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On the Uniqueness of Generic Representations in an L-Packet

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In this paper, we give a simple and short proof of the uniqueness of generic representations in an L-packet for a quasi-split connected classical group over a non-archimedean local field.

1 Introduction

Let G be a quasi-split connected reductive group defined over a non-archimedean local field F of characteristic zero. We denote the center of G by Z. A Whittaker datum for G is a conjugacy class of pairs $\mathfrak{w} = (B, \mu)$, where $B = TU$ is an F-rational Borel subgroup of G, T is a maximal F-torus, U is the unipotent radical of B, and μ is a generic character of $U(F)$. Here, $T(F)$ acts on $U(F)$ by conjugation, and we say that a character μ of $U(F)$ is generic if the stabilizer of μ in $T(F)$ is equal to $Z(F)$. Let $\mathrm{Irr}(G(F))$ be the set of equivalence classes of irreducible smooth representations of $G(F)$. We say that $\pi \in \mathrm{Irr}(G(F))$ is $\mathfrak{w}$-generic if $\mathrm{Hom}_{U(F)}(\pi, \mu) \neq 0$.

The local Langlands conjecture predicts a canonical partition

$$\mathrm{Irr}(G(F)) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi,$$

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where $\Phi(G)$ is the set of L-parameters of G, which are $\widehat{G}$-conjugacy classes of admissible homomorphisms

$$\phi\colon \mathrm{WD}_F \to {}^L G$$

from the Weil–Deligne group $\mathrm{WD}_F = \mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C})$ of F to the L-group ${}^L G = \widehat{G} \rtimes \mathrm{W}_F$ of G. Here, W_F is the Weil group of F and $\widehat{G}$ is the Langlands dual group of G. The set Π_ϕ is called the L-packet of ϕ. In addition, for a Whittaker datum $\mathfrak{w}$ for G, there would exist a bijection

$$\iota_{\mathfrak{w}}\colon \Pi_\phi \to \mathrm{Irr}(\mathcal{S}_\phi),$$

which satisfies certain character identities (see e.g., [15, Section 2]). Here, $\mathcal{S}_\phi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi), \widehat{G})/Z(\widehat{G})^{\mathrm{W}_F})$ is the component group of ϕ, which is a finite group.

As a relationship of L-packets and $\mathfrak{w}$-generic representations, the following is expected, and called the generic packet conjecture (or the tempered packet conjecture).

Desideratum 1.1. Let $\phi \in \Phi(G)$.

(1) If ϕ is tempered (see Section 2.2 below), then Π_ϕ contains a $\mathfrak{w}$-generic representation.

(2) If $\pi \in \Pi_\phi$ is $\mathfrak{w}$-generic, then $\iota_{\mathfrak{w}}(\pi)$ is the trivial representation of $\mathcal{S}_\phi$. □

Desideratum 1.1 (1) is a conjecture of Shahidi [28, Conjecture 9.4]. Desideratum 1.1 (2) asserts that each Π_ϕ has at most one $\mathfrak{w}$-generic representation. Hence the tempered $\mathfrak{w}$-generic representations should be canonically parameterized by the tempered L-parameters.

When $G = \mathrm{GL}_n$, the local Langlands conjecture has been established by Harris–Taylor [10], Henniart [13], and Scholze [26]. In this case, Desideratum 1.1 (1) follows from a result of Zelevinsky [36, Theorem 9.7], and (2) is trivial since Π_ϕ is a singleton for any $\phi \in \Phi(\mathrm{GL}_n)$.

When G is a quasi-split classical group, that is, G is a symplectic, special orthogonal or unitary group, the local Langlands conjecture is known by the recent works of Arthur [1] and Mok [23]. The L-packets are characterized by endoscopic character relations (ECR) [1, Theorem 2.2.1], [23, Theorem 3.2.1] and satisfy local intertwining relations (LIRs) (Theorem 2.2 below). Also, Arthur and Mok proved Desideratum 1.1 (1) by using a global argument [1, Proposition 8.3.2] and [23, Corollary 9.2.4]. Before these results, Konno showed that ECR implies Desideratum 1.1 (1) when the residual characteristic of F is not two [17, Theorem 8.4]. Jiang and Soudry [14] showed a weaker version of Desideratum 1.1 (2) for $G = \mathrm{SO}(2n+1)$, which asserts that there is a unique canonical bijection between the set of all equivalence classes of irreducible generic supercuspidal representations of $\mathrm{SO}(2n+1, F)$ and a suitable subset of $\Phi(\mathrm{SO}(2n+1))$. Their method is to prove the local converse theorem, which highly relies on the theory of the local descent. Recently, Varma [30, Corollary 6.16] extended Konno's method and established both Desideratum 1.1 (1) and (2) by using ECR even when the residual characteristic of F is equal to two.

Desideratum 1.1 is a special case of a more general conjecture on restriction problems, the so-called local Gan–Gross–Prasad conjecture [9, Conjecture 17.3]. To state this conjecture, one requires Vogan L-packets (the result of Kaletha et al. [16] and its analogue for orthogonal groups together with results of Arthur [1] and Mok [23]). This conjecture, at least for tempered L-parameters, has been proven by Waldspurger [32–35], Beuzart-Plessis [3–5], Gan–Ichino [8], and the author [2]. In the proofs, one of the main tools of Waldspurger and Beuzart-Plessis is ECR, and that of Gan–Ichino and the author is LIR.

The purpose of this paper is to give another proof of Desideratum 1.1 (2) for quasi-split classical groups. For our proof, the prerequisites are some properties of L-packets containing LIR (Theorems 2.1 and 2.2 below). The proofs in the previous works require additional inputs after assuming ECR, LIR (results in [1, 16, 23]) or the theory of the local descent, whereas, surprisingly, our proof of Desideratum 1.1 (2) is a formal consequence of LIR. The LIR is a relation between a normalized self-intertwining operator on an induced representation and the local Langlands correspondence (Theorem 2.2). In our proof, we use LIR together with Shahidi's result (Theorem 3.2), which describes the action of the intertwining operator on a canonical Whittaker functional on the induced representation. Both of LIR and Shahidi's result focus on induced representations, so that they cannot be applied directly to discrete series representations. The idea of our proof is as follows: For a given tempered $\mathfrak{w}$-generic representation π of $G(F)$, take a representation τ of $\mathrm{GL}_k(E)$ with an extension E/F, and consider the induced representation $\mathrm{Ind}_{P(F)}^{G'(F)}(\tau \boxtimes \pi)$ of a bigger group $G'(F)$ of the same type as $G(F)$. Then we can apply two statements to this induced representation. Taking several representations τ and considering the associated induced representations of several bigger groups, we obtain Desideratum 1.1 (2) for π.

Finally, we remark on the archimedean case. When F is an archimedean local field, the local Langlands conjecture for any quasi-split connected reductive group has been established by Langlands [21] himself. Desideratum 1.1 was proven by [18, 29, 31].

Notations

Let F be a non-archimedean local field of characteristic zero. For a finite extension E/F, we denote by W_E and $\mathrm{WD}_E = \mathrm{W}_E \times \mathrm{SL}_2(\mathbb{C})$ the Weil and Weil–Deligne groups of E, respectively. The normalized absolute value on E is denoted by $|\cdot|_E$.

2 Local Langlands correspondence for classical groups

The local Langlands correspondence (the local Langlands conjecture) for quasi-split classical groups has been established by Arthur [1] and Mok [23] under some assumption on the stabilization of twisted trace formulas. For this assumption, see also two books of Mœglin–Waldspurger [22]. In this section, we summarize some properties of the local Langlands correspondence which are used in this paper.

2.1 Generic representations

Let G be a quasi-split (connected) classical group, that is, G is a unitary, symplectic, or special orthogonal group. We denote by Z the center of G. A Whittaker datum for G is a conjugacy class of pairs $\mathfrak{w} = (B, \mu)$, where $B = TU$ is an F-rational Borel subgroup of G and μ is a generic character of $U(F)$. Here, $T(F)$ acts on $U(F)$ via the adjoint action so that $T(F)$ acts on the set of characters of $U(F)$. We say that a character μ of $U(F)$ is generic if the stabilizer of μ in $T(F)$ is equal to $Z(F)$.

Let $\mathrm{Irr}(G(F))$ be the set of equivalence classes of irreducible smooth representations of $G(F)$, and $\mathrm{Irr}_{\mathrm{temp}}(G(F))$ be the subset of $\mathrm{Irr}(G(F))$ consisting of classes of irreducible tempered representations. For a Whittaker datum $\mathfrak{w} = (B, \mu)$ with $B = TU$, we say that $\pi \in \mathrm{Irr}(G(F))$ is $\mathfrak{w}$-generic if

$$\mathrm{Hom}_{U(F)}(\pi, \mu) \neq 0.$$

2.2 L-parameters and component groups

Let G be a quasi-split (connected) classical group. If G is a unitary group, we denote by E the splitting field of G, which is a quadratic extension of F. If G is a symplectic group or a special orthogonal group, we set $E = F$. We denote by $\widehat{G}$ the Langlands dual group of G, and by ${}^L G = \widehat{G} \rtimes \mathrm{W}_F$ the L-group of G. An L-parameter φ of G is a $\widehat{G}$-conjugacy class of an admissible homomorphism

$$\varphi \colon \mathrm{WD}_F \to {}^L G.$$

When G is a symplectic or special orthogonal group, there is a standard representation ${}^LG \to \mathrm{GL}_N(\mathbb{C})$ for a suitable N (see [9, Section 7]). By composing an L-parameter φ with this map, we obtain a self-dual representation ϕ of $\mathrm{WD}_F = \mathrm{WD}_E$. When G is a unitary group, the dual group $\widehat{G}$ is isomorphic to $\mathrm{GL}_N(\mathbb{C})$ for some N, and the action of W_F on $\widehat{G}$ factors through $\mathrm{Gal}(E/F)$. By composing the restriction of an L-parameter φ to WD_E with the projection map ${}^LG \to \widehat{G} = \mathrm{GL}_N(\mathbb{C})$, we obtain a conjugate self-dual representation ϕ of WD_E. For more precisions, see [9, Sections 3 and 8]. We denote by $\Phi(G)$ the set of equivalence classes of (conjugate) self-dual representations of WD_E of suitable type and determinant as in [9, Theorem 8.1]. Then $\varphi \mapsto \phi$ gives a surjective map

$$\{L\text{-parameters of } G\} \to \Phi(G),$$

which is bijective unless $G = \mathrm{SO}(2n)$ in which case, the cardinality of each fiber of this map is one or two.

We say that $\phi \in \Phi(G)$ is tempered if $\phi(\mathrm{W}_E)$ is bounded. We denote by $\Phi_{\mathrm{temp}}(G)$ the subset of $\Phi(G)$ consisting of classes of tempered representations.

If $\phi \in \Phi(G)$ is given by an L-parameter φ, we define the component group $\mathcal{S}_\phi$ of ϕ by

$$\mathcal{S}_\phi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\varphi), \widehat{G})/Z(\widehat{G})^{\mathrm{W}_F}).$$

Here, $Z(\widehat{G})$ is the center of $\widehat{G}$. Note that $\mathcal{S}_\phi$ does not depend on the choice of φ, and $\mathcal{S}_\phi$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some non-negative integer r. As in [9, Section 4], $\mathcal{S}_\phi$ is described explicitly as follows: We denote by $\mathcal{B}_\phi$ the set of equivalence classes of representations ϕ' of WD_E such that

- ϕ' is contained in ϕ;
- ϕ' is a multiplicity-free sum of irreducible (conjugate) self-dual representations of WD_E of the same type as ϕ.

Also we put

$$\mathcal{B}_\phi^+ = \begin{cases} \mathcal{B}_\phi & \text{if } G = \mathrm{U}(m) \text{ or } G = \mathrm{SO}(2n+1), \\ \{\phi' \in \mathcal{B}_\phi \mid \dim(\phi') \in 2\mathbb{Z}\} & \text{if } G = \mathrm{Sp}(2n) \text{ or } G = \mathrm{SO}(2n). \end{cases}$$

When ϕ_0 is an irreducible representation of WD_E, the multiplicity of ϕ_0 in ϕ is denoted by $m(\phi_0; \phi)$, that is,

$$m(\phi_0; \phi) = \dim \mathrm{Hom}(\phi_0, \phi).$$

For $\phi_1, \phi_2 \in \mathcal{B}_\phi$, we write $\phi_1 \sim \phi_2$ if

$$m(\phi_0; \phi_1) + m(\phi_0; \phi_2) \equiv m(\phi_0; \phi) \bmod 2$$

for any irreducible (conjugate) self-dual representation ϕ_0 of WD_E of the same type as ϕ. For $a \in \mathcal{S}_\phi$, choose a semisimple representative $s \in \mathrm{Cent}(\mathrm{Im}(\varphi), \widehat{G})$ of a. We regard s as an automorphism on the space of ϕ, which commutes with the action of WD_E. We denote by $\phi^{s=-1}$ the (-1)-eigenspace of s, which is a representation of WD_E. We define $\phi^a \in \mathcal{B}_\phi^+$ so that

$$m(\phi_0; \phi^a) \equiv m(\phi_0; \phi^{s=-1}) \bmod 2$$

for any irreducible (conjugate) self-dual representation ϕ_0 of WD_E of the same type as ϕ. Then the image of ϕ^a in $\mathcal{B}_\phi^+/\sim$ does not depend on the choice of s, and the map

$$\mathcal{S}_\phi \to \mathcal{B}_\phi^+/\sim,\ a \mapsto \phi^a$$

is bijective.

2.3 Local Langlands correspondence for classical groups

In this subsection, we introduce $\Pi(G)$, which is a quotient of $\mathrm{Irr}(G(F))$, and state some properties of the local Langlands correspondence which we need.

First, we consider an even special orthogonal group $G = \mathrm{SO}(2n)$. Choose $\varepsilon \in \mathrm{O}(2n, F) \setminus \mathrm{SO}(2n, F)$. For $\pi \in \mathrm{Irr}(\mathrm{SO}(2n, F))$, its conjugate π^ε is defined by $\pi^\varepsilon(h) = \pi(\varepsilon^{-1} h \varepsilon)$ for $h \in \mathrm{SO}(2n, F)$. We define an equivalence relation $\sim_\varepsilon$ on $\mathrm{Irr}(\mathrm{SO}(2n, F))$ by

$$\pi \sim_\varepsilon \pi^\varepsilon$$

for $\pi \in \mathrm{Irr}(\mathrm{SO}(2n, F))$. In [1], Arthur has parameterized not $\mathrm{Irr}(\mathrm{SO}(2n, F))$ but $\mathrm{Irr}(\mathrm{SO}(2n, F))/\sim_\varepsilon$. Note that π is tempered (respectively, $\mathfrak{w}$-generic) if and only if so is π^ε.

We return to the general setting. Let G be a quasi-split (connected) classical group. We define $\Pi(G)$ by

$$\Pi(G) = \begin{cases} \mathrm{Irr}(G(F))/\sim_\varepsilon & \text{if } G = \mathrm{SO}(2n), \\ \mathrm{Irr}(G(F)) & \text{otherwise.} \end{cases}$$

For $\pi \in \mathrm{Irr}(G(F))$, we denote the image of π under the canonical map $\mathrm{Irr}(G(F)) \to \Pi(G)$ by $[\pi]$. We say that $[\pi] \in \Pi(G)$ is $\mathfrak{w}$-generic (respectively, tempered) if so is any representative π. Also, we put $\Pi_{\mathrm{temp}}(G)$ to be the image of $\mathrm{Irr}_{\mathrm{temp}}(G(F))$ in $\Pi(G)$.

Now we are ready to describe the local Langlands correspondence for G, which has been established by Arthur [1] and Mok [23].

Theorem 2.1. Let G be a quasi-split (connected) classical group. We fix a Whittaker datum $\mathfrak{w}$ for G.

(1) There exists a canonical surjection (not depending on $\mathfrak{w}$)

$$\Pi(G) \to \Phi(G).$$

For $\phi \in \Phi(G)$, we denote by Π_ϕ the inverse image of ϕ under this map, and call Π_ϕ the L-packet of ϕ.

(2) There exists a bijection (depending on $\mathfrak{w}$)

$$\iota_{\mathfrak{w}} : \Pi_\phi \to \widehat{\mathcal{S}_\phi},$$

which satisfies certain character identities. Here, $\widehat{\mathcal{S}_\phi}$ is the Pontryagin dual of $\mathcal{S}_\phi$.

(3) We have

$$\Pi_{\mathrm{temp}}(G) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G)} \Pi_\phi.$$

(4) Assume that $\phi = \phi_\tau \oplus \phi_0 \oplus {}^c\phi_\tau^\vee$, where
- ϕ_0 is an element in $\Phi_{\mathrm{temp}}(G_0)$ with a classical group G_0 of the same type as G;
- ϕ_τ is a tempered representation of WD_E of dimension k.

Here, ${}^c\phi_\tau$ is the Galois conjugate of ϕ_τ (see [9, Section 3]). Let τ be the irreducible tempered representation of $\mathrm{GL}_k(E)$ corresponding to ϕ_τ. Then for a representative π_0 of an element in Π_{ϕ_0}, the induced representation

$$\mathrm{Ind}_{P(F)}^{G(F)}(\tau \otimes \pi_0)$$

decomposes into a direct sum of irreducible tempered representations of $G(F)$, where $P = M_P U_P$ is a parabolic subgroup of G with Levi subgroup

$M_P(F) = \mathrm{GL}_k(E) \times G_0(F)$. The L-packet Π_ϕ is given by

$$\Pi_\phi = \{[\pi] \mid \pi \subset \mathrm{Ind}_{P(F)}^{G(F)}(\tau \otimes \pi_0), [\pi_0] \in \Pi_{\phi_0}\}.$$

Moreover, there is a canonical inclusion $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_\phi$. If π is a direct summand of $\mathrm{Ind}_{P(F)}^{G(F)}(\tau \otimes \pi_0)$ and $\mathfrak{w}_0$ is the Whittaker datum for G_0 given by the restriction of $\mathfrak{w}$, then $\iota_{\mathfrak{w}}([\pi])|\mathcal{S}_{\phi_0} = \iota_{\mathfrak{w}_0}([\pi_0])$.

(5) Assume that

$$\phi = \phi_1|\cdot|^{s_1} \oplus \cdots \oplus \phi_r|\cdot|^{s_r} \oplus \phi_0 \oplus {}^c\phi_r^\vee|\cdot|^{-s_r} \oplus \cdots \oplus {}^c\phi_1^\vee|\cdot|^{-s_1},$$

where

- ϕ_0 is an element in $\Phi_{\mathrm{temp}}(G_0)$ with a classical group G_0 of the same type as G;
- ϕ_i is a tempered representation of WD_E of dimension k_i for $1 \leq i \leq r$;
- s_i is a real number such that $s_1 \geq \cdots \geq s_r > 0$.

Let τ_i be the irreducible tempered representation of $\mathrm{GL}_{k_i}(E)$ corresponding to ϕ_i. Then the L-packet Π_ϕ consists of (the equivalence classes of) the unique irreducible quotient π of the standard module

$$\mathrm{Ind}_{P(F)}^{G(F)}(\tau_1|\det|_E^{s_1} \otimes \cdots \otimes \tau_r|\det|_E^{s_r} \otimes \pi_0),$$

where π_0 runs over (representatives of) elements in Π_{ϕ_0}. Here, $P = M_P U_P$ is a parabolic subgroup of G with Levi subgroup $M_P(F) = \mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_r}(E) \times G_0(F)$. Moreover, there is a canonical inclusion $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_\phi$, which is in fact bijective. If π is the unique irreducible quotient of the above standard module and $\mathfrak{w}_0$ is the Whittaker datum for G_0 given by the restriction of $\mathfrak{w}$, then $\iota_{\mathfrak{w}}([\pi])|\mathcal{S}_{\phi_0} = \iota_{\mathfrak{w}_0}([\pi_0])$. □

2.4 Local intertwining relations

The purpose of this paper is to give a simple and short proof of Desideratum 1.1 (2) when G is a quasi-split classical group. To do this, we need one more technical result, the so-called local intertwining relation (LIR). This is a relation between a normalized self-intertwining operator and the local Langlands correspondence.

Let G be a quasi-split (connected) classical group and $\mathfrak{w} = (B, \mu)$ be a Whittaker datum for G. We denote the unipotent radical of B by U. Fix a positive integer k, and put

$G' = \mathrm{U}(m+2k)$, $\mathrm{Sp}(2n+2k)$ or $\mathrm{SO}(m+2k)$ when $G = \mathrm{U}(m)$, $\mathrm{Sp}(2n)$ or $\mathrm{SO}(m)$, respectively. If $G = \mathrm{Sp}(2n)$ or $G = \mathrm{SO}(2n)$, we assume that k is even. Let $\mathfrak{w}' = (B', \mu')$ be a Whittaker datum for G' such that $B = B' \cap G$ and $\mu = \mu'|U(F)$. We consider a maximal F-parabolic subgroup $P = M_P U_P$ of G' containing B' such that the Levi subgroup M_P of P is of the form $M_P(F) \cong \mathrm{GL}_k(E) \times G(F)$. Here U_P is the unipotent radical of P, so that U_P is contained in the unipotent radical U' of B'. We denote by δ_P the modulus character of P. Let π (respectively, τ) be an irreducible tempered representation of $G(F)$ (respectively, $\mathrm{GL}_k(E)$) on a space $\mathcal{V}_\pi$ (respectively, $\mathcal{V}_\tau$). We consider the normalized induction

$$I_0(\tau \boxtimes \pi) = \mathrm{Ind}_{P(F)}^{G'(F)}(\tau \boxtimes \pi),$$

which consists of smooth functions $f_0 \colon G'(F) \to \mathcal{V}_\tau \otimes \mathcal{V}_\pi$ such that

$$f_0(uagg') = \delta_P^{\frac{1}{2}}(a)(\tau(a) \boxtimes \pi(g)) f_0(g')$$

for $u \in U_P(F)$, $a \in \mathrm{GL}_k(E)$, $g \in G(F)$ and $g' \in G'(F)$.

We denote by A_P the split component of the center of M_P and by $W(M_P) = \mathrm{Norm}(A_P, G')/M_P$ the relative Weyl group for M_P. Note that $W(M_P) \cong \mathbb{Z}/2\mathbb{Z}$ (unless G is the split $\mathrm{SO}(2)$ and $k = 1$). Let $w \in W(M_P)$ be the unique non-trivial element which induces an automorphism of $M_P(F) \cong \mathrm{GL}_k(E) \times G(F)$ whose restriction to $G(F)$ is trivial. Fixing a splitting of G' and a non-trivial additive character of F which give the Whittaker datum $\mathfrak{w}'$, we obtain a representative $\widetilde{w} \in G'(F)$ of w as in [1, Section 2.3] and [23, Section 3.3].

Now suppose that $w(\tau \boxtimes \pi) \cong \tau \boxtimes \pi$, where $w(\tau \boxtimes \pi)(m) = (\tau \boxtimes \pi)(\widetilde{w}^{-1} m \widetilde{w})$ for $m \in M_P(F)$. Then Arthur [1, Section 2.3] and Mok [23, Section 3.3] have defined a normalized intertwining operator

$$R_{\mathfrak{w}'}(w, \tau \boxtimes \pi) \colon I_0(\tau \boxtimes \pi) \to I_0(w(\tau \boxtimes \pi))$$

which depends on the Whittaker datum $\mathfrak{w}'$ for G'. Let $\widetilde{\tau \boxtimes \pi}(\widetilde{w}) \colon \mathcal{V}_\tau \otimes \mathcal{V}_\pi \to \mathcal{V}_\tau \otimes \mathcal{V}_\pi$ be the unique linear map satisfying

- $\widetilde{\tau \boxtimes \pi}(\widetilde{w}) \circ (\tau \boxtimes \pi)(\widetilde{w}^{-1} m \widetilde{w}) = (\tau \boxtimes \pi)(m) \circ \widetilde{\tau \boxtimes \pi}(\widetilde{w})$;
- $\widetilde{\tau \boxtimes \pi}(\widetilde{w}) = \widetilde{\tau}(\widetilde{w}) \otimes \widetilde{\pi}(\widetilde{w})$, where $\widetilde{\pi}(\widetilde{w}) \colon \mathcal{V}_\pi \to \mathcal{V}_\pi$ is the identity map, and $\widetilde{\tau}(\widetilde{w}) \colon \mathcal{V}_\tau \to \mathcal{V}_\tau$ is the unique linear map which preserves a Whittaker functional on $\mathcal{V}_\tau$ (with respect to the Whittaker datum for $\mathrm{GL}_k(E)$ given by the restriction of $\mathfrak{w}'$).

Note that τ is generic since τ is a tempered representation of $\mathrm{GL}_k(E)$. Finally, we define a normalized self-intertwining operator $R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi})\colon I_0(\tau \boxtimes \pi) \to I_0(\tau \boxtimes \pi)$ by

$$R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi}) = \widetilde{\tau \boxtimes \pi}(\widetilde{w}) \circ R_{\mathfrak{w}'}(w, \tau \boxtimes \pi).$$

Suppose that $[\pi] \in \Pi_\phi$ for $\phi \in \Phi_{\mathrm{temp}}(G)$. Let ϕ_τ be the tempered representation of WD_E corresponding to τ. Note that ${}^c\phi_\tau^\vee \cong \phi_\tau$ since $w(\tau \boxtimes \pi) \cong \tau \boxtimes \pi$ so that τ is (conjugate) self-dual. Put $\phi' = \phi_\tau \oplus \phi \oplus \phi_\tau \in \Phi_{\mathrm{temp}}(G')$. Let π' be an irreducible direct summand of $I_0(\tau \otimes \pi)$. Then we have $[\pi'] \in \Pi_{\phi'}$.

For the proof of Desideratum 1.1 (2), we use the following extra property, which is due to Arthur [1] and Mok [23]. This property plays an important role in the proof of the local Langlands correspondence (Theorem 2.1).

Theorem 2.2 (LIR)**.** Let the notations be as above. Assume that $\phi_\tau \in \mathcal{B}^+_{\phi'}$, so that there exists a unique element $a \in \mathcal{S}_{\phi'}$ such that $\phi'^a = \phi_\tau$ in $\mathcal{B}^+_{\phi'}/\sim$. Then

$$R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi})|\pi' = \iota_{\mathfrak{w}'}([\pi'])(a) \cdot \mathrm{id}. \qquad \square$$

The LIR immediately follows from Theorems 2.2.1, 2.2.4 in [1] when $G = \mathrm{Sp}(2n)$ or $G = \mathrm{SO}(m)$ and Theorems 3.2.1, 3.4.3 in [23] when $G = \mathrm{U}(m)$. For the convenience of the reader, we shall explain how these theorems imply Theorem 2.2.

Proof of Theorem 2.2. We prove only the case when $G = \mathrm{SO}(2n)$. The other cases are similar.

Suppose that $G = \mathrm{SO}(2n)$. In this case, since ϕ_τ and ϕ are orthogonal representations, there are non-degenerated symmetric matrices A and B of size k and $2n$ such that

$${}^t\phi_\tau(x) A \phi_\tau(x) = A \quad \text{and} \quad {}^t\phi(x) B \phi(x) = B$$

for $x \in \mathrm{WD}_F$, respectively. We regard $\widehat{G'} = \mathrm{SO}(2n+2k, \mathbb{C})$ as the special orthogonal group with respect to the symmetric matrix $\mathrm{diag}(A, B, -A)$. The image of $\phi' = \phi_\tau \oplus \phi \oplus \phi_\tau$ is contained in $\mathrm{O}(2n+2k, \mathbb{C})$. Let $\{e'_1, \dots, e'_k, e_1, \dots, e_{2n}, e''_1, \dots, e''_k\}$ be the canonical basis of $\mathbb{C}^{2n+2k}$. Then $\widehat{M}_P$ is realized as the Levi subgroup of $\widehat{G'}$ stabilizing two isotropic subspaces

$$\mathrm{span}\{e'_i + e''_i \mid i = 1, \dots, k\} \quad \text{and} \quad \mathrm{span}\{e'_i - e''_i \mid i = 1, \dots, k\}.$$

The image of $\phi' = \phi_\tau \oplus \phi \oplus \phi_\tau$ stabilizes these two subspaces.

Let $u \in \widehat{G'}$ be the element which acts on $\{e'_1, \dots, e'_k, e_1, \dots, e_{2n}\}$ by $+1$, and on $\{e''_1, \dots, e''_k\}$ by -1. Then $u \in \mathrm{Cent}(\mathrm{Im}(\phi'), \widehat{G'}) \cap \mathrm{Norm}(\widehat{M}_P, \widehat{G'})$. Note that $a \in \mathcal{S}_{\phi'}$ and $w \in W(M_P) \cong W(\widehat{M}_P)$ are the images of u under the canonical maps $\mathrm{Cent}(\mathrm{Im}(\phi'), \widehat{G'}) \to \mathcal{S}_{\phi'}$ and $\mathrm{Norm}(\widehat{M}_P, \widehat{G'}) \to W(\widehat{M}_P)$, respectively. We apply Theorems 2.2.1 (b) and 2.4.1 in [1] to u. They state that

$$\mathrm{Trans}(f')(\phi') = \sum_{[\pi'] \in \Pi_{\phi'}} \iota_{\mathfrak{w}'}([\pi'])(a) \cdot \mathrm{tr}(\pi'(f')),$$

$$\mathrm{Trans}(f')(\phi') = \sum_{[\pi] \in \Pi_{\phi}} \iota_{\mathfrak{w}}(\widetilde{\pi})(\widetilde{u}) \cdot \mathrm{tr}(R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi}) I_0(\tau \otimes \pi)(f'))$$

for any $f' \in \widetilde{\mathcal{H}}(G')$, respectively. Here,

- $\widetilde{\mathcal{H}}(G')$ is the subalgebra of the Hecke algebra $\mathcal{H}(G')$ of G' consisting of $\mathrm{Out}(G')$-invariant functions;
- $\mathrm{Trans}(f')(\phi')$ is the image of a "transfer" $\mathrm{Trans}(f')$ of f' to $\mathrm{Cent}(u, \widehat{G'})^\circ$ under the stable linear form with respect to

$$\phi' \colon \mathrm{WD}_F \to \mathrm{Cent}(u, \widehat{G'})^\circ \rtimes \mathrm{W}_F$$

  defined in [1, Theorem 2.2.1 (1)];
- $\iota_{\mathfrak{w}}(\widetilde{\pi})(\widetilde{u}) \in \mathbb{C}^\times$ is a constant.

(In [1], $\iota_{\mathfrak{w}'}([\pi'])(a)$ and $\iota_{\mathfrak{w}}(\widetilde{\pi})(\widetilde{u})$ are denoted by $\langle x, \pi' \rangle$ and $\langle \widetilde{u}, \widetilde{\pi} \rangle$, respectively.) Hence we obtain the character identity

$$\sum_{[\pi'] \in \Pi_{\phi'}} \iota_{\mathfrak{w}'}([\pi'])(a) \cdot \mathrm{tr}(\pi'(f')) = \sum_{[\pi] \in \Pi_{\phi}} \iota_{\mathfrak{w}}(\widetilde{\pi})(\widetilde{u}) \cdot \mathrm{tr}(R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi}) I_0(\tau \otimes \pi)(f'))$$

for any $f' \in \widetilde{\mathcal{H}}(G')$.

The constant $\iota_{\mathfrak{w}}(\widetilde{\pi})(\widetilde{u})$ is defined as follows (see also [16, Section 2.4.1]): Put

$$\mathfrak{N}_{\phi'} = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi'), \widehat{G'}) \cap \mathrm{Norm}(\widehat{M}_P, \widehat{G'}) / Z(\widehat{G'})^{\mathrm{W}_F}).$$

Then $\mathfrak{N}_{\phi'}$ contains $\mathcal{S}_\phi$ (see also the diagram (2.4.3) in [1]). We set $\widetilde{u} \in \mathfrak{N}_{\phi'}$ to be the image of u. Since k is even, for $u' \in \mathrm{Norm}(\widehat{M}_P, \widehat{G'})$, the action of u' on $\{e_1, \dots, e_{2n}\}$ gives an element $u'_0 \in \widehat{G}$. The map $u' \mapsto u'_0$ induces a section

$$s' \colon \mathfrak{N}_{\phi'} \to \mathcal{S}_\phi.$$

We define the map $\iota_{\mathfrak{w}}(\widetilde{\pi})\colon \mathfrak{N}_{\phi'} \to \{\pm 1\}$ by

$$\iota_{\mathfrak{w}}(\widetilde{\pi}) = \iota_{\mathfrak{w}}([\pi]) \circ s'.$$

In particular, since $s'(\widetilde{u}) = 1$, we have $\iota_{\mathfrak{w}}(\widetilde{\pi})(\widetilde{u}) = \iota_{\mathfrak{w}}([\pi])(1) = 1$.

Hence the character identity implies that

$$\sum_{[\pi']\in\Pi_{\phi'}} \iota_{\mathfrak{w}'}([\pi'])(a) \cdot \mathrm{tr}(\pi'(f')) = \sum_{[\pi]\in\Pi_\phi} \sum_{\pi'\subset I_0(\tau\otimes\pi)} \mathrm{tr}(R_{\mathfrak{w}'}(w, \widetilde{\tau\boxtimes\pi})\pi'(f'))$$

for $f' \in \widetilde{\mathcal{H}}(G')$. Therefore we have $R_{\mathfrak{w}'}(w, \widetilde{\tau\boxtimes\pi})|\pi' = \iota_{\mathfrak{w}'}([\pi'])(a) \cdot \mathrm{id}$. ■

Remark 2.3.

(1) If $G = \mathrm{SO}(2n)$ but k were odd, there would be no canonical choice of $\widetilde{\pi}(\widetilde{w})$. In this case, for each choice, the constant $\iota_{\mathfrak{w}}(\widetilde{\pi})(\widetilde{u}) \in \mathbb{C}^\times$ is defined by using the pairing of [1, Theorem 2.2.4].

(2) The LIRs in [1, Theorems 2.2.1, 2.4.1] and [23, Theorems 3.2.1, 3.4.3] are local statements. However, their proofs use both local and global arguments, and are completed only at the ends of the books after long inductions. □

3 Proof of Desideratum 1.1 (2) for classical groups

Now we give a proof of Desideratum 1.1 (2) when G is a quasi-split classical group. Namely we prove the following:

Theorem 3.1. Let G be a quasi-split classical group. For $\phi \in \Phi(G)$, if $[\pi] \in \Pi_\phi$ is $\mathfrak{w}$-generic, then $\iota_{\mathfrak{w}}([\pi])$ is the trivial representation of $\mathcal{S}_\phi$. □

The proof of Theorem 3.1 is a formal consequence of the LIR (Theorem 2.2) together with Shahidi's result. We recall Shahidi's result in Section 3.1.

3.1 Canonical Whittaker functional

In this subsection, we recall canonical Whittaker functionals of induced representations.

Let G be a quasi-split (connected) classical group and $\mathfrak{w} = (B, \mu)$ be a Whittaker datum for G. Fix a positive integer k. If $G = \mathrm{Sp}(2n)$ or $G = \mathrm{SO}(2n)$, we assume that k is even. Let G', $\mathfrak{w}' = (B', \mu')$, $P = M_P U_P$, $w \in W(M_P)$, and $\widetilde{w} \in G'(F)$ be as in Section 2.4. Let τ be an irreducible tempered representation of $\mathrm{GL}_k(E)$ on a space $\mathcal{V}_\tau$. For $s \in \mathbb{C}$, we realize $\tau_s = \tau|\det|_E^s$ on $\mathcal{V}_\tau$ by setting $\tau_s(a)v := |\det(a)|_E^s\tau(a)v$ for $v \in \mathcal{V}_\tau$ and $a \in \mathrm{GL}_k(E)$. Let π be an irreducible tempered representation of $G(F)$ on a space $\mathcal{V}_\pi$. We consider the normalized induction

$$I_s(\tau \boxtimes \pi) = \mathrm{Ind}_{P(F)}^{G'(F)}(\tau_s \boxtimes \pi).$$

Now we assume that π is $\mathfrak{w}$-generic. We regard $\mathfrak{w}'$ as a Whittaker datum for M_P by the restriction. Since τ is tempered, we see that $\tau \boxtimes \pi$ is $\mathfrak{w}'$-generic. Let $\omega \colon \mathcal{V}_\tau \otimes \mathcal{V}_\pi \to \mathbb{C}$ be a nonzero $\mathfrak{w}'$-Whittaker functional, that is,

$$\omega((\tau \boxtimes \pi)(u)v) = \mu'(u)\omega(v)$$

for $u \in U'(F) \cap M_P(F)$ and $v \in \mathcal{V}_\tau \otimes \mathcal{V}_\pi$. The representative $\widetilde{w} \in G'(F)$ of $w \in W(M_P)$ and the linear map $\widetilde{\tau \boxtimes \pi}(\widetilde{w}) \colon \mathcal{V}_\tau \otimes \mathcal{V}_\pi \to \mathcal{V}_\tau \otimes \mathcal{V}_\tau$ satisfy that

- $\widetilde{w}^{-1}(B' \cap M_P)\widetilde{w} = B' \cap M_P$;
- $\mu'(\widetilde{w}^{-1}u\widetilde{w}) = \mu'(u)$ for any $u \in U'(F) \cap M_P(F)$;
- $\omega \circ \widetilde{\tau \boxtimes \pi}(\widetilde{w}) = \omega$.

See also [1, Section 2.5] and [23, Section 3.5].

We define the Jacquet integral $\mathcal{W}_{\mu',\omega}(g', f_s)$ for $f_s \in I_s(\tau \boxtimes \pi)$ by

$$\mathcal{W}_{\mu',\omega}(g', f_s) = \int_{U_P(F)} \omega(f_s(\widetilde{w}^{-1}ug'))\mu'(u)^{-1}\mathrm{d}u,$$

where $\mathrm{d}u$ is a Haar measure on $U_P(F)$. By [7, Proposition 2.1] and [27, Proposition 3.1], the integral $\mathcal{W}_{\mu',\omega}(g', f_s)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$, and has an analytic continuation as an entire function of $s \in \mathbb{C}$. The map $f_0 \mapsto \mathcal{W}_{\mu',\omega}(1, f_0)$ gives a nonzero $\mathfrak{w}'$-Whittaker functional

$$\Omega_{\mu',\omega} \in \mathrm{Hom}_{U'(F)}(I_0(\tau \boxtimes \pi), \mu').$$

The following theorem follows from Shahidi's results [27, 28]. See also [1, Theorem 2.5.1] and [23, Proposition 3.5.3].

Theorem 3.2. Let $R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi}) \colon I_0(\tau \boxtimes \pi) \to I_0(\tau \boxtimes \pi)$ be the normalized self-intertwining operator and $\Omega_{\mu',\omega} \colon I_0(\tau \boxtimes \pi) \to \mathbb{C}$ be the $\mathfrak{w}'$-Whittaker functional defined as above. Then we have

$$\Omega_{\mu',\omega} \circ R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi}) = \Omega_{\mu',\omega}. \qquad \square$$

Recall that to define $\Omega_{\mu',\omega}$, we have to choose a Haar measure du on $U_P(F)$. We observe that Theorem 3.2 is independent of this choice.

3.2 Proof

Now we prove Theorem 3.1. First, we consider the tempered case. Let $\phi \in \Phi_{\mathrm{temp}}(G)$. Suppose that $[\pi] \in \Pi_\phi$ is $\mathfrak{w}$-generic. Fix a non-trivial element $a \in \mathcal{S}_\phi$. It is enough to show that $\iota_{\mathfrak{w}}([\pi])(a) = 1$. Choose a representative $\phi_\tau \in \mathcal{B}_\phi^+$ of $\phi^a \in \mathcal{B}_\phi^+/\sim$. Then we have ${}^c\phi_\tau^\vee \cong \phi_\tau$. Let $\tau \in \mathrm{Irr}(\mathrm{GL}_k(E))$ be the irreducible tempered representation corresponding to ϕ_τ, where $k = \dim(\phi_\tau)$. Note that k is even if $G = \mathrm{Sp}(2n)$ or $G = \mathrm{SO}(2n)$. Let G' and $\mathfrak{w}'$ be as in Section 3.1. Put

$$\phi' = \phi_\tau \oplus \phi \oplus \phi_\tau \in \Phi_{\mathrm{temp}}(G').$$

Then the canonical inclusion $\mathcal{S}_\phi \hookrightarrow \mathcal{S}_{\phi'}$ is in fact bijective (cf. Section 2.2). Consider the induced representation

$$\pi' = I_0(\tau \boxtimes \pi).$$

Then by Theorem 2.1 (4), we see that π' is irreducible, $[\pi'] \in \Pi_{\phi'}$ and $\iota_{\mathfrak{w}'}([\pi'])|\mathcal{S}_\phi = \iota_{\mathfrak{w}}([\pi])$. Moreover, by Theorem 2.2, we have

$$R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi}) = \iota_{\mathfrak{w}'}([\pi'])(a) \cdot \mathrm{id}.$$

On the other hand, by Theorem 3.2, we have

$$\Omega_{\mu',\omega} \circ R_{\mathfrak{w}'}(w, \widetilde{\tau \boxtimes \pi}) = \Omega_{\mu',\omega}.$$

Since $\Omega_{\mu',\omega}$ is a nonzero functional on π', we must have

$$\iota_{\mathfrak{w}'}([\pi'])(a) = 1.$$

This implies that $\iota_{\mathfrak{w}}([\pi])(a) = 1$.

Next, we consider the general case. Let $\phi \in \Phi(G)$ and assume that $[\pi] \in \Pi_\phi$ is $\mathfrak{w}$-generic. We may decompose

$$\phi = \phi_1|\cdot|^{s_1} \oplus \cdots \oplus \phi_r|\cdot|^{s_r} \oplus \phi_0 \oplus {}^c\phi_r^\vee|\cdot|^{-s_r} \oplus \cdots \oplus {}^c\phi_1^\vee|\cdot|^{-s_1}$$

as in Theorem 2.1 (5). Then π is the unique Langlands quotient of

$$\mathrm{Ind}_{P(F)}^{G(F)}(\tau_1|\det|_E^{s_1} \otimes \cdots \otimes \tau_r|\det|_E^{s_r} \otimes \pi_0)$$

for some $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(G_0(F))$ such that $[\pi_0] \in \Pi_{\phi_0}$. Here τ_i is the irreducible tempered representation of $\mathrm{GL}_{k_i}(E)$ corresponding to ϕ_i. Moreover, we have $\iota_{\mathfrak{w}}([\pi])|\mathcal{S}_{\phi_0} = \iota_{\mathfrak{w}_0}([\pi_0])$ for the Whittaker datum $\mathfrak{w}_0 = (B \cap G_0, \mu|U_0(F))$ for G_0, where $U_0 = U \cap G_0$ is the unipotent radical of the Borel subgroup $B_0 = B \cap G_0$ of G_0. By a result of Rodier [25] and [7, Corollary 1.7], there is an isomorphism

$$\mathrm{Hom}_{U(F)}(\mathrm{Ind}_{P(F)}^{G(F)}(\tau_1|\det|_E^{s_1} \otimes \cdots \otimes \tau_r|\det|_E^{s_r} \otimes \pi_0), \mu) \cong \mathrm{Hom}_{U_0(F)}(\pi_0, \mu|U_0(F)).$$

However, since π is $\mathfrak{w}$-generic, so is $\mathrm{Ind}_{P(F)}^{G(F)}(\tau_1|\det|_E^{s_1} \otimes \cdots \otimes \tau_r|\det|_E^{s_r} \otimes \pi_0)$. (In fact, the standard module conjecture proved in [6, 11, 12, 24] says that the standard module $\mathrm{Ind}_{P(F)}^{G(F)}(\tau_1|\det|_E^{s_1} \otimes \cdots \otimes \tau_r|\det|_E^{s_r} \otimes \pi_0)$ is irreducible, so that it is isomorphic to π.) This implies that π_0 is $\mathfrak{w}_0$-generic. Hence $\iota_{\mathfrak{w}_0}([\pi_0])$ is the trivial character of $\mathcal{S}_{\phi_0}$. Since the inclusion $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_{\phi}$ is bijective, we see that $\iota_{\mathfrak{w}}([\pi])$ is the trivial character of $\mathcal{S}_{\phi}$. This completes the proof of Theorem 3.1.

3.3 Remark

Finally, we remark on [1, Lemma 2.5.5]. This lemma is the "converse" of our proof of Theorem 3.1. Roughly speaking, this lemma asserts that when the residual characteristic of F is not two, the uniqueness of generic representations in an L-packet (Theorem 3.1) for all proper Levi subgroups M of G implies [1, Theorem 2.4.1], which we used to prove the LIR for G (Theorem 2.2). On the other hand, to prove Theorem 3.1 for G, we used the LIR for a bigger group G', which has a Levi subgroup $M_P(F) \cong \mathrm{GL}_k(E) \times G(F)$. If [1, Lemma 2.5.5] were used to prove [1, Theorem 2.4.1] for G', our proof of Theorem 3.1 would be a circular argument.

However, this is not the case. Indeed, Arthur and Mok have applied [1, Lemma 2.5.5] only to the basic cases [1, Lemmas 6.4.1, 6.6.2] and [23, Proposition 7.4.3], when the Levi subgroup M is a torus or a product of torus and SL(2). Hence only for tori and SL(2), our proof of Theorem 3.1 is actually a circular argument. As noted in the proof of [1, Lemma 6.4.1], Theorem 3.1 has already been established for these basic cases. For tori, it is trivial since the component groups are always trivial.

Hence one only needs to treat the SL(2) case. In this case, for an element ϕ in $\Phi_{\mathrm{temp}}(\mathrm{SL}(2))$ and a Whittaker datum $\mathfrak{w}$, it is easy to see that the L-packet Π_ϕ has a unique

$\mathfrak{w}$-generic representation. Indeed, it was shown by Kottwitz–Shelstad (the end of Section 5.3 in [19]) that $\pi \in \Pi_\phi$ is $\mathfrak{w}$-generic if and only if $\iota_{\mathfrak{w}}(\pi)$ is the trivial representation of $\mathcal{S}_\phi$. See also [20, Section 2].

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References

[1] Arthur, J. "The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups." American Mathematical Society Colloquium Publications, 61, Providence, RI, American Mathematical Society, 2013.

[2] Atobe, H. "The local theta correspondence and the local Gan–Gross–Prasad conjecture for the symplectic-metaplectic case." arXiv:1502.03528v2.

[3] Beuzart-Plessis, R. "La conjecture locale de Gross–Prasad pour les représentations tempérés des groupes unitaires." arXiv:1205.2987v2.

[4] Beuzart-Plessis, R. "Expression d'un facteur epsilon de paire par une formule intégrale." *Canadian Journal of Mathematics* 66, no. 5 (2014): 993–1049.

[5] Beuzart-Plessis, R. "Endoscopie et conjecture raffinée de Gan–Gross–Prasad pour les groupes unitaires." *Compositio Mathematica* 151, no. 7 (2015): 1309–71.

[6] Casselman, W., and F. Shahidi. "On irreducibility of standard modules for generic representations." *Annales scientifiques de l'École normale supérieure* 31, no. 4 (1998): 561–89.

[7] Casselman, W. and J. Shalika, "The unramified principal series of p-adic groups. II. The Whittaker function." *Compositio Mathematica* 41 (1980): 207–31.

[8] Gan, W. T., and A. Ichino. "The Gross–Prasad conjecture and local theta correspondence." arXiv:1409.6824v2.

[9] Gan, W. T., B. H. Gross and D. Prasad. "Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups (Sur les conjectures de Gross et Prasasd. I.)" *Astérisque* 346 (2012): 1–109.

[10] Harris, M., and R. Taylor. "The Geometry and Cohomology of Some Simple Shimura Varieties." With an appendix by Vladimir G. Berkovich. Annals of Mathematics Studies, vol. 151, Princeton, NJ, Princeton University Press, 2001.

[11] Heiermann, V., and G. Muić. "On the standard modules conjecture." *Mathematische Zeitschrift* 255 (2007): 847–53.

[12] Heiermann, V., and E. Opdam. "On the tempered L-function conjecture." *American Journal of Mathematics* 135 (2013): 777–800.

[13] Henniart, G. "Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique." *Inventiones mathematicae* 139 (2000): 439–55.

[14] Jiang, D., and D. Soudry. "The local converse theorem for $\mathrm{SO}(2n+1)$ and applications." *Annals of Mathematics* 157, no. 3 (2003): 743–806.

[15] Kaletha, T. "Genericity and contragredience in the local Langlands correspondence." *Algebra Number Theory* 7, no. 10 (2013): 2447–74.

[16] Kaletha, T., A. Mínguez, S. W. Shin, and P.-J. White. "Endoscopic classification of representations: inner forms of unitary groups." arXiv:1409.3731v3.

[17] Konno, T. "Twisted endoscopy and the generic packet conjecture." *Israel Journal of Mathematics* 129 (2002): 253–89.

[18] Kostant, B. "On Whittaker vectors and representation theory." *Inventiones mathematicae* 48, no. 2 (1978): 101–84.

[19] Kottwitz, R., and D. Shelstad. "Foundations of twisted endoscopy." *Astérisque* 255 (1999): vi+190.

[20] Labesse, J.-P., and R. P. Langlands. "L-indistinguishability for $\mathrm{SL}(2)$." *Canadian Journal of Mathematics* 31, no. 4 (1979): 726–85.

[21] Langlands, R. P. "On the Classification of Irreducible Representations of Real Algebraic Groups." *Representation theory and harmonic analysis on semisimple Lie groups*, 101–170, Math. Surveys Monogr., 31, Providence, RI, American Mathematical Society, 1989.

[22] Mœglin, C., and J.-L. Waldspurger, *Stabilisation de la formule des traces tordue*, Progress in Mathematics, vol. 316/317. Birkhäuser/Springer, 2017 (to appear).

[23] Mok, C. P. "Endoscopic classification of representations of quasi-split unitary groups." *Memoirs of the American Mathematical Society* 235, no. 1108 (2015).

[24] Muić, G. "A proof of Casselman–Shahidi's conjecture for quasi-split classical groups." *Canadian Mathematical Bulletin* 44, no. 3 (2001): 298–312.

[25] Rodier, F. "Whittaker Models for Admissible Representations of Reductive p-adic Split Groups." *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, pp. 425–30. Providence, RI, American Mathematical Society, 1973.

[26] Scholze, P. "The local Langlands correspondence for GL_n over p-adic fields." *Inventiones mathematicae* 192 (2013): 663–715.

[27] Shahidi, F. "On certain L-functions." *American Journal of Mathematics* 103, no. 2 (1981): 297–355.

[28] Shahidi, F. *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, *Annals of Mathematics* 132 (1990), 273–330.

[29] Shelstad, D. *Tempered endoscopy for real groups.* III*: inversion of transfer and L-packet structure*, *Representation Theory* 12 (2008), 369–402.

[30] Varma, S. "On descent and the generic packet conjecture." *Forum Mathematicum* (2016).

[31] Vogan, D. "Gel'fand–Kirillov dimension for Harish-Chandra modules." *Inventiones mathematicae* 48, no. 1 (1978): 75–98.

[32] Waldspurger, J.-L. "Une formule intégrale reliée à la conjecture locale de Gross–Prasad." *Compositio Mathematica* 146, no. 5 (2010): 1180–290.

[33] Waldspurger, J.-L. "Une formule intégrale reliée à la conjecture locale de Gross–Prasad, 2e partie: extension aux représentations tempérées." *Astérisque* 346 (2012): 171–312.

[34] Waldspurger, J.-L. "Calcul d'une valeur d'un facteur ε par une formule intégrale." *Astérisque* 347 (2012): 1–102.

[35] Waldspurger, J.-L. "La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux." *Astérisque* 347 (2012): 103–65.

[36] Zelevinsky, A. V. "Induced representations of reductive p-adic groups II: on irreducible representations of $\mathrm{GL}(n)$." *Annales scientifiques de l'École normale supérieure* 13, no. 4 (1980): 165–210.