On h-transforms of one-dimensional diffusions stopped upon hitting zero

Kouji Yano and Yuko Yano

Abstract For a one-dimensional diffusion on an interval for which 0 is the regular-reflecting left boundary, three kinds of conditionings to avoid zero are studied. The limit processes are h-transforms of the process stopped upon hitting zero, where h's are the ground state, the scale function, and the renormalized zero-resolvent. Several properties of the h-transforms are investigated.

1 Introduction

For the reflecting Brownian motion $\{(X_t),(\mathbb{P}_x)_{x\in[0,\infty)}\}$ and its excursion measure n away from 0, it is well-known that $\mathbb{P}^0_x[X_t]=x$ for all $x\geq 0$ and all $t>0$, where $\{(X_t),(\mathbb{P}^0_x)_{x\in[0,\infty)}\}$ denotes the process stopped upon hitting 0, and $t\mapsto n[X_t]$ is constant in $t>0$. Here and throughout this paper we adopt the notation $\mu[F]=\int F\mathrm{d}\mu$ for a measure μ and a function F. The process conditioned to avoid zero may be regarded as the h-transform with respect to $h(x)=x$ of the Brownian motion stopped upon hitting zero. The obtained process coincides with the 3-dimensional Bessel process and appears in various aspects of n (see, e.g., [11] and [21]).

We study three analogues of conditioning to avoid zero for one-dimensional diffusion processes. Adopting the natural scale $s(x)=x$, we let $M=\{(X_t)_{t\geq 0},(\mathbb{P})_{x\in I}\}$ be a D_mD_s-diffusion on I where $I'=[0,l')$ or $[0,l']$ and $I=I'$ or $I'\cup\{l\}$; the choices of I' and I depend on m (see Section 2). We suppose that 0 for M is regular-reflecting. Let $M^0=\{(X_t)_{t\geq 0},(\mathbb{P}^0)_{x\in I}\}$ denote the process M stopped upon hitting zero. We focus on three functions which are involved in conditionings to avoid zero. The first one is the natural scale $s(x)=x$. The second one is given as follows. When

Kouji Yano
Graduate School of Science, Kyoto University, JAPAN, e-mail: kyano@math.kyoto-u.ac.jp

Yuko Yano
Department of Mathematics, Kyoto Sangyo University, JAPAN e-mail: yyano@cc.kyoto-su.ac.jp

l' is natural, we set $\gamma_* = 0$ and $h_* = s$. When l' is not natural, it was shown in [17, Theorem 3.1] that the q-resolvent operator G_q^0 on $L^2(\mathrm{d}m)$ for M^0 is compact and is represented by the eigenfunction expansion $G_q^0 = \sum_n (q-\gamma_n)^{-1} f_n \otimes f_n$ with $0 \ge \gamma_1 > \gamma_2 > \cdots \downarrow -\infty$; in this case we write $\gamma_* = \gamma_1$ and $h_* = f_1$. The obtained function h_* is the second one. The third one is

$$h_0(x) = \lim_{q \downarrow 0} \{ r_q(0,0) - r_q(x,0) \}, \tag{1.1}$$

where $r_q(x,y)$ denotes the resolvent density with respect to the speed measure. We will prove h_0 always exists and we call h_0 the *renormalized zero-resolvent*.

We now state three theorems concerning conditionings of M to avoid zero. Their proofs will be given in Section 5. We write $(\mathscr{F}_t)_{t \ge 0}$ for the natural filtration. Let T_a denote the first hitting time of a. The first conditioning is a slight generalization of a formula found in [24, Section 2.2].

Theorem 1.1 *Let $x \in I' \setminus \{0\}$. Let T be a stopping time and F_T be a bounded $\mathscr{F}_T$-measurable functional. Then*

$$\lim_{a \uparrow \sup I} \frac{\mathbb{P}_x[F_T; T < T_a < T_0]}{\mathbb{P}_x(T_a < T_0)} = \mathbb{P}_x^0 \left[F_T \frac{X_T}{x}; T < T_* \right], \tag{1.2}$$

where $T_* = \sup_{a \in I} T_a$. (If l is an isolated point in I, we understand that the symbol $\lim_{a \uparrow \sup I}$ means the evaluation at $a = l$.)

The second conditioning is essentially due to McKean [17],[18].

Theorem 1.2 *Let $x \in I' \setminus \{0\}$. Let T be a stopping time and F_T be a bounded $\mathscr{F}_T$-measurable functional. Then*

$$\lim_{t \to \infty} \frac{\mathbb{P}_x[F_T; T < t < T_0]}{\mathbb{P}_x(t < T_0)} = \mathbb{P}_x^0 \left[F_T \frac{\mathrm{e}^{-\gamma_* T} h_*(X_T)}{h_*(x)}; T < \infty \right]. \tag{1.3}$$

The third conditioning is an analogue of Doney [6, Section 8] (see also Chaumont–Doney [3]) for Lévy processes. For $q > 0$, we write e_q for the exponential variable independent of M.

Theorem 1.3 *Let $x \in I' \setminus \{0\}$. Let T be a stopping time and F_T be a bounded $\mathscr{F}_T$-measurable functional. Then*

$$\lim_{q \downarrow 0} \frac{\mathbb{P}_x[F_T; T < e_q < T_0]}{\mathbb{P}_x(e_q < T_0)} = \mathbb{P}_x^0 \left[F_T \frac{h_0(X_T)}{h_0(x)}; T < \infty \right]. \tag{1.4}$$

The aim of this paper is to investigate several properties of the three functions h_*, h_0 and s and of the corresponding h-transforms.

We summarize some properties of the h-transforms of M^0 as follows (See Section 2 for the definition of the boundary classification and see the end of Section 4 for the classification of recurrence of 0; here we note that $m(\infty) < \infty$ if and only if 0 is positive recurrent):

(i) If $m(\infty) = \infty$, we have that s, h_* and h_0 all coincide. If l' for M is natural with $m(\infty) < \infty$, we have that s and h_* coincide.
(ii) For the h-transform of M^0 for $h = s$, h_* or h_0, the boundary 0 is entrance.
(iii) For the h-transform of M^0 for $h = s$,

a. the process explodes to ∞ in finite time when l' for M is entrance;
b. the process has no killing inside the interior of I and is elastic at l' when l' for M is regular-reflecting;
c. the process is conservative otherwise.

(iv) For the h-transform of M^0 for $h = h_*$, the process is conservative.
(v) For the h-transform of M^0 for $h = h_0$ when $m(\infty) < \infty$, the process has killing inside.

Let us give an example where the three functions are distinct from each other. Let M be a reflecting Brownian motion on $[0,l']$ where both boundaries 0 and l' are regular-reflecting. Then we have

$$h_*(x) = \frac{2l'}{\pi} \sin \frac{\pi x}{2l'}, \quad h_0(x) = x - \frac{x^2}{2l'}, \quad x \in [0,l']. \tag{1.5}$$

We shall come back to this example in Example 4.2.

We give several remarks about earlier studies related to the h-transforms for the three functions.

1°). The h-transform of M^0 for $h = s$ is sometimes used to obtain a integral representation of the excursion measure: see Salminen [23], Yano [29] and Salminen–Vallois–Yor [24].
2°). The penalization problems for one-dimensional diffusions which generalize Theorem 1.2 were studied in Profeta [19],[20].
3°). The counterpart of h_0 for one-dimensional symmetric Lévy processes where every point is regular for itself has been introduced by Salminen–Yor [25] who proved an analogue of the Tanaka formula. Yano–Yano–Yor [33] and Yano [30] [31] investigated the h-transform of M^0 and studied the penalisation problems and related problems. For an approach to asymmetric cases, see Yano [32].

This paper is organized as follows. We prepare notation and several basic properties for one-dimensional generalized diffusions in Section 2 and for excursion measures in Section 3. In Section 4, we prove existence of h_0. Section 5 is devoted to the proofs of Theorems 1.1, 1.2 and 1.3. In Section 6, we study invariance and excessiveness of h_0 and s. In Section 7, we study several properties of the h-transforms.

Acknowledgements The authors are thankful to Professor Masatoshi Fukushima for drawing their attention to the paper [4]. They also thank Professor Matsuyo Tomisaki and Dr. Christophe Profeta for their valuable comments.

The research of the first author, Kouji Yano, was supported by KAKENHI (26800058) and partially by KAKENHI (24540390). The research of the second author, Yuko Yano, was supported by KAKENHI (23740073).

2 Notation and basic properties for generalized diffusions

Let $\widetilde{m}$ and $\widetilde{s}$ be strictly-increasing functions $(0,l') \to \mathbb{R}$ such that $\widetilde{m}$ is right-continuous and $\widetilde{s}$ is continuous. We fix a constant $0 < c < l'$ (the choice of c does not affect the subsequent argument at all). We set

$$F_1 = \iint_{l'>y>x>c} \mathrm{d}\widetilde{m}(x)\mathrm{d}\widetilde{s}(y), \quad F_2 = \iint_{l'>y>x>c} \mathrm{d}\widetilde{s}(x)\mathrm{d}\widetilde{m}(y). \tag{2.1}$$

We adopt Feller's classification of the right boundary l' with a slight refinement as follows:

(i) If $F_1 < \infty$ and $F_2 < \infty$, then l' is called *regular*. In this case we have $\widetilde{s}(l'-) < \infty$.
(ii) If $F_1 < \infty$ and $F_2 = \infty$, then l' is called *exit*. In this case we have $\widetilde{s}(l'-) < \infty$.
(iii) If $F_1 = \infty$ and $F_2 < \infty$, then l' is called *entrance*. In this case we have $\widetilde{m}(l'-) < \infty$.
(iv) If $F_1 = \infty$ and $F_2 = \infty$, then l' is called *natural*. In this case we have either $\widetilde{s}(l'-) = \infty$ or $\widetilde{m}(l'-) = \infty$. There are three subcases as follows:

   a. If $\widetilde{s}(l'-) = \infty$ and $\widetilde{m}(l'-) = \infty$, then l' is called *type-1-natural*.
   b. If $\widetilde{s}(l'-) = \infty$ and $\widetilde{m}(l'-) < \infty$, then l' is called *type-2-natural*.
   c. If $\widetilde{s}(l'-) < \infty$ and $\widetilde{m}(l'-) = \infty$, then l' is called *type-3-natural* or *natural-approachable*.

The classification of the left boundary 0 is defined in a similar way.

Let m be a function $[0,\infty) \to [0,\infty]$ which is non-decreasing, right-continuous and $m(0) = 0$. We assume that there exist l' and l with $0 < l' \le l \le \infty$ such that

$$m \text{ is } \begin{cases} \text{strictly-increasing on } [0,l'), \\ \text{flat and finite on } [l',l), \\ \text{infinite on } [l,\infty). \end{cases} \tag{2.2}$$

We take $\widetilde{m} = m|_{(0,l')}$ and the natural scale $\widetilde{s}(x) = s(x) = x$ on $(0,l')$ to adopt the classification of the boundaries 0 and l'. We choose the intervals I' and I as follows:

(i) If l' is regular, there are three subcases related to the boundary condition as follows:

   a. If $l' < l = \infty$, then l' is called *regular-reflecting* and $I' = I = [0,l']$.
   b. If $l' < l < \infty$, then l' is called *regular-elastic*, $I' = [0,l']$ and $I = [0,l'] \cup \{l\}$.
   c. If $l' = l < \infty$, then l' is called *regular-absorbing*, $I' = [0,l)$ and $I = [0,l]$.

(ii) If l' is exit, then $l' = l < \infty$, $I' = [0,l)$ and $I = [0,l]$.
(iii) If l' is entrance, then $l' = l = \infty$ and $I' = I = [0,\infty)$.
(iv) If l' is natural, then $l' = l \le \infty$ and $I' = I = [0,l)$.

We always write $(X_t)_{t\ge 0}$ for the coordinate process on the space of paths $\omega : [0,\infty) \to \mathbb{R} \cup \{\partial\}$ with $\zeta(\omega) \in [0,\infty)$ such that $\omega : [0,\zeta(\omega)) \to \mathbb{R}$ is continuous and $\omega(t) = \partial$ for all $t \ge \zeta(\omega)$. We always adopt the canonical representation for each process and the right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ defined by $\mathscr{F}_t = \bigcap_{s>t} \sigma(X_u : u \leq s)$.

We study a $D_m D_s$-generalized diffusion on I where 0 is the regular-reflecting boundary (see Watanabe [28, Section 3]). Such a process can be constructed from the Brownian motion via the time-change method. Let $\{(X_t)_{t\geq 0}, (\mathbb{P}_x^B)_{x\in\mathbb{R}}\}$ denote the Brownian motion on $\mathbb{R}$ and let $\ell(t,x)$ denote its jointly-continuous local time. Set $A(t) = \int_I \ell(t,x)\mathrm{d}m(x)$ and write A^{-1} for the right-continuous inverse of A. Then the process $\{(X_{A^{-1}(t)})_{t\geq 0}, (\mathbb{P}_x^B)_{x\in I}\}$ is a realization of the desired generalized diffusion.

Let $M = \{(X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in I}\}$ denote the $D_m D_s$-generalized diffusion. We denote the resolvent operator of M by

$$R_q f(x) = \mathbb{P}_x\left[\int_0^\infty \mathrm{e}^{-qt} f(X_t)\mathrm{d}t\right], \quad q > 0. \tag{2.3}$$

For $x \in I$, we write

$$T_x = \inf\{t > 0 : X_t = x\}. \tag{2.4}$$

Then, for $a,x,b \in I$ with $a < x < b$, we have

$$\mathbb{P}_x(T_a > T_b) = \frac{x-a}{b-a}. \tag{2.5}$$

Note that, whenever $l \in I$, we have $\mathbb{P}_x(T_l < \infty) = 1$ for all $x \in I$ and l is a trap for M.

For a function $f : [0,l) \to \mathbb{R}$, we define

$$Jf(x) = \int_0^x \mathrm{d}y \int_{(0,y]} f(z)\mathrm{d}m(z). \tag{2.6}$$

We sometimes write $s(x) = x$ to emphasize the natural scale. For $q \in \mathbb{C}$, we write ϕ_q and ψ_q for the unique solutions of the integral equations

$$\phi_q = 1 + qJ\phi_q \quad \text{and} \quad \psi_q = s + qJ\psi_q \quad \text{on } [0,l), \tag{2.7}$$

respectively. They can be represented as

$$\phi_q = \sum_{n=0}^\infty q^n J^n 1 \quad \text{and} \quad \psi_q = \sum_{n=0}^\infty q^n J^n s. \tag{2.8}$$

Let $q > 0$. Note that ϕ_q and ψ_q are non-negative increasing functions. Set

$$H(q) = \lim_{x\uparrow l} \frac{\psi_q(x)}{\phi_q(x)} = \int_0^l \frac{1}{\phi_q(x)^2}\mathrm{d}x. \tag{2.9}$$

Then there exist σ-finite measures σ and σ^* on $[0,\infty)$ such that

$$H(q) = \int_{[0,\infty)} \frac{1}{q+\xi}\sigma(\mathrm{d}\xi) \quad \text{and} \quad \frac{1}{qH(q)} = \int_{[0,\infty)} \frac{1}{q+\xi}\sigma^*(\mathrm{d}\xi). \tag{2.10}$$

Note that

$$l = \lim_{q \downarrow 0} H(q) = \int_{[0,\infty)} \frac{\sigma(\mathrm{d}\xi)}{\xi} = \frac{1}{\sigma^*(\{0\})} \in (0,\infty]. \tag{2.11}$$

If we write $m(\infty) = \lim_{x\to\infty} m(x)$, we have

$$\pi_0 := \lim_{q \downarrow 0} qH(q) = \sigma(\{0\}) = \frac{1}{\int_{[0,\infty)} \frac{\sigma^*(\mathrm{d}\xi)}{\xi}} = \frac{1}{m(\infty)} \in [0,\infty). \tag{2.12}$$

Note that $\pi_0 = 0$ whenever $l < \infty$. We define

$$\rho_q(x) = \phi_q(x) - \frac{1}{H(q)} \psi_q(x). \tag{2.13}$$

Then the function ρ_q is a non-negative decreasing function on $[0,l)$ which satisfies

$$\rho_q = 1 - \frac{s}{H(q)} + qJ\rho_q. \tag{2.14}$$

We define

$$r_q(x,y) = r_q(y,x) = H(q)\phi_q(x)\rho_q(y) \quad 0 \le x \le y,\ x,y \in I'. \tag{2.15}$$

In particular, we have $r_q(0,x) = r_q(x,0) = H(q)\rho_q(x)$ and $r_q(0,0) = H(q)$. It is well-known (see, e.g., [13]) that

$$\mathbb{P}_x[\mathrm{e}^{-qT_y}] = \frac{r_q(x,y)}{r_q(y,y)}, \quad x,y \in I',\ q > 0. \tag{2.16}$$

In particular, we have

$$\rho_q(x) = \frac{r_q(x,0)}{r_q(0,0)} = \mathbb{P}_x[\mathrm{e}^{-qT_0}], \quad x \in I',\ q > 0. \tag{2.17}$$

We write $M' = \{(X_t)_{t\ge 0}, (\mathbb{P}'_x)_{x\in I'}\}$ for the process M killed upon hitting l. We write R'_q for the resolvent operator of M'. It is well-known (see, e.g., [13]) that $r_q(x,y)$ is the resolvent density of M' with respect to dm, or in other words,

$$R'_q f(x) = \int_{I'} f(y) r_q(x,y) \mathrm{d}m(y). \tag{2.18}$$

We have the resolvent equation

$$\int_{I'} r_q(x,y) r_p(y,z) \mathrm{d}m(y) = \frac{r_p(x,z) - r_q(x,z)}{q-p}, \quad x,z \in I',\ q,p > 0. \tag{2.19}$$

If $l \in I$, we define

$$r_q(l,y) = 0 \quad \text{for } y \in I', \tag{2.20}$$

$$r_q(x,l) = \frac{1}{q} - R'_q 1(x) \quad \text{for } x \in I', \tag{2.21}$$

$$r_q(l,l) = \frac{1}{q}, \tag{2.22}$$

and define a measure $\widetilde{m}$ on I by

$$\widetilde{m}(\mathrm{d}y) = 1_{I'}(y)\mathrm{d}m(y) + \delta_l(\mathrm{d}y). \tag{2.23}$$

We emphasize that $r_q(x,y)$ is no longer symmetric when either x or y equals l.

Proposition 2.1 *The formulae* (2.16) *and* (2.18) *extend to*

$$\mathbb{P}_x[\mathrm{e}^{-qT_y}] = \frac{r_q(x,y)}{r_q(y,y)}, \quad x,y \in I,\ q > 0, \tag{2.24}$$

$$R_q f(x) = \int_I f(y) r_q(x,y) \widetilde{m}(\mathrm{d}y), \quad x \in I,\ q > 0. \tag{2.25}$$

Proof. Suppose $l \in I$.

First, we let $x = l$. Then we have $\mathbb{P}_l[\mathrm{e}^{-qT_y}] = 0 = \frac{r_q(l,y)}{r_q(y,y)}$ for $y \in I'$ and $\mathbb{P}_l[\mathrm{e}^{-qT_l}] = 1 = \frac{r_q(l,l)}{r_q(l,l)}$. We also have $R_q f(l) = \mathbb{P}_l[\int_0^\infty \mathrm{e}^{-qt} f(X_t)\mathrm{d}t] = f(l)/q = f(l) r_q(l,l) \widetilde{m}(\{l\})$. Hence we obtain (2.24) and (2.25) in this case.

Second, we assume $x \in I'$. On one hand, we have

$$\int_0^\infty \mathrm{e}^{-qt} \mathbb{P}_x(t \geq T_l)\mathrm{d}t = \frac{1}{q}\mathbb{P}_x[\mathrm{e}^{-qT_l}] = r_q(l,l)\mathbb{P}_x[\mathrm{e}^{-qT_l}]. \tag{2.26}$$

On the other hand, we have

$$\int_0^\infty \mathrm{e}^{-qt} \mathbb{P}_x(t \geq T_l)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt} \left\{ 1 - \mathbb{P}_x(X_t \in I') \right\} \mathrm{d}t = \frac{1}{q} - R'_q 1(x) = r_q(x,l). \tag{2.27}$$

Hence we obtain (2.24) for $y = l$. Using (2.18), we obtain

$$R_q f(x) = R'_q f(x) + f(l) \int_0^\infty \mathrm{e}^{-qt} \mathbb{P}_x(t \geq T_l)\mathrm{d}t \tag{2.28}$$

$$= \int_{I'} f(y) r_q(x,y)\mathrm{d}m(y) + f(l) r_q(x,l) \widetilde{m}(\{l\}), \tag{2.29}$$

which implies (2.25). □

3 The excursion measure away from 0

For $y \in I$, we write $(L_t(y))_{t \geq 0}$ for the local time at y normalized as follows (see [9]):

$$\mathbb{P}_x\left[\int_0^\infty \mathrm{e}^{-qt} \mathrm{d}L_t(y)\right] = r_q(x,y), \quad x \in I, \ q > 0. \tag{3.1}$$

We write L_t for $L_t(0)$. Let n denote the excursion measure away from 0 corresponding to $(L_t)_{t \geq 0}$ (see [1]), where we adopt the convention that

$$X_t = 0 \text{ for all } t \geq T_0, \quad n\text{-a.e.} \tag{3.2}$$

We define the functional N_q by

$$N_q f = n\left[\int_0^\infty \mathrm{e}^{-qt} f(X_t) \mathrm{d}t\right], \quad q > 0. \tag{3.3}$$

Then it is well-known (see [22]) that n can be characterized by the following identity:

$$N_q f = \frac{R_q f(0)}{r_q(0,0)} \quad \text{whenever } f(0) = 0. \tag{3.4}$$

In particular, taking $f = 1_{I \setminus \{0\}}$, we have

$$n\left[1 - \mathrm{e}^{-qT_0}\right] = \frac{1}{r_q(0,0)} = \frac{1}{H(q)} \tag{3.5}$$

and, by (2.9), we have

$$n(T_0 = \infty) = \lim_{q \downarrow 0} \frac{1}{H(q)} = \frac{1}{l}. \tag{3.6}$$

We write $M^0 = \{(X_t)_{t \geq 0}, (\mathbb{P}_x^0)_{x \in I}\}$ for the process M stopped upon hitting 0 and write R_q^0 for the resolvent operator of M^0. By the strong Markov property of M, we have

$$R_q f(x) = R_q^0 f(x) + \mathbb{P}_x[\mathrm{e}^{-qT_0}] R_q f(0). \tag{3.7}$$

The resolvent density with respect to $\widetilde{m}(\mathrm{d}y)$ is given as

$$r_q^0(x,y) = r_q(x,y) - \frac{r_q(x,0) r_q(0,y)}{r_q(0,0)} \quad \text{for } x,y \in I. \tag{3.8}$$

Note that $r_q^0(x,y) = \psi_q(x) \rho_q(y)$ for $x \leq y$ and that

$$\mathbb{P}_x^0[\mathrm{e}^{-qT_y}] = \frac{r_q^0(x,y)}{r_q^0(y,y)} = \frac{\psi_q(x)}{\psi_q(y)} \quad \text{for } x,y \in I,\, x \le y. \tag{3.9}$$

Note also that $(L_t(y))_{t\ge0}$ is the local time at y such that

$$\mathbb{P}_x^0\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t(y)\right] = r_q^0(x,y), \quad x,y \in I\setminus\{0\},\ q>0. \tag{3.10}$$

The strong Markov property of n may be stated as

$$n[F_T\, G\circ\theta_T] = n\left[F_T \mathbb{P}_{X_T}^0[G]\right], \tag{3.11}$$

where T is a stopping time, F_T is a non-negative $\mathscr{F}_T$-measurable functional, G is a non-negative measurable functional such that $0 < n[F_T] < \infty$ or $0 < n[G\circ\theta_T] < \infty$.

Let $x,y \in I$ be such that $0<x<y$. Because of the properties of excursion paths of a generalized diffusion, we see that X under n hits y if and only if X hits x and in addition $X\circ\theta_{T_x}$ hits y. Hence, by the strong Markov property of n, we have,

$$\begin{aligned} n(T_y<\infty) =& n(\{T_y<\infty\}\circ\theta_{T_x}\cap\{T_x<\infty\}) && (3.12)\\ =& \mathbb{P}_x^0(T_y<\infty)n(T_x<\infty) && (3.13)\\ =& \mathbb{P}_x(T_y<T_0)n(T_x<\infty) && (3.14)\\ =& \frac{x}{y}n(T_x<\infty). && (3.15)\end{aligned}$$

This shows that $xn(T_x<\infty)$ equals a constant C in $x \in I\setminus\{0\}$, so that we have

$$n(T_x<\infty) = \frac{C}{x}, \quad x \in I\setminus\{0\}. \tag{3.16}$$

If $l \in I$, then we have

$$C = ln(T_l<\infty) = ln(T_0=\infty) = 1. \tag{3.17}$$

The following theorem generalizes this fact and a result of [4].

Theorem 3.1 (see also [4]) *In any case, $C=1$.*

Theorem 3.1 will be proved at the end of Section 6.

The following lemma is the first step of the proof of Theorem 3.1.

Lemma 3.2 (see also [4]) *The constant C may be represented as*

$$C = \lim_{t\downarrow0} n[X_t]. \tag{3.18}$$

Proof. By definition of C, we have

$$C = \sup_{x\in I\setminus\{0\}} xn(T_x<\infty). \tag{3.19}$$

Since $n(t < T_x < \infty) \uparrow n(T_x < \infty)$ as $t \downarrow 0$, we have

$$C = \sup_{x \in I \setminus \{0\}} x \sup_{t>0} n(t < T_x < \infty) \tag{3.20}$$
$$= \sup_{t>0} \sup_{x \in I \setminus \{0\}} xn(t < T_x < \infty) \tag{3.21}$$
$$= \lim_{t \downarrow 0} \sup_{x \in I \setminus \{0\}} xn(t < T_x < \infty). \tag{3.22}$$

Because of the properties of excursion paths of a generalized diffusion, we see that X under n hits T_x after t if and only if X does not hit x nor 0 until t and $X \circ \theta_t$ hits x. Hence, by the strong Markov property of n, we have,

$$\sup_{x \in I \setminus \{0\}} xn(t < T_x < \infty) = \sup_{x \in I \setminus \{0\}} xn(\{T_x < \infty\} \circ \theta_t \cap \{t < T_x \wedge T_0\}) \tag{3.23}$$
$$= \sup_{x \in I \setminus \{0\}} n[x\mathbb{P}_{X_t}(T_x < T_0); t < T_x \wedge T_0] \tag{3.24}$$
$$= \sup_{x \in I \setminus \{0\}} n[X_t; t < T_x \wedge T_0] . \tag{3.25}$$

We divide the remainder of the proof into three cases.

(i) The case $l < \infty$. Since $T_x \le T_l$ for $x \in I \setminus \{0\}$, we have

$$(3.25) = n[X_t; t < T_l \wedge T_0] \tag{3.26}$$
$$= n[X_t; t < T_0] - n[X_t; T_l \le t < T_0] . \tag{3.27}$$

Since $n(T_l < \infty) < \infty$, we may apply the dominated convergence theorem to obtain

$$n[X_t; T_l \le t < T_0] \le n[X_t; T_l < \infty] \underset{t \downarrow 0}{\longrightarrow} 0, \tag{3.28}$$

which implies Equality (3.18), since $n[X_t; t < T_0] = n[X_t]$.

(ii) The case $l' < l = \infty$. The proof of Case (i) works if we replace l by l'.

(iii) The case $l' = l = \infty$. Since $T_x \uparrow \infty$ as $x \to \infty$, we have

$$(3.25) = \lim_{x \to \infty} n[X_t; t < T_x \wedge T_0] = n[X_t; t < T_0] \tag{3.29}$$

by the monotone convergence theorem. This implies Equality (3.18). □

4 The renormalized zero resolvent

For $q > 0$ and $x \in I$, we set

$$h_q(x) = r_q(0,0) - r_q(x,0). \tag{4.1}$$

Note that $h_q(x)$ is always non-negative, since we have, by (2.24),

$$\frac{h_q(x)}{H(q)} = \mathbb{P}_x[1 - \mathrm{e}^{-qT_0}]. \tag{4.2}$$

The following theorem asserts that the limit $h_0 := \lim_{q\downarrow 0} h_q$ exists, which will be called the *renormalized zero resolvent*.

Theorem 4.1 *For $x \in I$, the limit $h_0(x) := \lim_{q\downarrow 0} h_q(x)$ exists and is represented as*

$$h_0(x) = s(x) - g(x) = x - g(x), \tag{4.3}$$

where

$$g(x) = \pi_0 J1(x) = \pi_0 \int_0^x m(y)\mathrm{d}y. \tag{4.4}$$

The function $h_0(x)$ is continuous increasing in $x \in I$, positive in $x \in I \setminus \{0\}$ and zero at $x = 0$. In particular, if $\pi_0 = 0$, then h_0 coincides with the scale function, i.e., $h_0(x) = s(x) = x$.

Proof. For $x \in I'$, we have

$$h_q(x) = H(q)\{1 - \rho_q(x)\} = x - qH(q)J\rho_q(x) \underset{q\downarrow 0}{\longrightarrow} x - \pi_0 J1(x), \tag{4.5}$$

where we used the facts that $0 \le \rho_q(x) \le 1$ and $\rho_q(x) \to 1 - \frac{x}{l}(= 1$ if $\pi_0 > 0)$ as $q \downarrow 0$ and used the dominated convergence theorem. If $l \in I$, we have

$$h_q(l) = r_q(0,0) = H(q) \underset{q\downarrow 0}{\longrightarrow} l, \tag{4.6}$$

and hence we obtain $h_0(l) = l$, which shows (4.3) for $x = l$, since $\pi_0 = 0$ in this case.

It is obvious that h_0 is continuous. If $\pi_0 = 0$, then $h_0(x) = x$ is increasing in $x \in I$ and positive in $x \in I \setminus \{0\}$. If $\pi_0 > 0$, then we have $\pi_0 m(y) \le 1$ for all $y \in I$ and $\pi_0 m(y) < 1$ for all $y < l'$, so that $h_0(x)$ is increasing in $x \in I$ and positive in $x \in I \setminus \{0\}$. The proof is now complete. □

Example 4.2 *Let $0 < l' < l = \infty$ and let $m(x) = \min\{x, l'\}$. In this case, M is a Brownian motion on $[0, l']$ where both boundaries 0 and l' are regular-reflecting. Then we have*

$$h_*(x) = \frac{2l'}{\pi} \sin \frac{\pi x}{2l'}, \quad h_0(x) = x - \frac{x^2}{2l'}, \quad x \in [0, l']. \tag{4.7}$$

Note that we have $\pi_0 = 1/m(\infty) = 1/l'$ and

$$\phi_q(x) = \begin{cases} \cosh \sqrt{q} x & \text{for } x \in [0,l'], \\ \phi_q(l') + \phi_q'(l')(x-l') & \text{for } x \in (l',\infty), \end{cases} \tag{4.8}$$

$$\psi_q(x) = \begin{cases} \sinh \sqrt{q} x / \sqrt{q} & \text{for } x \in [0,l'], \\ \psi_q(l') + \psi_q'(l')(x-l') & \text{for } x \in (l',\infty), \end{cases} \tag{4.9}$$

$$H(q) = \frac{1}{\sqrt{q} \tanh \sqrt{q} l'}. \tag{4.10}$$

We study recurrence and transience of 0.

Theorem 4.3 *For M, the following assertions hold:*

(i) 0 is transient if and only if $l < \infty$. In this case, it holds that

$$\mathbb{P}_x(T_0 = \infty) = \frac{x}{l} \quad \text{for } x \in I. \tag{4.11}$$

(ii) 0 is positive recurrent if and only if $\pi_0 > 0$. In this case, it holds that

$$\mathbb{P}_x[T_0] = \frac{h_0(x)}{\pi_0} \quad \text{for } x \in I. \tag{4.12}$$

(iii) 0 is null recurrent if and only if $l = \infty$ and $\pi_0 = 0$.

Although this theorem seems well-known, we give the proof for completeness of the paper.

Proof. (i) By the formula (2.16), we have, for $x \in I'$,

$$\mathbb{P}_x(T_0 = \infty) = \lim_{q \downarrow 0} \mathbb{P}_x[1 - \mathrm{e}^{-qT_0}] = \lim_{q \downarrow 0} \left\{ \frac{\psi_q(x)}{H(q)} - \{\phi_q(x) - 1\} \right\} = \frac{x}{l}. \tag{4.13}$$

Hence 0 is transient if and only if $l < \infty$. If $x = l \in I$, it is obvious that $\mathbb{P}_l(T_0 = \infty) = 1$. This proves the claim.

(ii) Since $(1 - \mathrm{e}^{-x})/x \uparrow 1$ as $x \downarrow 0$, we may apply the monotone convergence theorem to see that

$$\mathbb{P}_x[T_0] = \lim_{q \downarrow 0} \frac{1}{q} \mathbb{P}_x[1 - \mathrm{e}^{-qT_0}] = \lim_{q \downarrow 0} \frac{h_q(x)}{q r_q(0,0)} = \frac{h_0(x)}{\pi_0}, \tag{4.14}$$

for $x \in I$. This shows that $\mathbb{P}_x[T_0] < \infty$ if and only if $\pi_0 > 0$, which proves the claim.

(iii) This is obvious by (i) and (ii). □

We illustrate the classification of recurrence of 0 of Theorem 4.3 as follows:

	$l = \infty$	$l < \infty$
$\pi_0 = 0$	(1) null recurrent	(3) transient
$\pi_0 > 0$	(2) positive recurrent	impossible

(1) l' is type-1-natural.
(2) l' is type-2-natural, entrance or regular-reflecting.
(3) l' is type-3-natural, exit, regular-elastic or regular-absorbing.

5 Various conditionings to avoid zero

We prove the three theorems concerning conditionings to avoid zero. We need the following lemma in later use.

Lemma 5.1 *For any stopping time T and for any $x \in I$, it holds that*

$$\mathbb{P}_x^0[X_T; T < \infty] \leq x. \tag{5.1}$$

Proof. By [2, Proposition II.2.8], it suffices to prove that $\mathbb{P}_x^0[X_t] \leq x$ for all $t > 0$.

Note that $x \leq \liminf_{t\downarrow 0} \mathbb{P}_x^0[X_t]$ for all $x \in I$ by Fatou's lemma. By the help of [2, Corollary II.5.3], it suffices to prove that

$$\mathbb{P}_x^0[X_{T_K}; T_K < \infty] \leq x \quad \text{for } x \in I \setminus K \tag{5.2}$$

for all compact subset K of I.

Let K be a compact subset of I and let $x \in I \setminus K$. Let $a = \sup(K \cap (0,x)) \cup \{0\}$ and $b = \inf(K \cap (x,l)) \cup \{l\}$. Since 0 and l are traps for $\mathbb{P}_x^0$, we have $T_K = T_a \wedge T_b$ on $\{T_K < \infty\}$, $\mathbb{P}_x^0$-a.e. and thus we obtain

$$\mathbb{P}_x^0[X_{T_K}; T_K < \infty] \leq \mathbb{P}_x^0[X_{T_a \wedge T_b}] = a\mathbb{P}_x(T_a < T_b) + b\mathbb{P}_x(T_a > T_b) = x, \tag{5.3}$$

which proves (5.2) for $x \notin K$. Hence we obtain the desired result. □

First, we prove Theorem 1.1.

Proof of Theorem 1.1. (i) Suppose that $l'(= l)$ is entrance or natural. By the strong Markov property, we have

$$a\mathbb{P}_x[F_T; T < T_a < T_0] = a\mathbb{P}_x[F_T \mathbb{P}_{X_T}(T_a < T_0); T < T_a \wedge T_0] \tag{5.4}$$
$$= \mathbb{P}_x[F_T X_T; T < T_a \wedge T_0] \tag{5.5}$$
$$= \mathbb{P}_x^0[F_T X_T; T < T_a] \tag{5.6}$$

since $X_T = 0$ on $\{T \geq T_0\}$, $\mathbb{P}_x^0$-a.s. By the fact that $1_{\{T<T_a\}} \to 1_{\{T<\infty\}}$, $\mathbb{P}_x^0$-a.s. and by Lemma 5.1, we may thus apply the dominated convergence theorem to see that (5.6) converges as $a \uparrow l$ to $\mathbb{P}_x^0[F_T X_T; T < \infty]$. Since $a\mathbb{P}_x(T_a < T_0) = x$, we obtain (1.2).

(ii) Suppose that l' is regular-elastic, regular-absorbing or exit. By the strong Markov property, we have

$$l\mathbb{P}_x[F_T; T < T_l < T_0] = l\mathbb{P}_x[F_T \mathbb{P}_{X_T}(T_l < T_0); T < T_l \wedge T_0] \tag{5.7}$$
$$= \mathbb{P}_x^0[F_T X_T; T < T_l]. \tag{5.8}$$

Since $\mathbb{P}_x(T_l < T_0) = x/l$, we obtain (1.2).

(iii) In the case where l' is regular-reflecting, the proof is the same as (ii) if we replace l by l', and so we omit it. □

Second, we prove Theorem 1.2.

Proof of Theorem 1.2. By McKean [17] (see also [29]), we have the following facts. For $\gamma \in \mathbb{R}$, let ψ_γ be the solution of the integral equation $\psi_\gamma = s + \gamma J \psi_\gamma$. Then we have the eigendifferential expansion

$$r_q(x,y) = \int_{(-\infty,0)} (q-\gamma)^{-1} \psi_\gamma(x) \psi_\gamma(y) \theta(\mathrm{d}\gamma) \tag{5.9}$$

for the spectral measure θ. We now have

$$\frac{\mathbb{P}_x(T_0 \in \mathrm{d}t)}{\mathrm{d}t} = \int_{(-\infty,0)} \mathrm{e}^{\gamma t} \psi_\gamma(x) \theta(\mathrm{d}\gamma), \quad \frac{n(T_0 \in \mathrm{d}t)}{\mathrm{d}t} = \int_{(-\infty,0)} \mathrm{e}^{\gamma t} \theta(\mathrm{d}\gamma), \tag{5.10}$$

and, for $r > 0$,

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(T_0 > t)}{n(T_0 > t)} = h_*(x), \quad \lim_{t\to\infty} \frac{n(T_0 > t - r)}{n(T_0 > t)} = \mathrm{e}^{-\gamma_* r}. \tag{5.11}$$

We note that γ_* equals the supremum of the support of θ and that $h_* = \psi_{\gamma_*}$. If l' is natural, exit, regular-absorbing or regular-elastic, we see that $\gamma_* = 0$ and $h_* = s$.

By the strong Markov property, we have

$$\mathbb{P}_x[F_T; T < t < T_0] = \mathbb{P}_x^0\left[F_T \, \mathbb{P}_{X_T}(T_0 > t - r)|_{r=T}; T < t\right]. \tag{5.12}$$

Since we have

$$n(T_0 > t) \geq n\left(T_y < T_0,\ T_0 \circ \theta_{T_y} > t\right) = \frac{1}{y} \mathbb{P}_y(T_0 > t), \tag{5.13}$$

we have $\mathbb{P}_y(T_0 > t - r) \leq y n(T_0 > t - r)$. Hence, by Lemma 5.1 and by the dominated convergence theorem, we obtain

$$\lim_{t\to\infty} \frac{1}{n(T_0 > t)} \mathbb{P}_x[F_T; T < t < T_0] = \mathbb{P}_x^0\left[F_T \mathrm{e}^{-\gamma_* T} h_*(X_T); T < \infty\right]. \tag{5.14}$$

Dividing both sides of (5.14) by those of the first equality of (5.11), we obtain (1.3). □

Third, we prove Theorem 1.3.

Proof of Theorem 1.3. By (4.2), we have

$$H(q) \mathbb{P}_x(e_q < T_0) = h_q(x) \underset{q\downarrow 0}{\longrightarrow} h_0(x). \tag{5.15}$$

Note that

$$
\begin{aligned}
\mathbb{P}_x[F_T; T < e_q < T_0] =& \mathbb{P}_x\left[F_T \int_T^\infty 1_{\{t<T_0\}} q\mathrm{e}^{-qt}\mathrm{d}t\right] && (5.16)\\
=& \mathbb{P}_x\left[F_T \mathrm{e}^{-qT} \int_0^\infty 1_{\{t+T<T_0\}} q\mathrm{e}^{-qt}\mathrm{d}t\right] && (5.17)\\
=& \mathbb{P}_x\left[F_T \mathrm{e}^{-qT}; e_q + T < T_0\right] && (5.18)\\
=& \mathbb{P}_x\left[F_T \mathrm{e}^{-qT} 1_{\{e_q<T_0\}} \circ \theta_T; T < T_0\right]. && (5.19)
\end{aligned}
$$

By the strong Markov property, we have

$$
\begin{aligned}
H(q)\mathbb{P}_x[F_T; T < e_q < T_0] =& H(q)\mathbb{P}_x\left[F_T \mathrm{e}^{-qT} \mathbb{P}_{X_T}(e_q < T_0); T < T_0\right] && (5.20)\\
=& \mathbb{P}^0_x\left[F_T \mathrm{e}^{-qT} h_q(X_T); T < \infty\right], && (5.21)
\end{aligned}
$$

since $h_q(X_T) = 0$ on $\{T \geq T_0\}$, $\mathbb{P}^0_x$-a.s. Once the interchange of the limit and the integration is justified, we see that (5.21) converges as $q \downarrow 0$ to $\mathbb{P}^0_x[F_T h_0(X_T); T < \infty]$, and hence we obtain (1.4).

Let us prove $h_q(x) \leq x$ for $q > 0$ and $x \in I$. If $x \in I'$, we use (2.14) and we have

$$
h_q(x) = H(q)\{1 - \rho_q(x)\} = x - qH(q)J\rho_q(x) \leq x. \tag{5.22}
$$

If $l \in I$, we have $h_q(l) = H(q) \leq l$. We thus see that the integrand of (5.21) is dominated by X_T. By Lemma 5.1, we thus see that we may apply the dominated convergence theorem, and therefore the proof is complete. □

6 Invariance and excessiveness

Let us introduce notation of invariance and excessiveness. Let h be a non-negative measurable function on E.

(i) We say h is *α-invariant* for M^0 (resp. for n) ($\alpha \in \mathbb{R}$) if $\mathrm{e}^{-\alpha t}\mathbb{P}^0_x[h(X_t)] = h(x)$ for all $x \in E$ and all $t > 0$ (resp. there exists a positive constant C such that $\mathrm{e}^{-\alpha t} n[h(X_t)] = C$ for all $t > 0$).

(ii) We say h is *α-excessive* for M^0 (resp. for n) ($\alpha \geq 0$) if $\mathrm{e}^{-\alpha t}\mathbb{P}^0_x[h(X_t)] \leq h(x)$ for all $x \in E$ and all $t > 0$ and $\mathrm{e}^{-\alpha t}\mathbb{P}^0_x[h(X_t)] \to h(x)$ as $t \downarrow 0$ (resp. there exists a positive constant C such that $\mathrm{e}^{-\alpha t} n[h(X_t)] \leq C$ for all $t > 0$ and $n[h(X_t)] \to C$ as $t \downarrow 0$).

(iii) We say h is *invariant* (resp. *excessive*) when h is 0-invariant (resp. 0-excessive).

We give the following remarks.

(i) As a corollary of Theorem 1.2, the function h_* is γ_*-invariant for M^0.

(ii) As a corollary of (i), the function s is invariant for M^0 when l' for M is natural, exit, regular-absorbing or regular-elastic.

(iii) As a corollary of Lemma 5.1, the function s is excessive for M^0 when l' for M is entrance or regular-reflecting.

(iv) As a corollary of Theorem 1.3, the function h_0 is excessive for M^0.

In this section, we prove several properties to complement these statements. Following [8, Section 2], we introduce the operators

$$D_m f(x) = \lim_{\varepsilon,\varepsilon' \downarrow 0} \frac{f(x+\varepsilon) - f(x-\varepsilon')}{m(x+\varepsilon) - m(x-\varepsilon')} \tag{6.1}$$

whenever the limit exist. Note that $f(x) = \psi_q(x)$ (resp. $f(x) = \rho_q(x)$) is an increasing (resp. decreasing) solution of the differential equation $D_m D_s f = qf$ satisfying $f(0) = 0$ and $D_s f(0) = 1$ (resp. $f(0) = 1$ and $D_s f(0) = -1/H(q)$).

Theorem 6.1 *The function h_* is γ_*-invariant for n when l' for M is entrance or regular-reflecting.*

Proof. By [7, Section 12]), we see that if $D_m D_s f = F$ and $D_m D_s g = G$ then

$$D_m\{gD_s f - fD_s g\} = gF - fG. \tag{6.2}$$

Hence we have

$$(q-\gamma_*)\psi_{\gamma_*}\rho_q = D_m\left\{\psi_{\gamma_*}D_s\rho_q - \rho_q D_s\psi_{\gamma_*}\right\}. \tag{6.3}$$

Integrate both sides on I' with respect to dm, we obtain

$$(q-\gamma_*)\int_{I'}\psi_{\gamma_*}(x)\rho_q(x)\mathrm{d}m(x) = 1. \tag{6.4}$$

where we used the facts that $\rho_q(0) = \psi'_{\gamma_*}(0) = 1$, $\psi_{\gamma_*}(0) = \psi'_{\gamma_*}(l') = 0$, $\psi_{\gamma_*}(l') < \infty$ and $\rho'_q(l') = 0$. This shows that

$$N_q h_* = \frac{R_q h_*(0)}{H(q)} = \int_{I'}\rho_q(x)\psi_{\gamma_*}(x)\mathrm{d}m(x) = \frac{1}{q-\gamma_*}. \tag{6.5}$$

Hence we obtain $\mathrm{e}^{-\gamma_* t}n[h_*(X_t)] = 1$ for a.e. $t > 0$. For $0 < s < t$, we see, by the γ_*-invariance of h_* for M^0, that

$$\mathrm{e}^{-\gamma_* t}n[h_*(X_t)] = \mathrm{e}^{-\gamma_* t}n[\mathbb{P}^0_{X_s}[h_*(X_{t-s})]] = \mathrm{e}^{-\gamma_* s}n[h_*(X_s)], \tag{6.6}$$

which shows that $t \mapsto \mathrm{e}^{-\gamma_* t}n[h_*(X_t)]$ is constant in $t > 0$. Thus we obtain the desired result. □

For later use, we need the following lemma.

Lemma 6.2 *For $0 < p < q$, it holds that*

$$\int_{(0,l')}\rho_q(y)\psi_p(y)\mathrm{d}m(y) \le \frac{H(p)}{H(q)(q-p)}. \tag{6.7}$$

Consequently, it holds that $R'_q\psi_p(x) < \infty$.

Proof. Let $x < l'$. Using the fact that $\rho_p \geq 0$ and the resolvent equation, we have

$$\int_{(0,x]} \rho_q(y)\psi_p(y)\mathrm{d}m(y) \leq \int_{(0,x]} \rho_q(y)H(p)\phi_p(y)\mathrm{d}m(y) \tag{6.8}$$

$$\leq \frac{1}{H(q)\rho_p(x)} \int_{I'} r_q(0,y)r_p(y,x)\mathrm{d}m(y) \tag{6.9}$$

$$= \frac{1}{H(q)\rho_p(x)} \cdot \frac{r_p(0,x) - r_q(0,x)}{q-p} \tag{6.10}$$

$$\leq \frac{r_p(0,x)}{H(q)\rho_p(x)(q-p)} = \frac{H(p)}{H(q)(q-p)}. \tag{6.11}$$

Letting $x \uparrow l'$, we obtain (6.7). □

We compute the image of the resolvent operators of h_0.

Proposition 6.3 *For $q > 0$ and $x \in I$, it holds that*

$$R_q h_0(x) = \frac{h_0(x)}{q} + \frac{r_q(x,0)}{q} - \frac{\pi_0}{q^2}, \tag{6.12}$$

$$R_q^0 h_0(x) = \frac{h_0(x)}{q} - \frac{\pi_0}{q^2}\mathbb{P}_x[1 - \mathrm{e}^{-qT_0}], \tag{6.13}$$

$$N_q h_0 = \frac{1}{q} - \frac{\pi_0}{q^2 H(q)}. \tag{6.14}$$

Proof. Suppose $x \in I'$. Let $0 < p < q/2$. On one hand, by the resolvent equation, we have

$$R_q h_p(x) = r_p(0,0)\int_I r_q(x,y)\widetilde{m}(\mathrm{d}y) - \int_I r_q(x,y)r_p(y,0)\widetilde{m}(\mathrm{d}y) \tag{6.15}$$

$$= \frac{r_p(0,0)}{q} - \frac{r_p(x,0) - r_q(x,0)}{q-p} \tag{6.16}$$

$$= \frac{h_p(x)}{q-p} + \frac{r_q(x,0)}{q-p} - \frac{pH(p)}{q(q-p)} \tag{6.17}$$

$$\xrightarrow[p\downarrow 0]{} \frac{h_0(x)}{q} + \frac{r_q(x,0)}{q} - \frac{\pi_0}{q^2}. \tag{6.18}$$

On the other hand, for $y \in I'$, we have

$$h_p(y) = H(p)\{1 - \rho_p(y)\} = \psi_p(y) - H(p)\{\phi_p(y) - 1\} \leq \psi_{q/2}(y). \tag{6.19}$$

By Lemma 6.2, we see by the dominated convergence theorem that $R_q h_p(x) \to R_q h_0(x)$ as $p \downarrow 0$. Hence we obtain (6.12) for $x \in I'$.

Suppose $l \in I$ and $x = l$. Then we have

$$qR_q h_0(l) = qh_0(l)r_q(l,l)\widetilde{m}(\{l\}) = h_0(l), \tag{6.20}$$

which shows (6.12) for $x = l$, since $r_q(l,0) = 0$ and $\pi_0 = 0$ in this case. Thus we obtain (6.12). Using (3.7), (3.4), (6.12) and (2.24), we immediately obtain (6.13) and (6.14). □

We now obtain the image of the transition operators of h_0.

Theorem 6.4 *For $t > 0$ and $x \in I$, it holds that*

$$\mathbb{P}^0_x[h_0(X_t)] = h_0(x) - \pi_0 \int_0^t \mathbb{P}_x(s < T_0)\mathrm{d}s, \tag{6.21}$$

$$n[h_0(X_t)] = 1 - \pi_0 \int_0^t \mathrm{d}s \int_{[0,\infty)} \mathrm{e}^{-s\xi}\sigma^*(\mathrm{d}\xi). \tag{6.22}$$

Consequently, for M^0 and n, it holds that h_0 is invariant when $\pi_0 = 0$ and that h_0 is excessive but non-invariant when $\pi_0 > 0$.

Proof. By (6.13), we have

$$R^0_q h_0(x) = \frac{h_0(x)}{q} - \frac{\pi_0}{q}\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(t < T_0)\mathrm{d}t, \tag{6.23}$$

which proves (6.21) for a.e. $t > 0$. By Fatou's lemma, we see that $\mathbb{P}^0_x[h_0(X_t)] \le h_0(x)$ holds for all $t > 0$ and all $x \in I$. For $0 < s < t$, we have

$$\mathbb{P}^0_x[h_0(X_t)] = \mathbb{P}^0_x\left[\mathbb{P}^0_{X_s}[h_0(X_{t-s})]\right] \le \mathbb{P}^0_x[h_0(X_s)]. \tag{6.24}$$

This shows that $t \mapsto \mathbb{P}^0_x[h_0(X_t)]$ is non-increasing. Since the right-hand side of (6.21) is continuous in $t > 0$, we see that (6.21) holds for all $t > 0$.

By (6.14), we have

$$\begin{aligned} N_q h_0 &= \frac{1}{q} - \frac{\pi_0}{q}\int_{[0,\infty)} \frac{1}{q+\xi}\sigma^*(\mathrm{d}\xi) \qquad (6.25)\\ &= \frac{1}{q} - \frac{\pi_0}{q}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-qt}\int_{[0,\infty)} \mathrm{e}^{-t\xi}\sigma^*(\mathrm{d}\xi), \qquad (6.26)\end{aligned}$$

which proves (6.22) for a.e. $t > 0$. For $0 < s < t$, we have

$$n[h_0(X_t)] = n\left[\mathbb{P}^0_{X_s}[h_0(X_{t-s})]\right] \le n[h_0(X_s)], \tag{6.27}$$

from which we can conclude that (6.22) holds for all $t > 0$. □

We have already proved that s is invariant for M^0 and n when $\pi_0 = 0$. We now study properties of s in the case where $\pi_0 > 0$. In the case $l'(= \infty)$ is entrance, the measure $\mathbb{P}_{l'}$ denotes the extension of M starting from l' constructed by a scale transform (see also Fukushima [10, Section 6]).

Theorem 6.5 *Suppose $\pi_0 > 0$. Then the following assertions hold:*

(i) If l' is type-2-natural, then the scale function $s(x) = x$ is invariant for M^0 and n.

(ii) If l' is entrance or regular-reflecting, then, for any $q>0$ and any $t>0$,

$$R_q^0 s(x) = \frac{x}{q} - \frac{\psi_q(x)}{q}\chi_q(l'), \tag{6.28}$$

$$N_q s = \frac{1}{q}\mathbb{P}_{l'}[1-\mathrm{e}^{-qT_0}], \tag{6.29}$$

$$n[X_t] = \mathbb{P}_{l'}(t<T_0), \tag{6.30}$$

where

$$\chi_q(l') = \begin{cases} \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}] & \text{if } l' \text{ for } M \text{ is entrance}, \\ \frac{1}{q}\left\{\frac{l'}{\psi_q(l')} - \rho_q(l')\right\} & \text{if } l' \text{ for } M \text{ is regular-reflecting}. \end{cases} \tag{6.31}$$

Consequently, $s(x)=x$ is excessive but non-invariant for M^0 and n.

Proof. By (2.14), we have, for $x\in I'$,

$$\rho_q'(x) = -\frac{1}{H(q)} + q\int_{(0,x]} \rho_q(y)\mathrm{d}m(y). \tag{6.32}$$

Since $I'=I$ when $\pi_0>0$, we have

$$\int_{I'} \rho_q(y)\mathrm{d}m(y) = \frac{1}{H(q)}R_q 1(0) = \frac{1}{qH(q)}. \tag{6.33}$$

We write $\rho_q(l') = \lim_{x\uparrow l'}\rho_q(x)$. Recalling g is defined by (4.4) and using (6.33), we obtain

$$N_q g = \pi_0 \int_0^{l'} \mathrm{d}x\, m(x) \int_{I'\setminus(0,x]} \rho_q(y)\mathrm{d}m(y) \tag{6.34}$$

$$= -\frac{\pi_0}{q}\int_0^{l'} \mathrm{d}x\, m(x)\rho_q'(x) \tag{6.35}$$

$$= -\frac{\pi_0}{q}\int_{I'} \mathrm{d}m(y)\int_y^{l'} \rho_q'(x)\mathrm{d}x \tag{6.36}$$

$$= \frac{\pi_0}{q}\int_{I'} \mathrm{d}m(y)\{\rho_q(y)-\rho_q(l')\} \tag{6.37}$$

$$= \frac{\pi_0}{q}\left\{\frac{1}{qH(q)} - m(\infty)\rho_q(l')\right\}. \tag{6.38}$$

(i) If l' is type-2-natural, then, by [12, Theorem 5.13.3], we have $\rho_q(l')=0$. By (6.14), we obtain $N_q s = 1/q$. Since $t\mapsto n[X_t]$ is non-decreasing, we obtain $n[X_t]=1$ for all $t>0$. We thus conclude that s is invariant for n. The invariance of s for M^0 has already been remarked in the beginning of this section.

(ii) We postpone the proof of (6.28) until the end of the proof of Theorem 7.5. Let us prove (6.29) and (6.30).

If l' is regular-reflecting, we have $\rho_q(l') = \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}]$. If l' is entrance, then we may take limit as $x \uparrow l'$ and obtain

$$\rho_q(l') := \lim_{x \uparrow l'} \rho_q(x) = \lim_{x \uparrow l'} \mathbb{P}_x[\mathrm{e}^{-qT_0}] = \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}] \tag{6.39}$$

(see Kent [14, Section 6]). Since $\pi_0 m(\infty) = 1$, we obtain

$$N_q s = N_q h_0 + N_q g = \frac{1}{q}\mathbb{P}_{l'}[1 - \mathrm{e}^{-qT_0}] = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_{l'}(t < T_0)\mathrm{d}t. \tag{6.40}$$

This proves (6.29) and $n[X_t] = \mathbb{P}_{l'}(t < T_0)$ for a.e. $t > 0$. Since $t \mapsto \mathbb{P}_{l'}(t < T_0)$ is continuous (see Kent [14, Section 6]) and by Lemma 5.1, we can employ the same argument as the proof of Theorem 6.4, and therefore we obtain (6.30).

Suppose that s were invariant for M^0. Then we would see that $n[X_t] = n[\mathbb{P}^0_{X_s}[X_{t-s}]] = n[X_s]$ for $0 < s < t$, which would lead to the invariance of s for n. This would be a contradiction. □

Remark 6.6 *An excessive function h is called* minimal *if, whenever u and v are excessive and $h = u + v$, both u and v are proportional to h. It is known (see Salminen [23]) that s is minimal. We do not know whether h_0 is minimal or not in the positive recurrent case.*

We now prove Theorem 3.1.

Proof of Theorem 3.1. In the case where $\pi_0 = 0$, we have $h_0(x) = x$ by Theorem 4.1. Hence, by Theorem 6.4, we see that $n[X_t] \to 1$ as $t \downarrow 0$, which shows $C = 1$ in this case.

In the case where $\pi_0 > 0$, we obtain $C = 1$ by Theorem 6.5 and Lemma 3.2. The proof is therefore complete. □

7 The h-transforms of the stopped process

We study h-transforms in this section. For a measure μ and a function f, we write $f\mu$ for the measure defined by $f\mu(A) = \int_A f \mathrm{d}\mu$.

Since h_* is γ_*-invariant, there exists a conservative strong Markov process $M^{h_*} = \{(X_t)_{t \geq 0}, (\mathbb{P}^{h_*}_x)_{x \in I}\}$ such that

$$\mathbb{P}^{h_*}_x = \frac{\mathrm{e}^{-\gamma_* t} h_*(X_t)}{h_*(x)}\mathbb{P}^0_x \quad \text{on } \mathscr{F}_t \text{ for } t > 0 \text{ and } x \in I \setminus \{0\}, \tag{7.1}$$

$$\mathbb{P}^{h_*}_0 = \mathrm{e}^{-\gamma_* t} h(X_t) n \quad \text{on } \mathscr{F}_t \text{ for } t > 0. \tag{7.2}$$

We set

$$m^{h_*}(x) = \int_{(0,x]} h_*(y)^2 \widetilde{m}(\mathrm{d}y), \quad s^{h_*}(x) = \int_c^x \frac{\mathrm{d}y}{h_*(y)^2}, \tag{7.3}$$

where $0 < c < l'$ is a fixed constant, We define, for $q > 0$,

$$r_q^{h_*}(x,y) = \begin{cases} \dfrac{r^0_{q+\gamma_*}(x,y)}{h(x)h(y)} & \text{for } x,y \in I \setminus \{0\}, \\ \dfrac{r_{q+\gamma_*}(0,y)}{h(y)r_{q+\gamma_*}(0,0)} & \text{for } x = 0 \text{ and } y \in I \setminus \{0\}. \end{cases} \tag{7.4}$$

Then, we see that $r_q^{h_*}(x,y)$ is a density of the resolvent $R_q^{h_*}$ for M^{h_*}.

Theorem 7.1 *For M^{h_*}, the following assertions hold:*

(i) For $q > 0$, $\phi_q^{h_*} = \frac{\psi_{q+\gamma_*}}{h_*}$ (resp. $\rho_q^{h_*} = \frac{\rho_{q+\gamma_*}}{h_*}$) is an increasing (resp. decreasing) solution of $D_{m^{h_*}}D_{s^{h_*}}f = qf$ satisfying $f(0) = 1$ and $D_{s^{h_*}}f(0) = 0$ (resp. $f(0) = \infty$ and $D_{s^{h_*}}f(0) = -1$).
(ii) M^{h_*} is the $D_{m^{h_*}}D_{s^{h_*}}$-diffusion.
(iii) 0 for M^{h_*} is entrance.
(iv) l' for M^{h_*} is entrance when l' for M is entrance;
l' for M^{h_*} is regular-reflecting when l' for M is regular-reflecting.

Proof. (i) For $q \in \mathbb{R}$ and for any function h such that D_mD_sh exists, we see that

$$D_{m^h}D_{s^h}\left(\frac{\psi_{q+\alpha}}{h}\right) = \frac{1}{h^2}D_m\left\{h^2D_s\left(\frac{\psi_{q+\alpha}}{h}\right)\right\} \tag{7.5}$$

$$= \frac{1}{h^2}D_m\left\{hD_s\psi_{q+\alpha} - \psi_{q+\alpha}D_sh\right\} \tag{7.6}$$

$$= \left(q + \alpha - \frac{D_mD_sh}{h}\right)\frac{\psi_{q+\alpha}}{h}. \tag{7.7}$$

Taking $h = h_*$ and $\alpha = \gamma_*$. we obtain $D_{m^{h_*}}D_{s^{h_*}}\phi_q^{h_*} = q\phi_q^{h_*}$. In the same way we obtain $D_{m^{h_*}}D_{s^{h_*}}\rho_q^{h_*} = q\rho_q^{h_*}$. The initial conditions can be obtained easily.

Claims (ii) and (iii) are obvious from (i).

(iv) Suppose that l' for M is entrance or regular-reflecting. Then h_* is bounded, so that l' for M^{h_*} is of the same classification as l' for M. Since M^{h_*} is conservative, we obtain the desired result. □

We now develop a general theory for the h-transform with respect to an excessive function. Let $\alpha \geq 0$ and let h be a function on I which is α-excessive for M^0 and n which is positive on $I \setminus \{0\}$. Then it is well-known (see, e.g., [5, Theorem 11.9]) that there exists a (possibly non-conservative) strong Markov process $M^h = \{(X_t)_{t\geq 0}, (\mathbb{P}_x^h)_{x\in I}\}$ such that

$$1_{\{t<\zeta\}}\mathbb{P}_x^h = \frac{e^{-\alpha t}h(X_t)}{h(x)}\mathbb{P}_x^0 \quad \text{on } \mathscr{F}_t \text{ for } t > 0 \text{ and } x \in I \setminus \{0\}, \tag{7.8}$$

$$1_{\{t<\zeta\}}\mathbb{P}_0^h = e^{-\alpha t}h(X_t)n \quad \text{on } \mathscr{F}_t \text{ for } t > 0. \tag{7.9}$$

We note that M^h becomes a diffusion when killed upon hitting l if $l \in I$. If $\alpha \geq 0$, we see by [5, Theorem 11.9] that the identities (7.8) and (7.9) are still valid if we replace the constant time t by a stopping time T and restrict both sides on $\{T<\infty\}$. We set

$$m^h(x) = \int_{(0,x]} h(y)^2 \widetilde{m}(\mathrm{d}y), \quad s^h(x) = \int_c^x \frac{\mathrm{d}y}{h(y)^2}, \tag{7.10}$$

where $0<c<l'$ is a fixed constant, We define, for $q>0$,

$$r_q^h(x,y) = \begin{cases} \dfrac{r_{q+\alpha}^0(x,y)}{h(x)h(y)} & \text{for } x,y \in I\setminus\{0\}, \\ \dfrac{r_{q+\alpha}(0,y)}{h(y)r_{q+\alpha}(0,0)} & \text{for } x=0 \text{ and } y\in I\setminus\{0\}. \end{cases} \tag{7.11}$$

Then, we see that $r_q^h(x,y)$ is a density of the resolvent R_q^h for M^h.

Lemma 7.2 *Suppose that $h(x)\le \psi_{q+\alpha}(x)$ for all $q>0$ and all $x\in I$. Define $L_t^h(y) = L_t(y)/h(y)^2$ for $y\in I\setminus\{0\}$. Then the process $(L_t^h(y))_{t\ge 0}$ is the local time at y for M^h normalized as*

$$\mathbb{P}_x^h\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right] = r_q^h(x,y), \quad x\in I,\ y\in I\setminus\{0\}. \tag{7.12}$$

It also holds that

$$\mathbb{P}_x^h\left[\mathrm{e}^{-qT_y}\right] = \frac{r_q^h(x,y)}{r_q^h(y,y)}, \quad x\in I,\ y\in I\setminus\{0\}. \tag{7.13}$$

Proof. Since $\mathbb{P}_x^h$ is locally absolutely continuous with respect to $\mathbb{P}_x^0$, we see that $(L_t^h(y))_{t\ge 0}$ is the local time at y for M^h. Let $x,y\in I\setminus\{0\}$. For $u\ge 0$, we note that $\eta_u(y) = \inf\{t>0 : L_t(y)>u\}$ is a stopping time and that $X_{\eta_u(y)} = y$ if $\eta_u(y)<\infty$. Let $0=u_0<u_1<\ldots<u_n$. Then, by the strong Markov property, we have

$$\mathbb{P}_x^h\left[\int_{\eta_{u_{j-1}}(y)}^{\eta_{u_j}(y)} f(t)\mathrm{d}L_t^h(y)\right] = \frac{1}{h(x)h(y)}\mathbb{P}_x^0\left[\mathrm{e}^{-\alpha\eta_{u_j}(y)}\int_{\eta_{u_{j-1}}(y)}^{\eta_{u_j}(y)} f(t)\mathrm{d}L_t(y)\right]; \tag{7.14}$$

in fact, we have (7.14) with restriction on $\{\eta_{u_n}(y)\le T_{\varepsilon x}\}$ and then we obtain (7.14) by letting $\varepsilon\downarrow 0$. Hence, by the monotone convergence theorem, we obtain

$$\mathbb{P}_x^h\left[\int_0^\infty f(t)\mathrm{d}L_t^h(y)\right] = \frac{1}{h(x)h(y)}\mathbb{P}_x^0\left[\int_0^\infty \mathrm{e}^{-\alpha t}f(t)\mathrm{d}L_t(y)\right]. \tag{7.15}$$

Letting $f(t)=\mathrm{e}^{-qt}$, we obtain (7.12) for $x\in I\setminus\{0\}$.

Let $x=0$ and $y\in I\setminus\{0\}$. For $p>0$, we write e_p for an independent exponential time of parameter p. By the strong Markov property, we have

$$\mathbb{P}_0^h\left[\int_{e_p}^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right] = \mathbb{P}_0^h\left[\mathrm{e}^{-qe_p}\, r_q^h(X_{e_p},y)\right]. \tag{7.16}$$

On one hand, we have

$$(7.16) \le \mathbb{P}_0^h\left[r_q^h(X_{e_p},y)\right] = p\int_I r_q^h(0,x)r_q^h(x,y)m^h(\mathrm{d}x) \tag{7.17}$$

$$= \frac{p}{p-q}\cdot\left\{r_q^h(0,y) - r_p^h(0,y)\right\} \underset{p\to\infty}{\longrightarrow} r_q^h(0,y). \tag{7.18}$$

On the other hand, since we have $h(x) \le \psi_{q+\alpha}(x)$, we have

$$(7.16) \ge \mathbb{P}_0^h\left[\mathrm{e}^{-qe_p}; e_p < T_y\right]\frac{\rho_{q+\alpha}(y)}{h(y)} \underset{p\to\infty}{\longrightarrow} r_q^h(0,y). \tag{7.19}$$

By the monotone convergence theorem, we obtain (7.12) for $x = 0$.

Using (7.12) and using the strong Markov property, we obtain

$$\mathbb{P}_x^h\left[\mathrm{e}^{-qT_y}\right] = \frac{\mathbb{P}_x^h\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right]}{\mathbb{P}_y^h\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right]} = \frac{r_q^h(x,y)}{r_q^h(y,y)}. \tag{7.20}$$

This shows (7.13). □

Theorem 7.3 *For M^s, i.e., the h-transform for $h = s$, the following assertions hold:*

(i) For $q > 0$, $\phi_q^s = \frac{\psi_q}{s}$ (resp. $\rho_q^s = \frac{\rho_q}{s}$) is an increasing (resp. decreasing) solution of $D_{m^s}D_{s^s}f = qf$ satisfying $f(0) = 1$ and $D_{s^s}f(0) = 0$ (resp. $f(0) = \infty$ and $D_{s^s}f(0) = -1$).

(ii) M^s is the $D_{m^s}D_{s^s}$-diffusion.

(iii) 0 for M^s is entrance.

(iv) l' for M^s is of the same classification as l' for M when $l' < \infty$, i.e., l' for M is exit, regular-absorbing, regular-elastic or type-3-natural;
l' for M^s is type-3-natural when l' for M is natural;
l' for M^s is exit when $l'(=\infty)$ for M is entrance with $\int_c^\infty x^2\mathrm{d}m(x) = \infty$;
l' for M^s is regular-absorbing when $l'(=\infty)$ for M is entrance with $\int_c^\infty x^2\mathrm{d}m(x) < \infty$;
l' for M^s is regular-elastic when l' for M is regular-reflecting.

Proof. Claim (i) can be obtained in the same way as the proof of (i) of Theorem 7.1. Claims (ii) and (iii) are obvious from (i).

(iv) Suppose l' for M is exit, regular-absorbing or regular-elastic. Then we have $l' < \infty$, and hence it is obvious that l' for M^s is of the same classification as l' for M.

Suppose l' for M is natural. Then we have

$$\iint_{l'>y>x>c}\mathrm{d}m^s(x)\mathrm{d}s^s(y) = \int_{l'>x>c} x\mathrm{d}m(x) \ge \iint_{l'>y>x>c}\mathrm{d}x\mathrm{d}m(y) = \infty \tag{7.21}$$

and

$$\iint_{l'>y>x>c} \mathrm{d}s^s(x)\mathrm{d}m^s(y) = \int_{l'>y>c} \left(\frac{1}{c} - \frac{1}{y}\right) y^2 \mathrm{d}m(y) \geq \int_{l'>y>2c} y\mathrm{d}m(y) = \infty. \tag{7.22}$$

Thus we see that l' for M^s is natural. Since $s^s(l') = 1/c - 1/l' < \infty$, we see that l' for M^s is type-3-natural.

Suppose $l'(= \infty)$ for M is entrance. Then we have

$$\iint_{\infty>y>x>c} \mathrm{d}m^s(x)\mathrm{d}s^s(y) = \iint_{\infty>y>x>c} \mathrm{d}x\mathrm{d}m(y) + c\{m(\infty) - m(c)\} < \infty. \tag{7.23}$$

In addition, we have

$$\iint_{\infty>y>x>c} \mathrm{d}s^s(x)\mathrm{d}m^s(y) = \int_{\infty>y>c} \left(\frac{1}{c} - \frac{1}{y}\right) y^2 \mathrm{d}m(y), \tag{7.24}$$

which is finite if and only if $\int_c^\infty x^2 \mathrm{d}m(x)$ is finite.

Suppose l' for M is regular-reflecting. Then it is obvious that l' for M^s is regular. Since M^s has no killing inside $[0, l')$ and since M^s is not conservative, we see that M^s has killing at l'. Since we have

$$\mathbb{P}^s_{l'}(T_x < \zeta) = \frac{x}{l'} \mathbb{P}^0_{l'}(T_x < T_0) = \frac{x}{l'} < 1 \quad \text{for all } x < l', \tag{7.25}$$

we see that M^s has killing at l'. Thus we see that l' for M^s is regular-elastic. $\square$

Remark 7.4 *When $l' = \infty$ and $\int_{\infty>x>c} x^2 \mathrm{d}m(x) < \infty$, the left boundary ∞ is called of* limit circle type. *See Kotani [15] for the spectral analysis involving Herglotz functions.*

Theorem 7.5 *Suppose l' for M is entrance or regular-reflecting. For M^s, it holds that*

$$\mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = \frac{\psi_q(x)}{x} \chi_q(l'), \quad q > 0,\ x \in I' \setminus \{0\}, \tag{7.26}$$

where $\chi_q(l')$ is given by (6.31).

Proof. Suppose l' is entrance. Then we have

$$\mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = \lim_{y \uparrow l'} \mathbb{P}^s_x[\mathrm{e}^{-qT_y}] = \lim_{y \uparrow l'} \frac{y}{x} \cdot \frac{\psi_q(x)}{\psi_q(y)}. \tag{7.27}$$

By [12, Theorem 5.13.3], we have

$$\lim_{y \uparrow l'} \frac{y}{\psi_q(y)} = \lim_{y \uparrow l'} \frac{1}{\psi'_q(y)} = \rho_q(l') = \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}]. \tag{7.28}$$

Suppose $l'(= \infty)$ is regular-reflecting. Then we have

$$\mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = \mathbb{P}^s_x[\mathrm{e}^{-qT_{l'}}]\mathbb{P}^s_{l'}[\mathrm{e}^{-q\zeta}] = \frac{r^s_q(x,l')}{r^s_q(l',l')} \cdot \frac{1}{l'} R^0_q s(l') \tag{7.29}$$

$$= \frac{\psi_q(x)}{x} \cdot \frac{\rho_q(l')}{\psi_q(l')} \cdot \int_{(0,l']} \psi_q(x) x \mathrm{d}m(x). \tag{7.30}$$

Since $D_m\{\psi'_q(x)x - \psi_q(x)\} = q\psi_q(x)x$, we obtain

$$\int_{(0,l']} \psi_q(x) x \mathrm{d}m(x) = \frac{1}{q}\left\{\psi'_q(l')l' - \psi_q(l')\right\} = \frac{1}{q}\left\{\frac{l'}{\rho_q(l')} - \psi_q(l')\right\}. \tag{7.31}$$

Thus we obtain (7.26). □

We now give the proof of (6.28).

Proof of (6.28). Note that

$$1 - \mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = q\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-qt}\mathbb{P}^s_x(\zeta > t) = \frac{q}{x}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-qt}\mathbb{P}^0_x[X_t] = \frac{1}{x} q R^0_q s(x). \tag{7.32}$$

Combining this fact with (7.26), we obtain (6.28). □

Theorem 7.6 *For M^{h_0}, i.e., the h-transform for $h = h_0$, the following assertions hold:*

(i) For $q > 0$, $\phi^{h_0}_q = \frac{\psi_q}{h_0}$ (resp. $\rho^{h_0}_q = \frac{\rho_q}{h_0}$) is an increasing (resp. decreasing) solution of $D_{m^{h_0}} D_{s^{h_0}} f = qf$ satisfying $f(0) = 1$ and $D_{s^{h_0}} f(0) = 0$ (resp. $f(0) = \infty$ and $D_{s^{h_0}} f(0) = -1$).

(ii) M^{h_0} is the $D_{m^{h_0}} D_{s^{h_0}}$-diffusion with killing measure $\frac{\pi_0}{h_0}\mathrm{d}m^{h_0}$.

(iii) 0 for M^{h_0} is entrance;

(iv) l' for M^{h_0} is natural when l' for M is type-2-natural;
l' for M^{h_0} is entrance when l' for M is entrance;
l' for M^{h_0} is regular when l' for M is regular-reflecting.

(For the boundary classifications for diffusions with killing measure, see, e.g., [13, Chapter 4].)

Proof. Claim (i) can be obtained in the same way as the proof of (i) of Theorem 7.1.

(ii) For $f = \frac{\psi_q}{h_0}$ or $f = \frac{\rho_q}{h_0}$, we have

$$\left(D_{m^{h_0}} D_{s^{h_0}} - \frac{\pi_0}{h_0}\right) f = qf, \tag{7.33}$$

since $D_m D_s h_0 = -\pi_0$. This shows (ii).

Claim (iii) is obvious from (i).

(iv) Suppose l' for M is type-2-natural. Then it is obvious that $\lim_{x\uparrow l'} \rho^{h_0}_q(x) = 0$. Since we have $D_m\{h_0\rho'_q - \rho_q h'_0\} = (qh_0 + \pi_0)\rho_q$, we have

$$D_{s^{h_0}} \rho^{h_0}_q(x) = (h_0\rho'_q - \rho_q h'_0)(x) = -1 + \int_{(0,x]} (qh_0(x) + \pi_0)\rho_q(x)\mathrm{d}m(x). \tag{7.34}$$

Hence, by Proposition 6.3, we obtain

$$\lim_{x\uparrow l'} D_{s^{h_0}} \rho_q^{h_0}(x) = -1 + \frac{1}{H(q)} R_q(qh_0 + \pi_0)(0) = 0. \tag{7.35}$$

Thus we see that l' for M^{h_0} is natural.

Suppose l' for M is entrance. Note that

$$\frac{h_0(x)}{\pi_0} = x \int_{(x,\infty)} \mathrm{d}m(z) + \int_{(0,x]} z\mathrm{d}m(z). \tag{7.36}$$

Since we have $\int_{(0,\infty)} z\mathrm{d}m(z) < \infty$, we see that

$$h_0(l') := \lim_{x\uparrow l'} h_0(x) = \pi_0 \int_{(0,\infty)} z\mathrm{d}m(z) < \infty. \tag{7.37}$$

This shows that l' for M^{h_0} is of the same classification as l' for M.

The last statement is obvious. □

Remark 7.7 *General discussions related to Theorems 7.3 and 7.6 can be found in Maeno [16], Tomisaki [27] and Takemura [26].*

References

1. R. M. Blumenthal. *Excursions of Markov processes*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1992.
2. R. M. Blumenthal and R. K. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968.
3. L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.*, 10:no. 28, 948–961, 2005.
4. Z.-Q. Chen and M. Fukushima. One-point reflection. *Stochastic Process. Appl.*, 125:no. 4, 1368–1393, 2015.
5. K. L. Chung and J. B. Walsh. *Markov processes, Brownian motion, and time symmetry*, volume 249 of *Grundlehren der Mathematischen Wissenschaften*. Springer, New York, second edition, 2005.
6. R. A. Doney. *Fluctuation theory for Lévy processes*, volume 1897 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005, Edited and with a foreword by Jean Picard.
7. W. Feller. On second order differential operators. *Ann. of Math. (2)*, 61:90–105, 1955.
8. W. Feller. Generalized second order differential operators and their lateral conditions. *Illinois J. Math.*, 1:459–504, 1957.
9. P. J. Fitzsimmons and R. K. Getoor. Smooth measures and continuous additive functionals of right Markov processes. In *Itô's stochastic calculus and probability theory*, pages 31–49. Springer, Tokyo, 1996.
10. M. Fukushima. On general boundary conditions for one-dimensional diffusions with symmetry. *J. Math. Soc. Japan*, 66(1):289–316, 2014.
11. N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.

12. K. Itô. *Essentials of stochastic processes*, volume 231 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2006. Translated from the 1957 Japanese original by Yuji Ito.
13. K. Itô and H. P. McKean, Jr. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.
14. J. T. Kent. Eigenvalue expansions for diffusion hitting times. *Z. Wahrsch. Verw. Gebiete*, 52(3):309–319, 1980.
15. S. Kotani. Krein's strings with singular left boundary. *Rep. Math. Phys.*, 59(3):305–316, 2007.
16. M. Maeno. One-dimensional h-path generalized diffusion processes. *Annual reports of Graduate School of Humanities and Sciences, Nara Women's University*, 21:167–185, 2006.
17. H. P. McKean, Jr. Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.*, 82:519–548, 1956.
18. H. P. McKean, Jr. Excursions of a non-singular diffusion. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 1:230–239, 1962/1963.
19. C. Profeta. Penalization of a positively recurrent diffusion by an exponential function of its local time. *Publ. Res. Inst. Math. Sci.*, 46(3):681–718, 2010.
20. C. Profeta. Penalizing null recurrent diffusions. *Electron. J. Probab.*, 17:no. 69, 23, 2012.
21. D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition, 1999.
22. L. C. G. Rogers. Itô excursion theory via resolvents. *Z. Wahrsch. Verw. Gebiete*, 63(2):237–255, 1983. Addendum: 67(4):473–476, 1984.
23. P. Salminen. One-dimensional diffusions and their exit spaces. *Math. Scand.*, 54(2):209–220, 1984.
24. P. Salminen, P. Vallois, and M. Yor. On the excursion theory for linear diffusions. *Jpn. J. Math.*, 2(1):97–127, 2007.
25. P. Salminen and M. Yor. Tanaka formula for symmetric Lévy processes. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 265–285. Springer, Berlin, 2007.
26. T. Takemura. State of boundaries for harmonic transforms of one-dimensional generalized diffusion processes. *Annual reports of Graduate School of Humanities and Sciences, Nara Women's University*, 25:285–294, 2010.
27. M. Tomisaki. Intrinsic ultracontractivity and small perturbation for one-dimensional generalized diffusion operators. *J. Funct. Anal.*, 251(1):289–324, 2007.
28. S. Watanabe. On time inversion of one-dimensional diffusion processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 31:115–124, 1974/75.
29. K. Yano. Excursion measure away from an exit boundary of one-dimensional diffusion processes. *Publ. Res. Inst. Math. Sci.*, 42(3):837–878, 2006.
30. K. Yano. Excursions away from a regular point for one-dimensional symmetric Lévy processes without Gaussian part. *Potential Anal.*, 32(4):305–341, 2010.
31. K. Yano. Two kinds of conditionings for stable Lévy processes. In *Probabilistic approach to geometry*, volume 57 of *Adv. Stud. Pure Math.*, pages 493–503. Math. Soc. Japan, Tokyo, 2010.
32. K. Yano. On harmonic function for the killed process upon hitting zero of asymmetric Lévy processes. *J. Math-for-Ind.*, 5A:17–24, 2013.
33. K. Yano, Y. Yano, and M. Yor. Penalising symmetric stable Lévy paths. *J. Math. Soc. Japan*, 61(3):757–798, 2009.