuous. In a remarkable contrast, our fine-graining limit of the Shannon entropy for such a map is strictly greater than its Lyapunov exponent.

We give another small remark. The induced Markov chain can always be realized as a random mapping dynamics. So one may want to adopt the Shannon entropy of the random mapping dynamics rather than that of the Markov chain. However, the former is not less than the latter, and, in addition, the way of such realizations is not unique; see Yano–Yasutomi [14] and [15] for related results.

This paper is organized as follows. In Section 2, we prepare notations of the finite-state Markov chain induced by coarse-graining. In Section 3, we define $H_\Delta(f)$ and prove that its fine-graining limits are invariants with respect to topological conjugate. Section 4 is devoted to the computation of the fine-graining limit. In Section 5, we examine the results in the case of skew tent maps.

2. Random chaotic maps with noise which causes coarse-graining

Let $f : [0,1] \to [0,1]$ be a measurable map with a unique non-atomic invariant probability measure μ on $[0,1]$ which is ergodic. For a positive integer N, we call $\Delta = \{A^{(1)}, \dots, A^{(N)}\}$ an *equivolume partition* if Δ consists of disjoint subintervals of $[0,1]$ such that $\bigcup_{n=1}^N A^{(n)} = [0,1]$ and $\mu(A^{(n)}) = 1/N$ for $n = 1, \dots, N$. Since μ is non-atomic, the function $[0,1] \ni x \mapsto \mu([0,x]) \in [0,1]$ is continuous, so that there exists an equivolume partition. We write $\|\Delta\| = 1/N$, which will be called the *noise level*. Let $U = (U^{(1)}, \dots, U^{(N)})$ be a vector-valued random variable whose marginal $U^{(n)}$ is distributed as μ conditional on $A^{(n)}$, i.e.,

$$P\big(U^{(n)} \in B\big) = \frac{\mu\big(B \cap A^{(n)}\big)}{\mu(A^{(n)})} = N\mu\big(B \cap A^{(n)}\big) \quad \text{for } B \in \mathcal{B}([0,1]). \tag{2.1}$$

We do not require any assumption for the joint distribution among $U^{(1)}, \dots, U^{(N)}$, because we only need the marginal distributions of U. Let $\pi^\Delta : [0,1] \to \{1, \dots, N\}$ be the projection map such that

$$\pi^\Delta[x] = n \text{ if and only if } x \in A^{(n)}. \tag{2.2}$$

We define a random map f^Δ from $[0,1]$ to itself by

$$f^\Delta(x) = f(U^{(\pi^\Delta[x])}). \tag{2.3}$$

We define a random map F^Δ from $\{1, \dots, N\}$ to itself by

$$F^\Delta(n) = \pi^\Delta[f(U^{(n)})]. \tag{2.4}$$

We note that

$$\pi^{\Delta}[f^{\Delta}(x)] = F^{\Delta}(\pi^{\Delta}[x]). \tag{2.5}$$

For $n, n' = 1, \ldots, N$, we write

$$p_{\Delta}(n'|n) := P(F^{\Delta}(n) = n') = N \cdot \mu\Big(f^{-1}\Big(A^{(n')}\Big) \cap A^{(n)}\Big). \tag{2.6}$$

We are now interested in the orbit of the iteration of the random maps repeated independently. Let $(U_t)_{t=1,2,\ldots}$ be a sequence of independent copies of U. Then we obtain the random maps $(f_t^{\Delta})_{t=1,2,\ldots}$ and $(F_t^{\Delta})_{t=1,2,\ldots}$ from (2.3) and (2.4). Let x_0 be a random variable taking values in $[0, 1]$ and being independent of $(U_t)_{t=1,2,\ldots}$ which obeys the law μ. Set $X_0 = \pi^{\Delta}[x_0]$, which is thus distributed uniformly on $\{1, \ldots, N\}$. We define $(x_t)_{t=1,2,\ldots}$ and $(X_t)_{t=1,2,\ldots}$ recursively by

$$x_t = f_t^{\Delta}(x_{t-1}), \quad t = 1, 2, \ldots \tag{2.7}$$

and

$$X_t = F_t^{\Delta}(X_{t-1}), \quad t = 1, 2, \ldots \tag{2.8}$$

We note that

$$X_t = \pi^{\Delta}[x_t], \quad t = 1, 2, \ldots \tag{2.9}$$

and it is immediate that $(X_t)_{t=1,2,\ldots}$ is a time-homogeneous Markov chain. Its transition probability is given as

$$P(X_t = n'|X_{t-1} = n) = p_{\Delta}(n'|n) \tag{2.10}$$

for $n, n' = 1, \ldots, N$ and $t = 1, 2, \ldots$, and its stationary distribution is the uniform distribution:

$$\mu_{\Delta}(n) = \frac{1}{N}, \quad n = 1, \ldots, N \tag{2.11}$$

3. Entropy of the induced Markov chain

We denote the Shannon entropy of the induced Markov chain $(X_t)_{t=1,2,\ldots}$ by

$$H_{\Delta}(f) = \sum_{n,n'=1}^{N} \mu_{\Delta}(n)\phi(p_{\Delta}(n'|n)), \tag{3.1}$$

where

$$\phi(t) = -t \log t \ (t > 0), \quad \phi(0) = 0. \tag{3.2}$$

Now we write its fine-graining limits as the noise level $\|\Delta\| = 1/N$ tends to zero by

$$\overline{H}(f) = \limsup_{\|\Delta\| \to 0} H_\Delta(f), \quad \underline{H}(f) = \liminf_{\|\Delta\| \to 0} H_\Delta(f). \tag{3.3}$$

Theorem 3.1. *Suppose that $f : [0,1] \to [0,1]$ has a unique non-atomic invariant probability measure μ on $[0,1]$ which is ergodic. Then the fine-graining limits $\overline{H}(f)$ and $\underline{H}(f)$ are invariants with respect to topological conjugate.*

Proof (Proof). Let $C : [0,1] \to [0,1]$ be a homeomorphism and write $g = C \circ f \circ C^{-1}$. Then the interval map g also has the unique non-atomic invariant probability measure given as $\nu := \mu \circ C^{-1}$ which is ergodic. For any partition $\Delta = \{A^{(1)}, \ldots, A^{(N)}\}$ equivolume with respect to μ, the partition $C(\Delta) = \{C(A^{(1)}), \ldots, C(A^{(N)})\}$ is equivolume with respect to ν. Let us denote the transition probability p_Δ for the dynamical system (f, μ) and the equivolume partition Δ as is defined in (2.6), and write $q_{C(\Delta)}$ for its counterpart for the dynamical system (g, ν) and the equivolume partition $C(\Delta)$. It is then obvious that

$$\begin{aligned} p_\Delta(n'|n) =& N \cdot \mu\Big(f^{-1}\Big(A^{(n')}\Big) \cap A^{(n)}\Big) & (3.4) \\ =& N \cdot \nu\Big(g^{-1}\Big(C(A^{(n')})\Big) \cap C(A^{(n)})\Big) & (3.5) \\ =& q_{C(\Delta)}(n'|n). & (3.6) \end{aligned}$$

Now we obtain

$$H_\Delta(f) = H_{C(\Delta)}(g). \tag{3.7}$$

If Δ varies all the equivolume partitions, so does $C(\Delta)$. Therefore, immediately from the definition (3.3), we obtain

$$\overline{H}(f) = \overline{H}(g), \quad \underline{H}(f) = \underline{H}(g). \tag{3.8}$$

The proof is now complete.

4. Existence of fine-graining limits for piecewise-linear maps

Let f be an interval map with a unique non-atomic invariant probability measure μ on $[0,1]$ which is ergodic. Suppose that f is piecewise C^1, i.e., there exists a finite partition of $[0,1]$, say $0 = a_0 < a_1 < \cdots < a_{r-1} < a_r = 1$, such that the restriction of f on each subinterval $[a_{i-1}, a_i]$ can be extended to a C^1 map defined on an open interval including $[a_{i-1}, a_i]$. The Lyapunov exponent of f is defined as

$$\lambda(f) = \int_0^1 \log |f'(x)| \mu(\mathrm{d}x). \tag{4.1}$$

Let us write

$$\{x\} = \min\{x + n : n \in \mathbb{Z},\ x + n \geq 0\}. \tag{4.2}$$

Theorem 4.1. *Suppose that μ is the Lebesgue measure on $[0,1]$. Suppose, in addition, that f is piecewise-linear, i.e., there exists a finite partition of $[0,1]$, say $0 = a_0 < a_1 < \cdots < a_{r-1} < a_r = 1$, such that f is linear on each subinterval $E_i = (a_{i-1}, a_i)$. Then one has*

$$H(f) := \overline{H}(f) = \underline{H}(f) = \lambda(f) + D(f), \tag{4.3}$$

where $D(f)$ is given as

$$D(f) = 2 \int_0^1 \frac{\rho(|f'(x)|)}{|f'(x)|} \mathrm{d}x \tag{4.4}$$

and the function ρ is defined as

$$\rho(m) = 0 \quad \text{if } m \in \mathbb{Z}, \tag{4.5}$$

$$\rho(m) = \frac{1}{p} \sum_{n=1}^{p-1} \phi\left(\frac{n}{p}\right) \quad \text{if } m = \frac{q}{p}\text{: irreducible, } p, q \in \mathbb{Z} \text{ and } p \geq 2, \tag{4.6}$$

$$\rho(m) = \frac{1}{4} \quad \text{if } m \text{ is irrational.} \tag{4.7}$$

Combining Theorem 4.1 with Theorem 3.1, we obtain the following.

Corollary 4.2. *Suppose that f is a piecewise-C^1 map which is topologically conjugate to a piecewise-linear map g with the Lebesgue measure being the unique non-atomic invariant probability measure μ on $[0,1]$ which is ergodic. Then one has*

$$\overline{H}(f) = \underline{H}(f) \geq \lambda(f). \tag{4.8}$$

Unless g' is integer valued, the inequality in (4.8) is strict.

Before proving Theorem 4.1, we need the following lemma.

Lemma 4.3. *Suppose that μ is the Lebesgue measure. Then the map f satisfies*

$$|f'| \geq 1 \quad a.e. \tag{4.9}$$

Proof (Proof of Lemma 4.3). Recall that the operator $L : L^1([0,1]) \to L^1([0,1])$ defined as

$$(L\varphi)(x) = \sum_{y: f(y)=x} \frac{1}{|f'(y)|}\varphi(y) \quad \text{for } \varphi \in L^1([0,1]). \tag{4.10}$$

is the Perron–Frobenius operator for the dynamical system (f, μ), i.e.,

$$\int_0^1 (L\varphi)(x)\psi(x)\mathrm{d}x = \int_0^1 \varphi(x)\psi(f(x))\mathrm{d}x \tag{4.11}$$

holds for all $\varphi \in L^1([0,1])$ and all $\psi \in L^\infty([0,1])$. If we take $\varphi(x) \equiv 1$, we have, since $\mu \circ f^{-1} = \mu$,

$$\int_0^1 (L1)(x)\psi(x)\mathrm{d}x = \int_0^1 \psi(f(x))\mathrm{d}x = \int_0^1 \psi(x)\mathrm{d}x, \tag{4.12}$$

and thus we obtain

$$(L1)(x) = \sum_{y: f(y)=x} \frac{1}{|f'(y)|} = 1 \quad \text{a.e.} \tag{4.13}$$

From this we obtain (4.9).

Now we prove Theorem 4.1.

Proof (Proof of Theorem 4.1). Let $\Delta = \{A^{(1)}, \ldots, A^{(N)}\}$ be an equivolume partition. Since μ is the Lebesgue measure, we may assume that $A^{(n)} = [x_{n-1}, x_n)$ for $n = 1, \ldots, N-1$ and $A^{(N)} = [x_{N-1}, x_N]$ where $x_n = n/N$ for $n = 1, \ldots, N$. For $i = 1, \ldots, r$, let $m_i = |f'(x)|$ for $x \in E_i$. Note that $m_i \geq 1$ by Lemma 4.3.

Let $i = 1, \ldots, r$ and $n = 1, \ldots, N$ be fixed such that $\overline{A^{(n)}} \subset E_i$. Let us write m simply for m_i. Since f' is constant on E_i, we may suppose without loss of generality that $m = f'(x) \geq 1$ for $x \in E_i$, and, consequently, f is increasing on E_i. We now have

$$\mu(f(A^{(n)})) = f(x_n) - f(x_{n-1}) = m(x_n - x_{n-1}) = \frac{m}{N}. \tag{4.14}$$

Let u and v be such that

$$f(x_{n-1}) \in A^{(u)} \quad \text{and} \quad f(x_n) \in A^{(v)}. \tag{4.15}$$

We then have

$$u = \lceil N f(x_{n-1}) \rceil \quad \text{and} \quad v = \lceil N f(x_n) \rceil \tag{4.16}$$

where

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\} \quad \text{for } x \in \mathbb{R}. \tag{4.17}$$

Set $B^{(i)} = A^{(i)}$ for $i = u+1, \ldots, v-1$ and set

$$B^{(u)} = [f(x_{n-1}), x_u) \quad \text{and} \quad B^{(v)} = [x_{v-1}, f(x_n)]. \tag{4.18}$$

We then have $f(A^{(n)}) = B^{(u)} \cup \cdots \cup B^{(v)}$, and hence

$$p_\Delta(n'|n) = 0 \quad \text{for } n' < u \text{ or } n' > v. \tag{4.19}$$

Since μ is the Lebesgue measure and since f is linear on $A^{(n)}$, we see that

$$p_\Delta(n'|n) = \frac{\mu(B^{(n')})}{\mu(f(A^{(n)}))} \quad \text{for } n' = u, \ldots, v. \tag{4.20}$$

Hence we obtain

$$p_\Delta(n'|n) = \begin{cases} 1/m & \text{if } n' = u+1, \ldots, v-1, \\ a/m & \text{if } n' = u, \\ b/m & \text{if } n' = v, \\ 0 & \text{otherwise,} \end{cases} \tag{4.21}$$

where

$$a =\lceil Nf(x_{n-1})\rceil - Nf(x_{n-1}) = 1 - \{Nf(x_{n-1})\}\,, \tag{4.22}$$

$$b =Nf(x_n) - \lceil Nf(x_n)\rceil + 1 = \{Nf(x_n)\}_+\,, \tag{4.23}$$

where $\{x\}_+ = \min\{x+n : n \in \mathbb{Z},\ x+n > 0\}$. Noting that $\phi(xy) = x\phi(y) + y\phi(x)$ for $x, y \geq 0$, we have

$$\sum_{n'} \phi(p_\Delta(n'|n)) =(v-u-1)\phi\left(\frac{1}{m}\right) + \phi\left(\frac{a}{m}\right) + \phi\left(\frac{b}{m}\right) \tag{4.24}$$

$$=(v-u-1+a+b)\phi\left(\frac{1}{m}\right) + \frac{1}{m}\phi(a) + \frac{1}{m}\phi(b) \tag{4.25}$$

$$=m\phi\left(\frac{1}{m}\right) + \frac{1}{m}\phi(a) + \frac{1}{m}\phi(b) \tag{4.26}$$

$$=\log m + \frac{1}{m}\phi(a) + \frac{1}{m}\phi(b). \tag{4.27}$$

Let $i = 1, \ldots, r$ be fixed and return to write m_i instead of m. We then have

$$\sum_{n:\overline{A^{(n)}}\in E_i} \phi(b) = \sum_{n:\overline{A^{(n)}}\in E_i} \phi(\{Nf(n/N)\}_+) = \sum_{n:\overline{A^{(n)}}\in E_i} \phi(\{m_i n\}), \tag{4.28}$$

where we note that $\phi(\{x\}_+) = \phi(\{x\})$ because $\phi(0) = \phi(1) = 0$. Let $c_i(N)$ denote the number of n's such that $n : \overline{A^{(n)}} \in E_i$. Then we see that

$$\frac{1}{c_i(N)} \sum_{n:\overline{A^{(n)}}\in E_i} \phi(b) \underset{N\to\infty}{\longrightarrow} \rho(m_i) \tag{4.29}$$

by the following arguments:

(i) If $m_i \in \mathbb{Z}$, then $\{m_i n\} = 0$ for all n so that we obtain (4.29).

(ii) If $m_i = q/p$: irreducible, $p, q \in \mathbb{Z}$ and $p \geq 2$, then the set $\{\{m_i n\} : n = kp+1, kp+2, \ldots, (k+1)p\}$ coincides with the set $\{0, 1/p, 2/p, \ldots, (p-1)/p\}$ for all $k = 0, 1, \ldots$, so that we obtain (4.29).

(iii) If m_i is irrational, then by Weyl's equidistribution theorem we obtain

$$\frac{1}{c_i(N)} \sum_{n:\overline{A^{(n)}}\in E_i} \phi(\{m_i n\}) \underset{N\to\infty}{\longrightarrow} \int_0^1 \phi(t)\mathrm{d}t = \frac{1}{4}, \tag{4.30}$$

so that we obtain (4.29).

In the same way, we obtain

$$\frac{1}{c_i(N)} \sum_{n:\overline{A^{(n)}} \in E_i} \phi(a) \underset{N\to\infty}{\longrightarrow} \rho(m_i). \tag{4.31}$$

Thus, using (4.27), we obtain

$$\begin{aligned}
&\sum_{i=1}^{r} \sum_{n:\overline{A^{(n)}} \subset E_i} \mu_\Delta(n) \sum_{n'} \phi(p_\Delta(n'|n)) && (4.32)\\
&= \sum_{i=1}^{r} \frac{c_i(N)}{N} \cdot \frac{1}{c_i(N)} \sum_{n:\overline{A^{(n)}} \subset E_i} \phi(p_\Delta(n'|n)) && (4.33)\\
&\underset{N\to\infty}{\longrightarrow} \sum_{i=1}^{r} (a_i - a_{i-1}) \cdot \left\{ \log m_i + \frac{\rho(m_i)}{m_i} + \frac{\rho(m_i)}{m_i} \right\} && (4.34)\\
&= \int_0^1 \left\{ \log|f'(x)| + 2\frac{\rho(|f'(x)|)}{|f'(x)|} \right\} \mathrm{d}x. && (4.35)
\end{aligned}$$

Note that if $\overline{A^{(n)}}$ does not included in any E_i, then $\overline{A^{(n)}}$ contains at least one of the points $a_1, \ldots, a_r$, so that the number of such n's is not greater than r. Since

$$\sum_{n'} \phi(p_\Delta(n'|n)) \leq \sum_{n'} \phi\left(\frac{1}{N}\right) = \log N, \tag{4.36}$$

we obtain

$$\sum_{i=1}^{r} \sum_{n:\overline{A^{(n)}} \not\subset E_i} \mu_\Delta(n) \sum_{n'} \phi(p_\Delta(n'|n)) \leq r \cdot \frac{1}{N} \cdot \log N \underset{N\to\infty}{\longrightarrow} 0. \tag{4.37}$$

Therefore, we conclude that

$$H_\Delta(f) = \sum_{i=1}^{r} \sum_{n,n'} \mu_\Delta(n)\phi(p_\Delta(n'|n)) \underset{N\to\infty}{\longrightarrow} \int_0^1 \left\{ \log|f'(x)| + 2\frac{\rho(|f'(x)|)}{|f'(x)|} \right\} \mathrm{d}x, \tag{4.38}$$

which completes the proof.

5. Examples: skew tent maps

For the illustration of Theorem 4.1, we compute the difference $D(f)$ for a *skew tent map* $f : [0,1] \to [0,1]$, which is defined as

$$f(x) = \begin{cases} mx & \text{if } 0 \le x \le 1/m, \\ l(1-x) & \text{if } 1/m < x \le 1 \end{cases} \tag{5.1}$$

for some $m > 1$ and $l > 1$ such that $1/m + 1/l = 1$. Note that the Lebesgue measure is the unique non-atomic invariant probability measure for f and is ergodic; see, e.g., Jetschke–Stiewe [5].

(1) Suppose that $m = 2$. In this case, we have $l = 2$ and hence we have $D(f) = 0$. Note that this map f is topologically conjugate to the *logistic map*

$$g(x) = \frac{1}{4}x(1-x) \quad \text{for } x \in [0,1], \tag{5.2}$$

so that we have

$$H(g) = \lambda(g) = H(f) = \lambda(f) = \log 2. \tag{5.3}$$

(2) Suppose that m is rational and $m \neq 2$. We represent $m = (p+q)/p$ as an irreducible fraction with $p, q \in \mathbb{Z}$, $p, q \ge 1$. In this case, we have $l = (p+q)/q$ and hence we obtain

$$\begin{aligned} D(f) =& 2 \cdot \frac{1}{m} \cdot \frac{\rho(m)}{m} + 2 \cdot \frac{1}{l} \cdot \frac{\rho(l)}{l} && (5.4) \\ =& 2\frac{p}{(p+q)^2} \sum_{n=1}^{p-1} \phi\left(\frac{n}{p}\right) + 2\frac{q}{(p+q)^2} \sum_{n=1}^{q-1} \phi\left(\frac{n}{q}\right), && (5.5) \end{aligned}$$

where the summation $\sum_{n=1}^{p-1}$ (resp. $\sum_{n=1}^{q-1}$) is discarded if $p = 1$ (resp. $q = 1$).

(3) Suppose that m is irrational. In this case, we see that l is also irrational and hence we obtain

$$\begin{aligned} D(f) =& 2 \cdot \frac{1}{m} \cdot \frac{1/4}{m} + 2 \cdot \frac{1}{l} \cdot \frac{1/4}{l} && (5.6) \\ =& \frac{m^2 - 2m + 2}{2m^2}. && (5.7) \end{aligned}$$

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