博　士　論　文

On the construction of twisted triple product p-adic L-functions

（捻れ三重積 p 進 L 関数の構成について）

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平成２９年３月

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A thesis presented
by

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to
The Division of Mathematics and Mathematical Sciences
Graduate School of Science
for the degree of Doctor of Science

Kyoto University
March 2017

To my family

Abstract

In this thesis, we study the construction of twisted triple product p-adic L-functions for automorphic forms on $\mathrm{GL}_2/F \times \mathrm{GL}_2/\mathbb{Q}$, where $F/\mathbb{Q}$ is a real quadratic field. By means of a special value formula proved by Ichino, we construct p-adic L-functions along Hida families of modular forms on $\mathrm{GL}_2/\mathbb{Q} \times \mathrm{GL}_2/F$, or on the multiplicative group of the product of definite quternion algebras $B/\mathbb{Q} \times B \otimes_{\mathbb{Q}} F/F$, which interpolate the central values of the twisted triple product L-functions.

Sometimes, the arithmetic of special values of L-functions can be studied via different constructions of the associated p-adic L-functions. For example, the Kubota-Leopodlt p-adic L-fucntion for a Dirichlet character can be constructed by using Stickelberger elements, or the constant term of an Eisenstein series on $\mathrm{GL}_2/\mathbb{Q}$, and different constructions lead to different proofs of the Iwasawa main conjecture for the Dirichlet character. Mazur-Tate-Teitelbaum p-adic L-function for an elliptic modular form can be constructed via the Rankin-Selberg method, or the constant term of an Eisenstein series on $\mathrm{U}(2,2)$. In many cases, the Iwasawa main conjecture for an elliptic modular form is eventually proved by combining works of Kato, who uses the Rankin-Selberg method, and Skinner-Urban, who use the Eisenstein series. An explicit interpolation formula is an essential ingredient in the comparison of p-adic L-functions constructed via different methods. In this thesis, we obtain interpolation formulas for the twisted triple product p-adic L-functions. We expect the formula will have applications to the arithmetic of special values of such L-functions and the Selmer groups of the tensor product of Galois representations associated with an pair of an elliptic modular form and a Hilbert modular form.

The main innovation of this thesis is to prove explicit interpolation formulas for twisted triple product p-adic L-functions by computing local period integrals which appear in Ichino's formula. In general, when the local representation at p of the specialization of a Hida family is highly ramified, it is difficult to compute the local period integrals directly. To overcome this difficulty, we prove a splitting formula which reduces the computation of the local period integrals to that of Rankin-Selberg type integrals which are more easily computed.

Acknowledgements

I would like to thank my adviser, Tetsushi Ito, for his encouragements and his total supports during my Ph. D. course. I would like to thank Ming-Lun Hsieh for his thorough supports, giving me this problem and its related problems and helpful discussions. I would like to thank Tadashi Ochiai and Masataka Chida for their supports, discussions and givving me valuable information which enable me to proceed the project of this thesis.

Contents

CHAPTER 1

Introduction

The special value of an L-function is one of the main themes in number theory, and the study of *p-adic L-functions* is an important branch. In this thesis, we construct p-adic L-functions for twisted triple product automorphic forms, which is autmorphic forms on $\mathrm{GL}_2/F \times \mathrm{GL}_2/\mathbb{Q}$ where $F/\mathbb{Q}$ is a real quadratic extension. More precisely, we construct p-adic L-functions along Hida families of modular forms on $\mathrm{GL}_2/\mathbb{Q}$ and GL_2/F, or of automorphic forms on the multiplicative groups of definite quternion algebras $B/\mathbb{Q}$ and $B \otimes_{\mathbb{Q}} F/F$, which interpolate the central values of the L-functions of twisted triple product automorphic forms along Hida families.

Our main innovation is to prove explicit interpolation formulas for p-adic L-functions by computing Ichino's formula (Section 1.1). Generally speaking, the local component at p of an automorphic representation obtained as a specialization of a Hida family is highly ramified, so that it is difficult to directly compute the local period integral which appears in Ichino's formula. To overcome the difficulty, we prove a splitting formula which reduce the computation of the local period integral to that of Rankin-Selberg type integrals which can be by far more easily computed (Section 1.1.1).

Recently, Darmon and Rogtger construct p-adic L-functions for split triple product automorphic forms, namely p-adic L-functions interpolating L-functions of modular forms on $\mathrm{GL}_2^3/\mathbb{Q}$ in [**DR14**]. They prove a formula which said a special value of the p-adic L-function at a point outside its interpolation range is described as the image of the p-adic Abel-Jacobi map of a diagonal cycle on a product of Kuga-Sato varieties. It is an important new aspect of p-adic L-functions, but they don't give an explicit interpolation formula in their paper. However, explicit interpolation formulas give us a lot of information. For example, it enables us to identify p-adic L-functions obtained by different constructions, to observe exceptional zero phenomenas, p-integrality of special values of L-functions, and so on. The explicit interpolation formulas of p-adic L-functions for ordinary split triple product automorphic forms are given by Hsieh [**Hsi**], and our result is its twisted analogue.

In [**GS15**], they constructe split triple product p-adic L-functions along Coleman families.

1.1. Ichino's formula

In order to construct p-adic L-functions, existence of a special value formula for a L-function is essential. In our case, we use Ichino's formula [**Ich08**]: Let $E := \mathbb{Q} \times F$. Let D be a quaternion algebra over $\mathbb{Q}$ of discriminant N^- which is the product of prime numbers at which B is ramified. We denote $D_F := D \otimes_{\mathbb{Q}} F$. Let $\Pi \cong \otimes'_v \Pi_v$ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ with central character trivial on $\mathbb{A}_{\mathbb{Q}}^\times$. Here, we diagonally embeds $\mathbb{A}_{\mathbb{Q}}$ into $\mathbb{A}_E$. We assume that there exists an irreducible unitary automorphic representation Π^D of $D^\times(\mathbb{A}_E)$ associated with Π through the Jacquet-Langlands correspondence. We define an element

$$I \in \mathrm{Hom}_{D^\times(\mathbb{A}_{\mathbb{Q}}) \times D^\times(\mathbb{A}_{\mathbb{Q}})}\left(\Pi^D \boxtimes (\Pi^D)^\vee, \mathbb{C}\right)$$

by

$$I(\phi\boxtimes\phi') := \int_{\mathbb{A}_{\mathbb{Q}}^{\times}D^{\times}(\mathbb{Q})\backslash D^{\times}(\mathbb{A}_{\mathbb{Q}})}\int_{\mathbb{A}_{\mathbb{Q}}^{\times}D^{\times}(\mathbb{Q})\backslash D^{\times}(\mathbb{A}_{\mathbb{Q}})}\phi(x)\phi'(y)\,dxdy,$$

for $\phi\in\Pi^D$ and $\phi'\in(\Pi^D)^\vee$, where $(*)^\vee$ denotes the contragredient representation of $(*)$, and dx and dy are the Tamagawa measures on $\mathbb{A}_{\mathbb{Q}}^{\times}\backslash D^{\times}(\mathbb{A}_{\mathbb{Q}})$. We define an element

$$\mathcal{B}\in\mathrm{Hom}_{D^{\times}(\mathbb{A}_E)}\big(\Pi^D\otimes(\Pi^D)^\vee,\mathbb{C}\big)$$

by

$$\mathcal{B}(\phi,\phi') := \int_{\mathbb{A}_E^{\times}D(E)\backslash D(\mathbb{A}_E)}\phi(x)\phi'(x)dx,$$

for $\phi\in\Pi^D$ and $\phi'\in(\Pi^D)^\vee$, where dx is the Tamawagawa measure. For each place v of $\mathbb{Q}$, we fix an element

$$\mathcal{B}_v\in\mathrm{Hom}_{D^{\times}(\mathbb{A}_{E_v})}\big(\Pi_v^D\otimes(\Pi_v^D)^\vee,\mathbb{C}\big),$$

where $E_v := E\otimes_{\mathbb{Q}}\mathbb{Q}_v$, and assume that for almost all v,

$$\mathcal{B}_v(\phi_v,\phi'_v) = 1$$

for any $\otimes'_v\phi_v\in\otimes'_v\Pi_v^D$ and $\otimes'_v\phi'_v\in\otimes'_v(\Pi_v^D)^\vee$. Then there exists $C_1\in\mathbb{C}^\times$ such that

$$\mathcal{B} = C_1\prod_v\mathcal{B}_v.$$

For $\otimes'_v\phi_v\in\otimes'_v\Pi_v^D$ and $\otimes'_v\phi'_v\in\otimes'_v(\Pi_v^D)^\vee$, we define

$$\mathcal{I}_{\Pi_v^D}(\phi) := \frac{\zeta_{\mathbb{Q}_v}(2)}{\zeta_{E_v}(2)}\frac{L(1,\mathrm{Ad}\Pi_v)}{L(1/2,\Pi_v)}\int_{\mathbb{Q}_v^{\times}\backslash D^{\times}(\mathbb{Q}_v)}\mathcal{B}_v(\Pi_v^D(g)\phi_v,\phi'_v)\,d_vg,$$

Here, $L(s,\mathrm{Ad}\Pi_v)$ and $L(s,\Pi_v)$ are the L-functions defined by representations $\mathbb{C}^{\otimes 3}$ and $\mathrm{Lie}(\widehat{G})/\mathrm{Lie}(Z(\widehat{G}))$ of ${}^LG = \mathrm{GL}_2(\mathbb{C})^3\rtimes\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, respectively, where $G := \mathrm{Res}_{E/F}\mathrm{GL}_2$, $\widehat{G}$ be the dual group of G, and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\widehat{G}$ as $\mathfrak{S}_3$ through the permutation of $\mathrm{Spec}(E\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})$. Ichino's formula [**Ich08**] is stated as follows:

Theorem 1.1.1 ([Ich08]). For $\phi=\otimes'_v\phi_v\in\Pi^D$ and $\phi'=\otimes'_v\phi'_v\in(\Pi^D)^\vee$ such that

$$\mathcal{B}(\phi,\phi')\neq 0,$$

we have

$$\frac{I(\phi,\phi')}{\mathcal{B}(\phi,\phi')} = \frac{C}{2^c}\cdot\frac{\zeta_E(2)}{\zeta_F(2)}\cdot\frac{L(1/2,\Pi)}{L(1,\mathrm{Ad}\Pi)}\cdot\prod_v\frac{\mathcal{I}_{\Pi_v^D}(\phi_v,\phi'_v)}{\mathcal{B}_v(\phi_v,\phi'_v)},$$

where c is the number of the connected component of $\mathrm{Spec}(E)$ and $C\in\mathbb{C}^\times$ is a constant depending on D and the choice of measures $\{d_vx\}_v$.

Regarding the precise description for the constant C, see the proceeding paragraph of Theorem 6.1.1. In our case, the quaternion algebra D is either a definite quaternion algebra, namely, $D(\mathbb{R})$ is a division algebra, or the matrix algebra. By the above theorem, we find that we have to do two things: to compute the local period integral $\mathcal{I}_{\Pi_v^D}$ on the right hand side, and to construct an element of an Iwasawa algebra interpolating the global period integral I on the left hand side.

1.1.1. Computations on local period integrals. In general situation, it is difficult to compute the the local period integral $\mathcal{I}_{\Pi_v^D}$ directly. At the following places: archimedean places, a places at which the exponent conductor of the local representation is at most one, and places dividing N^-, it is computed directly by Chen-Cheng [**CC16**], so we have to compute it when the local representation at v is highly ramified at a non-archimedean place. To compute the local period integral, we prove a *splitting formula*. Let us explain it:

Let F_2/F_1 be a quadratic extension of fields which are finite extensions over $\mathbb{Q}_p$ for a prime number p. We fix an element $\xi \in F_2^\times$ with $\mathrm{tr}_{F_2/F_1}(\xi) = 0$. We fix a non-trivial additive character $\psi : F_1 \to \mathbb{C}^\times$, and we define $\psi_\xi(x) := \psi(\mathrm{tr}_{F_2/F_1}(\xi x))$ for $x \in F_2^\times$. For each $i = 1, 2$, we denote by $|\cdot|_{F_i}$ the non-archimedean absolute value on F_i such that $|p|_{F_i} = \#\left(\mathcal{O}_{F_i}/p\mathcal{O}_{F_i}\right)^{-1}$. Let q_i be the order of the residue field of F_i, and we define

$$\zeta_{F_i}(s) := \frac{1}{1 - q_i^{-s}}.$$

Let $\mu, \nu : F_1^\times \to \mathbb{C}^\times$ be quasi-characters. We assume that $\mu = \chi_1|\cdot|_{F_1}^{\lambda_1}$ and $\nu = \chi_2|\cdot|_{F_1}^{\lambda_2}$ for some unitary characters χ_1, χ_2 on $F_1^\times$, and for some complex numbers $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfying

$$|\mathrm{Re}(\lambda_1)|, |\mathrm{Re}(\lambda_2)| < \frac{1}{2}.$$

Let $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$ and $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu^{-1} \boxtimes \nu^{-1})$ be the induced representations normalized by the modulus character of the group of upper triangular matrices $B(F_1) = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$, which are models of the principal series representations $\pi(\mu, \nu)$ and $\pi(\mu^{-1}, \nu^{-1})$, respectively. Namely, $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$ and $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu^{-1} \boxtimes \nu^{-1})$ are $\mathbb{C}$-vector spaces of locally constant functions f on $\mathrm{GL}_2(F_1)$ such that

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu(a)\nu(d)|ad^{-1}|_{F_1}^{1/2} f(g),$$
$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu^{-1}(a)\nu^{-1}(d)|ad^{-1}|_{F_1}^{1/2} f(g)$$

for $g \in \mathrm{GL}_2(F_1)$, $a, d \in F_1^\times$ and $b \in F_1$, respectively. For $f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$, $\widetilde{f} \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu^{-1} \boxtimes \nu^{-1})$, and $g \in \mathrm{GL}_2(F_1)$, we define

$$\Phi_{f,\widetilde{f}}(g) := \int_K f(kg)\widetilde{f}(k)\, dk,$$

where $K := \mathrm{PGL}_2(\mathcal{O}_{F_1})$, and dk is the invariant measure on $\mathrm{PGL}_2(F_1)$ with $\mathrm{vol}(K, dk) = 1$.

Let π_2 be an irreducible tempered admissible representation of $\mathrm{GL}_2(F_2)$ with central character $\omega_2 : F_2^\times \to \mathbb{C}^\times$. We assume that $(\omega_2)|_{F_1^\times}\mu\nu$ is trivial on $F_1^\times$. We denote by $\mathscr{W}(\pi_2, \psi_\xi)$ and $\mathscr{W}(\pi_2^\vee, \psi_\xi)$ the Whittaker models of π_2 and $\pi_2^\vee$ associated with ψ_ξ, respectively, where $\pi_2^\vee$ denotes the contragredient representation of π_2. For $W \in \mathscr{W}(\pi_2, \psi_\xi)$, $\widetilde{W} \in \mathscr{W}(\pi_2^\vee, \psi_\xi)$, and $g \in \mathrm{GL}_2(F_2)$, we define

$$\Phi_{W,\widetilde{W}}(g) := \int_{F_2^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \widetilde{W}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times a,$$

where $d^\times a$ is the invariant measure on $F_2^\times$ with $\mathrm{vol}(\mathcal{O}_{F_2}^\times, d^\times a) = 1$. For $W \in \mathscr{W}(\pi_2, \psi_\xi)$, $\widetilde{W} \in \mathscr{W}(\pi_2^\vee, \psi_\xi)$, $f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$, and $\widetilde{f} \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu^{-1} \boxtimes \nu^{-1})$, we define

$$\Psi(W, f) := \int_{N(F_1)\backslash \mathrm{PGL}_2(F_1)} W(g) f(g)\, dg,$$
$$\widetilde{\Psi}(\widetilde{W}, \widetilde{f}) := \int_{N(F_1)\backslash \mathrm{PGL}_2(F_1)} \widetilde{W}(\eta g) \widetilde{f}(g)\, dg,$$

where $\eta := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $N(F_1) := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.

We put

$$\Pi := \mathscr{W}(\pi_2, \psi_\xi) \boxtimes \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu),$$

which is an irreducible admissible representation of $\mathrm{GL}_2(F_2) \times \mathrm{GL}_2(F_1)$. We define two parings

$$I_{\mathrm{GP}},\, I_{\mathrm{RS}} \colon \Pi \times \Pi^\vee \longrightarrow \mathbb{C}$$

by

$$I_{\mathrm{GP}}(W \boxtimes f, \widetilde{W} \boxtimes \widetilde{f}) := |\xi \mathcal{D}_{F_2/F_1}|_{F_2}^{1/2} \frac{\zeta_{F_1}(1)}{\zeta_{F_2}(1)} \int_{\mathrm{PGL}_2(F_1)} \Phi_{W, \widetilde{W}}(g) \Phi_{f, \widetilde{f}}(g)\, dg,$$
$$I_{\mathrm{RS}}(W \boxtimes f, \widetilde{W} \boxtimes \widetilde{f}) := \Psi(W, f) \widetilde{\Psi}(\widetilde{W}, \widetilde{f}),$$

where $\mathcal{D}_{F_2/F_1}$ is a generator of the different ideal of F_2/F_1. Here,"GP" (resp. "RS") stands for "Gross-Prasad" (resp. "Rankin-Selberg"). The main result is the equality between these parings:

Theorem 1.1.2 (see Theorem 5.1.1). We have

$$I_{\mathrm{GP}} = I_{\mathrm{RS}}.$$

We note that the I_{GP} is the same up to scalar as $\mathcal{I}_{\Pi_v^D}$ and the Ψ and $\widetilde{\Psi}$ of the right hand side is much easier to compute (see also Theorem 5.4.1). In the split case (i.e. $F_2 = F_1 \times F_1$), an analogue of Theorem 1.1.2 was proved by Hsieh in [**Hsi**], which generalizes the result of Michel-Venkatesh [**MV10**, Lemma 3.4.2].

1.2. Hida families and main results

For simplicity, we assume that the class number of F is one, and the prime number p is inert in F for ease of notations. We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, where we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\mathfrak{H}$ be the upper half plane. Let $f(z)$ be an elliptic cusp form of weight k_1 and let $g(z_1, z_2)$ be a Hilbert cusp form of weight (k_2, k_3) (for the precise definition, see Section 4.1.1). They are analytic functions on $z \in \mathfrak{H}$ and $(z_1, z_2) \in \mathfrak{H}^2$, respectively. We assume that f and g are normalized cuspidal Hecke eigenforms with trivial central characters, new outside p, and ordinary at p, namely, their Fourier coefficients at p are p-adic units as elements of $\mathbb{C}_p$ via the above embedding. In this thesis, we construct a p-adic L-function interpolating $L(1/2, f \times g)$ when one of the following two conditions holds:

(1) $k_1 \geqslant k_2 + k_3$ (called the unbalanced condition with respect to k_1)
(2) $k_1 < k_2 + k_3,\ k_2 < k_1 + k_3,\ k_3 < k_1 + k_2$ (called the balanced condition) and f has a Jacquet Langlands lift of a definite quaternion algebra.

We note that each case of (1) and (2) corresponds to the sign of the local root number at the archimedean place of the twisted triple product L-function associated with f and g. In the case (1), the sign is $+1$, and in the another case (2), it is -1. By Loke's theorem [**Lok01**], we see that the sign controls on which algebraic group the local period integral is zero at the archimedean place. Thus to construct a nonzero p-adic L-function, we need different Hida theories on a different algebraic groups for cases (1) and (2).

1.2.1. A Main result for the unbalanced case. In this case, we use the Hida theory on GL_2. We assume that f is spherical outside p. It is known that

$$a\left(1, 1_f\mathcal{H}(\delta_{z_1}^{(k_1-k_2+k_3)/2}g|_{\mathfrak{H}})\right) = \frac{(f, \mathcal{H}(\delta_{z_1}^{(k_1-k_2+k_3)/2}g|_{\mathfrak{H}}))}{(f,f)},$$

where $a(1,\cdot)$ denotes the first Fourier coefficient of the modular form, δ_{z_1} is the Maass-Shimura operator along the variable z_1, $\mathcal{H}$ is the holomorphic projection, 1_f is the idempotent of the Hecke ring associated with f, and $(\cdot,\cdot)$ is the Petersson inner product. By definition, $(f, \mathcal{H}(\delta_{z_1}^{(k_1-k_2+k_3)/2}g|_{\mathfrak{H}}))^2$ is just the global period integral which appear in Ichino's formula. Thus we develop the nearly ordinary Hida theory by means of Wiles' formulation [**Wi88**], which regards the Hida family as the p-adic deformation of Fourier coefficients. For Hida families $\mathscr{F}$ and $\mathscr{G}$ run through f and g respectively, we define the (square root of) p-adic L-function by

$$\mathscr{L}_p(\mathscr{F}\otimes\mathscr{G}) := a(1, 1_{\mathscr{F}}\Theta(\mathscr{G})) \in \mathrm{Frac}(\mathbb{I}_1\otimes\mathbb{I}_2),$$

where Θ is a deformation of the Maass-Shimura operators to adjust weights (Definition 4.5.1), and $\mathbb{I}_1$ and $\mathbb{I}_2$ are coefficient rings of $\mathscr{F}$ and $\mathscr{G}$, respectively. For any arithmetic point P (resp. Q) (see Section 4.4.1) of $\mathbb{I}_1$ (resp. $\mathbb{I}_2$) , we denote the specialization at P (resp. Q) by $\mathscr{F}_P$ (resp. $\mathscr{G}_Q$). Let $\pi_{\mathscr{F}_P}$ and $\pi_{\mathscr{G}_Q}$ are unitary cuspidal representations associated with $\mathscr{F}_P$ and $\mathscr{G}_Q$, respectively. We have the following result:

Theorem 1.2.1 (Theorem 6.2.5). Let $P\otimes Q \in \mathscr{X}(\mathbb{I}_1\otimes\mathbb{I}_2)$ be an element such that $P|_{\mathbf{G}} = P_{k_P,w_P,\omega_1,\mathbf{1}}$ and $Q|_{\mathbf{G}} = P_{k_P-2r\sigma-\underline{t}_F,w_P-r\sigma,\omega_2,\mathbf{1}}$ for some $r \geqslant 0$, we have

$$\begin{aligned}
&(P\otimes Q)(\mathscr{L}_p\left(\mathscr{F}\otimes\mathscr{G}\right)^2) \\
&= 2^{r+4}\sqrt{D}_{\mathbf{p}}^{2w_P|_{\mathbb{Q}}-2\underline{t}_F-r\sigma}\, a(\mathscr{F}_P,p)^{2c(\omega_1)}\varepsilon_{\mathrm{RS}}(1/2, \mathrm{As}\pi_{\mathscr{G}_{Q,p}}\otimes\mu_{\mathscr{F}_{P,p}},\psi,\sqrt{D}^{-1}) \\
&\quad\times\left(\frac{L(1,\mu_{\mathscr{F}_P}\nu_{\mathscr{F}_P}^{-1})L(0,\mu_{\mathscr{F}_P}\nu_{\mathscr{F}_P}^{-1})}{L(1/2,\mathrm{As}\pi_{\mathscr{G}_{Q,p}}\otimes\mu_{\mathscr{F}_{P,p}})}\right)^2\cdot\frac{L(1/2,\pi_{\mathscr{F}_P}\otimes\pi_{\mathscr{G}_Q}\otimes\sqrt{\omega_P\omega_Q}^{-1})}{D\cdot\Omega(P)^2},
\end{aligned}$$

where $\mathcal{D}\in\mathbb{Z}_{>0}$ is the discriminant of $F/\mathbb{Q}$, and $c(\omega_1)$ is the exponent of p of conductor of ω_1. We assume that $\pi_{\mathscr{F}_P,p}$ is the irreducible subquotient of $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu_P\boxtimes\nu_P)$. The complex number $\Omega(P)\in\mathbb{C}^\times$ is nonzero and defined by

$$\Omega(P) := 2^{k_P|_{\mathbb{Q}}}\,p^{c(\omega_1)((k_P|_{\mathbb{Q}})/2-1)}\,\varepsilon(1/2,\pi_{\mathscr{F}_P})\left(\widetilde{\mathscr{F}_P},\widetilde{\mathscr{F}_P}\right)_{\Gamma_0(p^{c(\omega_1)})},$$

where we define

$$\widetilde{\mathscr{F}}\colon \text{ the new form associated with the ordinary form } \mathscr{F}_P,$$

$$\Gamma_0(p^{c(\omega_1)}) := \left\{ x \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p^{c(\omega_1)} \right\},$$

$$\left(\widetilde{\mathscr{F}_P}, \widetilde{\mathscr{F}_P}\right)_{\Gamma_0(p^{c(\omega_1)})} := \int_{\Gamma_0(p^{c(\omega_1)})\backslash\mathfrak{H}} \left|\widetilde{\mathscr{F}_P}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\right|^2 \frac{dx\,dy}{y^2}.$$

For the notations we don't explain, see Section 6.2. We remark that we can see the Euler factor which can cause the exceptional zero in the above theorem. We note that the denominator of the p-adic L-function is controlled by the congruence number (see Remark 6.2.6).

1.2.2. A main result for the balanced case. In this case, we need Hida theory for the multiplicative group on a definite quaternion algebra. Let f^B and g^B be the Jacquet-Langlands lifts on the definite quaternion algebra B. Mainly, we follow the method of [**GS15**]. In their article, they construct a triple product p-adic L-function for general finite slope modular forms along a Coleman's famiy, but they treat only the case that the base field is $\mathbb{Q}$, and we can't consider the integrality of the special value of L-functions in their frame work. Thus we develop a theory measure valued form theory for definite quaternion algebra over any general totally real field. For Hida families Φ_1 and Φ_2 running through f^B, g^B respectively, we construct an element $\mathscr{L}_p(\Phi_1 \otimes \Phi_2)$ in the fractional field of an Iwasawa algebra $\mathbb{I}$. The interpolation formula is as follows:

Theorem 1.2.2 (Theorem 6.3.1). Let $P \in \mathscr{X}(\mathbb{I})$ such that $P|_{\mathbf{G}_E} = P_{k_1,w_1,\boldsymbol{\omega}_1} \times P_{k_2,w_2,\boldsymbol{\omega}_2}$ with $\boldsymbol{\omega}_i = (\omega_i, \mathbf{1})$ and $k_1 < k_{2,\sigma} + k_{2,\rho}$, $k_{2,\sigma} < k_1 + k_{2,\rho}$, and $k_{2,\rho} < k_1 + k_{2,\sigma}$ hold. We have

$$\begin{aligned}
&P(\mathscr{L}_p(\Phi_1 \otimes \Phi_2)) \\
=&C'\mathcal{D}_{\mathfrak{p}}^{-k_1^*-1} \cdot \prod_{q|N^-} e_q(F/\mathbb{Q}) \\
&\times \frac{\mathcal{E}_p(\Pi_P)}{\mathcal{E}(\pi_{1,P},\mathrm{Ad})\mathcal{E}(\pi_{2,P},\mathrm{Ad})} \cdot \left(\frac{L(1/2, \mu_{1,P}\nu_{2,P})}{L(1/2, \mathrm{As}\pi_{2,P} \otimes \mu_{1,P})L(1/2, \mu_{1,P}^{-1}\nu_{2,P}^{-1})}\right)^2 \cdot \frac{L(1/2, \Pi_P)}{L(1, \mathrm{Ad}\Pi_P)}.
\end{aligned}$$

Here, C'' is a nonzero rational number depending only on Φ_1 and Φ_2, $e_q(F/\mathbb{Q})$ is the ramified index of $F/\mathbb{Q}$ at q,

$$\mathcal{E}_p(\Pi_P) := \frac{\varepsilon_{\mathrm{RS}}(1/2, \mathrm{As}\pi_2 \otimes \mu_1, \psi_{\sqrt{D}^{-1}})\varepsilon(1/2, \mu_1^{-1}\nu_{2,P}^{-1}, \psi^{-1})}{\varepsilon(1/2, \mu_1\nu_{2,P}, \psi)}$$

and $\mathcal{E}(\pi_{i,P}, \mathrm{Ad})$ is that defined in Proposition 5.3.1.

For the notations we don't explain, see Section 6.3. We remark that we can see the Euler factor which can cause the exceptional zero in the above theorem. We note that the denominator of $\mathscr{L}_p$ has an explicitly constructed and we can see its behavior.

1.3. Basic notations

Let F be a totally real field and p be a prime. Let $\mathbb{A}_{\mathbb{Q}}$ be the adèle ring over $\mathbb{Q}$. Let $\mathbb{A}_{\mathbb{Q},f}$ the finite part of $\mathbb{A}_{\mathbb{Q}}$. We fix an algebraic closure of F denoted by $\overline{\mathbb{Q}}$ and we denote by $\mathbb{C}$ the fields of complex numbers. We fix $\mathbb{C}_p$ which is the completion of an algebraic closure of $\mathbb{Q}_p$. We define the additive valuation

$$\mathrm{ord}_p\colon \mathbb{C}_p \longrightarrow \mathbb{Q} \cup \{\infty\} \text{ such that } \mathrm{ord}_p(p) = 1,\ \mathrm{ord}_p(0) = \infty$$

and define the multiplicative valuation

$$|x|_p := p^{-\mathrm{ord}_p(x)} \in \mathbb{R}_{\geqslant 0}.$$

For a finite $\mathbb{Q}_p$-algebra L and for $x \in L$, we define

$$|x|_L := |\,\mathrm{N}_{L/\mathbb{Q}_p}(x)|_p.$$

We fix embeddings $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p,\ \iota_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\iota\colon \mathbb{C}_p \cong \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} & & \mathbb{C}_p \\ & \overset{\iota_p}{\nearrow} & \\ F \hookrightarrow \overline{\mathbb{Q}} & & \wr\|\ \iota \\ & \underset{\iota_\infty}{\searrow} & \\ & & \mathbb{C} \end{array}$$

is commutative.

We denote by $I_F := \{\,\sigma\colon F \hookrightarrow \mathbb{C}_p\,\}$ (or I if there occurs no confusion) the set of the embeddings from F into $\mathbb{C}_p$. We identify the set I_F with the set of the embeddings from F into $\mathbb{C}$ via the isomorphism $\mathbb{C}_p \cong \mathbb{C}$. We denote by $\mathbb{Z}[I_F]$ the free abelian group generated by I_F:

$$\mathbb{Z}[I_F] := \bigoplus_{\sigma\in I_F} \mathbb{Z}\sigma.$$

We denote by $k_\sigma \in \mathbb{Z}$ the σ-component of $k \in \mathbb{Z}[I_F]$, namely, $k = \sum_\sigma k_\sigma\sigma$. We define an element $\underline{t}_F \in \mathbb{Z}[I_F]$ (we denote it by $\underline{t}$ if there occurs no confusion) by

$$\underline{t}_F := \sum_{\sigma\in I_F} \sigma.$$

Let F_1/F_2 be two totally real fields. For $k \in \mathbb{Z}[I_{F_2}]$, we denote by $k|_{F_1} \in \mathbb{Z}[I_{F_1}]$ the following element

$$k|_{F_1} := \sum_{\sigma\in I_{F_1}} \left(\sum_{\substack{\tau\in I_{F_2} \\ \tau|_{F_1}=\sigma}} k_\tau \right)\sigma.$$

For any $z = (z_\sigma)_{\sigma\in I} \in F \otimes_{\mathbb{Q}} \mathbb{C}_p \cong \prod_{\sigma\in I_F} \mathbb{C}_p$, we define

$$z^k := \prod_{\sigma\in I} z_\sigma^{k_\sigma} \in \mathbb{C}_p$$

We define several rings as follows

$$\begin{aligned} F_p &:= F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{\mathfrak{p}\mid p} F_{\mathfrak{p}} \\ \mathcal{O}_{F_p} &:= \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{\mathfrak{p}\mid p} \mathcal{O}_{F_{\mathfrak{p}}} \\ k_{F_p} &:= \mathcal{O}_{F_p}/J(\mathcal{O}_{F_p}) \cong \prod_{\mathfrak{p}\mid p} \mathcal{O}_{F_{\mathfrak{p}}}/\varpi_{\mathfrak{p}}\mathcal{O}_{F_{\mathfrak{p}}}, \end{aligned}$$

where $J(\mathcal{O}_{F_p})$ is the Jacobson radical of $\mathcal{O}_{F_p}$ and $\varpi_{\mathfrak{p}} \in \mathcal{O}_{F_{\mathfrak{p}}}$ is a uniformizer.

For any abelian profinte group H, we denote by $H(p)$ a unique p-Sylow group. We define the projection from H to $H(p)$ by

$$\langle \cdot \rangle \colon H \longrightarrow H(p).$$

Let

$$\mathbf{e} \colon \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \longrightarrow \mathbb{C}$$

be a unique additive character such that for $x_\infty \in \mathbb{R}$,

$$\mathbf{e}(x_\infty) = e^{2\pi\sqrt{-1}x_\infty}.$$

Let

$$\begin{aligned} \mathrm{Pic}_{\mathrm{cyc}} \colon \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times &\longrightarrow \mathbb{Z}_p^\times; \qquad x \mapsto x_p^{-1}|x|_{\mathbb{A}_{\mathbb{Q},f}}^{-1} \\ \tau \colon \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times &\longrightarrow \mathbb{Z}_p^\times; \qquad x \mapsto \langle\, \epsilon_{\mathrm{cyc}}(x) \,\rangle\, \epsilon_{\mathrm{cyc}}(x)^{-1} \end{aligned}$$

be the cyclotomic character and Teichmüller character, respectively.

For a $\mathbb{Q}$ (resp. $\mathbb{Q}_p$)-algebra L, we denote by $\mathcal{O}_L$ the integral closure of $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) in L.

For any finite $\mathbb{Q}$-algebra L, we define

$$\begin{aligned} \mathbb{A}_L &:= \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} L, \\ \mathbb{A}_{L,f} &:= \mathbb{A}_{\mathbb{Q},f} \otimes_{\mathbb{Q}} L, \\ L_+^\times &:= \{\, x \in L^\times \mid \sigma(x) \in \mathbb{R}_{>0} \text{ for any } \mathbb{Q}\text{-algebra homomorphism } \sigma \colon L \to \mathbb{C} \,\}, \\ \mathcal{O}_{L,+}^\times &:= \mathcal{O}_L^\times \cap L_+^\times, \\ \mathbf{e}_L \colon \mathbb{A}_L/L &\xrightarrow{\mathrm{Tr}_{L/\mathbb{Q}}} \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \xrightarrow{\mathbf{e}} \mathbb{C} \\ \epsilon_{cyc,L} \colon \mathbb{A}_L^\times/L^\times &\xrightarrow{\mathrm{N}_{L/\mathbb{Q}}} \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \xrightarrow{\epsilon_{\mathrm{cyc}}} \mathbb{Z}_p^\times, \\ \tau_L \colon \mathbb{A}_L^\times/L^\times &\xrightarrow{\mathrm{N}_{L/\mathbb{Q}}} \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \xrightarrow{\tau} \mathbb{Z}_p^\times. \end{aligned}$$

For any place v of $\mathbb{Q}$, we define

$$\mathbf{e}_v := \mathbf{e}|_{\mathbb{Q}_v} \colon \mathbb{Q}_v \longrightarrow \mathbb{C}^\times.$$

For a finite $\mathbb{Q}_q$-algebra L', though it's rather abuse of notation, we define

$$\mathbf{e}_L \colon L \xrightarrow{\mathrm{Tr}_{L/\mathbb{Q}_v}} \mathbb{Q}_v \xrightarrow{\mathbf{e}_v} \mathbb{C}^\times.$$

We define several algebraic groups over $\mathbb{Z}$ as follows: for any $\mathbb{Z}$-algebra R,

$$\begin{aligned} N(R) &:= \left\{ \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix} \right\}, \\ {}^tN(R) &:= \left\{ \begin{pmatrix} 1 & 0 \\ R & 1 \end{pmatrix} \right\}, \\ T(R) &:= \left\{ \begin{pmatrix} R^\times & 0 \\ 0 & R^\times \end{pmatrix} \right\}. \end{aligned}$$

For any algebraic group G over F, we use the following notation: let $U \subset G(\mathbb{A}_F)$ be a subgroup. For any nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$, we define subgroups of U by

$$\begin{aligned} U_{\mathfrak{a}} &:= \{\, u \in U \mid u_{\mathfrak{l}} = 1 \text{ for any prime ideal } \mathfrak{l} \mid \mathfrak{a} \,\}, \\ U^{\mathfrak{a}} &:= \{\, u \in U \mid u_{\mathfrak{l}} = 1 \text{ for any prime ideal } \mathfrak{l} \nmid \mathfrak{a} \,\}. \end{aligned}$$

When $\mathfrak{a} = (a)$ for some $a \in \mathcal{O}_F$, we usually omit to write "(" and ")," namely, we denote $U^{(a)}$ and $U_{(a)}$ by U^a and U_a, respectively.

Let R be a ring, G a group and let M be a $R[G]$-module. For a group homomorphism $\epsilon\colon G \to \mathrm{Aut}_{R[G]}(M)$, we denote by $M[\epsilon]$ the space

$$\{\, x \in M \mid g \cdot x = \epsilon(g)x \,\}.$$

CHAPTER 2

A review of $\mathbb{I}$-adic forms on definite quaternion algebras over totally real fields

2.1. Quaternioic automorphic forms

2.1.1. The generity of quaternionic automorphic forms. Let B be a definite quaternion algebra over F of discriminant $\mathfrak{d}$, which is ramified at all of infinite places and prime ideal of F dividing $\mathfrak{n}^-$. We assume that p is prime to $\mathfrak{d}$. Suppose that $\mathfrak{n}^-$ is prime to p. We denote $B \otimes_F \mathbb{A}_{F,f}$ by $\widehat{B}$, where $\mathbb{A}_{F,f}$ is the finite adèle ring over F. For any subgroup $S \subset (\widehat{B}^\times)$, we denote by $X(S)$ the following quotient space:

$$X(S) := B^\times \backslash \widehat{B}/S.$$

In the case $S_p = 1$, we define a right action of $\mathrm{GL}_2(F_p)$ on $X(S)$ by a natural way. For any nonzero prime ideal $\mathfrak{q}$ not dividing $\mathfrak{d}$, we fix an isomorphism and embedding

$$i_{\mathfrak{q}} \colon B \otimes_F F_{\mathfrak{q}} \cong \mathrm{M}_2(F_{\mathfrak{q}}) \hookrightarrow \mathrm{M}_2(\mathbb{C}), \tag{2.1.1}$$

where $\mathrm{M}_2(\cdot)$ means the matrix ring. We always identify $B \otimes_F F_{\mathfrak{q}}$ with $\mathrm{M}_2(F_{\mathfrak{q}})$ and occasionally omit to write $i_{\mathfrak{q}}$.

We define the most general form of quaternionic automorphic forms:

Definition 2.1.1. Let $U \subset \widehat{B}^\times$ be an open compact subgroup and M an left U_p-module. The *M-valued p-adic quaternionic modular form of level U* is a map ϕ from $X(U^p)$ to M satisfying

$$\phi(bu) = u^{-1}\phi(b),$$

where $b \in X(U^p)$, and $u \in U_p$. We denote by $S(U, M)$ the space of M-valued p-adic quaternionic modular forms of level U.

Definition 2.1.2. Let $U, U' \subset \widehat{B}^\times$ be open compact subgroups and $g \in \widehat{B}$. Let M be a $\mathbb{Z}[U_p, U'_p, g_p]$-module. We define an homomorphism

$$[U'gU] \colon S(U, M) \longrightarrow S(U', M)$$

by

$$\big([U'gU]\phi\big)(b) := \sum_i (g_i)_p \phi(bg_i),$$

where $b \in \widehat{B}^\times$, $\phi \in S(U, M)$ and g_i are defined by the following finite decomposition:

$$U'gU = \bigsqcup_i g_i U. \tag{2.1.2}$$

Let A be a commutative ring and $\mathrm{Sym}^m(A)$ the space of two variable A-coefficient homogeneous polynomials of degree m. This module has a left action of semigroup $\mathrm{M}_2(A)$ define by

$$\gamma f(X, Y) := f\left((X, Y)\gamma\right) = f(aX + cY, bX + dY),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(A)$ and $f \in \mathrm{Sym}^m(A)$. Suppose that A is a $\mathbb{Z}_{(q)}$-algebra with $m < q$. We define a perfect paring

$$\langle\,\cdot,\cdot\,\rangle_m \colon \mathrm{Sym}^m(A) \times \mathrm{Sym}^m(A) \longrightarrow A \tag{2.1.3}$$

by

$$\langle\, X^iY^{m-i},\, X^jY^{m-j}\,\rangle_m := \begin{cases} \dfrac{(-1)^i i!j!}{m!} & (i+j=m) \\ 0 & (i+j \neq 0) \end{cases}$$

This pairing satisfies

$$\langle\, \gamma f, \gamma g\,\rangle_m = \det(\gamma)^m \langle\, f, g\,\rangle_m$$

for any $\gamma \in \mathrm{M}_2(A)$ and $f, g \in \mathrm{Sym}^m(A)$. Let M be an A-module, and we define

$$\mathrm{Sym}^m(M) := \mathrm{Sym}(A) \otimes_A M.$$

As above, $\mathrm{Sym}^m(M)$ has the action of $\mathrm{M}_2(A)$ and an M-valued pairing.

Let A be a ring and M be an A-module. For any $n = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[I]$, we define

$$\mathrm{Sym}^n(M) := \bigotimes_{\sigma \in I} \mathrm{Sym}^{n_\sigma}(M),$$

where the tensor products are taken over A. We denote the indeterminates by X^σ, Y^σ. This module has a natural left action of semigroup $\prod_{\sigma \in I} \mathrm{M}_2(A)$. We also define the paring for A which is a $\mathbb{Z}_{(q)}$-algebra with $m < q$

$$\langle\,\cdot,\cdot\,\rangle_n := \bigotimes_\sigma \langle\,\cdot,\cdot\,\rangle_{n_\sigma} \colon \mathrm{Sym}^n(M) \otimes_A \mathrm{Sym}^n(M) \longrightarrow M.$$

Note that when $A = \mathbb{C}_p$, the natural embedding $F \otimes_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{C}_p$ induces an embedding

$$\prod_{\mathfrak{p} \mid p} \mathrm{M}_2(F_\mathfrak{p}) \hookrightarrow \prod_{\sigma \in I} \mathrm{M}_2(\mathbb{C}_p).$$

In particular, via this embedding and $i_\mathfrak{p}$ above, $\mathrm{Sym}^n(M)$ has an action of $\widehat{B}_p^\times$.

We fix a non empty (not necessarily whole) set $\{\, \mathfrak{p}_1, \ldots, \mathfrak{p}_m \,\}$ of prime ideals above p, and define

$$\mathbf{p} := \prod_{i=1}^m \mathfrak{p}_i.$$

For $n \in \mathbb{Z}[I]$, the perfect pairing $\langle\,\cdot,\cdot\,\rangle_{n_\sigma}$ for $\sigma \in I_\mathbf{p}$ induce

$$\langle\,\cdot,\cdot\,\rangle_\mathbf{p} \colon \mathrm{Sym}^n \times \mathrm{Sym}^n \longrightarrow \mathrm{Sym}^{n^\mathbf{p}}.$$

We fix a open compact subgroup $\Sigma \subset \widehat{B}^\times$ such that

$$\Sigma_p \subset \prod_{\mathfrak{p} \mid p} i_\mathfrak{p}^{-1}\big(\,\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})\big),$$

$$\Sigma_\mathbf{p} = \prod_{\mathfrak{p} \mid \mathbf{p}} i_\mathfrak{p}^{-1}\big(\,\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})\big).$$

We always denote by s or s' the elements of

$$\bigoplus_{\mathfrak{p} \mid \mathbf{p}} \mathbb{Z}\mathfrak{p} \ \big(\cong F_\mathbf{p}^\times / \mathcal{O}_{F_\mathbf{p}}^\times\big).$$

For $s = \sum_{\mathfrak{p} \mid \mathbf{p}} s_{\mathfrak{p}} \mathfrak{p}$, we define

$$\mathbf{p}^s := \prod_{\mathfrak{p} \mid \mathbf{p}} \mathfrak{p}^{s_{\mathfrak{p}}}$$

and denote by $s' \geqslant s$ (resp. $s > s'$) if

$$s' - s \in \bigoplus_{\mathfrak{p} \mid \mathbf{p}} \mathbb{Z}_{\geqslant 0} \mathfrak{p} \qquad \text{(resp. } \bigoplus_{\mathfrak{p} \mid \mathbf{p}} \mathbb{Z}_{>0} \mathfrak{p}).$$

We define the following open compact subgroups for s as above

$$\begin{aligned} \Sigma_0(\mathbf{p}^s) &:= K_0^B(\mathbf{p}^s) \cap \Sigma, \\ \Sigma_1(\mathbf{p}^s) &:= K_1^B(\mathbf{p}^s) \cap \Sigma \\ \Sigma(\mathbf{p}^s) &:= K^B(\mathbf{p}^s) \cap \Sigma, \end{aligned}$$

where

$$K_0^B(\mathbf{p}^s) := \left\{ u \in \widehat{B} \;\middle|\; \begin{array}{ll} \begin{cases} i_{\mathfrak{q}}(u_{\mathfrak{q}}) \in \mathrm{M}_2(\mathcal{O}_{F_{\mathfrak{q}}}) \\ i_{\mathfrak{q}}(u_{\mathfrak{q}}) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \bmod \mathbf{p}^s \mathrm{M}_2(\mathcal{O}_{F_{\mathfrak{q}}}) \end{cases} & \text{if } \mathfrak{q} \nmid \mathfrak{d} \\ u_{\mathfrak{q}} \in \{ \text{maximal order of } B \otimes_F F_{\mathfrak{q}} \} & \text{if } \mathfrak{q} \mid \mathfrak{d} \end{array} \right\}$$

$$K_1^B(\mathbf{p}^s) := \left\{ u \in K_0^B(\mathfrak{a}) \middle| i_{\mathfrak{q}}(u_{\mathfrak{q}}) \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \quad \bmod \mathbf{p}^s \mathrm{M}_2(\mathcal{O}_{F_{\mathfrak{q}}}) \text{ for } \mathfrak{q} \nmid \mathfrak{d} \right\}$$

$$K^B(\mathbf{p}^s) := \left\{ u \in K_0^B(\mathfrak{a}) \;\middle|\; i_{\mathfrak{q}}(u_{\mathfrak{q}}) \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \bmod \mathbf{p}^s \mathrm{M}_2(\mathcal{O}_{F_{\mathfrak{q}}}) \text{ for } \mathfrak{q} \nmid \mathfrak{d} \right\}.$$

Let

$$\Delta(\mathbf{p})_{\mathbf{p}} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}}) \;\middle|\; c \in \mathbf{p}\mathcal{O}_{F_{\mathbf{p}}},\ d \in \mathcal{O}_{F_{\mathbf{p}}}^{\times},\ ad - bc \neq 0 \right\}.$$

We define

$$\begin{aligned} \mathrm{Cl}_F^+(\Sigma(\mathbf{p}^s)) &:= \mathbb{A}_{F,f}^{\times} / F_+ (\mathbb{A}_{F,f}^{\times} \cap \Sigma(\mathbf{p}^s)) \mathbb{R}_{>0}^I \\ \mathbf{G}_s &:= \mathrm{Cl}_F^+(\Sigma(\mathbf{p}^s)) \times \Sigma_1(\mathbf{p}^s) / \Sigma(\mathbf{p}^s). \end{aligned}$$

Definition 2.1.3. Let $R \subset \mathbb{C}_p$ be a subring and let A be a R-module. Let $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t} \geqslant 0$ and $2w - k \in \mathbb{Z}\underline{t}$. Let $\boldsymbol{\omega} := (\omega, \omega') : \mathbf{G}_s \to \mathrm{Aut}_R(A)$ be a pair of group homomorphisms. For any $s \geqslant 0$, we define the spaces of A-coefficient p-adic quaternionic automorphic forms of weight (k, w) by $S(\Sigma(\mathbf{p}^s), \mathrm{Sym}^{k-2\underline{t}}(A) \otimes \det^{\underline{t}-w})$. We denote it by $S_{k,w}(\Sigma(\mathbf{p}^s); A)$. We also define the space of A-coefficient p-adic quaternionic forms of weight (k, w) and fo character ω or $\boldsymbol{\omega}$ by.

$$\begin{aligned} &S_{k,w}(\Sigma(\mathbf{p}^s), \omega; A) \\ &:= \left\{ f \in S_{k,w}(\Sigma(\mathbf{p}^s); A) \;\middle|\; f(bz) = \epsilon_{\mathrm{cyc},F}(z)^{[k-2w]} \omega(z) f(b) \text{for } z \in \mathbb{A}_F^{\times} \right\} \\ &S_{k,w}(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; A) \\ &= S_{k,w}(\Sigma(\mathbf{p}^s), \omega, \omega'; A) \\ &:= \left\{ f \in S_{k,w}(\Sigma(\mathbf{p}^s), \omega; A) \;\middle|\; f(bu) = \omega'(\det(u)) u^{-1} f(b) \text{ for } u \in \Sigma_1(\mathbf{p}^s) \right\} \end{aligned}$$

Remark 2.1.4 (The relation to usual automorphic forms). The relation between p-adic automorphic forms and automorphic forms over $B^{\times}$ is given as follows: let $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t} \geqslant 0$ and $2w - k \in \mathbb{Z}\underline{t}$ and let $\boldsymbol{\omega} = (\omega, \omega') : \mathbf{G}_s \to \mathbb{C}^{\times}$ be a pair of group homomorphims. For any commutative ring A, we denote by $\rho^A_{k,w}$ the action of $\mathrm{M}_2(A)$ on $\mathrm{Sym}^{k-2\underline{t}}(A)\otimes \det^{\underline{t}-w}$. For an open compact subgroup $U\subset \widehat{B}^\times$, we define

$$\begin{aligned}&\mathscr{A}^u_k(\Sigma(\mathbf{p}^s),\omega)\\ &:=\left\{f\colon B^\times\backslash B^\times(\mathbb{A}_F)\longrightarrow \mathrm{Sym}^{k-2\underline{t}}(\mathbb{C})\;\middle|\; \begin{array}{lll} f(bu) &= f(b) & \text{for any } u\in\Sigma(\mathbf{p}^s)\\ f(zb) &=\omega(z)f(b) & \text{for any } z\in\mathbb{A}_F^\times\end{array}\right\}.\\ &\mathscr{A}^u_k(\Sigma(\mathbf{p}^s),\boldsymbol{\omega}):=\left\{f\in\mathscr{A}^u_k\;\middle|\; f(bu) \;=\omega'(\det(u))f(b)\quad \text{for any } u\in\Sigma_1(\mathbf{p}^s)\right\}.\end{aligned}$$

Using the identification $\iota\colon \mathbb{C}_p\cong\mathbb{C}$, the following two morphism are inverses to each other:

$$\begin{array}{ccc} S_{k,w}(\Sigma(\mathbf{p}^s),\omega;\mathbb{C}_p) & \longrightarrow & \mathscr{A}^u_k(\Sigma(\mathbf{p}^s),\omega)\\ \Downarrow & & \Downarrow\\ \phi & \longmapsto & \phi^u(b):=\rho^{\mathbb{C}}_{k,w}(b_\infty^{-1})\iota\big(\rho^{\mathbb{C}_p}_{k,w}(b_p)\phi(b_f)\big)|\det(b)|_{\mathbb{A}_F}^{\frac{[k-2w]}{2}}, \end{array}$$

$$\begin{array}{ccc} \mathscr{A}^u_k(\Sigma(\mathbf{p}^s),\omega) & \longrightarrow & S_{k,w}(\Sigma(\mathbf{p}^s),\omega;\mathbb{C}_p)\\ \Downarrow & & \Downarrow\\ ;f & \longmapsto & \widehat{f}(b):=\rho^{\mathbb{C}_p}_{k,w}(b_p^{-1})\iota^{-1}\big(f(b_f)\big)|\det(b_f)|_{\mathbb{A}_{F,f}}^{-\frac{[k-2w]}{2}}, \end{array}$$

where we use the same notation ι to describe the identifcation $\mathrm{Sym}^{k-2\underline{t}}(\mathbb{C}_p)\cong\mathrm{Sym}^{k-2\underline{t}}(\mathbb{C})$ induced from $\iota\colon\mathbb{C}_p\cong\mathbb{C}$. We note that for $\phi_1,\phi_2\in S_{k,w}(\Sigma(\mathbf{p}^s),\omega)$, we have

$$\langle\phi_1^u(b),\phi_2^u(b)\rangle_{k-2\underline{t}}=\epsilon^{[2w-k]}_{\mathrm{cyc},F}(\mathrm{Nrd}_{B/F}(b))\langle\phi_1(b),\phi_2(b)\rangle_{k-2\underline{t}}. \tag{2.1.4}$$

2.2. Hida theory for definite quaternion algebras for totally real fields I

We fix a finite flat $\mathbb{Z}_p$-algebra $\mathcal{O}\subset\mathbb{C}_p$ containing $\sigma(\mathcal{O}_F)$ for all $\sigma\in I$ and a uniformizer $\varpi\in\mathcal{O}$. We denote by K the fraction field. For a prime ideal $\mathfrak{p}\mid p$ of F,

$$I_{\mathfrak{p}}:=\{\,\sigma\in I\mid\sigma\text{ factor through }F_{\mathfrak{p}}\,\}.$$

For $n\in\mathbb{Z}[I]$ and for any ideal $\mathfrak{a}\mid p$, we denote

$$\begin{aligned} I_{\mathfrak{a}}&:=\bigsqcup_{\mathfrak{p}\mid\mathfrak{a}} I_{\mathfrak{p}}\\ n_{\mathfrak{a}}&:=\sum_{\sigma\in I_{\mathfrak{a}}} n_\sigma\sigma,\\ n^{\mathfrak{a}}&:=\sum_{\sigma\notin I_{\mathfrak{a}}} n_\sigma\sigma.\end{aligned}$$

In particular, we have $n=\sum_{\mathfrak{p}\mid p} n_{\mathfrak{p}}$.

2.2.1. Normalized Hecke operators.

We define the *normalized Hecke operators*:

Definition 2.2.1. Let A be am $\mathcal{O}$-module. Let $k,w\in\mathbb{Z}[I]$ satisfying $k-2\underline{t}\geqslant 0$, $s>0$ and x an element of $\mathcal{O}_{F_{\mathfrak{p}}}$ such that $x^{\underline{t}_{\mathfrak{p}}}\neq 0$. For $\phi\in S_{k,w}(\Sigma(\mathbf{p}^s);A)$, we define the *normalized Hecke operator* $T_0(x)$ as follows:

$$\big(T_0(x)\phi\big)(b) := \sum_i \big(\det(\gamma_i)x^{-1}\big)^{\underline{t}-w}\,\phi(b\gamma_i)((X,Y)\gamma_i),$$

where $b \in \widehat{B}^\times$ and γ_i is determined from the following decomposition:

$$\Sigma(\mathbf{p}^s)\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\Sigma(\mathbf{p}^s) = \bigsqcup_i \gamma_i \Sigma(\mathbf{p}^s).$$

Note that $\big(\det(\gamma_i)x^{-1}\big)^{\underline{t}-w} \in \mathcal{O}^\times$ is regarded as a scalar morphsim in $\mathrm{Aut}_{\mathcal{O}}(A)$.

Lemma 2.2.2. Let $? = 0{,}1$ or $\varnothing$. The right coset decomposition of (2.1.2) for $U' = \Sigma_?(\mathbf{p}^s)$, $U = \Sigma_0(\mathbf{p}^{s'}) \cap \Sigma_?(\mathbf{p}^s)$ with $s' \geqslant s \geqslant 0$ and $g = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}_{\mathbf{p}}$ with $x \in \mathbf{p}^{s'-s}\mathcal{O}_{F_{\mathbf{p}}}$ such that $x^{\underline{t}_{\mathbf{p}}} \neq 0$ is explicitly given as follows:

$$\Sigma_?(\mathbf{p}^s)\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}_{\mathbf{p}}\Sigma_0(\mathbf{p}^{s'}) \cap \Sigma_?(\mathbf{p}^s) = \bigsqcup_{(c_{\mathfrak{p}})_{\mathfrak{p}\mid\mathbf{p}}} \prod_{\mathfrak{p}\mid\mathbf{p}} \gamma_{c_{\mathfrak{p}}}\Sigma_0(\mathbf{p}^{s'}) \cap \Sigma_?(\mathbf{p}^s),$$

where the index $(c_{\mathfrak{p}})_{\mathfrak{p}\mid\mathbf{p}}$ runs over

$$\prod_{s_{\mathfrak{p}}>0}\mathcal{O}_{F_{\mathfrak{p}}}/x\mathcal{O}_{F_{\mathfrak{p}}} \times \prod_{s_{\mathfrak{p}}=0}\mathcal{O}_{F_{\mathfrak{p}}}/x\mathcal{O}_{F_{\mathfrak{p}}} \bigsqcup \prod_{s_{\mathfrak{p}}>0}\mathcal{O}_{F_{\mathfrak{p}}}/x\mathcal{O}_{F_{\mathfrak{p}}} \times \prod_{s_{\mathfrak{p}}=0}\mathbf{p}\mathcal{O}_{F_{\mathfrak{p}}}/x\mathcal{O}_{F_{\mathfrak{p}}}.$$

We define $\gamma_{\mathfrak{p}}$ for $\mathfrak{p} \mid \mathbf{p}$ with $s_{\mathfrak{p}} > 0$ by

$$\gamma_{c_{\mathfrak{p}}} := \begin{pmatrix} x_{\mathfrak{p}} & c_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}_{\mathbf{p}} \qquad \text{for } c_{\mathfrak{p}} \in \mathcal{O}_{F_{\mathfrak{p}}}/x\mathcal{O}_{F_{\mathfrak{p}}}.$$

The $\gamma_{\mathfrak{p}}$ for $\mathfrak{p} \mid \mathbf{p}$ such that $s_{\mathfrak{p}} = 0$ are defined by

$$\gamma_{c_{\mathfrak{p}}} := \begin{pmatrix} x_{\mathfrak{p}} & c_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}_{\mathbf{p}} \qquad \text{for } c_{\mathfrak{p}} \in \big(\text{left hand side of } \mathcal{O}_{F_{\mathfrak{p}}}/x\mathcal{O}_{F_{\mathfrak{p}}}\big)$$

$$\gamma_{c_{\mathfrak{p}}} := \begin{pmatrix} 1 & 0 \\ c_{\mathfrak{p}} & x_{\mathfrak{p}} \end{pmatrix}_{\mathbf{p}} \qquad \text{for } c_{\mathfrak{p}} \in \big(\text{right hand side of } \mathbf{p}\mathcal{O}_{F_{\mathfrak{p}}}/x\mathcal{O}_{F_{\mathfrak{p}}}\big).$$

The point is that γ_c are independet of s and $? = 0, 1.\varnothing$.

PROOF. Put $U' = \Sigma_?(\mathbf{p}^s)$, $U = \Sigma_0(\mathbf{p}^{s'}) \cap \Sigma_?(\mathbf{p}^s)$ and $g = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}_{\mathbf{p}}$. We have bijections

$$U'/U' \cap gUg^{-1} \cong U'gU/U; \quad [u'] \mapsto [u'g].$$

On the other hand, we have

$$U'/U' \cap gUg^{-1} = \{\gamma_c\}_c\,.$$

Combining these explicit bijections, we have the formula. □

Theorem 2.2.3. Let A be an $\mathcal{O}$-module and $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t} \geqslant 0$ and $s \geqslant 0$. Let $\boldsymbol{\omega} = (\omega, \omega') : \mathbf{G}_s \to \mathrm{Aut}_{\mathcal{O}}(A)$ be a pair of group homomorphisms. Then the space $S_{k,w}(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; A)$ is stable under the normalized Hecke operator $T_0(x)$. Moreover, For $s' \geqslant s$, if $x \in \mathbf{p}^{s'-s}\mathcal{O}_{F_{\mathbf{p}}}$ such that $x^{\underline{t}_{\mathbf{p}}} \neq 0$, we have

$$T_0(x)\big(S_{k,w}(\Sigma(\mathbf{p}^{s'}), \boldsymbol{\omega}; A)\big) \subset S_{k,w}(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; A)$$

PROOF. It follows by direct computation and Lemma 2.2.2. □

Let r be a positive integer such that $s \geqslant \sum_{\mathfrak{p}\,|\,\mathbf{p}} r\mathbf{p} > 0$. Let

$$\omega\colon \mathrm{Cl}_F^+(\Sigma_0(\mathbf{p}^s)) \longrightarrow (\mathcal{O}/\varpi^r\mathcal{O})^\times = \mathrm{Aut}_{\mathcal{O}}\left(\varpi^{-r}\mathcal{O}/\mathcal{O}\right),$$
$$\omega'\colon \mathcal{O}_{F_\mathbf{p}}^\times \longrightarrow (\mathcal{O}/\varpi^r\mathcal{O})^\times.$$

For $n \in \mathbb{Z}[I_\mathbf{p}]$, we define

$$\chi_n\colon \mathcal{O}_{F_\mathbf{p}}^\times \longrightarrow (\mathcal{O}/\varpi^r\mathcal{O})^\times; x \mapsto x^n \mod \varpi^r\mathcal{O}.$$

Theorem 2.2.4. For $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t}$ and $2w - k = 2(\alpha-1)\underline{t}$ for some $\alpha \in \mathbb{Z}$. Fix an element $x_0 \in \mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}^\times$. We define two homomorphisms between spaces of automorphic forms of different weights

$$\iota\colon S_{k,w}\left(\Sigma(\mathbf{p}^s), \omega, \omega'; \varpi^{-r}\mathcal{O}/\mathcal{O}\right) \longrightarrow S_{2\underline{t}_\mathbf{p}+k^\mathbf{p}, \alpha\underline{t}_\mathbf{p}+w^\mathbf{p}}\left(\Sigma(\mathbf{p}^s), \omega, \omega'\chi_{w_\mathbf{p}-\alpha\underline{t}_\mathbf{p}}; \varpi^{-r}\mathcal{O}/\mathcal{O}\right)$$
$$\pi\colon S_{2\underline{t}_\mathbf{p}+k^\mathbf{p}, \alpha\underline{t}_\mathbf{p}+w^\mathbf{p}}\left(\Sigma(\mathbf{p}^s), \omega, \omega'\chi_{w_\mathbf{p}-\alpha\underline{t}_\mathbf{p}}; \varpi^{-r}\mathcal{O}/\mathcal{O}\right) \longrightarrow S_{k,w}\left(\Sigma(\mathbf{p}^s), \omega, \omega'; \varpi^{-r}\mathcal{O}/\mathcal{O}\right)$$

as follows:

$$\iota(\phi)(b) := \langle\, \phi(b),\, X^{k_\mathbf{p}-2\underline{t}_\mathbf{p}} \,\rangle_\mathbf{p},$$
$$\pi(\phi)(b) := \sum_{c \in \mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}} \phi\left(b\begin{pmatrix} x_0 & c \\ 0 & 1 \end{pmatrix}_\mathbf{p}\right) \cdot (cX+Y)^{k-2\underline{t}}.$$

The homomorphisms ι and π are well-defined. Moreover they satisfy the following formula:

$$\iota \circ \pi = T_0(x_0),$$
$$\pi \circ \iota = T_0(x_0).$$

Proof. Put

$$\omega'' := \omega'\chi_{w_\mathbf{p}-\alpha\underline{t}_\mathbf{p}}$$

For the first statement, note that since $s \geqslant \sum_{\mathfrak{p}\,|\,\mathbf{p}} r\mathfrak{p} > 0$, consider two $\Sigma_1(\mathbf{p}^s)$- homomorphisms

$$\begin{array}{rcl} i\colon \mathrm{Sym}^{k-2\underline{t}}(\varpi^{-r}\mathcal{O}/\mathcal{O}) \otimes \det^{\underline{t}-w} & \longrightarrow & \mathrm{Sym}^{k^\mathbf{p}-2\underline{t}^\mathbf{p}}(\varpi^{-r}\mathcal{O}/\mathcal{O}) \otimes \det^{\underline{t}-w} \\ \Psi & & \Psi \\ f & \longmapsto & f^\mathbf{p} \end{array}$$

$$\begin{array}{rcl} j\colon \mathrm{Sym}^{k^\mathbf{p}-2\underline{t}^\mathbf{p}}(\varpi^{-r}\mathcal{O}/\mathcal{O}) \otimes \det^{\underline{t}-w} & \longrightarrow & \mathrm{Sym}^{k-2\underline{t}}(\varpi^{-r}\mathcal{O}/\mathcal{O}) \otimes \det^{\underline{t}-w}; \\ \Psi & & \Psi \\ g & \longmapsto & gX^{k_\mathbf{p}-2\underline{t}_\mathbf{p}}, \end{array}$$

where $f^\mathbf{p}$ is an element of $\mathbb{Z}[\{X_\sigma, Y_\sigma\}_{\sigma\in I^\mathbf{p}}]$ given by substituting 0 for X_σ and 1 for Y_σ for $\sigma \in I_\mathbf{p}$ and $\tau_{x_0} := \begin{pmatrix} 0 & 1 \\ x_0 & 0 \end{pmatrix}_\mathbf{p}$. It immediately follows that ι is well-defined since ι is induced by i. For π, consider

$$\begin{array}{rcl} W\colon S_{2\underline{t}_\mathbf{p}+k^\mathbf{p},w}\left(\Sigma(\mathbf{p}^s), \omega, \omega''; \varpi^{-r}\mathcal{O}/\mathcal{O}\right) & \longrightarrow & S_{2\underline{t}_\mathbf{p}+k^\mathbf{p},w}\left(\Sigma(\mathbf{p}^s); \omega, \omega|_{\mathcal{O}_{F_\mathbf{p}}^\times}(\omega'')^{-1}; \varpi^{-r}\mathcal{O}/\mathcal{O}\right) \\ \Psi & & \Psi \\ \phi & \longmapsto & [b \mapsto \tau_{x_0}\phi(b\tau_{x_0})], \end{array}$$

We claim that

$$\left[\Sigma(\mathbf{p}^s)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_\mathbf{p}\Sigma(\mathbf{p}^s)\right]$$

induces

$$S_{k,w}\big(\Sigma(\mathbf{p}^s);\omega,\omega|_{\mathcal{O}_{F_\mathbf{p}}^\times}(\omega'')^{-1};\varpi^{-r}\mathcal{O}/\mathcal{O}\big)\longrightarrow S_{k,w}\big(\Sigma(\mathbf{p}^s),\omega,\omega'';\varpi^{-r}\mathcal{O}/\mathcal{O}\big)$$

In fact, since

$$\Sigma(\mathbf{p}^s)\begin{pmatrix}0&1\\1&0\end{pmatrix}_{\mathbf{p}}\Sigma(\mathbf{p}^s)=\bigsqcup_{c\in\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}}\begin{pmatrix}c&1\\1&0\end{pmatrix}_{\mathbf{p}}\Sigma(\mathbf{p}^s)$$

for any $u\in\Sigma_1(\mathbf{p}^s)$, there exists $c'\in\mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}$ and $u_c\in\Sigma_1(\mathbf{p}^s)$ such that

$$u\begin{pmatrix}c&1\\1&0\end{pmatrix}_{\mathbf{p}}=\begin{pmatrix}c'&1\\1&0\end{pmatrix}_{\mathbf{p}}\begin{pmatrix}0&1\\1&0\end{pmatrix}_{\mathbf{p}}u_c\begin{pmatrix}0&1\\1&0\end{pmatrix}_{\mathbf{p}},$$

$$\left[\Sigma(\mathbf{p}^s)\begin{pmatrix}0&1\\1&0\end{pmatrix}_{\mathbf{p}}\Sigma(\mathbf{p}^s)\right]\phi(bu)=\sum_c\omega(\omega\omega''^{-1})^{-1}(\det(u_c))u^{-1}\begin{pmatrix}c'&1\\1&0\end{pmatrix}_{\mathbf{p}}\phi\Big(b\begin{pmatrix}c'&1\\1&0\end{pmatrix}_{\mathbf{p}}\Big).$$

Since $\det(u_c)=\det(u)$, we have the claim. Clearly,

$$\varpi=\left[\Sigma(\mathbf{p}^s)\begin{pmatrix}0&1\\1&0\end{pmatrix}_{\mathbf{p}}\Sigma(\mathbf{p}^s)\right]\circ j_*\circ W,$$

where j_* is a homomorphism between the space of automorphic forms induced by j. Thus ϖ is well-defined. Since $s\geqslant\sum_{\mathfrak{p}\,|\,\mathbf{p}}r\mathbf{p}$, the second assertion follows immediately from the definition of $T_0(x_0)$ (Definition 2.2.1)

□

Lemma 2.2.5. Let $k,w\in\mathbb{Z}[I]$ such that $k-2\underline{t}\geqslant 0$. Suppose that $2w-k\in\mathbb{Z}\underline{t}$ and s is sufficiently large. Then the natural homomorphism

$$S_{k,w}(\Sigma(\mathbf{p}^s);\mathcal{O})\otimes_{\mathcal{O}}\varpi^{-r}\mathcal{O}/\mathcal{O}\longrightarrow S_{k,w}(\Sigma(\mathbf{p}^s);\varpi^{-r}\mathcal{O}/\mathcal{O})$$

is an isomorphism.

Proof. Let $A=\mathcal{O}$ or $\varpi^{-r}\mathcal{O}/\mathcal{O}$. Since we have a finite decomposition

$$\widehat{B}^\times=\bigsqcup_{i=1}^m B^\times t_i\Sigma(\mathbf{p}^s),$$

we have an injection of $\mathcal{O}$-modules

$$S_{k,w}(\Sigma(\mathbf{p}^s);A)\hookrightarrow\bigoplus_{i=1}^m\mathrm{Sym}^{k-2\underline{t}}(\mathcal{O});\qquad\phi\mapsto(\phi(t_i))_{i=1}^m.$$

The image of it is actually

$$\bigoplus_{i=1}^m\big(\mathrm{Sym}^{k-2\underline{t}}(A)\otimes\det{}^{\underline{t}-w}\big)^{\Delta_i},$$

where $\Delta_i:=t_i^{-1}B^\times t_i\cap\Sigma(\mathbf{p}^s)$. Put

$$\overline{\Delta}_i:=\Delta_i/\Sigma(\mathbf{p}^s)\cap\mathcal{O}_{F,+}^\times.$$

Since $2w - k \in \mathbb{Z}\underline{t}$, $\Sigma(\mathbf{p}^s) \cap \mathcal{O}_{F,+}^\times$ trivially acts on $\mathrm{Sym}^{k-2\underline{t}} \otimes \det^{\underline{t}w}$. The action of Δ_i factors through $\overline{\Delta}_i$. By considering the group cohomology of $\overline{\Delta}_i$, we have an exact sequence

$$\begin{aligned} 0 &\longrightarrow S_{k,w}(\Sigma(\mathbf{p}^s); \mathcal{O}) \otimes_{\mathcal{O}} \varpi^{-r}\mathcal{O}/\mathcal{O} \longrightarrow S_{k,w}(\Sigma(\mathbf{p}^s); \varpi^{-r}\mathcal{O}/\mathcal{O}) \\ &\longrightarrow \bigoplus_{i=1}^{m} H^1\left(\overline{\Delta}_i, \mathrm{Sym}^{k-2\underline{t}}(\mathcal{O}) \otimes \det{}^{\underline{t}-w}\right). \end{aligned}$$

For any s and i,

$$\Sigma(\mathbf{p}^s)\mathbb{A}_{F,f}^\times/\mathbb{A}_{F,f}^\times \bigcap t_i^{-1}\widehat{B}^\times t_i \mathbb{A}_{F,f}^\times/\mathbb{A}_{F,f}^\times$$

is a finite subset of $\widehat{B}^\times/\mathbb{A}_{F,f}^\times$. Thus for sufficiently large s, we have

$$\overline{\Delta}_i \subset \Sigma(\mathbf{p}^s)\mathbb{A}_F^\times/\mathbb{A}_F^\times \bigcap t_i^{-1}B^\times t_i \mathbb{A}_F^\times/\mathbb{A}_F^\times = \{1\}$$

and we've proved the required result. □

2.2.2. Hida's ordinary idempotents.

Proposition 2.2.6. Suppose that $k - 2\underline{t} \geqslant 0$. Let A be a finite $\mathcal{O}$-algebra and $x \in \mathcal{O}_{F_\mathbf{p}}$ such that $x^{\underline{t}_\mathbf{p}} \neq 0$. We have the following assertions:

(1) the limit $\lim_{n\to\infty} T_0(x)^{n!}$ exists in $\mathrm{End}_{\mathcal{O}}\left(S_{k,w}(\Sigma(\mathbf{p}^s); A)\right)$. We denote the limit by e_x call it *Hida's ordinary idempotent* associated with x,
(2) the endomorphism e_x is an idempotent, and
(3) the endomorphism e_x depend only on the class of x in a quotient set

$$k_{F_\mathbf{p}}/k_{F_\mathbf{p}}^\times \cong \prod_{\mathfrak{p}\mid\mathbf{p}} \{0, 1\}.$$

Proof. For the statement (1) and (2), let H be the $\mathcal{O}$-algebra generated by $T_0(x)$ in $\mathrm{End}_{\mathcal{O}}\left(S_{k,w}(\Sigma(\mathbf{p}^s); A)\right)$. Since H is a finite over $\mathcal{O}$, we have a decomposition

$$H \cong H_1 \times \cdots \times H_m,$$

where H_i is a finite local $\mathcal{O}$-algebra. Let $\epsilon_i \in H$ be am idempotent element corresponding to H_i and describe $T_0(x)$ as $\sum_{i=1}^m h_i\epsilon_i$. Since

$$\lim_{n\to\infty} h_i^{n!} = \begin{cases} 1 & h_i \in H_i^\times \\ 0 & \text{otherwise}, \end{cases}$$

the limit of $T_0(x)^{n!}$ exists and is clearly idempotent. For the assertion (3), since if $x, y \in \mathcal{O}_{F_\mathbf{p}}$ such that $x^{\underline{t}}, y^{\underline{t}} \neq 0$, we have

$$T_0(xy) = T_0(x)T_0(y) = T_0(y)T_0(x).$$

Thus we have

$$e_{xy} = e_x e_y = e_y e_x.$$

For $z \in \mathcal{O}_{F_\mathbf{p}}^\times$, we have $e_z = 1$. Thus we have the assertions. □

Definition 2.2.7. Let $\boldsymbol{\omega} = (\omega, \omega') : \mathbf{G}_s \to \mathrm{Aut}_{\mathcal{O}}(A)$ be pair of group homomorphisms. Let A be a finite $\mathcal{O}$-module, $k, w \in \mathbb{Z}[I]$ such that $k - 2t \geqslant 0$ and $s > 0$. We define

$$S_{k,w}^{\mathrm{ord}}\big(\Sigma(\mathbf{p}^s); A\big) := e_{\mathbf{p}} S_{k,w}\big(\Sigma(\mathbf{p}^s); A\big),$$
$$S_{k,w}^{\mathrm{ord}}\big(\Sigma(\mathbf{p}^s), \omega; A\big) := e_{\mathbf{p}} S_{k,w}\big(\Sigma(\mathbf{p}^s), \omega; A\big),$$
$$S_{k,w}^{\mathrm{ord}}\big(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; A\big) := e_{\mathbf{p}} S_{k,w}\big(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; A\big),$$

where $e_{\mathbf{p}} = \lim_{n\to\infty} T_0(x)^{n!}$ is the Hida's idempotent associated with $x \in \mathbf{p}\mathcal{O}_{F_{\mathbf{p}}}$ such that $x^{\underline{t}_{\mathbf{p}}} \neq 0$. Note that $e_{\mathbf{p}}$ is independent of the choice of x by Proposition 2.2.6 (3).

In the ordinary part, we have the following important theorems:

Theorem 2.2.8. Let A be a finite $\mathcal{O}$-module and $s' \geqslant s > 0$. be group homomorphisms. Then for $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t} \geqslant 0$ and $2w - k \in \mathbb{Z}\underline{t}$ we have an isomorphism

$$S_{k,w}^{\mathrm{ord}}\big(\Sigma(\mathbf{p}^s); A\big) \cong S_{k,w}^{\mathrm{ord}}\big(\Sigma(\mathbf{p}^{s'}); A\big).$$

Proof. It follows immediately from Thorem 2.2.3. ☐

Theorem 2.2.9. Let $\boldsymbol{\omega} = (\omega, \omega') : \mathbf{G}_s \to (\mathcal{O}/\varpi^r\mathcal{O})^\times$ be pair of group homomorphisms. Let $s \geqslant \sum_{\mathfrak{p} \mid \mathbf{p}} r\mathfrak{p}$ and let For $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t} \geqslant 0$ and $2w - k = 2(\alpha - 1)\underline{t}$ for some $\alpha \in \mathbb{Z}$, we have an isomorphism

$$S_{k,w}^{\mathrm{ord}}\big(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; \varpi^{-r}\mathcal{O}/\mathcal{O}\big) \cong S_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}}, \alpha\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}^{\mathrm{ord}}\big(\Sigma(\mathbf{p}^s), \omega, \omega'\chi_{w_{\mathbf{p}}-\alpha\underline{t}_{\mathbf{p}}}; \varpi^{-r}\mathcal{O}_{\boldsymbol{\omega}}/\mathcal{O}_{\boldsymbol{\omega}}\big)$$

induced from ι defined in Theorem 2.2.4.

Proof. It follows immediately from Theorem 2.2.4 ☐

2.2.3. The control theorem.

Definition 2.2.10. We define

$$\mathbf{G} := \varprojlim_s \mathbf{G}_s.$$

We have a $\mathbf{G}$-action on p-adic automorphic forms as follows:

Definition 2.2.11. Let $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t} \geqslant 0$ and $2w - k \in \mathbb{Z}\underline{t}$. For $s \geqslant 0$ and an $\mathcal{O}$-module A, we define a continuous action of $(z, u) \in \mathbf{G}$ on $\phi \in S_{k,w}\big(\Sigma_0(\mathbf{p}^s); A\big)$ by

$$(z, u)\phi(b) := \epsilon_{\mathrm{cyc},F}^{[2w^{\mathbf{p}} - w^{\mathbf{p}}]}(zz_{\mathbf{p}}^{-1})\phi(bzu)$$

Definition 2.2.12. For any ring R, we define the complete group ring $\widetilde{\Lambda}_R$ by

$$\widetilde{\Lambda}_R := R[[\mathbf{G}]] = \varprojlim_s R[\mathbf{G}_s].$$

Remark 2.2.13. Since $\mathbf{G} \cong \mathbb{Z}_p^m \times \{\,\text{torsion elements}\,\}$, $\widetilde{\Lambda}_{\mathcal{O}}$ is a finite flat extension of a ring isomorphic to $\mathcal{O}[[X_1, \ldots, X_m]]$. In particular, $\widetilde{\Lambda}_{\mathcal{O}}$ is a finite product of complete noetherian local rings with finite residue fields.

Definition 2.2.14. We define the weight space

$$\mathscr{X} := \mathrm{Hom}_{\mathrm{conti}}(\mathbf{G}, \mathbb{C}_p^\times).$$

For any subring $R \subset \mathcal{O}_{\mathbb{C}_p}$, we have a natural bijection

$$\mathscr{X} \cong \mathrm{Hom}_{R\text{-conti}}\left(\widetilde{\Lambda}_R, \mathbb{C}_p\right).$$

For $P \in \mathscr{X}$, we denote by P_R the kernel of R-algebra homomorphism corresponding to P.

Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ such that $2w^{\mathbf{p}} - k^{\mathbf{p}} \in 2(\alpha-1)\underline{t}^{\mathbf{p}}$. For $k_{\mathbf{p}}, w_{\mathbf{p}} \in \mathbb{Z}[I_{\mathbf{p}}]$ with $2w_{\mathbf{p}} + w^{\mathbf{p}} - k_{\mathbf{p}} - k^{\mathbf{p}} \in \mathbb{Z}\underline{t}$, the homomorphism

$$P_{k_{\mathbf{p}},w_{\mathbf{p}}} : \mathbf{G} \ni (z,a) \mapsto \ \epsilon_{\mathrm{cyc},F}^{[k_{\mathbf{p}}-2w_{\mathbf{p}}]}(z_{\mathbf{p}})a^{w_{\mathbf{p}}-\underline{t}_{\mathbf{p}}} \in \mathcal{O}^{\times}$$

induces an element of $\mathscr{X}$ (in case $I^{\mathbf{p}} = \varnothing$, we set $\alpha = 1$). For any finite order character

$$\boldsymbol{\omega} = (\omega, \omega') \colon \mathbf{G} \longrightarrow \mathbb{C}_p^{\times},$$

We denote $P_{k_{\mathbf{p}},w_{\mathbf{p}}}(z,a)\boldsymbol{\omega}(z,a)$ by $P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}}(z,a)$. We define $s(\boldsymbol{\omega}) > 0$ by

$$s(\boldsymbol{\omega}) := \min_{s>0}\left\{ s \;\middle|\; \boldsymbol{\omega} \text{ factors through } \mathbf{G}_s \right\}.$$

Definition 2.2.15. We define

$$\mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}},w^{\mathbf{p}}} \subset \mathscr{X}$$

by

$$\mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}},w^{\mathbf{p}}} := \left\{ P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}} \;\middle|\; \begin{array}{l} k_{\mathbf{p}}, w_{\mathbf{p}} \in \mathbb{Z}[I_{\mathbf{p}}] \text{ such that} \\ k - 2\underline{t} \geqslant 0 \text{ and } 2w - k \in 2\mathbb{Z}\underline{t} \quad (k := k_{\mathbf{p}} + k^{\mathbf{p}},\ w := w_{\mathbf{p}} + w^{\mathbf{p}}) \\ \boldsymbol{\omega} \colon \mathbf{G} \longrightarrow \mathbb{C}_p^{\times} \ : \text{ finite order character} \end{array} \right\}.$$

We call $P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}}$ an *arithmetic point* of weight $(k_{\mathbf{p}} + k^{\mathbf{p}}, w_{\mathbf{p}} + w^{\mathbf{p}})$ and character $\boldsymbol{\omega} = (\omega, \omega')$.

Lemma 2.2.16. Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $2w^{\mathbf{p}} - k^{\mathbf{p}} \in 2\mathbb{Z}\underline{t}_{\mathbf{p}}$ and let

$$\mathcal{F} \subset \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}},w^{\mathbf{p}}}$$

be a subset defined by

$$\mathcal{F} := \left\{ P_{2\underline{t}_{\mathbf{p}},([2w_{\mathbf{p}}-k_{\mathbf{p}}]/2+1)\underline{t}_p,\boldsymbol{\omega}} \middle| \boldsymbol{\omega} \colon \mathbf{G} \longrightarrow \mathbb{C}_p^{\times} \ : \ \text{finite order character} \right\}$$

Then we have

$$\bigcap_{P \in \mathcal{F}} P_R = \{0\}$$

in $\widetilde{\Lambda}_R$ for any subring $R \subset \mathcal{O}_{\mathbb{C}_p}$.

Proof. Note that $\widetilde{\Lambda}_R$ is regarded as the space of R-valued measures on $\mathbf{G}$. By the (p-adic) Stone-Weierstrass theorem [**?**], the $\mathbb{C}_p$-algebra generated by $\mathcal{F}$ is dense in the space of $\mathbb{C}_p$-valued functions on $\mathbf{G}$. Thus we have the lemma. □

Definition 2.2.17. For $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $2w^{\mathbf{p}} - k^{\mathbf{p}} \in \mathbb{Z}\underline{t}^{\mathbf{p}}$, we define

$$\mathscr{V}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}} := \varinjlim_{r} \varinjlim_{s} S_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}},\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}\left(\Sigma(\mathbf{p}^s), \varpi^{-r}\mathcal{O}/\mathcal{O}\right),$$

$$\mathscr{V}^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}} := \varinjlim_{r} \varinjlim_{s} S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}},\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}\left(\Sigma(\mathbf{p}^s), \varpi^{-r}\mathcal{O}/\mathcal{O}\right).$$

If $I^{\mathbf{p}} = \varnothing$, we simply denote them by $\mathscr{V}(\Sigma)_{\mathcal{O}}$ and $\mathscr{V}^{\mathrm{ord}}(\Sigma)_{\mathcal{O}}$, respectively.

Theorem 2.2.18. Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ such that $2w^{\mathbf{p}} - k^{\mathbf{p}} = 2(\alpha-1)\underline{t}^{\mathbf{p}}$ for some $\alpha \in \mathbb{Z}$ Let $P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}} \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}},w^{\mathbf{p}}}$ be an arithmetic point. Then we have

$$\big(\mathscr{V}^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_{\boldsymbol{\omega}}\big)[P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}}] \cong \Big(S^{\mathrm{ord}}_{k_{\mathbf{p}}+k^{\mathbf{p}},w_{\mathbf{p}}+w^{\mathbf{p}}}\big(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}); \mathcal{O}_{\boldsymbol{\omega}}\big) \otimes_{\mathcal{O}} K/\mathcal{O}\Big)[\boldsymbol{\omega}],$$

where $\mathcal{O}_{\boldsymbol{\omega}} \subset \mathbb{C}_p$ is an $\mathcal{O}$-algebra generated by the image of $\boldsymbol{\omega}$.

Proof. Put $\boldsymbol{\omega} := (\omega, \omega')$. Fix $r > 0$ and put $k := k_{\mathbf{p}} + k^{\mathbf{p}}, w := w_{\mathbf{p}} + w^{\mathbf{p}}$. Let $s \geqslant \max\{s(\boldsymbol{\omega}), r\}$ be sufficiently large. Note that

$$\begin{aligned} &S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}},\alpha\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}\big(\Sigma(\mathbf{p}^s), \omega, \omega'\chi_{w_{\mathbf{p}}-\alpha\underline{t}_{\mathbf{p}}}; \varpi^{-r}\mathcal{O}_{\boldsymbol{\omega}}/\mathcal{O}_{\boldsymbol{\omega}}\big) \\ &= S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}},\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}\big(\Sigma(\mathbf{p}^s); \varpi^{-r}\mathcal{O}_{\boldsymbol{\omega}}/\mathcal{O}_{\boldsymbol{\omega}}\big)[P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}}]. \end{aligned}$$

By Theorem 2.2.9, we have

$$S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}},\alpha\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}\big(\Sigma(\mathbf{p}^s), \omega, \omega'\chi_{w_{\mathbf{p}}-\alpha\underline{t}_{\mathbf{p}}}; \varpi^{-r}\mathcal{O}_{\boldsymbol{\omega}}/\mathcal{O}_{\boldsymbol{\omega}}\big) \cong S^{\mathrm{ord}}_{k,w}\big(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; \varpi^{-r}\mathcal{O}_{\boldsymbol{\omega}}/\mathcal{O}_{\boldsymbol{\omega}}\big).$$

For large s, by Lemma 2.2.5, we have

$$S^{\mathrm{ord}}_{k,w}\big(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; \varpi^{-r}\mathcal{O}_{\boldsymbol{\omega}}/\mathcal{O}_{\boldsymbol{\omega}}\big) \cong \big(S^{\mathrm{ord}}_{k,w}\big(\Sigma(\mathbf{p}^s); \mathcal{O}_{\boldsymbol{\omega}}\big) \otimes_{\mathcal{O}_{\boldsymbol{\omega}}} \varpi^{-r}\mathcal{O}_{\boldsymbol{\omega}}/\mathcal{O}_{\boldsymbol{\omega}}\big)[\boldsymbol{\omega}].$$

On the other hand, by applying Theorem 2.2.8, we have

$$S^{\mathrm{ord}}_{k,w}\big(\Sigma(\mathbf{p}^s); \mathcal{O}_{\boldsymbol{\omega}}\big) \cong S^{\mathrm{ord}}_{k,w}\big(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}); \mathcal{O}_{\boldsymbol{\omega}}\big).$$

Thus by taking limit along s and r, we have the theorem. □

The space $\mathscr{V}^{\mathrm{ord}}(\Sigma)_{\mathcal{O}}$ has a continuous action of $\mathbf{G}$ and is regarded as a $\widetilde{\Lambda}$-module.

Definition 2.2.19. For $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $2w^{\mathbf{p}} - k^{\mathbf{p}} \in 2\mathbb{Z}\underline{t}^{\mathbf{p}}$, we define the following $\widetilde{\Lambda}_{\mathcal{O}}$-module:

$$\begin{aligned} V_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}} &:= \mathrm{Hom}_{\mathrm{cont}}\big(\mathscr{V}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}},\ \mathbb{Q}_p/\mathbb{Z}_p\big) \\ \mathbf{S}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}) &:= \mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O}}}\Big(V_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}},\ \widetilde{\Lambda}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{D}^{-1}_{K/\mathbb{Q}_p}\Big) \\ V^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}} &:= \mathrm{Hom}_{\mathrm{cont}}\Big(\mathscr{V}^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}),\ \mathbb{Q}_p/\mathbb{Z}_p\Big) \\ \mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}) &:= \mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O}}}\Big(V^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}),\ \widetilde{\Lambda}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{D}^{-1}_{K/\mathbb{Q}_p}\Big) \end{aligned}$$

If $I^{\mathbf{p}} = \varnothing$, we simply denote them by $V(\Sigma; \widetilde{\Lambda}_{\mathcal{O}})$, $\mathbf{S}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}})$, $V^{\mathrm{ord}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}})$ and $\mathbf{S}^{\mathrm{ord}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}})$, respectively.

Theorem 2.2.20. Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $k^{\mathbf{p}} - 2\underline{t}^{\mathbf{p}} \geqslant 0$ and $2w^{\mathbf{p}} - k^{\mathbf{p}} = 2(\alpha-1)\underline{t}^{\mathbf{p}}$. Assume that for any finite order character $\boldsymbol{\omega} \colon \mathbf{G} \longrightarrow \mathbb{C}_p^{\times}$,

(2.2.1) $$S_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}},\alpha\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}\big(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}), \boldsymbol{\omega}; \mathcal{O}_{\boldsymbol{\omega}}\big) \otimes_{\mathcal{O}} K/\mathcal{O} \text{ is divisivle.}$$

Let

$$\widetilde{\Lambda}_{\mathcal{O}} \cong \prod_j \Lambda_j$$

be the finite decomposition such that Λ_j are local rings and put $u_j \in \widetilde{\Lambda}_{\mathcal{O}}$ as the idempotent corresponding to Λ_j. Then the space $u_j\mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}})$ is free of finite rank over Λ_j. For any $P = P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}} \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}},w^{\mathbf{p}}}$, we have an isomorphism

$$\mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}\big)/P_{\mathcal{O}}\mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{p}},w^{\mathbf{p}}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}\big) \cong S^{\mathrm{ord}}_{k_{\mathbf{p}}+k^{\mathbf{p}},w_{\mathbf{p}}+w^{\mathbf{p}}}\big(\Sigma(p^{s(\boldsymbol{\omega})}), \boldsymbol{\omega}; \mathcal{O}_{\boldsymbol{\omega}}\big),$$

where $\mathcal{O}_{\boldsymbol{\omega}} \subset \mathbb{C}_p$ is the algebra generated by the image of $\boldsymbol{\omega}$.

Proof. For the first assertion, it suffices to prove that $u_j V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)$ is free of finite rank over Λ_j. By Lemma 2.2.16, there exists an arithmetic point $Q := P_{2\underline{t},\alpha\underline{t}_{\mathbf{p}},\eta}$ on $\mathbf{G}$ such that

$$Q_{\mathcal{O}} \in \mathrm{Spec}(\Lambda_j).$$

The dual of Theorem 2.2.18 is described as

$$\begin{aligned} V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}/Q_{\mathcal{O}}V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}} &\cong V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}_\eta}/Q_{\mathcal{O}_\eta}V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}_\eta} \\ &\cong \frac{S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{P}},(\alpha+1)\underline{t}_{\mathbf{p}}+w^{\mathbf{P}}}\left(\Sigma(\mathbf{p}^{s(\eta)});\mathcal{O}_\eta\right)^\vee}{\langle \eta(g)-g\rangle_{g\in\mathbf{G}} S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{P}},(\alpha+1)\underline{t}_{\mathbf{p}}+w^{\mathbf{P}}}\left(\Sigma(\mathbf{p}^{s(\eta)});\mathcal{O}_\eta\right)^\vee}, \end{aligned}$$

where $(\cdot)^\vee$ means $\mathrm{Hom}_{\mathcal{O}}(\cdot,\mathcal{D}^{-1}_{K/\mathcal{O}})$. Here, $\mathcal{D}_{K/\mathbb{Q}_p}$ is the different ideal of K over $\mathbb{Q}_p$ and the trace $\mathrm{Tr}_{K/\mathbb{Q}_p}$ induces a natural isomorphism

$$\mathcal{D}^{-1}_{K/\mathbb{Q}_p} \cong \mathrm{Hom}_{\mathrm{cont}}(K/\mathcal{O},\mathbb{Q}_p/\mathbb{Z}_p). \tag{2.2.2}$$

Thus we conclude that, for any j, $u_j V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}$ is finite over Λ_j. Let κ_j be the residue field of Λ_j and let

$$m := \dim_{\kappa_j} V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}} \otimes_{\Lambda_j} \kappa_j.$$

Then for any finite character $\rho\colon \mathbf{G} \longrightarrow \mathbb{C}_p^\times$, we put

$$R_{\mathcal{O}} := (P_{2\underline{t}_{\mathbf{p}},\alpha\underline{t}_{\mathbf{p}},\rho})_{\mathcal{O}} \in \mathrm{Spec}(\Lambda_j).$$

By the assumption (2.2.1), the module

$$V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}/R_{\mathcal{O}}V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}$$

is torsion free and we have

$$m = \mathrm{rank}_{\mathcal{O}} V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}/R_{\mathcal{O}}V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}.$$

Take a surjection

$$\Psi\colon \Lambda_j^m \longrightarrow u_j V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}},$$

Since for any ρ, we have proved

$$\mathrm{Ker}(\Psi) \subset R_{\mathcal{O}}\Lambda_j^m,$$

by Lemma 2.2.16, we have

$$\mathrm{Ker}(\Psi) = \{0\}.$$

It is the required result. For the second assertion, suppose $P = P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}} \in \mathrm{Spec}(\Lambda_j)$, the dual of Theorem 2.2.18 for P is described as follows:

$$u_j V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}/P_{\mathcal{O}}u_j V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}} \cong S^{\mathrm{ord}}_{k_{\mathbf{p}}+k^{\mathbf{P}},w_{\mathbf{p}}+w^{\mathbf{P}}}\left(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}),\boldsymbol{\omega};\mathcal{O}_{\boldsymbol{\omega}}\right)^\vee$$

Since $u_j V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}$ is free of finite rank over Λ_i, we have

$$\begin{aligned} &\mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma;\widetilde{\Lambda}_{\mathcal{O}})/P_{\mathcal{O}}\mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma;\widetilde{\Lambda}_{\mathcal{O}}) \\ &\cong \mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O}}}\left(V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}}/P_{\mathcal{O}}V^{\mathrm{ord}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\Sigma)_{\mathcal{O}},\Lambda_{\mathcal{O}}/P_{\mathcal{O}}\otimes\mathcal{D}^{-1}_{K/\mathbb{Q}_p}\right) \\ &\cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{\omega}}}\left(\frac{S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{P}},(\alpha+1)\underline{t}_{\mathbf{p}}+w^{\mathbf{P}}}\left(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})});\mathcal{O}_{\boldsymbol{\omega}}\right)^\vee}{\langle \boldsymbol{\omega}(g)-g\rangle_{g\in\mathbf{G}} S^{\mathrm{ord}}_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{P}},(\alpha+1)\underline{t}_{\mathbf{p}}+w^{\mathbf{P}}}\left(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})});\mathcal{O}_{\boldsymbol{\omega}}\right)^\vee},\mathcal{D}^{-1}_{K/\mathbb{Q}_p}\right) \\ &\cong S^{\mathrm{ord}}_{k_{\mathbf{p}}+k^{\mathbf{P}},w_{\mathbf{p}}+w^{\mathbf{P}}}\left(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}),\boldsymbol{\omega};\mathcal{O}_{\boldsymbol{\omega}}\right) \end{aligned}$$

□

Remark 2.2.21. Let
$$\widehat{B}^\times = \bigsqcup_{i=1}^m B^\times t_i \Sigma$$
be a decomposition. Then the assumption (2.2.1) satisfies if each
$$\Delta_i := \Sigma \cap t_i^{-1} B t_i$$
is torsin free modulo center, namely $\Delta_i = \mathcal{O}_F^\times$. and p is odd. In fact, for $P_{\underline{t}_{\mathbf{p}}, \alpha \underline{t}_{\mathbf{p}}, \psi} \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}}, w^{\mathbf{p}}}$, the module
$$S_{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}}, \alpha\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}\big(\Sigma(\mathbf{p}^{s(\psi)}); \mathcal{O}_{\boldsymbol{\omega}}\big) \otimes_{\mathcal{O}} K/\mathcal{O}[\boldsymbol{\omega}]$$
is isomorphic to
$$\bigoplus_{i=1}^m \left(\mathrm{Sym}^{2\underline{t}_{\mathbf{p}}+k^{\mathbf{p}}, \alpha\underline{t}_{\mathbf{p}}+w^{\mathbf{p}}}(\mathcal{O}_{\boldsymbol{\omega}}) \otimes_{\mathcal{O}} (K/O) \otimes \det{}^{(\alpha-1)\underline{t}_{\mathbf{p}}+\underline{t}^{\mathbf{p}}-w^{\mathbf{p}}-k^{\mathbf{p}}} \boldsymbol{\omega}\right)^{\mathcal{O}_F^\times}.$$
and is divisible since p is odd and $\psi(\mathcal{O}_{F,+}^\times \cap \mathbf{G}) = 1$. If we were to define the $\mathbf{G}$ as
$$\mathbf{G}_s := \mathrm{Cl}_F(\Sigma(\mathbf{p}^s)) \times (\mathcal{O}_{F_{\mathbf{p}}}/\mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}})^\times,$$
$$\mathbf{G} := \varprojlim_s \mathbf{G}_s,$$
where $\mathrm{Cl}_F(\Sigma(\mathbf{p}^s)) := \mathbb{A}_{F,f}^\times / F^\times \mathbb{A}_{F,f} \cap \Sigma(\mathbf{p}^s)$, we could prove a weaker version of Theorem 2.2.20 in the same manner. (We have to modify the definition of arithmetic points by changing the condition "$k - 2w \in \mathbb{Z}\underline{t}$" to "$k - 2w \in 2\mathbb{Z}\underline{t}$ if $I^{\mathbf{p}} = \varnothing$.") Under this modification, we prove the control theorem even if p is even.

Definition 2.2.22. Let $\mathbb{I}$ be a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra. We define the space of $\mathbb{I}$-*adic forms* by
$$\mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{p}}, w^{\mathbf{p}}}(\Sigma; \mathbb{I}) := \mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O}}}\big(V^{\mathrm{ord}}_{k^{\mathbf{p}}, w^{\mathbf{p}}}(\Sigma)_{\mathcal{O}}, \mathbb{I}\big).$$

Remark 2.2.23. Under the assumption of Theorem 2.2.20, we have
$$\mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{p}}, w^{\mathbf{p}}}(\Sigma; \mathbb{I}) \cong \mathbf{S}^{\mathrm{ord}}_{k^{\mathbf{p}}, w^{\mathbf{p}}}(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}) \otimes_{\widetilde{\Lambda}_{\mathcal{O}}} \mathbb{I}.$$

2.2.4. A reformulation of $\mathbf{S}(\Sigma; \mathbb{I})$. In this section, we assume that every prime above p divides $\mathbf{p}$. It means that all of the conditions of upper $\mathbf{p}$ (for example, $2w^{\mathbf{p}} - k^{\mathbf{p}} \in 2\mathbb{Z}\underline{t}^{\mathbf{p}}$) are empty. We omit to write notations involved with upper $\mathbf{p}$ such a $k^{\mathbf{p}}, w^{\mathbf{p}}, \underline{t}^{\mathbf{p}}$ and we simply write $k, w, \underline{t}, \ldots$ instead of $k_{\mathbf{p}}, w_{\mathbf{p}}, \underline{t}_{\mathbf{p}}, \ldots$. We fix a finite product of noetherian complete local $\widetilde{\Lambda}_{\mathcal{O}}$-algebras with finite residue fields and denote it by $\mathbb{I}$. Let
$$P_s := \mathrm{Ker}\left(\widetilde{\Lambda}_{\mathcal{O}} \longrightarrow \mathcal{O}[[\mathbf{G}_s]]\right)$$
We define
$$\mathrm{Pic}_{\mathcal{O}} X(\Sigma(\mathbf{p}^s)) := \bigoplus_{x \in X(\Sigma(\mathbf{p}^s))} \mathcal{O}x$$
be a free abelian group generated by elements of $X(\Sigma(\mathbf{p}^s))$. Since we have a natural perfect pairing
$$\mathrm{Pic}_{\mathcal{O}} X(\Sigma(\mathbf{p}^s)) \times S_{2\underline{t}, \underline{t}}\big(\Sigma(\mathbf{p}^s); K/\mathcal{O}\big) \longrightarrow K/\mathcal{O}; \quad (x, \phi) \mapsto \phi(x),$$
the following isomorphism holds:
$$V(\Sigma)_{\mathcal{O}} \cong \varprojlim_s \mathrm{Pic}_{\mathcal{O}} X(\Sigma(\mathbf{p}^s)).$$

Therefore, for a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra $\mathbb{I}$, we have

$$\mathbf{S}(\Sigma;\mathbb{I}) \cong \mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O}}}\big(\varprojlim_{s} \mathrm{Pic}_{\mathcal{O}}\, X(\Sigma(\mathbf{p}^s)),\ \mathbb{I}\big).$$

Theorem 2.2.24. We have isomorphisms

$$\begin{aligned}\mathbf{S}(\Sigma,\mathbb{I}) &\cong \varprojlim_{s} \mathrm{Hom}_{\mathcal{O}[\mathbf{G}_s]}\big(\mathrm{Pic}_{\mathcal{O}}\, X(\Sigma(\mathbf{p}^s)),\ \mathbb{I}/P_s\mathbb{I}\big)\\ &\cong \mathcal{C}^0(X(\Sigma^p N(\mathcal{O}_{F_p}));\mathbb{I})^{\mathbf{G}}\\ &:= \left\{ \begin{array}{l} \mathbf{f}\colon X(\Sigma^p N(\mathcal{O}_{F_p})) \longrightarrow \mathbb{I}\\ \text{continuous}\end{array} \,\middle|\, \begin{array}{l} t\cdot \mathbf{f}(xt) = \mathbf{f}(x)\\ \text{for } x \in X(\Sigma^p N(\mathcal{O}_{F_p})),\ t \in \mathbf{G}\end{array} \right\}\end{aligned}$$

where the topology on $\mathbb{I}$ is defined by giving the discrete topology to each $\mathbb{I}/P_s\mathbb{I}$.

PROOF. For the first isomorphism,

$$\begin{aligned}\mathbf{S}(\Sigma;\mathbb{I}) &\cong \mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O}}}\big(\varprojlim_{s'} \mathrm{Pic}_{\mathcal{O}}\, X(\Sigma(\mathbf{p}^{s'})),\ \mathbb{I}\big)\\ &\cong \mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O}}}\big(\varprojlim_{s'} \mathrm{Pic}_{\mathcal{O}}\, X(\Sigma(\mathbf{p}^{s'})),\ \varprojlim_{s} \mathbb{I}/P_s\mathbb{I}\big)\\ &\cong \varprojlim_{s} \mathrm{Hom}_{\mathcal{O}[\mathbf{G}_s]}\big(\varprojlim_{s'} \mathrm{Pic}_{\mathcal{O}}\, X(\Sigma(\mathbf{p}^{s'})),\ \mathbb{I}/P_s\mathbb{I}\big)\\ &\cong \varprojlim_{s} \mathrm{Hom}_{\mathcal{O}[\mathbf{G}_s]}\big(\mathrm{Pic}_{\mathcal{O}}\, X(\Sigma(\mathbf{p}^{s})),\ \mathbb{I}/P_s\mathbb{I}\big).\end{aligned}$$

For the second isomorphism, since

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}[\mathbf{G}_s]}\big(\mathrm{Pic}_{\mathcal{O}}\, \Sigma(\mathbf{p}^s),\ \mathbb{I}/P_s\mathbb{I}\big)\\ &\quad\cong \left\{ \mathbf{f}\colon X(\Sigma(\mathbf{p}^s)) \longrightarrow \mathbb{I}/P_s\mathbb{I} \,\middle|\, \begin{array}{l} \mathbf{f}(xt) = t^{-1}\mathbf{f}(x)\\ \text{for } x \in X(\Sigma(\mathbf{p}^s)),\ t \in \mathbf{G}_s\end{array} \right\},\end{aligned}$$

we prove that the canonical inclusion

$$\begin{aligned}&\varprojlim_{s} \left\{ f_s\colon X(\Sigma(\mathbf{p}^s)) \longrightarrow \mathbb{I}/P_s\mathbb{I} \,\middle|\, f_s(xt) = t^{-1} f_s(x) \text{ for } x \in X(\Sigma(\mathbf{p}^s)), t \in \mathbf{G}_s \right\}\\ &\quad\hookrightarrow \left\{ \begin{array}{l} \mathbf{f}\colon X(\Sigma^p N(\mathcal{O}_{F_p})) \longrightarrow \mathbb{I}\\ \text{continuous}\end{array} \,\middle|\, \begin{array}{l} \mathbf{f}(xt) = t^{-1}\mathbf{f}(x)\\ \text{for } x \in X(\Sigma^p N(\mathcal{O}_{F_p})),\ t \in \mathbf{G}\end{array} \right\}.\end{aligned}$$

is surjective. In fact, let $\mathbf{f}$ be an element of right hand side. Consider the right action of $\mathbf{G}_s$ on $X(\Sigma(\mathbf{p}^s))$ and describe the right coset decomposition as

$$X(\Sigma(\mathbf{p}^s)) = x_1\mathbf{G}_s \sqcup \cdots \sqcup x_r\mathbf{G}_s.$$

We choose a lift $y_i \in X(\Sigma^p N(\mathcal{O}_{F_p}))$ of x_i and define for $t \in \mathbf{G}_s$

$$f_s\colon X(\Sigma(\mathbf{p}^s)) \longrightarrow \mathcal{O}[\mathbf{G}_s]; \qquad x_i t \mapsto t^{-1} f(y_i) \mod P_s.$$

Clearly, f_s commutes with the action of $\mathbf{G}_s$. We claim that, for $s' > s$, there exists a $\mathbf{G}_{s'}$-equivariant function $f_{s'}\colon X(\Sigma(\mathbf{p}^{s'})) \longrightarrow \mathbb{I}/P_{s'}\mathbb{I}$ such that

$$\begin{array}{ccc} X(\Sigma(\mathbf{p}^{s'})) & \xrightarrow{f_{s'}} & \mathbb{I}/P_{s'}\mathbb{I}\\ \downarrow & & \downarrow\\ X(\Sigma(\mathbf{p}^{s})) & \xrightarrow{f_{s}} & \mathbb{I}/P_{s}\mathbb{I}\end{array}$$

is commutative. Let $\widetilde{x}_i \in X(\Sigma(\mathbf{p}^{s'}))$ be a lift of x_i. We also have

$$X(\Sigma(\mathbf{p}^{s'})) = \widetilde{x}_1 \mathbf{G}_{s'} \sqcup \cdots \sqcup \widetilde{x}_r \mathbf{G}_{s'}.$$

We define f'_s by

$$f_{s'}(\widetilde{x}_i t') = {t'}^{-1} f(y_i)$$

for $t' \in \mathbf{G}_{s'}$. Thus we have a projective system (f_s) of the left hand side. Since $\mathbf{f}$ is a continuous function, we conclude that $\varprojlim f_s = \mathbf{f}$. ☐

Let $x \in \mathcal{O}_{F_p}$ such that $x^{\underline{t}} \neq 0$. For any function $f \in \mathcal{C}^0\big(X(\Sigma^p N(\mathcal{O}_{F_p})); \mathbb{I}\big)$, we define $T_0(x)f$ by

$$T_0(x)f(b) := \sum_{c \in \mathcal{O}_{F_p}/x\mathcal{O}_{F_p}} f\left(b \begin{pmatrix} x & c \\ 0 & 1 \end{pmatrix}\right).$$

If $f \in \mathbf{S}(\Sigma; \mathbb{I})$, $T_0(x)f$ is also in $\mathbf{S}(\Sigma; \mathbb{I})$. Moreover, $T_0(x)^{n!} f$ converges for any f and induces an operator e_x on $S(\Sigma; \mathbb{I})$, which is in fact the limit of Hida's idempotents e_x. In particular, $f \in S^{\mathrm{ord}}(\Sigma; \mathbb{I})$ if and only if $e_p f = f$, where $e_p := \lim_n T_0(p)^{n!}$.

Remark 2.2.25. Let $P = P_{k,w,\boldsymbol{\omega}} \in \mathscr{X}^{\mathrm{arith}}$. The inverse of the isomorphism of Theorem 2.2.20

$$\mathbf{S}^{\mathrm{ord}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}_{\boldsymbol{\omega}}}\big)/P_{\mathcal{O}_{\boldsymbol{\omega}}} \mathbf{S}^{\mathrm{ord}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}_{\boldsymbol{\omega}}}\big) \cong S^{\mathrm{ord}}_{k,w}\big(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}), \boldsymbol{\omega}; \mathcal{O}_{\boldsymbol{\omega}}\big)$$

is described as follows: let $\phi \in S^{\mathrm{ord}}_{k,w}\big(\Sigma(\mathbf{p}^{s(\psi)}); \mathcal{O}_\psi\big)[\psi]$. We define $\mathbf{f}_\phi \in \mathbf{S}^{\mathrm{ord}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}/P_{\mathcal{O}} \widetilde{\Lambda}_{\mathcal{O}}\big)$ by

$$\mathbf{f}_\phi(b) := \langle X^{k-2\underline{t}}, \phi(b) \rangle_{k-2\underline{t}} \;\in\; \mathcal{O}_\psi \cong \widetilde{\Lambda}_{\mathcal{O}}/P_{\mathcal{O}} \widetilde{\Lambda}_{\mathcal{O}}$$

for $b \in X(\Sigma^p N_p)$. As in Remark 2.2.23,

$$\mathbf{S}^{\mathrm{ord}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}/P_{\mathcal{O}} \widetilde{\Lambda}_{\mathcal{O}}\big) \cong \mathbf{S}^{\mathrm{ord}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}}\big) \otimes_{\widetilde{\Lambda}_{\mathcal{O}}} \widetilde{\Lambda}_{\mathcal{O}}/P_{\mathcal{O}}.$$

Thus the required inverse is the following correspondence

$$\phi \mapsto \mathbf{f}_\phi \mod P_{\mathcal{O}_\psi} \mathbf{S}^{\mathrm{ord}}\big(\Sigma; \widetilde{\Lambda}_{\mathcal{O}_\psi}\big).$$

2.3. Hida theory for definite quaternion algebras for totally real fields II

In this section, we treat general $\mathbf{p}$. We fix a finite product of noetherian complete local $\mathbb{Z}_p[[\mathbf{G}]]$-algebras with finite residue fields and denote it by $\mathbb{I}$.

2.3.1. The generality of measure valued form. Let $T(\mathcal{O}_{F_\mathbf{p}})$ act $\mathbb{I}$ via the following homomorphism:

$$T(\mathcal{O}_{F_\mathbf{p}}) \ni (t_1, t_2) \mapsto (\overline{t_2}^{-1}, t_1^{-1} t_2) \in \mathbf{G},$$

where $\overline{t_2}$ is the image of t_2 in $\mathrm{Cl}_F^+(\Sigma(\mathbf{p}^s))$. We fix a uniformizer $\varpi_\mathfrak{p}$ of $\mathcal{O}_{F_\mathfrak{p}}$ for each $\mathfrak{p} \mid p$ and denote $(\varpi_\mathfrak{p})_{\mathfrak{p} \mid \mathbf{p}}$ by $\varpi_\mathbf{p}$. We define a subgroup

$$\varPi := \prod_{\mathfrak{p} \mid \mathbf{p}} \begin{pmatrix} \varpi_\mathfrak{p}^{\mathbb{Z}} & 0 \\ 0 & 1 \end{pmatrix} \subset \mathrm{GL}_2(F_\mathbf{p}).$$

Definition 2.3.1. We define

$$\begin{array}{l} \mathcal{M} := \mathrm{GL}_2(\mathcal{O}_{F_\mathbf{p}})/{}^t N(\mathcal{O}_{F_\mathbf{p}}) \\ \cap \\ M := \mathrm{GL}_2(F_\mathbf{p})/\varPi\, {}^t N(F_\mathbf{p}) \end{array}$$

Then we have the following fibration

$$T(\mathcal{O}_{F_{\mathbf{p}}}) \longrightarrow \mathcal{M} \longrightarrow \mathbb{P}^1(F_{\mathbf{p}}).$$

We write $\mathbf{p} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$ and define

$$\mathcal{S} = \{\, \mathfrak{p}_1, \ldots, \mathfrak{p}_m \,\}.$$

As a set, we consider $\mathbb{P}^1(F_{\mathbf{p}})$ as $\big(F_{\mathbf{p}} \times F_{\mathbf{p}} \backslash \{\,(0,0)\,\}\big)/F_{\mathbf{p}}^\times$ and we take open coverings $\{U_A\}_{A \subset \mathcal{S}}$ and $\{U_A^\circ\}_{A \subset \mathcal{S}}$ of $\mathbb{P}^1(F_{\mathbf{p}})$ as follows: for any subset $A \subset \mathcal{S}$, we define anate system of $\mathbf{P}^1(F_{\mathbf{p}})$ by

$$\varphi_A: \qquad U_A := \prod_{\mathfrak{p} \in A} \mathcal{O}_{F_\mathfrak{p}} \times \prod_{\mathfrak{p} \in \mathcal{S} \backslash A} \mathcal{O}_{F_\mathfrak{p}} \xrightarrow{\cong} \prod_{\mathfrak{p} \in A} \big[\mathcal{O}_{F_\mathfrak{p}} \times \{1\}\big] \times \prod_{\mathfrak{p} \in \mathcal{S} \backslash A} \big[\{1\} \times \mathcal{O}_{F_\mathfrak{p}}\big],$$

$$U_A^\circ := \prod_{\mathfrak{p} \in A} \mathcal{O}_{F_\mathfrak{p}} \times \prod_{\mathfrak{p} \in \mathcal{S} \backslash A} \mathfrak{p}\mathcal{O}_{F_\mathfrak{p}} \xrightarrow{\cong} \prod_{\mathfrak{p} \in A} \big[\mathcal{O}_{F_\mathfrak{p}} \times \{1\}\big] \times \prod_{\mathfrak{p} \in \mathcal{S} \backslash A} \big[\{1\} \times \mathfrak{p}\mathcal{O}_{F_\mathfrak{p}}\big]$$

Let $\mu \colon \mathcal{M} \longrightarrow \mathbb{P}^1(F_{\mathbf{p}})$ be the projection. Then $\mathcal{M}$ is a locally trivial $T(\mathcal{O}_{F_{\mathbf{p}}})$-bundle described by

$$\begin{aligned} U_A \times T(\mathcal{O}_{F_{\mathbf{p}}}) &\cong \mu^{-1}\big(U_A\big) \\ (z, [t_1, t_2]) &\mapsto \left[\begin{pmatrix} 1 & z_\mathfrak{p} \\ 0 & 1 \end{pmatrix}_{\mathfrak{p} \in A} \begin{pmatrix} 0 & 1 \\ 1 & z_\mathfrak{p} \end{pmatrix}_{\mathfrak{p} \in \mathcal{S} \backslash A} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right], \end{aligned}$$

$$\begin{aligned} \varphi_A^{-1}(\varphi_A(U_A) \cap \varphi_B(U_B)) \times T(\mathcal{O}_{F_{\mathbf{p}}}) &\cong \varphi_B^{-1}(\varphi_A(U_A) \cap \varphi_B(U_B)) \times T(\mathcal{O}_{F_{\mathbf{p}}}) \\ (z, t) &\mapsto \big((z_\mathfrak{p})_{\mathfrak{p} \in \mathcal{S} \backslash (A \triangle B)} (z_\mathfrak{p}^{-1})_{\mathfrak{p} \in A \triangle B},\ (-z_\mathfrak{p}^{-1}, z_\mathfrak{p})_{\mathfrak{p} \in A \triangle B} t \big), \end{aligned}$$

where $A, B \subset \mathcal{S}$ and $A \triangle B := A \cup B \backslash A \cap B$. *From now on, we use these local coordinates for local computations.* We define an $\mathbb{I}$-bundle $\mathscr{E}$ over $\mathbb{P}^1(F_{\mathbf{p}})$ by

$$\mathscr{E} := \mathcal{M} \times_{T(\mathcal{O}_{F_{\mathbf{p}}})} \mathbb{I}.$$

(The topology on $\mathbb{P}^1(F_{\mathbf{p}})$ is the usual totally disconnected topology.) The $\mathbb{I}$-bundle $\mathscr{E}$ is also described as follows: let

$$\nu \colon \mathscr{E} \longrightarrow \mathbb{P}^1(F_{\mathbf{p}})$$

be the structure morphism. Then $\mathscr{E}$ is also defined as

$$\begin{aligned} \nu^{-1}\big(U_A\big) &:= U_A \times \mathbb{I} \\ i_B^A \colon \nu^{-1}(U_A \cap U_B) &\cong \nu^{-1}(U_B \cap U_A); \\ (z, \lambda) &\mapsto \big((z_\mathfrak{p})_{\mathfrak{p} \in S \backslash (A \triangle B)} (z_\mathfrak{p}^{-1})_{\mathfrak{p} \in A \triangle B},\ (z_\mathfrak{p}^{-1}, -z_\mathfrak{p}^2)_{\mathfrak{p} \in A \triangle B} \lambda \big). \end{aligned}$$

Note that $\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}), \mathscr{E}\big)$ is described by the following exact sequence:

$$\begin{array}{ccccc} 0 \longrightarrow \Gamma(\mathbb{P}^1(F_{\mathbf{p}}), \mathscr{E}) \longrightarrow & \prod_{A \subset \mathcal{S}} \mathcal{C}_c^0(U_A; \mathbb{I}) & \longrightarrow & \prod_{A,B \subset \mathcal{S}} \mathcal{C}_c^0(U_A \cap U_B; \mathbb{I}) \\ & \uplus & & \uplus \\ & (f_A)_{A \subset \mathcal{S}} & \longmapsto & \big(f_A|_{U_A \cap U_B}(z) - (i_B^A)^* f_B|_{U_A \cap U_B} \big)_{A,B \subset \mathcal{S}}. \end{array}$$

Let $M \times_{T(\mathcal{O}_{F_{\mathbf{p}}})} \mathbb{I}$ be a $T(F_{\mathbf{p}}) \times_{T(\mathcal{O}_{F_{\mathbf{p}}})} \mathbb{I}$-bundle. Then $\mathrm{GL}_2(F_{\mathbf{p}})$ acts on $\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}), M \times_{T(\mathcal{O}_{F_{\mathbf{p}}})} \mathbb{I}\big)$ from the right naturally. Now $\Gamma\big(\mathbb{P}^1(\mathcal{O}_{F_{\mathbf{p}}}), \mathscr{E}\big)$ has an action of $\mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}}) \cap \mathrm{GL}_2(F_{\mathbf{p}})$ as in Definition 2.3.2 below.

Definition 2.3.2. We regard $\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}),\mathscr{E}\big)$ as a subspace of $\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}), M\times_{T(\mathcal{O}_{F_{\mathbf{p}}})}\mathbb{I}\big)$. For $u\in \mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}})\cap \mathrm{GL}_2(F_{\mathbf{p}})$ and $f\in\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}),\mathscr{E}\big)$, we define

$$(fu)(z) := (u^{-1}\times 1)f(uz)\mathbf{1}_{f(uz)\in(u\times 1)\mathscr{E}}(z).$$

Since

$$\mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}})\cap \mathrm{GL}_2(F_{\mathbf{p}}) = \left\{ u\in \mathrm{GL}_2(F_{\mathbf{p}}) \;\middle|\; (u^{-1}\times 1)(M\backslash\mathcal{M})\subset M\backslash\mathcal{M} \right\},$$

the associativity of the action of Definition 2.3.2 follows.

Proposition 2.3.3. Let $f\in\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}),\mathscr{E}\big)$. The action of $u\in \mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}})\cap \mathrm{GL}_2(F_{\mathbf{p}})$ is described as follows: for $u=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Sigma_0(\mathbf{p})$, we have

$$(fu)|_{U_A^\circ}(z) = \left((cz_{\mathfrak{p}}+d)^{-1},\ \frac{(cz_{\mathfrak{p}}+d)^2}{\det(u)}\right)_{\mathfrak{p}\in A}\left((bz_{\mathfrak{p}}+a)^{-1},\ \frac{(bz_{\mathfrak{p}}+a)^2}{\det(u)}\right)_{\mathfrak{p}\in\mathcal{S}\backslash A}$$
$$\times f|_{U_A^\circ}\left(\left[\left(\frac{az_{\mathfrak{p}}+b}{cz_{\mathfrak{p}}+d}\right)_{\mathfrak{p}\in A};\ \left(\frac{dz_{\mathfrak{p}}+c}{bz_{\mathfrak{p}}+a}\right)_{\mathfrak{p}\in\mathcal{S}\backslash A}\right]\right).$$

For $u=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$, we have

$$(fu)|_{U_A}(z) = f|_{U_{\mathcal{S}\backslash A}}(z).$$

For $u=\begin{pmatrix} \varpi_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}$, we have

$$(fu)|_{U_A}(z) = \begin{cases} f|_{U_A^\circ}\left([(\varpi_{\mathfrak{p}}z_{\mathfrak{p}})_{\mathfrak{p}\in A};(\varpi_{\mathfrak{p}}^{-1}z_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{S}\backslash A}]\right)\mathbf{1}_{U_A^\circ}(z) & \text{if } \mathfrak{p}\in A\\ 0 & \text{if } \mathfrak{p}\notin A.\end{cases}$$

Let $\Gamma(U_{\mathcal{S}};\mathscr{E})$ and $\Gamma(U_{\varnothing}^\circ;\mathscr{E})$ be the sets of sections of $\mathscr{E}\to\mathbb{P}^1$ on $U_{\mathcal{S}}$ and $U_{\varnothing}^\circ$, respectively. We regard them as sub $\mathbb{I}$-modules of $\Gamma(\mathbb{P}^1(F_{\mathbf{p}}),\mathscr{E})$ by zero-extension. Then both space are stable under the action of $\Delta(\mathbf{p})$. Let $J(\mathbb{I})$ be the Jacobson radical of $\mathbb{I}$. For any open subset $U\subset\mathbb{P}^1(F_{\mathbf{p}})$, we define a system of neighborhoods of 0 of $\Gamma(U,\mathscr{E})$ by $\{\,\Gamma(U,\mathscr{E}\otimes_{\mathbb{I}} J(\mathbb{I})^n)\,\}_{m\geqslant 0}$ and regard $\Gamma(U,\mathscr{E})$ as a continuous $\mathbb{I}$-module.

Remark 2.3.4. We regard $\Gamma(\mathbb{P}^1,\mathscr{E})$ as a $\mathcal{C}^0(\mathbb{P}^1;\mathbb{I})$-module. Here $\mathcal{C}^0(\mathbb{P}^1;\mathbb{I})$ has a natural right action of $\mathrm{GL}_2(F_{\mathbf{p}})$ induced from the left action on $\mathbb{P}^1$. Let $f\in\Gamma(\mathbb{P}^1,\mathscr{E})$ and $\xi\in\mathcal{C}^0(\mathbb{P}^1;\mathbb{I})$. For $u\in \mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}})\cap \mathrm{GL}_2(F_{\mathbf{p}})$, we have

$$(\xi f)u = \xi u\cdot fu.$$

Remark 2.3.5. For open subset $U\subset\mathbb{P}^1(F_{\mathbf{p}})$, we have

$$\Gamma(U,\mathscr{E}\otimes_{\mathbb{I}} J(\mathbb{I})^n) = J(\mathbb{I})^n\Gamma(U,\mathscr{E}).$$

In fact, since $\mathbb{I}$ is noetherian, $J(\mathbb{I})^n$ is finitely generated. Let $J(\mathbb{I})^n=(a_1,\dots,a_m)$. Then we have a surjection

$$\mathbb{I}^m \overset{(a_1,\dots,a_m)}{\longrightarrow} J(\mathbb{I})^n\longrightarrow 0.$$

In general, for an $\mathbb{I}$-module M, since $\mathcal{E}\otimes_{\mathbb{I}} M$ is flasque, we have $H^1(U,\mathcal{E}\otimes_{\mathbb{I}} M)=0$. Thus we have an exact sequence

$$\Gamma(U,\mathscr{E})^m \overset{(a_1,\dots,a_m)}{\longrightarrow} \Gamma(U,\mathscr{E}\otimes J(\mathbb{I})^n)\longrightarrow 0,$$

namely, we have $\Gamma(U,\mathscr{E}\otimes_{\mathbb{I}} J(\mathbb{I})^n) = J(\mathbb{I})^n\Gamma(U,\mathscr{E})$.

Definition 2.3.6. We define

$$\mathscr{D}(\mathbb{I}) := \mathrm{Hom}_{\mathbb{I}\text{-cont}}\Big(\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}), \mathscr{E}\big),\ \mathbb{I}\Big),$$
$$\mathscr{D}(\mathcal{O}_{F_{\mathbf{p}}}, \mathbb{I}) := \mathrm{Hom}_{\mathbb{I}\text{-cont}}\Big(\Gamma\big(U_{\mathcal{S}}, \mathscr{E}\big),\ \mathbb{I}\Big).$$

The space $\mathscr{D}(\mathbb{I})$ (resp. $\mathfrak{D}(U_{\mathscr{S}}, \mathbb{I})$ has a natural left action of $\mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}}) \cap \mathrm{GL}_2(F_{\mathbf{p}})$ (resp. $\Delta(\mathbf{p})$). For $\mu \in \mathscr{D}(\mathbb{I})$, $U \subset \mathbb{P}^1(F_{\mathbf{p}})$ and $f \in \Gamma(\mathbb{P}^1, \mathscr{E})$, we usually denote $\mu(f\mathbf{1}_U)$ by

$$\int_U f\ d\mu.$$

Thus we have the spaces for big quaternionic automorphic forms:

Definition 2.3.7. For $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $k^{\mathbf{p}} - 2\underline{t}^{\mathbf{p}} \geqslant 0$, we define

$$\mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I}) := \mathscr{D}(\mathbb{I}) \otimes_{\mathbb{Z}_p} \mathrm{Sym}^{k^{\mathbf{p}}-2\underline{t}^{\mathbf{p}}}(\mathcal{O}) \otimes \det{}^{\underline{t}^{\mathbf{p}}-w^{\mathbf{p}}},$$
$$\mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathcal{O}_{F_{\mathbf{p}}}, \mathbb{I}) := \mathscr{D}(\mathcal{O}_{F_{\mathbf{p}}}; \mathbb{I}) \otimes_{\mathbb{Z}_p} \mathrm{Sym}^{k^{\mathbf{p}}-2\underline{t}^{\mathbf{p}}}(\mathcal{O}) \otimes \det{}^{\underline{t}^{\mathbf{p}}-w^{\mathbf{p}}}.$$

We consider $\mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}$ (resp. $\mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathcal{O}_{F_{\mathbf{p}}}, \mathbb{I})$) as a left $\mathrm{GL}_2(\mathcal{O}_{F_{\mathbf{p}}})$ (resp. $\Sigma_0(\mathbf{p})$)-module by the usual action.

We define the Hecke operators as follows:

Definition 2.3.8. Let $\Phi \in S(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I}))$ (resp. $\Phi \in S(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathcal{O}_{F_{\mathbf{p}}}, \mathbb{I}))$). For $g \in \mathrm{M}_2(\mathcal{O}_{F_{\mathbf{p}}}) \cap \mathrm{GL}_2(F_{\mathbf{p}})$ (resp. $g \in \Delta(\mathbf{p})$), we define

$$U(g)\Phi(b) := \sum_i g_i \Phi(b g_i),$$

where g_i are defined by the decomposition

$$\Sigma g \Sigma = \bigsqcup_i g_i \Sigma \qquad \Big(\text{resp. } \Sigma_0(\mathbf{p}) g \Sigma_0(\mathbf{p}) = \bigsqcup_i g_i \Sigma_0(\mathbf{p})\Big).$$

In particular, when $g = \begin{pmatrix} \pi_{\mathbf{p}}^s & 0 \\ 0 & 1 \end{pmatrix}$ for the $\pi_{\mathbf{p}}$ fixed in the beginning of the section, we denote $U(g)$ by $T_0(\pi_{\mathbf{p}}^s)$ and call it the *normalized Hecke operator*.

For $f \in \Gamma(U_{\mathcal{S}}, \mathscr{E})$, we define $\widetilde{f} \in \Gamma(\mathbb{P}^1(F_{\mathbf{p}}), \mathscr{E})$ by

$$\widetilde{f}|_{U_A} = \begin{cases} f & \text{if } A = \mathcal{S} \\ 0 & \text{if } A \neq \mathcal{S}. \end{cases}$$

The correspondence $f \mapsto \widetilde{f}$ is the section of the restriction $\Gamma(\mathbb{P}^1, \mathscr{E}) \longrightarrow \Gamma(U_{\mathcal{S}}, \mathscr{E})$. It induces a $\Sigma_0(\mathbf{p})$-homomorphism

$$\mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I}) \longrightarrow \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(U_{\mathcal{S}}, \mathbb{I}). \tag{2.3.1}$$

Theorem 2.3.9. The natural homomorphism induced from (2.3.1)

$$\mathbf{r}\colon S\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big) \longrightarrow S\big(\Sigma_0(\mathbf{p}), \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(U_{\mathcal{S}}, \mathbb{I})\big)$$

is an isormopshim and commutes with $T_0(\pi_{\mathbf{p}}^s)$ for $s \geqslant 0$.

PROOF. We construct the inverse of $\mathbf{r}$. Let $\Phi \in S\big(\Sigma_0(\mathbf{p}), \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(U_{\mathcal{S}}, \mathbb{I})\big)$. Let

$$\Sigma_{\mathbf{p}} = \bigsqcup_i \gamma_i \Sigma_0(\mathbf{p})_{\mathbf{p}}$$

be a decomposition. For $\phi \in \Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}), \mathscr{E}\big)$, we define $\widetilde{\Phi}$ by

$$\int_{\mathbb{P}^1} \phi \; d\widetilde{\Phi}(b) = \sum_i \int_{U_{\mathcal{S}}} \phi|\gamma_i^{-1}(z) \; d\Phi(b\gamma_i)(z).$$

Then $\widetilde{\Phi} \in S\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)$ and $\Phi \mapsto \widetilde{\Phi}$ is the inverse to $\mathbf{r}$. For the commutativity with $T_0(x)$, it follows from Lemma 2.2.2. □

We also define the ordinary part of $S\big(\Sigma, \mathscr{D}(\mathbb{I})\big)$ by

Definition 2.3.10. Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $k^{\mathbf{p}} - 2\underline{t}^{\mathbf{p}} \geqslant 0$. Let $x \in \mathbf{p}\mathcal{O}_{F_{\mathbf{p}}}$ with $x^{\underline{t}} \neq 0$. We define the *ordinary part* of $S\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)$ by

$$S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big) = \bigcap_{n=1}^{\infty} T_0(\varpi_{\mathbf{p}})^n S\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big).$$

Note that the ordinary part is independent of the choice of x.

Theorem 2.3.11. Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $k^{\mathbf{p}} - 2\underline{t}^{\mathbf{p}} \geqslant 0$. Let $\Phi \in S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)$. If

$$\int_{[\mathcal{O}_{F_{\mathbf{p}}} \times \{1\}]} 1 \; d\Phi = 0,$$

we have

$$\Phi = 0.$$

PROOF. For $s \in \bigoplus_{\mathfrak{p}|\mathbf{p}} \mathbb{Z}_{>0}\mathfrak{p}$, let

$$P_s = \mathrm{Ker}\left(\widetilde{\Lambda}_{\mathcal{O}} \longrightarrow \mathcal{O}[\mathbf{G}_s]\right).$$

Since

$$S\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big) \cong \varprojlim_s S\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I}/P_s\mathbb{I})\big)$$

We can replace $\mathbb{I}$ with $\mathbb{I}/P_s\mathbb{I}$. Thus we fix s and assume that

$$P_s\mathbb{I} = 0.$$

In addition, by Theorem 2.3.9, it suffices to prove that $\mathbf{r}(\Phi) = 0$. For $t \in \bigoplus_{\mathfrak{p}|\mathbf{p}} \mathbb{Z}_{>0}\mathfrak{p}$, we put

$$L_t := \sum_{\gamma \in \Delta(\mathbf{p})} \mathbb{I}\mathbf{1}_{[\mathbf{p}^t\mathcal{O}_F \times \{1\}]}(\gamma z) \subset \Gamma(U_{\mathcal{S}}, \mathscr{E}).$$

Since $P_s\mathbb{I}$=0, L_t is stable under the action of $\Delta(\mathbf{p}^s)$ (see 2.3.3), where

$$\Delta(\mathbf{p}^s) := \left\{ \begin{pmatrix} \mathcal{O}_{F_{\mathbf{p}}} & \mathcal{O}_{F_{\mathbf{p}}} \\ \mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}} & \mathcal{O}_{F_{\mathbf{p}}}^{\times} \end{pmatrix} \right\}.$$

We define

$$M_t := S\left(\Sigma_0(\mathbf{p}^s),\ \mathrm{Hom}_{\mathbb{I}}(L_t, \mathbb{I})) \otimes_{\mathbb{Z}_p} \mathrm{Sym}^{k^{\mathbf{p}} - 2\underline{t}^{\mathbf{p}}}(\mathcal{O}) \otimes \det{}^{\underline{t}^{\mathbf{p}} - w^{\mathbf{p}}}\right)$$

and $T_0(\varpi_{\mathbf{p}})$ on M_s in the similar manner. By Lemma 2.2.2, the natural homomorphism

$$c_t\colon S\big(\Sigma_0(\mathbf{p}), \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathcal{O}_{F_{\mathbf{p}}};\mathbb{I})\big) \longrightarrow M_t$$

satisfies

$$c_t \circ T_0(\varpi_{\mathbf{p}}) = T_0(\varpi_{\mathbf{p}}) \circ c_t.$$

Moreover, since $\bigcup_t L_t$ is dense in $\Gamma\big(U_{\mathcal{S}}, \mathscr{E}\big)$,

$$\bigcap_t \mathrm{Ker}(c_t) = 0.$$

The assumption for Φ means $\Phi \in \mathrm{Ker}(c_0)$. Let $x \in \mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}$ with $x^{\underline{t}} \neq 0$. For any σ_i of the decomposition

$$\Sigma_0(\mathbf{p}^s)\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\Sigma_0(\mathbf{p}^s) = \bigsqcup_i \sigma_i\Sigma_0(\mathbf{p}^s),$$

we have

$$\sigma_i L_s \subset L_0.$$

Thus $T_0(x)c_s(\Phi) = 0$. Since L_t is free of finite rank over $\mathbb{I}$, the limit

$$e_x := \lim_n T_0(\varpi_{\mathbf{p}})^{n!}$$

exists at M_t and $T_0(x)$ acts on $c_s\big(S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)\big)$ as an isomorphism. Thus we have $c_t(\Phi) = 0$ and conclude that $\Phi \in \bigcap_t \mathrm{Ker}(c_t) = \{0\}$, namely, $\Phi = 0$. ☐

Remark 2.3.12. By the proof above, Hida's idempotent

$$e_{\mathbf{p}} : S\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big) \longrightarrow S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)$$

exists as a limit of a normalized Hecke operator $\lim_n T_0(x)^{m!}$.

Corollary 2.3.13. Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $k^{\mathbf{p}} - 2\underline{t}^{\mathbf{p}} \geqslant 0$. On $S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)$, for any $x \in \mathbf{p}\mathcal{O}_{F_{\mathbf{p}}}$ with $x^{\underline{t}} \neq 0$, $T_0(x)$ acts as an isomorphism.

Proof. It suffice to prove $T_0(x)$ acts as an injective homomorphism. Let

$$\Phi \in S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)$$

and suppose

$$T_0(\varpi_{\mathbf{p}})\Phi = 0.$$

Let $x \in \mathbf{p}\mathcal{O}_{F_{\mathbf{p}}}$ with $x^{\underline{t}} \neq 0$. Since

$$0 = \int_{[\,\mathbf{p}\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}\,]} 1\, dT_0(x)\Phi(b) = \int_{[\,\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}\,]} 1\, d\Phi\left(b\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right),$$

we have $\Phi = 0$ by Theorem 2.3.11. ☐

On the ordinary part, we have good expressions for integrations:

Theorem 2.3.14. Let $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ with $k^{\mathbf{p}} - 2\underline{t}^{\mathbf{p}} \geqslant 0$. Let $\Phi \in S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})\big)$. For $f \in \Gamma\big(\mathbb{P}^1(F_{\mathbf{p}}); \mathscr{E}\big)$, we have

$$\int_{[\,\{1\}\times\varpi_{\mathbf{p}}^s\mathcal{O}_{F_{\mathbf{p}}}\,]} f([1;z])\, d\Phi(b)(z) = (1,-1)\int_{[\,\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}\,]} f(\tau_{-\varpi_{\mathbf{p}}^s}[z;1])\, dT_0(\varpi_{\mathbf{p}}^s)^{-1}\Phi(b\tau_{-\varpi_{\mathbf{p}}^s})(z),$$

$$\int_{[\,\varpi_{\mathbf{p}}^s\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}\,]} f([z;1])\, d\Phi(b)(z) = \int_{[\,\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}\,]} f([\varpi_{\mathbf{p}}^s z;1])\, dT_0(\varpi_{\mathbf{p}}^s)^{-1}\Phi(b)(z),$$

where $\tau_{-\varpi_{\mathbf{p}}^s} := \begin{pmatrix} 0 & 1 \\ \varpi_{\mathbf{p}}^s & 0 \end{pmatrix}$.

Proof. We only need to prove the first formula. It suffice to prove that

$$\int_{[\{1\}\times\varpi_{\mathbf{p}}^s\mathcal{O}_{F_{\mathbf{p}}}]} \phi([1;z])\, dT_0(x)\Phi(b)(z) = (1,-1)\int_{[\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}]} \phi(\tau_{-\varpi_{\mathbf{p}}^s}[z;1])\, d\Phi(b\tau_{-\varpi_{\mathbf{p}}^s})(z).$$

It follows from Lemma 2.2.2 and Proposition 2.3.3. □

In the end of this section, we define the specialization map. For $r \geqslant 0$, we define

$$\begin{aligned} \mathscr{X}(\mathbb{I}) &:= \mathrm{Hom}_{\mathcal{O}\text{-conti}}(\mathbb{I}, \mathbb{C}_p), \\ \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I}) &:= \left\{ P \in \mathscr{X}(\mathbb{I}) \,\middle|\, P|_{\mathbf{G}} \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}} \right\} \\ \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I})_{\geqslant r} &:= \left\{ P \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I}) \,\middle|\, P|_{\mathbf{G}} = P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}} \text{ with } k_{\mathbf{p}} \geqslant r\underline{t}_{\mathbf{p}} \right\}, \end{aligned}$$

where, we denote by $P|_{\mathbf{G}}$ the composition $\mathbf{G} \to \mathbb{I}^\times \xrightarrow{P} \mathbb{C}^\times$. We define

Definition 2.3.15. Let $\mathbb{I}$ be a topological $\widetilde{\Lambda}_{\mathcal{O}}$-algebra. Let $\Phi \in S\big(\Sigma; \mathscr{D}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I})\big)$ and $P \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I})$ with $P|_{\mathbf{G}} = P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}}$ ($\boldsymbol{\omega} = (\omega,\omega')$). We define

$$\mathrm{Sp}_P(\Phi) := P\left(\int_{[\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}]} (zX+Y)^{k_{\mathbf{p}}-2\underline{t}_{\mathbf{p}}}\, d\Phi \right) \in S_{k,w}\big(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}), \omega, \omega'; \mathcal{O}_{\boldsymbol{\omega}}\big).$$

For $s \geqslant s(\boldsymbol{\omega})$, we define

$$\widehat{\mathrm{Sp}}^{(s)}_P(\Phi) = P\left(\int_{[\{1\}\times\mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}]} (X+zY)^{k_{\mathbf{p}}-2\underline{t}_{\mathbf{p}}}\, d\Phi \right) \in S_{k,w}\big(\Sigma(\mathbf{p}^s), \omega, \omega\omega'^{-1}; \mathcal{O}_{\boldsymbol{\omega}}\big).$$

Proposition 2.3.16. For the specialization map , we have

$$\mathrm{Sp}_P \circ T_0(\varpi_{\mathbf{p}}^s) = T_0(\varpi_{\mathbf{p}}^s) \circ \mathrm{Sp}_P.$$

Proof. It follows by direct computations. □

Proposition 2.3.17. Let $\Phi \in S(\Sigma; \mathscr{D}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I}))$ and $P \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I})$ with $P|_{\mathbf{G}} = P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}}$. For a polynomial $h(T) \in \mathbb{C}_p[\,\{T_\sigma\}_{\sigma\in I_{\mathbf{p}}}\,]$ in $(\#I_{\mathbf{p}})$-variables such that the degree of z_σ is smaller than or equal to k_σ, we have

$$P\left(\int_{[\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}]} h((z^\sigma)_{\sigma\in I_{\mathbf{p}}})\, d\Phi \right) = \left\langle\, h(-Y/X)X^{k_{\mathbf{p}}-2\underline{t}_{\mathbf{p}}},\, \mathrm{Sp}_P(\Phi) \,\right\rangle_{k-2\underline{t}}.$$

For $s \geqslant s(\psi)$, we have

$$P\left(\int_{[\{1\}\times\mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}]} h((z^\sigma)_{\sigma\in I_{\mathbf{p}}})\, d\Phi \right) = \left\langle\, \widehat{\mathrm{Sp}}^{(s)}_P(\Phi),\, h(-X/Y)Y^{k_{\mathbf{p}}-2\underline{t}_{\mathbf{p}}} \,\right\rangle_{k-2\underline{t}}.$$

In the ordinary part, we have the following propositions:

Proposition 2.3.18. Let $\Phi \in S^{\mathrm{ord}}\big(\Sigma; \mathscr{D}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I})\big)$ and $P \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I})$ with $P|_{\mathbf{G}} = P_{k_{\mathbf{p}},w_{\mathbf{p}},\boldsymbol{\omega}}$. We have

$$\widehat{\mathrm{Sp}}^{(s)}_P(\Phi)(b) = (\varpi_{\mathbf{p}}^s)^{w_{\mathbf{p}}-\underline{t}_{\mathbf{p}}}\tau_{-\varpi_{\mathbf{p}}^s}\,\mathrm{Sp}_P\left(T_0(\varpi_{\mathbf{p}}^s)^{-1}\Phi(b\tau_{-\varpi_{\mathbf{p}}^s})\right),$$

where $\tau_{-\varpi_{\mathbf{p}}^s} := \begin{pmatrix} 0 & 1 \\ -\varpi_{\mathbf{p}}^s & 0 \end{pmatrix}$. In particular, we have

$$P\left(\int_{[\{1\}\times\mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}]} h(z)\,d\Phi(b)\right) = \langle\, \tau_{-\varpi_{\mathbf{p}}^s}\, \mathrm{Sp}_P(T_0(\varpi_{\mathbf{p}}^s)^{-1}\Phi(b\tau_{-\varpi_{\mathbf{p}}^s}),\ h(-X/Y)Y^{k_{\mathbf{p}}-2\underline{t}_{\mathbf{p}}}\,\rangle_{k-2\underline{t}}.$$

Proof. It follows immediately from Theorem 2.3.14 $\square$

Proposition 2.3.19. We assume that there exists a subset $\mathscr{Y} \subset \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})$ satisfying

$$\bigcap_{P\in\mathscr{Y}} \mathrm{Ker}(P) = 0, \tag{2.3.2}$$

Then for any $z \in \mathbb{A}_{F,f}$ and $\Phi \in S^{\mathrm{ord}}(\Sigma; \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I}))$, we have

$$\Phi(bz) = (z,1)\Phi(b). \tag{2.3.3}$$

Moreover, for any $r \geqslant 0$, if

$$\bigcap_{P\in\mathscr{Y}\cap\mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I})_{\geqslant r}} \mathrm{Ker}(P) = 0, \tag{2.3.4}$$

The formula (2.3.3) holds for not necessarily ordinary $\Phi \in S(\Sigma; \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I}))$.

Proof. For any $b \in \widehat{B}$, $P \in \mathscr{Y}$, we have

$$\mathrm{Sp}_P(\Phi)(bz) - \mathrm{Sp}_P((z,1)\Phi)(b) = 0.$$

Thus by Theorem 2.3.11, we have the first assertion. The second assertion follows from the density of polynomials by p-adic Weirestrass theorem ([**?**]). $\square$

2.4. 𝕀-adic Petersson inner product for measure valued forms

In this subsection, we fix $k^{\mathbf{p}}, w^{\mathbf{p}} \in \mathbb{Z}[I_{\mathbf{p}}]$ with $k^{\mathbf{p}} - 2w^{\mathbf{p}} \geqslant 0,\ 2w^{\mathbf{p}} - k^{\mathbf{p}} \in \mathbb{Z}\underline{t}^{\mathbf{p}}$, and a finite product of noetherian complete local $\mathbb{Z}_p[[\mathbf{G}]]$-algebras with finite residue fields and denote it by $\mathbb{I}$. We define

$$\begin{aligned} G &:= B^\times/Z(B^\times), \\ G_\infty &:= G(F\otimes_{\mathbb{Q}}\mathbb{R}), \\ \Gamma_H^b &:= b^{-1}G(F)b \cap HG_\infty \qquad (H \subset G(\mathbb{A}_{F,f}) :\ \text{open compact subgroup}), \end{aligned}$$

where $Z(B^\times)$ is the center of $B^\times$. Throughout this section, we make the following assumption:

Assumption 2.4.1. For any $\Phi \in S(\Sigma, \mathscr{D}_{k^{\mathbf{p}},w^{\mathbf{p}}}(\mathbb{I}))$ and $z \in \mathbb{A}_{F,f}^\times$,

$$\Phi(bz) = (z,1)\Phi(b)$$

for all $b \in \widehat{B}^\times$ (See Proposition 2.3.19).

2.4.1. $\mathbb{I}$-adic Petersson inner products.

Definition 2.4.2. We define a $\mathrm{GL}_2(\mathcal{O}_{F_{\mathbf{p}}})$-invariant open compact subset of $\mathbb{P}^1(F_{\mathbf{p}}) \times \mathbb{P}^1(F_{\mathbf{p}})$ by

$$V := \left\{ \big((x;y),\,(z;w)\big) \in \mathbb{P}^1(\mathcal{O}_{F_{\mathbf{p}}})^2 \;\middle|\; xw - yz \in \mathcal{O}_{F_{\mathbf{p}}}^\times \right\}.$$

Let

$$j\colon \mathcal{O}_{F_{\mathbf{p}}}^\times \longrightarrow \mathbf{G}; \qquad x \mapsto (x^{-1}, x^2).$$

Definition 2.4.3. We define

$$D \in \Gamma\big(\mathbb{P}^1(F_{\mathbf{p}})^2,\, \mathscr{E} \otimes_{\mathbb{I}} \mathscr{E}\big)$$

by the zero extension of an element of $\Gamma(V, \mathscr{E} \otimes_{\mathbb{I}} \mathscr{E})$ defined by

$$D|_{V \cap U_A \times U_{\mathcal{S}\setminus A}}(z, w) := j\big((1 - z_{\mathfrak{p}} w_{\mathfrak{p}})_{\mathfrak{p}\in A} \cdot (z_{\mathfrak{p}} w_{\mathfrak{p}} - 1)_{\mathfrak{p}\in \mathcal{S}\setminus A}\big),$$

where $A \subset \mathcal{S}$. We note that $\{V \cap U_A \times U_{\mathcal{S}\setminus A}\}_{A\subset \mathcal{S}}$ is an open covering of V.

Proposition 2.4.4. Let $\mathrm{GL}_2(\mathcal{O}_{F_{\mathbf{p}}})$ act on $\Gamma\big(\mathbb{P}^1(F_{\mathbf{p}})^2, \mathscr{E} \otimes_{\mathbb{I}} \mathscr{E}\big)$ diagonally. For any $u \in \mathrm{GL}_2(\mathcal{O}_{F_{\mathbf{p}}})$, we have

$$D|u = \big(\det(u)^{-1}, 1\big) D.$$

Remark 2.4.5. The section D satisfying the formula in Proposition 2.4.4 is determined unique up to scalar. In fact, suppose D' satisfies the same formula as D. Then we have

$$\begin{aligned} D'\big((1;0),(0;1)\big)\, D\big((a;c),(b;d)\big) &= (ad - bc,\, 1) \begin{pmatrix} a & b \\ c & d \end{pmatrix} D'\big((1;0),(0;1)\big) \\ &= D'\big((a;c),(b;d)\big). \end{aligned}$$

Thus

$$D' = D'\big((1;0),(0;1)\big) \cdot D.$$

Definition 2.4.6. Let $\Phi, \Psi \in S\big(\Sigma, \mathscr{D}_{k^{\mathbf{P}}, w^{\mathbf{P}}}(\mathbb{I})\big)$. For open compact subgroups $H_1, H_2 \subset \Sigma_{\mathbf{p}}$ and $\mathbf{z} \in \mathbb{P}^1(F_{\mathbf{p}}) \times \mathbb{P}^1(F_{\mathbf{p}})$, we define

$$\beta^{\mathbf{z}}_{H_1,H_2}(\Phi,\Psi)(b) := \langle\,\cdot,\cdot\,\rangle_{k^{\mathbf{P}}-2\underline{t}^{\mathbf{P}}} \circ \left(\int_{H_1\times H_2\mathbf{z}} D(z,w)\, d\Phi(b)(z) \otimes \Psi(b)(w) \right) \in \mathbb{I}.$$

Here, we denote by $\langle\,\cdot,\cdot\,\rangle_{k^{\mathbf{P}}-2\underline{t}^{\mathbf{P}}}$ the homomorphism induced from the pairing

$$\mathrm{Sym}^{k^{\mathbf{P}}-2\underline{t}^{\mathbf{P}}}(\mathbb{I}) \times \mathrm{Sym}^{k^{\mathbf{P}}-2\underline{t}^{\mathbf{P}}}(\mathbb{I}) \longrightarrow \mathbb{I}$$

of (2.1.3).

The function $\beta^{\mathbf{z}}_{H_1,H_2}(\Phi,\Psi)$ satisfies the following formula.

Proposition 2.4.7. Let $\mathbf{z} \in \mathbb{P}^1(F_{\mathbf{p}})$ and $H_1, H_2 \subset \Sigma_{\mathbf{p}}$ open compact subgroups. Let $\Phi, \Psi \in S(\Sigma, \mathscr{D}_{k^{\mathbf{P}}, w^{\mathbf{P}}}(\mathbb{I}))$. For any $u \in \Sigma_{\mathbf{p}}$ such that $u(H_1 \times H_2\mathbf{z}) = H_1 \times H_2\mathbf{z}$, we have

$$\beta^{\mathbf{z}}_{H_1,H_2}(\Phi,\Psi)(bu) = \big(\det(u_{\mathbf{p}}),\, 1\big) \det(u^{\mathbf{P}})^{-2w^{\mathbf{P}}+k^{\mathbf{P}}} \beta^{\mathbf{z}}_{H_1,H_2}(\Phi,\Psi)(b).$$

Proof. We have the formula by simple computation using Proposition 2.3.3. ☐

Definition 2.4.8 ($\mathbb{I}$-adic Petersson inner product). For $\Phi, \Psi \in S(\Sigma, \eta, \mathscr{D}_{k^{\mathbf{P}}, w^{\mathbf{P}}}(\mathbb{I}))$, we define

$$\mathcal{B}_{\mathbb{I}}(\Phi, \Psi) := \sum_{b \in G(F)\backslash G(\mathbb{A}_F)/\Sigma} \frac{1}{\#\Gamma^b_\Sigma} \big(\mathrm{Nrd}_{B/F}(b^{-1}), 1\big)\, \beta^{\mathbf{z}}_{\Sigma\times\Sigma}(\Phi, \Psi)(b)$$
$$\in \quad \mathbb{I}\left[\left\{\frac{1}{\#\Gamma^b_\Sigma}\right\}_b\right],$$

where $\mathbf{z} \in \mathbb{P}^1(F_{\mathbf{p}}) \times \mathbb{P}^1(F_{\mathbf{p}})$ and $\mathcal{B}_{\mathbb{I}}$ is independent of the choice of $\mathbf{z}$.

We have the following key lemma:

Lemma 2.4.9. Let $\mathbf{z} = (z_1, z_2) \in \mathbb{P}^1(F_{\mathbf{p}})^2$. Then, for $\Phi, \Psi \in S\big(\Sigma, \mathscr{D}_{k^{\mathbf{P}}, w^{\mathbf{P}}}(\mathbb{I})\big)$, the sum

$$\sum_{b \in B^\times \mathbb{A}_F^\times \backslash \widehat{B}/\Sigma_0(\mathbf{p}^s)} \frac{1}{\#\Gamma^b_{\Sigma_0(\mathbf{p}^s)}} \big(\mathrm{Nrd}_{B/F}(b^{-1}), 1\big)\, \beta^{\mathbf{z}}_{\Sigma_0(\mathbf{p}^s)_{\mathbf{p}}, \Sigma_{\mathbf{p}}}(\Phi, \Psi)(b)$$

is independent of s. Similarly, the sum

$$\sum_{b \in B^\times \mathbb{A}_F^\times \backslash \widehat{B}/\Sigma_0(\mathbf{p}^s)} \frac{1}{\#\Gamma^b_{\Sigma_0(\mathbf{p}^s)}} \big(\mathrm{Nrd}_{B/F}(b^{-1}), 1\big)\, \beta^{\mathbf{z}}_{\Sigma_{\mathbf{p}}, \Sigma_0(\mathbf{p}^s)_{\mathbf{p}}}(\Phi, \Psi)(b),$$

is independent of s.

Proof. $H_s := \Sigma_0(\mathbf{p}^s)$. We only prove the first formula. The second formula can be proved in the same way. Denote the first sum by I_{H_s}. We prove, for any s,

$$I_\Sigma = I_{H_s}.$$

Define a finite subset $\{ g_i \} \subset \Sigma_{\mathbf{p}}$ by

$$\Sigma_{\mathbf{p}} = \bigsqcup_{i=1}^m g_i H_s.$$

By definition, we have

$$\sum_{i=1}^m \big(\mathrm{Nrd}_{B/F}(bg_i), 1\big)\beta^{\mathbf{z}}_{H_s, \Sigma_{\mathbf{p}}}(\Phi, \Psi)(bg_i) = \big(\mathrm{Nrd}_{B/F}(b), 1\big)\beta^{\mathbf{z}}_{\Sigma_{\mathbf{p}}, \Sigma_{\mathbf{p}}}(\Phi, \Psi)(b).$$

Thus

$$I_{H_s} = \sum_{i,j} \frac{1}{\#\Gamma^{t_i g_j}_{H_s}} \frac{\#\Gamma^{t_i}_\Sigma \cap g_j H_s g_j^{-1}}{\#\Gamma^{t_i g_j}_\Sigma} \big(\mathrm{Nrd}_{B/F}(t_i g_j), 1\big)\beta^{\mathbf{z}}_{H_s, \Sigma_{\mathbf{p}}}(\Phi, \Psi)(t_i g_i)$$
$$= I_\Sigma.$$

□

Corollary 2.4.10. Let the assumptions be as in Definition 2.4.8. Let Φ, Ψ be elements of $S\big(\Sigma, \mathscr{D}_{k^{\mathbf{P}}, w^{\mathbf{P}}}(\mathbb{I})\big)$. For any $P \in \mathscr{X}(\mathbb{I})$ with $P|_{\mathbf{G}} = P_{k_{\mathbf{p}}, w_{\mathbf{p}}, \boldsymbol{\omega}} \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}}, w^{\mathbf{P}}}$ and for any $s \geqslant s(\psi)$, we have

$$P(\mathcal{B}_{\mathbb{I}}(\Phi, \Psi)) = \sum_{b \in B^\times \mathbb{A}_F^\times \backslash \widehat{B}/\Sigma_0(\mathbf{p}^s)} \omega^{-1} \epsilon^{[2w_{\mathbf{p}} - k_{\mathbf{p}}]}_{\mathrm{cyc}, F}(\mathrm{Nrd}_{B/F}(b)) \Big\langle \mathrm{Sp}_P(\Phi)(b),\, \widehat{\mathrm{Sp}}^{(s)}_P(\Psi)(b) \Big\rangle_{k - 2\underline{t}}$$
$$= \frac{1}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s))} \int_{B^\times \mathbb{A}_F^\times \backslash B(\mathbb{A}_F)} \omega^{-1}(\mathrm{Nrd}_{B/F}(b)) \Big\langle \mathrm{Sp}_P(\Phi)^u(b), \widehat{\mathrm{Sp}}^{(s)}_P(\Psi)^u(b) \Big\rangle_{k-2\underline{t}} db,$$

where the notation $(\cdot)^u$ is as in Remark 2.1.4.

Proof. It follows from the following elementary formula: for $f, g \in \mathrm{Sym}^r(\mathbb{Q})$ for some integer $r \geqslant 0$, we have

$$\langle f(X_1, Y_1)g(X_2, Y_2), (X_1Y_2 - X_2Y_1)^r\rangle_{r,r} = \langle f, g\rangle_r,$$

which follows by the uniqueness of a invariant paring on $\mathrm{Sym}^r(\mathbb{Q})$. □

Corollary 2.4.11. Let the assumptions be as in Definition 2.4.8. Let $\Phi \in S\big(\Sigma, \mathscr{D}_{k^{\mathbf{P}},w^{\mathbf{P}}}(\mathbb{I})\big)$ and $\Psi \in S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^{\mathbf{P}},k^{\mathbf{P}}}(\mathbb{I})\big)$. For any $P \in \mathcal{X}(\mathbb{I})$ such that $P|_{\mathbf{G}} = P_{k_{\mathbf{p}},w_{\mathbf{p}},\omega,\omega'} \in \mathscr{X}^{\mathrm{arith}}_{k^{\mathbf{P}},w^{\mathbf{P}}}$ and for any $s \geqslant s(\omega, \omega')$, we have

$$\begin{aligned}
&P(\mathcal{B}_{\mathbb{I}}(\Phi, \Psi))\\
&= \sum_{b \in B^\times \mathbb{A}_F^\times \backslash \widehat{B}/\Sigma_0(\mathbf{p}^s)} (\varpi_{\mathbf{p}}^s)^{w_{\mathbf{p}}-\underline{t}_{\mathbf{p}}}\omega^{-1}\epsilon_{\mathrm{cyc},F}^{[2w_{\mathbf{p}}-k_{\mathbf{p}}]}(\mathrm{Nrd}_{B/F}(b))\\
&\quad \times \big\langle\, \mathrm{Sp}_P(\Phi)(b),\ \tau_{-\varpi_{\mathbf{p}}^s}\mathrm{Sp}_P(T_0(\varpi_{\mathbf{p}}^s)^{-1}\Psi(b\tau_{-\varpi_{\mathbf{p}}^s}))\,\big\rangle_{k-2\underline{t}}\\
&= \int_{B^\times \mathbb{A}_F^\times \backslash B(\mathbb{A}_F)} \omega^{-1}(\mathrm{Nrd}_{B/F}(b))\Big\langle\, \mathrm{Sp}_P(\Phi)^u(b),\ T_k(\varpi_{\mathbf{p}}^s)^{-1}\mathrm{Sp}_P\big(\Psi\big)^u(b\tau_{-\varpi_{\mathbf{p}}^s})\big)\Big\rangle_{k-2\underline{t}} db,
\end{aligned}$$

where $(\cdot)^u$ is as in Remark 2.1.4, which is an element of $\mathscr{A}_k^u(\Sigma(\mathbf{p}^s), \omega)$ and for any $\phi \in \mathscr{A}_k^u(\Sigma(\mathbf{p}^s), \omega)$ and $x \in \mathcal{O}_{F_{\mathbf{p}}}$ such that $x^{\underline{t}_{\mathbf{p}}} \neq 0$, we define

$$T_k(x)\phi(g) := |\varpi_{\mathbf{p}}|_{\mathbb{A}_F}^{[2w-k]/2}\sum_i \phi(g\sigma_i),$$

where

$$\Sigma(\mathbf{p}^s)\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix}\Sigma(\mathbf{p}^s) = \bigsqcup_i \sigma_i\Sigma(\mathbf{p}^s)$$

2.4.2. The lifing to Hida families of quaternionic automorphic forms. We discuss about the Hecke equivalence of the $\mathbb{I}$-adic Petersson inner product. Let $\mathfrak{a} \subset \mathcal{O}_F$ be a nonzero ideal. We define an order $\widehat{R}(\mathfrak{a})$ called the *Eichler order* of level $\mathfrak{n}$ by

$$\begin{aligned}
\widehat{R}(\mathfrak{a}) := \prod_{\mathfrak{q}\nmid\mathfrak{d}} i_{\mathfrak{q}}^{-1}\left(\left\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{M}_2(\mathcal{O}_{F_{\mathfrak{q}}}) \,\middle|\, c \equiv 0 \mod \mathfrak{a}\mathcal{O}_{F_{\mathfrak{q}}}\right\}\right)\\
\times \prod_{\mathfrak{q}|\mathfrak{d}} \big\{\text{the maximal order of } B \otimes_F F_{\mathfrak{q}}\big\}.
\end{aligned}$$

We define open compact subgroups of $\widehat{B}^\times$. For any nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$ prime to $p\mathfrak{n}^-$, we define

$$K_0^B(\mathfrak{a}) := \widehat{R}(\mathfrak{a})^\times, \tag{2.4.1}$$

$$K_1^B(\mathfrak{a}) := \left\{ u \in K_0^B(\mathfrak{a}) \,\middle|\, i_{\mathfrak{q}}(u) \equiv \begin{pmatrix} * & *\\ 0 & 1\end{pmatrix} \pmod{\mathfrak{a}\mathrm{M}_2(\mathcal{O}_{F_{\mathfrak{q}}})} \quad \text{for } \mathfrak{q} \nmid \mathfrak{d}\right\}, \tag{2.4.2}$$

$$K^B(\mathfrak{a}) := \left\{ u \in K_0^B(\mathfrak{a}) \,\middle|\, i_{\mathfrak{q}}(u) \equiv \begin{pmatrix} 1 & *\\ 0 & 1\end{pmatrix} \pmod{\mathfrak{a}\mathrm{M}_2(\mathcal{O}_{F_{\mathfrak{q}}})} \quad \text{fo } \mathfrak{q} \nmid \mathfrak{d}\right\}. \tag{2.4.3}$$

We fix a nonzero ideal $\mathfrak{n} \subset \mathcal{O}_F$ prime to $\mathbf{p}\mathfrak{n}^-$, and assume that

$$K_1(\mathfrak{n})^B \subset \Sigma \subset K_0^B(\mathfrak{n}).$$

Let $\mathfrak{a} \subset \mathcal{O}_F$ be a nonzero ideal. Let $\Delta(\mathfrak{a}) \subset \widehat{B}$ be a semigroup defined by

$$\Delta(\mathfrak{a}) := \left\{ x \in \widehat{B} \,\middle|\, i_{\mathfrak{q}}(x) = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \text{ such that } c \in \mathfrak{a}\mathcal{O}_{F_{\mathfrak{q}}},\ d \in \mathcal{O}_{F_{\mathfrak{q}}}^\times \text{ for all } \mathfrak{q} \mid \mathfrak{a}\right\}.$$

We recall the definition of Hecke operators on the space $S\big(\Sigma(\mathbf{p}^s), M\big)$ (Definition 2.1.1) for $s > 0$ and a $\Delta(\mathbf{p}^s\mathfrak{n})$-module M.

Definition 2.4.12. Let M be a $\Delta(\mathbf{p}^s\mathfrak{n})$-module. We define operators $T(\mathfrak{a})$, $U(\mathfrak{b})$ and $U(x)$ acting on the space $S(\Sigma(\mathbf{p}^s), M)$ for nonzero ideals satisfying $(\mathfrak{a}, p\mathfrak{n}) = 1$, $\mathfrak{b}|\mathfrak{n}$ and $x \in \mathcal{O}_{F_p}$, $x^{\underline{t}} \neq 0$ as follows:

- (*Definition of* $T(\mathfrak{a})$): Let $(\mathfrak{a}, p\mathfrak{n}) = 1$. Let $a^{p\mathfrak{d}} \in \left(\mathbb{A}_{F,f}^{p\mathfrak{d}}\right)^\times$ and $b_\mathfrak{n} \in \widehat{B}_\mathfrak{n}^\times$ such that
$$a^{p\mathfrak{d}} Nrd_{B_\mathfrak{n}}(b_\mathfrak{n})\mathcal{O}_F = \mathfrak{a}.$$
We define
$$T(\mathfrak{a}) := \left[\Sigma(\mathbf{p}^s)\begin{pmatrix} a^{p\mathfrak{d}} & 0 \\ 0 & 1 \end{pmatrix} b_\mathfrak{n}\, \Sigma(\mathbf{p}^s)\right].$$
- (*Definition of* $U(\mathfrak{b})$): Let $\mathfrak{b}|\mathfrak{n}$. Let $a^{p\mathfrak{n}\mathfrak{d}} \in \left(\mathbb{A}_{F,f}^{p\mathfrak{n}\mathfrak{d}}\right)^\times$ with
$$a^{p\mathfrak{n}\mathfrak{d}}\mathcal{O}_F = \mathfrak{b}.$$
We define
$$U(\mathfrak{b}) := \left[\Sigma(\mathbf{p}^s)\begin{pmatrix} a^{p\mathfrak{n}\mathfrak{d}} & 0 \\ 0 & 1 \end{pmatrix} \Sigma(\mathbf{p}^s)\right].$$
- (*Definition of* $U(x)$): Let $x \in \mathcal{O}_{F_\mathbf{p}}$ with $x^{\underline{t}_\mathbf{P}} \neq 0$. We define
$$U(x) := \left[\Sigma(\mathbf{p}^s)\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \Sigma(\mathbf{p}^s)\right].$$

Proposition 2.4.13. Let M be a $\Delta(\mathbf{p}^s\mathfrak{n})$-module. The assocative algebra generated by $T(\mathfrak{a}), T(\mathfrak{a},\mathfrak{a}), U(\mathfrak{b})$ and $U(x)$ over $\mathbb{Z}$ in $\mathrm{End}(S(\Sigma(\mathbf{p}^s), M)$ is commutative.

PROOF. See [**Hi91**, Proposition 1.1]. □

Definition 2.4.14. We define $\mathbb{I}$-adic Hecke algebra $\mathbf{h}_B(\mathfrak{n};\mathbb{I})$ by the $\mathbb{I}$-algebra generated by $T(\mathfrak{a}), U(\mathfrak{b})$ and $U(x)$ in $\mathrm{End}_\mathbb{I}(S(\Sigma(\mathbf{p}^s), \mathscr{D}_{k^\mathbf{P}, w^\mathbf{P}}(\mathbb{I}))$.

We also define the ordinary part of $\mathbb{I}$-adic Hecke algebra $\mathbf{h}_B^{\mathrm{ord}}(\mathfrak{n};\mathbb{I})$ by the $\mathbb{I}$-algebra generated by $T(\mathfrak{a}), U(\mathfrak{b})$ and $U(x)$ in $\mathrm{End}_\mathbb{I}(S^{\mathrm{ord}}(\Sigma(\mathbf{p}^s), \mathscr{D}_{k^\mathbf{P}, w^\mathbf{P}}(\mathbb{I}))$. According to Remark 2.3.12, if we put $e_x = \lim_n T_0(x)^{n!} \in \mathbf{h}_B(\mathfrak{n};\mathbb{I})$ is the ordinary idempotent ($x \in \mathbf{p}\mathcal{O}_{F_\mathbf{p}}$ with $x^{\underline{t}_\mathbf{P}} \neq 0$),
$$e_x\mathbf{h}(\mathfrak{n};\mathbb{I}) = \mathbf{h}^{\mathrm{ord}}(\mathfrak{n};\mathbb{I})$$

Theorem 2.4.15. We assume there exists $\mathscr{Y} \subset \mathscr{X}^{\mathrm{arith}}_{k^\mathbf{P}, w^\mathbf{P}}$ with (2.3.2). For $\Phi, \Psi \in S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}_{k^\mathbf{P}, w^\mathbf{P}}(\mathbb{I})\big)$ and $T \in \mathbf{h}_B^{\mathrm{ord}}(\mathfrak{n};\mathbb{I})$, we have
$$\mathcal{B}_\mathbb{I}(T\Phi, \Psi) = \mathcal{B}_\mathbb{I}(\Phi, T\Psi).$$
If for any $r \geqslant 0$, (2.3.4) holds, we have the same formula for any $\Phi, \Psi \in S\big(\Sigma, \mathscr{D}_{k^\mathbf{P}, w^\mathbf{P}}(\mathbb{I})\big)$ and $T \in \mathbf{h}_B(\mathfrak{n};\mathbb{I})$ $\Phi \in S\big(\Sigma, \mathscr{D}_{k^\mathbf{P}, w^\mathbf{P}}(\mathbb{I})\big)$.

PROOF. It follows from Corollary 2.4.10. □

Theorem 2.4.16. Let $\phi \in S^{\mathrm{ord}}_{k,w}(\Sigma(\mathbf{p}^s), \boldsymbol{\omega}; \mathcal{O}_O[\boldsymbol{\omega}])$ be a p-adic quaterninic automorphic form which is new at each place dividing $\mathfrak{n}$. There exists finite $\widetilde{\Lambda}_\mathcal{O}$-algebra $\mathbb{I}$ which is integrally closed domain and $\Phi \in S(\Sigma(\mathbf{p}^s); \mathscr{D}_{k_\mathbf{p}, w_\mathbf{p}}(\mathbb{I}))$ such that there exists $P \in \mathscr{X}_{k^\mathbf{P}, w^\mathbf{P}}(\mathbb{I})$ with $P|_\mathbf{G} = P_{k_\mathbf{p}, w_\mathbf{p}, \boldsymbol{\omega}}$ such that
$$\mathrm{Sp}_P(\Phi) = \phi.$$

PROOF. It follows from the same argument preceding [**Wi88**, Theorem 1.4.1]. □

2.5. The relation between measure valued form and $\mathbf{S}(\Sigma;\mathbb{I})$ and the control theorem

In this section we assume that every prime above p divides $\mathbf{p}$. It means that all of the conditions on upper $\mathbf{p}$ (for example, $2w^{\mathbf{p}} - k^{\mathbf{p}} \in 2\mathbb{Z}\underline{t}^{\mathbf{p}}$) are empty. We omit to write notations involved with upper $\mathbf{p}$ such as $k^{\mathbf{p}}, w^{\mathbf{p}}, \underline{t}^{\mathbf{p}}$, and we simply write $k, w, \underline{t}, \ldots$ instead of $k_{\mathbf{p}}, w_{\mathbf{p}}, \underline{t}_{\mathbf{p}}, \ldots$. We fix a finite product of noetherian complete local $\widetilde{\Lambda}_{\mathcal{O}}$-algebras with finite residue fields, and denote it by $\mathbb{I}$.

Recall the identification

$$\mathbf{S}(\Sigma;\mathbb{I}) = \left\{ \begin{array}{l} \mathbf{f}\colon X(\Sigma^p N(\mathcal{O}_{F_p})) \longrightarrow \mathbb{I} \\ \text{continuous} \end{array} \;\middle|\; t\cdot \mathbf{f}(xt) = \mathbf{f}(x) \text{ for } x \in X(\Sigma^p N_p),\ t \in \mathbf{G} \right\}$$

as in Theorem 2.2.24.

Definition 2.5.1. We define a $\mathbb{I}$-module homomorphism

$$\mathbf{Sp}\colon S\left(\Sigma, \mathscr{D}(\mathbb{I})\right) \longrightarrow \mathbf{S}(\Sigma;\mathbb{I})$$

by

$$\Phi \;\longmapsto\; \left[b \mapsto \int_{[\,\mathcal{O}_{F_p}\times\{1\}\,]} 1\, d\Phi(b) \right].$$

Since $\mathbf{Sp}$ commutes with normalized Hecke operators, we also define

$$\mathbf{Sp}^{\mathrm{ord}}\colon S^{\mathrm{ord}}\left(\Sigma, \mathscr{D}(\mathbb{I})\right) \longrightarrow \mathbf{S}^{\mathrm{ord}}(\Sigma;\mathbb{I}).$$

Proposition 2.5.2. Let $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$ such that $P|_{\mathbf{G}} = P_{k,w,\boldsymbol{\omega}}$. The following diagram

$$\begin{array}{ccc} S\left(\Sigma;\mathscr{D}(\mathbb{I})\right) & \xrightarrow{\quad \mathbf{Sp} \quad} & \mathbf{S}(\Sigma;\mathbb{I}) \\ \Big\downarrow {\scriptstyle \mathrm{Sp}_P} & & \Big\downarrow {\scriptstyle P_*} \\ S_{k,w}\big(\Sigma_0(\mathbf{p}^{s(\boldsymbol{\omega})}), \mathcal{O}_{\boldsymbol{\omega}}\big)[\psi] & \xrightarrow{\;\langle X^{k-2\underline{t}}, \cdot\rangle\;} & \mathbf{S}(\Sigma;\mathbb{I}/P\mathbb{I}) \end{array}$$

is commutative. Here, for $f \in \mathrm{Sym}^{k-2\underline{t}}(\mathcal{O}_{\boldsymbol{\omega}})$, $\langle X^{k-2\underline{t}}, \cdot\rangle$ means the value $f(0,1) \in \mathcal{O}_{\boldsymbol{\omega}}$. (It is actually abuse of notation. In fact, the pairing $\langle \cdot, \cdot \rangle_{k-2\underline{t}}$ of (2.1.3) is only defined over $\mathcal{O}_{\psi}$ under the assumption $k \leqslant p\underline{t} + 1$.)

Proof. It follows immediately form the definitions. □

We focus on the ordinary part. Then we have the following result:

Theorem 2.5.3. The homomorphism $\mathbf{Sp}^{\mathrm{ord}}$ is an isomorphism.

Proof. The injectivity follows by Theorem 2.3.11. We prove the surjectivity. Let $\mathbf{f} \in \mathbf{S}^{\mathrm{ord}}(\Sigma;\mathbb{I})$. By Theorem 2.2.24, $\mathbf{f}$ is described as an element

$$(f_s)_{s\in\mathbb{Z}_{>0}} \;\in\; \varprojlim_{s} \mathrm{Hom}_{\mathcal{O}[\mathbf{G}_s]}\left(e_{\mathbf{p}}\, \mathrm{Pic}_{\mathcal{O}}\, X\big(\Sigma(\mathbf{p}^s)\big),\ \mathbb{I}/P_s\mathbb{I} \right),$$

where $e_{\mathbf{p}}$ is Hida's ordinary idempotent. For $s' \in \bigoplus_{\mathfrak{p}\mid\mathbf{p}} \mathbb{Z}_{>0}\mathfrak{p}$, let

$$L_{s'} := \sum_{\gamma\in\Delta(\mathbf{p})} \mathbb{I}/P_s\mathbb{I}\,\mathbf{1}_{[\,\mathbf{p}^{s'}\mathcal{O}_F\times\{1\}\,]}(\gamma z) \;\subset\; \Gamma(\mathcal{O}_{F_{\mathbf{p}}}, \mathscr{E}\otimes_{\mathbb{I}} \mathbb{I}/P_s\mathbb{I}).$$

The space $L_{s'}$ is free of finite rank over $\mathbb{I}/P_s\mathbb{I}$, in fact

$$L_{s'} := \bigoplus_{\gamma\in\Sigma_0(\mathbf{p})/\Sigma_0(\mathbf{p})\cap{}^t\Sigma_0(\mathbf{p}^{s'})} \mathbb{I}/P_s\mathbb{I}\,\mathbf{1}_{[\,\mathbf{p}^{s'}\mathcal{O}_F\times\{1\}\,]}(\gamma z).$$

Let $\delta_0^{s'} \in \mathrm{Hom}_{\mathbb{I}}(L_{s'}, \mathbb{I}/P_{s'})$ defined by

$$\delta_0^{s'}\big(\mathbf{1}_{[\mathbf{p}^{s'}\mathcal{O}_F \times \{1\}]}(\gamma z)\big) = \begin{cases} 1 & \text{if } \gamma \in \Sigma_0(\mathbf{p}) \cap {}^t\Sigma_0(\mathbf{p}^{s'}), \\ 0 & \text{if } \gamma \notin \Sigma_0(\mathbf{p}) \cap {}^t\Sigma_0(\mathbf{p}^{s'}). \end{cases}$$

For $b \in X(\Sigma^p)$, we define

$$\Phi_s(b) := \sum_{c \in \mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}} \big(T_0(\varpi_p^s)^{-1} f_s\big)\left(b\begin{pmatrix} \varpi_p^s & c \\ 0 & 1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\delta_0^s\right) \in \mathrm{Hom}_{\mathbb{I}}\big(L_s, \mathbb{I}/P_s\mathbb{I}\big),$$

where, we denote $(\varpi_\mathfrak{p}^{s_\mathfrak{p}})_{\mathfrak{p}|p} \in \mathcal{O}_{F_p}$ by ϖ_p^s. By Lemma 2.2.2, for any $u \in \Sigma_0(\mathbf{p})$, we have

$$\Phi_s(bu) = u^{-1}\Phi_s(b).$$

We prove

$$\big(\Phi_s(b)\big)_s \in \varprojlim_s \mathrm{Hom}_{\mathbb{I}}\big(L_s, \mathbb{I}/P_s\mathbb{I}\big) \cong \mathscr{D}(\mathcal{O}_{F_\mathbf{p}}; \mathbb{I}).$$

Let $s' > s$. By using Lemma 2.2.2, the image of $\Phi_{s'}$ in $\mathrm{Hom}_{\mathbb{I}}(L_s, \mathbb{I}/P_s\mathbb{I})$ is

$$\begin{aligned} &\sum_{c \in \mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}} \sum_{d \in \mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^{s'-s}\mathcal{O}_{F_\mathbf{p}}} \big(T_0(\varpi_p^{s'})^{-1} f_s\big)\left(b\begin{pmatrix} \varpi_p^s & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varpi_p^{s'-s} & d \\ 0 & 1 \end{pmatrix}\right) \\ &\qquad \times \left(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \varpi_p^s d \\ 0 & 1 \end{pmatrix}\delta_0^s\right) \\ &= \sum_{c \in \mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}} \sum_{d \in \mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^{s'-s}\mathcal{O}_{F_\mathbf{p}}} \big(T_0(\varpi_p^{s'})^{-1} f_s\big)\left(b\begin{pmatrix} \varpi_p^s & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varpi_p^{s'-s} & d \\ 0 & 1 \end{pmatrix}\right) \\ &\qquad \times \left(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\delta_0^s\right) \\ &= \Phi_s. \end{aligned}$$

Thus have

$$\Phi := \varprojlim_s \Phi_s \in S\big(\Sigma_0(\mathbf{p}), \mathscr{D}(\mathcal{O}_{F_\mathbf{p}}; \mathbb{I})\big)$$

satisfying $\mathbf{Sp}(\Phi) = \mathbf{f}$. Let $\Phi^{(n)}$ be a measure valued form obtained from the $T_0(x)^{-n}\mathbf{f}$ by using the same construction above. Clearly $T_0(x)^n\Phi^{(n)} = \Phi$, thus

$$\Phi \in S^{\mathrm{ord}}\big(\Sigma_0(\mathbf{p}), \mathscr{D}(\mathcal{O}_{F_\mathbf{p}}, \mathbb{I})\big).$$

□

Corollary 2.5.4. Let $P \in \mathscr{X}(\mathbb{I})$ with $P|_{\mathbf{G}} = P_{k,w,\boldsymbol{\omega}}$. The ordinary specialization map induces the following isomorphism:

$$\mathrm{Sp}_P^{\mathrm{ord}} \colon S^{\mathrm{ord}}(\Sigma, \mathscr{D}(\mathbb{I})) \otimes_{\mathbb{I}} \mathbb{I}/\mathrm{Ker}(P) \xrightarrow{\cong} S_{k,w,\psi}(\Sigma(\mathbf{p}^{s(\boldsymbol{\omega})}), \boldsymbol{\omega}; P(\mathbb{I}))$$

Proof. It follows from Proposition 2.5.2, Theorem 2.5.3 and Theorem 2.2.20. (See Remark 2.2.25.) □

Remark 2.5.5. Corollary 2.5.4 is the control theorem for measure valued forms. The following formula

$$\mathrm{Ker}(\mathrm{Sp}_P^{\mathrm{ord}}) = \mathrm{Ker}(P) S^{\mathrm{ord}}\big(\Sigma, \mathscr{D}(\mathbb{I})\big)$$

can be proved for any $\mathbb{I}$. However, for the surjectivity of Sp_P, it seems difficult to prove it without Therem 2.5.3 and Theorem 2.2.20. In fact, since $\mathscr{D}(\mathcal{W};\mathcal{O})$ is generated by Dirac measures, the image of

$$\rho_{k,w,\boldsymbol{\omega}}\colon \mathscr{D}(\mathcal{W};\mathcal{O}) \longrightarrow \mathrm{Sym}^{k-2\underline{t}}(\mathcal{O}_{\boldsymbol{\omega}})$$

is the same as

$$\mathcal{O}_{\boldsymbol{\omega}}[\Sigma_0(\mathbf{p})]Y^{k-2\underline{t}} \subset \mathrm{Sym}^{k-2\underline{t}}(\mathcal{O}_{\boldsymbol{\omega}}).$$

In general, the left hand side is not equal to $\mathrm{Sym}^{k-2\underline{t}}(\mathcal{O}_{\boldsymbol{\omega}})$.

CHAPTER 3

Construction of three-variable p-adic L-functions for balanced triple product

In this chapter we assume that every prime above p divides $\mathbf{p}$. It means that all of the conditions on upper $\mathbf{p}$ (for example, $2w^{\mathbf{p}} - k^{\mathbf{p}} \in 2\mathbb{Z}\underline{t}^{\mathbf{p}}$) are empty. We omit to write notations involved with upper $\mathbf{p}$ such as $k^{\mathbf{p}}, w^{\mathbf{p}}, \underline{t}^{\mathbf{p}}$, and we simply write $k, w, \underline{t}, \ldots$ instead of $k_{\mathbf{p}}, w_{\mathbf{p}}, \underline{t}_{\mathbf{p}}, \ldots$. We fix a finite product of noetherian complete local $\widetilde{\Lambda}_{\mathcal{O}}$-algebras with finite residue fields, and denote it by $\mathbb{I}$.

Assume that p is odd. Let E is a totally real cubic étale algebra over F, namely, E is one of the following F-algebras:

$$E = \begin{cases} F_1 \times F_1 \times F_1 & (F_1 = F), \\ F_1 \times F_2 & (F_1 = F \text{ and } F_2 \text{ is a totally real quadratic extension over } F), \\ F_3 & (F_1 = F \text{ and } F_3 \text{ is a totally real cubic extension over } F). \end{cases}$$

We denote $B_{F_i} := B \otimes_F F_i$. Let

$$\begin{aligned} I_F &:= I = \{\sigma\colon F \hookrightarrow \mathbb{C}_p : \text{field embedding}\} \\ I_{F_i} &:= \{\sigma\colon F_i \hookrightarrow \mathbb{C}_p : \text{field embedding}\} \\ \mathbb{Z}[I_{F_i}] &:= \bigoplus_{\sigma \in I_{F_i}} \mathbb{Z}\sigma \\ \underline{t}_{F_i} &:= \sum_{\sigma \in I_{F_i}} \sigma \in \mathbb{Z}[I_{F_i}]. \end{aligned}$$

Let

$$\begin{aligned} E_{\mathfrak{p}} &:= E \otimes_F F_{\mathfrak{p}} \\ E_p &:= E \otimes_{\mathbb{Q}} \mathbb{Q}_p \end{aligned}$$

For each $\mathfrak{p} \mid p$, $E_{\mathfrak{p}}$ is isomorphic to $F_{\mathfrak{p}}^3$ or $F_{\mathfrak{p}} \times K'_{\mathfrak{p}}$ or $K''_{\mathfrak{p}}$, where $K'_{\mathfrak{p}}$ (resp. $K''_{\mathfrak{p}}$) is a quadratic (resp. cubic) extension over $F_{\mathfrak{p}}$. We fix an isomorphism between them (we fix them more precisely in Section 3.3) We fix a finite flat $\mathbb{Z}_p$-algebra $\mathcal{O} \subset \mathbb{C}_p$ containing all conjugation of $\mathcal{O}_E$. Fix nonzero ideals $\mathfrak{n}_{1,1}, \mathfrak{n}_{1,2}, \mathfrak{n}_{1,3}, \mathfrak{n}_1 \subset \mathcal{O}_{F_1}$, $\mathfrak{n}_2 \subset \mathcal{O}_{F_2}$ and $\mathfrak{n}_3 \subset \mathcal{O}_{F_3}$, which are prime to p. We define an open compact subgroup of $B^\times(\mathbb{A}_{F,f} \otimes_F E)$ by

$$\begin{aligned} \Sigma_E &:= \begin{cases} K_1^{B_{F_1}}(\mathfrak{n}_{1,1}) \times K_1^{B_{F_1}}(\mathfrak{n}_{1,2}) \times K_1^{B_{F_1}}(\mathfrak{n}_{1,3}) & \text{if } E = F_1 \times F_1 \times F_1, \\ K_1^{B_{F_1}}(\mathfrak{n}_1) \times K_1^{B_{F_2}}(\mathfrak{n}_2) & \text{if } E = F_1 \times F_2, \\ K_1^{B_{F_3}}(\mathfrak{n}_3) & \text{if } E = F_3, \end{cases} \\ \Sigma &:= \Sigma_E \cap \widehat{B}^\times. \end{aligned}$$

We assume that (by taking sufficiently small Σ_E) for any $b \in \widehat{B}$,

$$\Gamma^b_{\Sigma_E \cap \widehat{B}^\times} := (b^{-1}B^\times b\mathbb{A}_{F,f}^\times \cap \Sigma_E\mathbb{A}_{F,f}^\times)/\mathbb{A}_{F,f}^\times = \{1\}. \tag{3.0.1}$$

Put

$$\mathrm{Cl}^+_{F_i}(\Sigma(p^\infty)) := \varprojlim_s \mathrm{Cl}^+_{F_i}(\Sigma(\mathbf{p}^s))$$

$$\mathrm{Cl}^+_E(\Sigma(p^\infty)) := \begin{cases} \mathrm{Cl}^+_{F_1}(\Sigma(p^\infty)) \times \mathrm{Cl}^+_{F_1}(\Sigma(p^\infty)) \times \mathrm{Cl}^+_{F_1}(\Sigma(p^\infty)) & \text{if } E = F_1 \times F_1 \times F_1, \\ \mathrm{Cl}^+_{F_1}(\Sigma(p^\infty)) \times \mathrm{Cl}^+_{F_2}(\Sigma(p^\infty)) & \text{if } E = F_2 \times F_1, \\ \mathrm{Cl}^+_{F_3}(\Sigma(p^\infty)) & \text{if } E = F_3. \end{cases}$$

We define

$$\mathbf{G}_{F_i} := \mathrm{Cl}^+_{F_i}(\Sigma(\mathbf{p}^\infty)) \times \mathcal{O}^\times_{F_p}$$

$$\mathbf{G}_E := \begin{cases} \mathbf{G}_{F_1} \times \mathbf{G}_{F_1} \times \mathbf{G}_{F_1} & \text{if } E = F_1 \times F_1 \times F_1, \\ \mathbf{G}_{F_2} \times \mathbf{G}_{F_1} & \text{if } E = F_2 \times F_1, \\ \mathbf{G}_{F_3} & \text{if } E = F_3, \end{cases}$$

We embed $\mathbf{G}_F$ into $\mathbf{G}_E$ diagonally. Let

$$\mathrm{Cl}^+_{F_i}(\Sigma(p^\infty)) = \mathrm{Cl}^+_{F_i}(\Sigma(p^\infty))(p) \oplus Z'_i$$

be a decomposition where $Z'_i \subset \mathrm{Cl}^+_{F_i}(\Sigma(p^\infty))$ is a finite group of order prime to p. For $j = 1, 2, 3$ and $i = 1, 2, 3$, we fix a character

$$\begin{aligned} \chi_{1,j} \colon\ & \mathrm{Cl}^+_{F_1}(\Sigma(p^\infty)) \longrightarrow Z'_1 \longrightarrow \mathbb{C}^\times_p, \\ \chi_i \colon\ & \mathrm{Cl}^+_{F_1}(\Sigma(p^\infty)) \longrightarrow Z'_i \longrightarrow \mathbb{C}^\times_p, \\ \chi \colon\ & \mathrm{Cl}^+_E(\Sigma(p^\infty)) \longrightarrow \mathbb{C}^\times_p \\ := & \begin{cases} (\chi_{1,1}, \chi_{1,2}, \chi_{1,3}) & \text{if } E = F_1 \times F_1 \times F_1, \\ (\chi_1, \chi_2) & \text{if } E = F_1 \times F_2, \\ \chi_3 & \text{if } E = F_1. \end{cases} \end{aligned}$$

We make the following assumption on χ:

$$\chi(\mathrm{Cl}^+_F(\Sigma(p^\infty))) = 1,$$

in particular, for any $x \in \mathrm{Cl}^+_F(\Sigma(p^\infty))$, the image of $x \in \mathrm{Cl}^+_E(\Sigma(p^\infty))$ is contained in the p-sylow group $\mathrm{Cl}^+_E(\Sigma(p^\infty))(p)$. Let

$$\chi_{\mathfrak{p}} \colon \mathcal{O}^\times_{E_{\mathfrak{p}}} \longrightarrow \mathrm{Cl}^+_E(\Sigma(p^\infty))(p) \xrightarrow{\ \chi\ } \mathbb{C}^\times_p.$$

We fix a finite $\mathcal{O}[[\mathbf{G}_{F_1}]]$-algebra $\mathbb{I}_{1,j}$ for $j = 1, 2, 3$ such that on each algebra Z'_1 acts as $\chi_{1,j}$ and we also fix a finite $\mathcal{O}[[\mathbf{G}_{F_i}]]$-algebra $\mathbb{I}_i$ for $i = 1, 2, 3$ such that on each algebra Z'_i acts through χ_i. We define

$$\mathbb{I} := \begin{cases} \mathbb{I}_{1,1} \widehat{\otimes}_{\mathcal{O}} \mathbb{I}_{1,1} \widehat{\otimes}_{\mathcal{O}} \mathbb{I}_{1,1} & \text{if } E = F_1 \times F_1 \times F_1, \\ \mathbb{I}_1 \widehat{\otimes}_{\mathcal{O}} \mathbb{I}_2 & \text{if } E = F_1 \times F_2, \\ \mathbb{I}_3 & \text{if } E = F_3. \end{cases}$$

Let

$$\mathscr{E}_{\mathfrak{p}} := \mathrm{GL}_2(\mathcal{O}_{E_{\mathfrak{p}}}) / {}^tN(\mathcal{O}_{E_{\mathfrak{p}}}) \times_{T(\mathcal{O}_{E_{\mathfrak{p}}})} \mathbb{I}$$

be the $\mathbb{I}$-bundle defined as in the same manner in Section 2.3.

3.1. The definition of the section δ

At first, for each type of $E_{\mathfrak{p}}$ listed as below, we define $\delta_{\mathfrak{p}} \in \Gamma(\mathbb{P}^1(E_{\mathfrak{p}}), \mathscr{E}_{\mathfrak{p}})$ and then we define a unique element (up to scalar)

$$\delta := \prod_{\mathfrak{p}\,|\,p} \delta_{\mathfrak{p}} \in \Gamma(\mathbb{P}^1(\widehat{E}_p), \mathscr{E}_E)$$

satisfying for $u \in \mathrm{GL}_2(\mathcal{O}_{F_p}) \subset \Sigma_{Ep}$,

$$\delta|u(z) = (\det(u)^{-\frac{1}{2}}, 1)\delta(z). \tag{3.1.1}$$

3.1.1. $E_{\mathfrak{p}} = F_{\mathfrak{p}} \times F_{\mathfrak{p}} \times F_{\mathfrak{p}}$ **case.** Let

$$V_{\mathfrak{p}}^E := \left\{ ((x_1; y_1), (x_2; y_2), (x_3; y_3)) \in \mathbb{P}^1(\mathcal{O}_{F_{\mathfrak{p}}})^3 \middle| x_i y_j - y_i x_j \in \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \text{ for } i, j = 1, 2, 3, i \neq j \right\}$$

and let

$$\begin{aligned} V_1 &:= V_{\mathfrak{p}}^E \cap \left[\{1\} \times \mathcal{O}_{F_{\mathfrak{p}}}\right] \times \left[\mathcal{O}_{F_{\mathfrak{p}}} \times \{1\}\right] \times \left[\mathcal{O}_{F_{\mathfrak{p}}} \times \{1\}\right] \\ V_2 &:= V_{\mathfrak{p}}^E \cap \left[\mathcal{O}_{F_{\mathfrak{p}}} \times \{1\}\right] \times \left[\{1\} \times \mathcal{O}_{F_{\mathfrak{p}}}\right] \times \left[\mathcal{O}_{F_{\mathfrak{p}}} \times \{1\}\right] \\ V_3 &:= V_{\mathfrak{p}}^E \cap \left[\mathcal{O}_{F_{\mathfrak{p}}} \times \{1\}\right] \times \left[\mathcal{O}_{F_{\mathfrak{p}}} \times \{1\}\right] \times \left[\{1\} \times \mathcal{O}_{F_{\mathfrak{p}}}\right] \end{aligned}$$

be an open covering. We define $\delta_{\mathfrak{p}} \in \Gamma(\mathbb{P}^1(E_{\mathfrak{p}}), \mathscr{E}_{\mathfrak{p}})$ as the zero extension of a unique element element of $\Gamma(V_{\mathfrak{p}}^E, \mathscr{E}_{\mathfrak{p}})$ satisfying (3.1.1) and

$$\begin{aligned} \delta_{\mathfrak{p}}|_{V_3}(z) =& \chi_{\mathfrak{p}} \left(\frac{z_1 - z_2}{1 - z_2 z_3}, \frac{z_1 - z_2}{1 - z_1 z_3}, 1 \right)^{-1} \\ & \times \left[\begin{array}{c} \left\langle \frac{(z_1-z_2)(1-z_1z_3)}{1-z_2z_3}, \frac{(z_1-z_2)(1-z_2z_3)}{1-z_1z_3}, \frac{(1-z_2z_3)(1-z_1z_3)}{z_1-z_2} \right\rangle^{-\frac{1}{2}}, \\ \left(\frac{(z_1-z_2)(1-z_1z_3)}{1-z_2z_3}, \frac{(z_1-z_2)(1-z_2z_3)}{1-z_1z_3}, \frac{(1-z_2z_3)(1-z_1z_3)}{z_1-z_2} \right) \end{array} \right] \in \mathbf{G}_E \times \mathcal{O}_{E_p}^{\times} \end{aligned}$$

where $z = (z_1, z_2, z_3) \in \mathcal{O}_{F_{\mathfrak{p}}}^3$.

3.1.2. $E_{\mathfrak{p}} = F_{\mathfrak{p}} \times K'_{\mathfrak{p}}$ **case (**$K'_{\mathfrak{p}}/F_{\mathfrak{p}}$ **is a quadratic extension).** We fix $\xi_{\mathfrak{p}} \in K'_{\mathfrak{p}}$ such $\mathcal{O}_{K'_{\mathfrak{p}}} = \mathcal{O}_{F_{\mathfrak{p}}}[\xi_{\mathfrak{p}}]$ and $\mathrm{tr}_{K'_{\mathfrak{p}}/F_{\mathfrak{p}}}(\xi_{\mathfrak{p}}) = 0$. We denote by $\varsigma \in \mathrm{Gal}(K'_{\mathfrak{p}}/F_{\mathfrak{p}})$ the generator. Let

$$V_{\mathfrak{p}}^E := \left\{ ((x_1; y_1), (x_2; y_2)) \in \mathbb{P}^1(\mathcal{O}_{F_{\mathfrak{p}}}) \times \mathbb{P}^1(\mathcal{O}_{K'_{\mathfrak{p}}}) \middle| \; x_2 y_1 - y_2 x_1, \; \frac{(x_2^{\varsigma} y_2 - y_2^{\varsigma} x_2)}{2\xi_{\mathfrak{p}}} \in \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \right\}$$

and let

$$\begin{aligned} V'_1 &:= V_{\mathfrak{p}}^E \cap \left[\mathcal{O}_{F_{\mathfrak{p}}} \times \{1\}\right] \times \left[\{1\} \times \mathcal{O}_{K'_{\mathfrak{p}}}\right] \\ V'_2 &:= V_{\mathfrak{p}}^E \cap \left[\{1\} \times \mathcal{O}_{F_{\mathfrak{p}}}\right] \times \left[\mathcal{O}_{K'_{\mathfrak{p}}} \times \{1\}\right] \end{aligned}$$

be an open covering. We define $\delta_{\mathfrak{p}} \in \Gamma(\mathbb{P}^1(E_{\mathfrak{p}}), \mathscr{E}_{\mathfrak{p}})$ as the zero extension of a unique element element of $\Gamma(V_{\mathfrak{p}}^E, \mathscr{E}_{\mathfrak{p}})$ satisfying (3.1.1) defined by

$$\begin{aligned} \delta_{\mathfrak{p}}|_{V'_1}(z) = \delta_{\mathfrak{p}}|_{V'_2}(z) =& \chi_{\mathfrak{p}} \left(1, \frac{z_2 - z_2^{\varsigma}}{2\xi_{\mathfrak{p}}(1 - z_2^{\varsigma} z_1)} \right)^{-1} \\ & \times \left[\begin{array}{c} \left(\frac{2\xi_{\mathfrak{p}}(1-z_2^{\varsigma}z_1)(1-z_2z_1)}{(z_2-z_2^{\varsigma})}, \frac{(z_2-z_2^{\varsigma})(1-z_2z_1)}{2\xi_{\mathfrak{p}}(1-z_2^{\varsigma}z_1)} \right)^{-\frac{1}{2}}, \\ \left(\frac{2\xi_{\mathfrak{p}}(1-z_2^{\varsigma}z_1)(1-z_2z_1)}{(z_2-z_2^{\varsigma})}, \frac{(z_2-z_2^{\varsigma})(1-z_2z_1)}{2\xi_{\mathfrak{p}}(1-z_2^{\varsigma}z_1)} \right) \end{array} \right] \in \mathbf{G}_E \times \mathcal{O}_{F_p}^{\times} \end{aligned}$$

where $z = (z_1, z_2) \in \mathcal{O}_{F_{\mathfrak{p}}} \times \mathcal{O}_{K'_{\mathfrak{p}}}$.

3.1.3. $E_{\mathfrak{p}} = K''_{\mathfrak{p}}$ case ($K''_{\mathfrak{p}}/F_{\mathfrak{p}}$ is a cubic extension). Let $\widetilde{K''_{\mathfrak{p}}}$ be a Galois closure of $K''_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$. Fix a generator ρ of unique normal subgroup of $\mathrm{Gal}(\widetilde{K}''/F_{\mathfrak{p}})$. Let $\theta_{\mathfrak{p}}$ be a element of $\mathcal{O}_{K''_{\mathfrak{p}}}$ such that $\mathcal{O}_{K''_{\mathfrak{p}}} = \mathcal{O}_{F_{\mathfrak{p}}}[\theta_{\mathfrak{p}}]$ Let

$$V_{\mathfrak{p}}^{E} := \left\{ (x;y) \in \mathbb{P}^1(\mathcal{O}_{K''_{\mathfrak{p}}}) \middle| \ (\theta_{\mathfrak{p}}^{\rho} - \theta_{\mathfrak{p}}^{\rho^2})^{-1} (xy^{\rho} - yx^{\rho^2}) \in \mathcal{O}_{K''_{@}}^{\times} \right\}$$

Let

$$V''_1 := V_{\mathfrak{p}}^{E} \cap [\mathcal{O}_{K''_{\mathfrak{p}}} \times \{1\}]$$
$$V''_2 := V_{\mathfrak{p}}^{E} \cap [\{1\} \times \mathcal{O}_{K''_{\mathfrak{p}}}]$$

be an open covering. We define $\delta_{\mathfrak{p}} \in \Gamma(\mathbb{P}^1(E_{\mathfrak{p}}), \mathscr{E}_{\mathfrak{p}})$ as the zero extension of an element of $\Gamma(V_{\mathfrak{p}}^{E}, \mathscr{E}_{\mathfrak{p}})$ defined by

$$\delta_{\mathfrak{p}}|_{V''_1}(z) = \delta_{\mathfrak{p}}|_{V''_2}(z) = \chi_{\mathfrak{p}} \left(\frac{\theta_{\mathfrak{p}}^{\rho} - \theta_{\mathfrak{p}}^{\rho^2}}{z^{\rho} - z^{\rho^2}} \right)^{-1}$$
$$\times \left[\left\langle \frac{(\theta_{\mathfrak{p}}^{\rho} - \theta_{\mathfrak{p}}^{\rho^2})(z - z^{\rho})(z - z^{\rho^2})}{(\theta_{\mathfrak{p}} - \theta_{\mathfrak{p}}^{\rho})(\theta_{\mathfrak{p}} - \theta_{\mathfrak{p}}^{\rho^2})(z^{\rho} - z^{\rho^2})} \right\rangle^{-\frac{1}{2}}, \left(\frac{(\theta_{\mathfrak{p}}^{\rho} - \theta_{\mathfrak{p}}^{\rho^2})(z - z^{\rho})(z - z^{\rho^2})}{(\theta_{\mathfrak{p}} - \theta_{\mathfrak{p}}^{\rho})(\theta_{\mathfrak{p}} - \theta_{\mathfrak{p}}^{\rho^2})(z^{\rho} - z^{\rho^2})} \right) \right] \in \mathbf{G}_E \times \mathcal{O}_{F_p}^{\times}$$

Note that $\delta_{\mathfrak{p}}$ is independent of the choice of the generator ρ.

3.2. The construction of Theta element

We give an important property of δ again:

Proposition 3.2.1. Let δ as above. For $u \in \mathrm{GL}_2(\mathcal{O}_{F_p}) \subset \Sigma_{Ep}$, we have

$$\delta|u(z) = (\det(u)^{-\frac{1}{2}}, 1)\delta(z)$$

Remark 3.2.2. Any section defined by zero extension of $\Gamma(\prod_{\mathfrak{p}} V_{\mathfrak{p}}^{E}, \mathscr{E} \otimes \mathscr{E} \otimes \mathscr{E})$ satisfying the property of this proposition is uniquely determined up to scalar.

Now we define the theta element.

Definition 3.2.3. Let

$$\Phi_{1,j} \in S\left(K_1^{B_{F_1}}(\mathfrak{n}_{1,j}); \mathscr{D}(\mathbb{I}_{1,j})\right)$$

for $j = 1, 2, 3$ and let

$$\Phi_i \in S\left(K_1^{B_{F_i}}(\mathfrak{n}_i); \mathscr{D}(\mathbb{I}_i)\right)$$

for $i = 1, 2, 3$. We put

$$\Phi := \begin{cases} \Phi_{1,1} \otimes \Phi_{1,2} \otimes \Phi_{1,3} & \text{if } E = F_1 \times F_1 \times F_1, \\ \Phi_1 \otimes \Phi_2 & \text{if } E = F_1 \times F_2, \\ \Phi_3 & \text{if } E = F_3. \end{cases}$$

We define the theta element $\Theta_{\Phi} \in \mathbb{I}$ by

$$\Theta_{\Phi} := \sum_{b \in \mathbb{A}_{F,f}^{\times} B^{\times} \backslash \hat{B}^{\times} / \Sigma} \left(\mathrm{Nrd}_{B/F}(b)^{-\frac{1}{2}}, 1 \right) \int_{\mathbb{P}^1(\mathcal{O}_{\hat{E}_p})} \delta(z) d(\Phi(b))(z).$$

We also define the square root of p-adic L-function

$$\mathscr{L}_p^B(\Phi) := \frac{\Theta_\Phi^2}{\mathcal{B}_{\mathbb{I}}(\Phi,\Phi)},$$

where

$$\mathcal{B}_{\mathbb{I}}(\Phi,\Phi) := \begin{cases} \prod_{j=1}^3 \mathcal{B}_{\mathbb{I}_{1,j}}(\Phi_{1,j},\Phi_{1,j}) & \text{if } E = F_1\times F_1\times F_1, \\ \mathcal{B}_{\mathbb{I}_1}(\Phi_1,\Phi_1)\mathcal{B}_{\mathbb{I}_2}(\Phi_2,\Phi_2) & \text{if } E = F_1\times F_2, \\ \mathcal{B}_{\mathbb{I}_3}(\Phi_3,\Phi_3) & \text{if } E = F_3. \end{cases}$$

3.3. The interpolation of Theta elements

We fix a uniformizer $\varpi_{\mathfrak{p}}$ of $\mathcal{O}_{F_{\mathfrak{p}}}$ for each $\mathfrak{p} \mid p$ and denote $(\varpi_{\mathfrak{p}})_{\mathfrak{p}\mid p} \in \mathcal{O}_{F_{\mathbf{p}}}$ by ϖ_p.

3.3.1. $E = F_1 \times F_1 \times F_1$ case. Let $\Sigma_{1,j} \subset \widehat{B}^\times$ be an open subgroup and $\Phi_{1,j} \in S\big(\Sigma_{1,j}, \eta_1^{(j)}, \mathscr{D}(\mathbb{I})\big)$ as in Definition 3.2.3 ($j = 1,2,3$). For each $\mathfrak{p}$, we choose a canonical isomorphism

$$E_{\mathfrak{p}} \cong F_{\mathfrak{p}}^3.$$

For $n_1, n_2, n_3 \in \mathbb{Z}_{\geqslant 0}[I_{F_1}]$, we denote the indeterminate by

$$\begin{aligned} &\mathrm{Sym}^{n_1}(\mathbb{C}_p)\otimes_{\mathbb{C}_p}\mathrm{Sym}^{n_2}(\mathbb{C}_p)\otimes_{\mathbb{C}_p}\mathrm{Sym}^{n_3}(\mathbb{C}_p) \\ &= \bigotimes_{\sigma\in I}\mathbb{C}_p[X_1^\sigma,Y_1^\sigma]_{n_{1\sigma}}\otimes_{\mathbb{C}_p}\mathbb{C}_p[X_2^\sigma,Y_2^\sigma]_{n_{2\sigma}}\otimes_{\mathbb{C}_p}\mathbb{C}_p[X_3^\sigma,Y_3^\sigma]_{n_{3\sigma}}. \end{aligned}$$

It has natural $\mathrm{GL}_2(E\otimes_F\mathbb{C}_p) \cong \prod_{\sigma\in I}\mathrm{GL}_2(\mathbb{C}_p)^3$-action.

Lemma 3.3.1. For any $s > 0$, we have

$$\Theta_\Phi = \sum_{b\in\mathbb{A}_{F,f}^\times B^\times\backslash\widehat{B}^\times/\Sigma_0(\mathbf{p}^s)} (\mathrm{Nrd}_{B/F}(b)^{-\frac{1}{2}},1)\int_{[\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}]\times[\mathcal{O}_{F_{\mathbf{p}}}\times\{1\}]\times[\{1\}\times\mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}]}\delta(z)d(\Phi(b))(z),$$

Proof. It's proved in the same way of the proof of Lemma 2.4.9 □

Theorem 3.3.2. Assume $\Phi_1^{(1)}, \Phi_1^{(2)}$ are ordinary and there exists $a_j \in \mathbb{I}^\times$ such that

$$T_0(\varpi_{\mathfrak{p}})\Phi_{1,j} = a_{j,\mathfrak{p}}\Phi_{1,j}$$

for $j = 1,2$. For any $P \in \mathscr{X}(\mathbb{I})$ such that $P|_{\mathbf{G}_E} = P_{k_1,w_1,\omega_1,\omega_1'} \times P_{k_2,w_2,\omega_2,\omega_2'} \times P_{k_3,w_3,\omega_3,\omega_3'}$ with $2w_i - k_i = \alpha_i \underline{t}_1$ and $\sum_{i=1}^3 \alpha_i \in 2\mathbb{Z}$. For any $s \geqslant s(\omega_j,\omega_j')$ ($j = 1,2,3$), we have

$$\begin{aligned} &P(\Theta_\Phi) \\ &= \frac{(\omega_1\omega_2\omega_3)^{1/2}(\varpi_p^s)\#\big(\mathcal{O}_{F_{\mathbf{p}}}/\mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}\big)^\times\#\big(\mathcal{O}_{F_{\mathbf{p}}}/\mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}\big)}{(-1)^{k_3}\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}\mid p}a_{1,\mathfrak{p}}^{s_{\mathfrak{p}}}(P)^u a_{2,\mathfrak{p}}^{s_{\mathfrak{p}}}(P)^u}\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)} \\ &\left\langle \Delta_k, \mathrm{Sp}_P(\Phi_{1,1})^u(x\begin{pmatrix}1 & \varpi_p^{-s}\\ 0 & 1\end{pmatrix})\otimes\mathrm{Sp}_P(\Phi_{1,2})^u(x)\otimes\widehat{\mathrm{Sp}}_P^{(s)}(\Phi_{1,3})^u(x\begin{pmatrix}\varpi_p^{-s} & 0\\ 0 & 1\end{pmatrix})\right\rangle_{k^*-2\underline{t}^*} \\ &\times(\omega_1\omega_2\omega_3)^{-1/2}(\mathrm{Nrd}_{B/F}(x))dx \end{aligned}$$

where the symbol $(\cdot)^u$ is that explained in Remark 2.1.4 and $k_* := k_1 + k_2 + k_3$, $\underline{t}^* := \sum_{\sigma E \to \mathbb{C}} \sigma$.

$$\begin{aligned}
\Delta_k &= (X_2Y_3 - X_3Y_2)^{k_1^* - \underline{t}}(X_1Y_3 - X_3Y_1)^{k_2^* - \underline{t}}(X_1Y_2 - X_2Y_1)^{k_3^* - \underline{t}} \\
&:= \prod_{\sigma \in I} (X_2^\sigma Y_3^\sigma - X_3^\sigma Y_2^\sigma)^{(k_1^*)_\sigma - 1}(X_1^\sigma Y_3^\sigma - X_3^\sigma Y_1^\sigma)^{(k_2^*)_\sigma - 1}(X_1^\sigma Y_2^\sigma - X_2^\sigma Y_1^\sigma)^{(k_3^*)_\sigma - 1} \\
&\text{with } k_i^* := \frac{k_1 + k_2 + k_3}{2} - k_i \\
&(\omega_1\omega_2\omega_3)^{1/2} := \left(\omega_1\omega_2\omega_3 \tau_F^{-(\alpha_1+\alpha_2+\alpha_3)}\right)^{1/2} \tau_F^{\alpha_1+\alpha_2+\alpha_3} \\
a_{i,\mathfrak{p}}(P)^u &:= \varpi_{\mathfrak{p}}^{t_F - w_i} |\varpi_{\mathfrak{p}}|_{\mathbb{A}_F}^{[2w_i - k_i]/2} a_{j,\mathfrak{p}}(P)
\end{aligned}$$

Proof. We denote by $F(x, y, z)$ the function on $B(\mathbb{A}_{F_1})^3$

$$\left\langle \Delta_k, \mathrm{Sp}_P(\Phi_1^{(1)})^u(x) \otimes \mathrm{Sp}_P(\Phi_1^{(2)})^u(y) \otimes \widehat{\mathrm{Sp}}_P^{(s)}(\Phi_1^{(3)})^u(z) \right\rangle.$$

Since for $(z_1, z_2, z_3) \in \mathcal{O}_{F_p}^2 \times \mathbf{p}^s \mathcal{O}_{F_p}$ such that $z_1 - z_2 \in \mathcal{O}_{F_p}^\times$,

$$\begin{aligned}
&P\left(\left\langle \frac{(z_1 - z_2)(1 - z_1z_3)}{1 - z_2z_3}, \frac{(z_1 - z_2)(1 - z_2z_3)}{1 - z_1z_3}, \frac{(1 - z_2z_3)(1 - z_1z_3)}{z_1 - z_2} \right\rangle^{-\frac{1}{2}}, 1\right)^2 \\
&= \omega_1\omega_2\omega_3^{-1}(z_1 - z_2)\chi\left(z_1 - z_2,\, z_1 - z_2,\, 1\right)^2 \\
&\times \epsilon_{\mathrm{cyc},F}^{2[2w_1^* - k_1^*]}(1 - z_2z_3)\epsilon_{\mathrm{cyc},F}^{2[2w_2^* - k_2^*]}(1 - z_1z_3)\epsilon_{\mathrm{cyc},F}^{2[2w_3^* - k_3^*]}(z_1 - z_2),
\end{aligned}$$

where $w_i^* := \frac{w_1 + w_2 + w_3}{2} - w_i$, we have

$$\begin{aligned}
&P\left(\left\langle \frac{(z_1 - z_2)(1 - z_1z_3)}{1 - z_2z_3}, \frac{(z_1 - z_2)(1 - z_2z_3)}{1 - z_1z_3}, \frac{(1 - z_2z_3)(1 - z_1z_3)}{z_1 - z_2} \right\rangle^{-\frac{1}{2}}, 1\right) \\
&= \pm \left(\begin{array}{l} (\omega_1\omega_2\omega_3^{-1})^{-1/2}(z_1 - z_2)\chi\left(z_1 - z_2,\, z_1 - z_2,\, 1\right)^{-1} \\ \times \epsilon_{\mathrm{cyc},F}^{[2w_1^* - k_1^*]}(1 - z_2z_3)\epsilon_{\mathrm{cyc},F}^{[2w_2^* - k_2^*]}(1 - z_1z_3)\epsilon_{\mathrm{cyc},F}^{[2w_3^* - k_3^*]}(z_1 - z_2) \end{array} \right),
\end{aligned}$$

where $(\omega_1\omega_2\omega_3^{-1})^{-1/2} := (\omega_1\omega_2\omega_3^{-1}\tau_F^{\alpha_1+\alpha_2-\alpha_3})^{1/2}\tau_F^{-\alpha_1-\alpha_2+\alpha_3}$. By substituting $(1, 0, 0)$ for $(z_i)_i$, we found the sign is $+$.

$$\begin{aligned}
P(\Theta_\Phi) = &\frac{1}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s)) \prod_{\mathfrak{p} \mid p} a_{1,\mathfrak{p}}^{s_{\mathfrak{p}}}(P)^u a_{2,\mathfrak{p}}^{s_{\mathfrak{p}}}(P)^u} \int_{\mathbb{A}_F^\times B^\times \backslash B^\times(\mathbb{A}_F)} \sum_{\substack{c_1, c_2 \in \mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p} \\ c_1 - c_2 \in (\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p})^\times}} \\
&\times (-1)^{k_3}(\omega_1\omega_2\omega_3^{-1})^{-1/2}\omega_1'\omega_2'\omega_3'^{-1}\omega_3'(c_1 - c_2) \\
&\times F(x\begin{pmatrix} \varpi_p^s & c_1 \\ 0 & 1 \end{pmatrix}, x\begin{pmatrix} \varpi_p^s & c_2 \\ 0 & 1 \end{pmatrix}, x)(\omega_1\omega_2\omega_3)^{-1/2}(\mathrm{Nrd}(x))dx
\end{aligned}$$

By using the right invariance of the measure, we have

$$P(\Theta_\Phi) = \frac{1}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}\mid p} a_{1,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u} \int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)} \sum_{\substack{c_1,c_2\in\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p} \\ c_1-c_2\in(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p})^\times}}$$
$$\times (-1)^{k_3}(\omega_1\omega_2\omega_3^{-1})^{-1/2}\omega_1'\omega_2'\omega_3^{-1}\omega_3'(c_1-c_2)$$
$$\times F(x\begin{pmatrix} \varpi_p^s & c_1-c_2 \\ 0 & 1\end{pmatrix}, x\begin{pmatrix} \varpi_p^s & 0 \\ 0 & 1\end{pmatrix}, x)(\omega_1\omega_2\omega_3)^{-1/2}(\mathrm{Nrd}(x))dx$$

$$= \frac{(-1)^{k_3}\#\left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}\right)^\times\#\left(\mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}\right)}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}\mid p} a_{1,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u}$$
$$\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)} F(x\begin{pmatrix} \varpi_p^s & 1 \\ 0 & 1\end{pmatrix}, x\begin{pmatrix} \varpi_p^s & 0 \\ 0 & 1\end{pmatrix}, x)(\omega_1\omega_2\omega_3)^{-1/2}(\mathrm{Nrd}(x))dx$$

$$= \frac{(-1)^{k_3}\omega_1\omega_2\omega_3)^{1/2}(\pi_p^s)\#\left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}\right)^\times\#\left(\mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}\right)}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}\mid p} a_{1,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u}$$
$$\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)} F(x\begin{pmatrix} 1 & \varpi_p^{-s} \\ 0 & 1\end{pmatrix}, x, x\begin{pmatrix} \varpi_p^{-s} & 0 \\ 0 & 1\end{pmatrix})(\omega_1\omega_2\omega_3)^{-1/2}(\mathrm{Nrd}(x))dx$$

□

Corollary 3.3.3. Assume $\Phi_1^{(1)}, \Phi_1^{(2)}$ and $\Phi_1^{(3)}$ are ordinary and

$$T_0(\varpi_\mathfrak{p})\Phi_{1,j} = a_{1,j,\mathfrak{p}}\Phi_{1,j}$$

for $j = 1,2,3$ and $\mathfrak{p} \mid p$. For any $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$ such that $P|_{\mathbf{G}_E} = P_{k_1,w_1,\omega_1,\omega_1'} \times P_{k_2,w_2,\omega_2,\omega_2'} \times P_{k_3,w_3,\omega_3,\omega_3'}$. Then we have

$$P(\Theta_\Phi)$$
$$= \frac{(\omega_1\omega_2\omega_3)^{1/2}(\varpi_p^s)\#\left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}\right)^\times\#\left(\mathcal{O}_{F_\mathbf{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathbf{p}}\right)}{(-1)^{k_3}\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}\mid p} a_{1,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u a_{3,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u}\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)}$$
$$\left\langle \Delta_k, \mathrm{Sp}_P(\Phi_{1,1})^u(x\begin{pmatrix} 1 & \varpi_p^{-s} \\ 0 & 1\end{pmatrix})\otimes \mathrm{Sp}_P(\Phi_{1,2})^u(x)\otimes\mathrm{Sp}_P(\Phi_{1,3})^u(x\begin{pmatrix} 0 & \varpi_p^{-s} \\ \varpi_p^s & 0\end{pmatrix})\right\rangle_{k^*-2\underline{t}^*}$$
$$\times (\omega_1\omega_2\omega_3)^{-1/2}(\mathrm{Nrd}_{B/F}(x))dx$$

where the notations are as in Theorem 3.3.2.

3.3.2. $E = F_1 \times F_2$ **case.** Let Σ_i be an open subgroup and $\Phi_i \in S(\Sigma_i;\mathscr{D}(\mathbb{I}))$ as in Definition 3.2.3 ($i = 1,2$). Let ς be the generator of $\mathrm{Gal}(F_2/F)$ and for $\sigma \in I_F$, fix $\overline{\sigma} \in I_{F_2}$ such that $\overline{\sigma}|_F = \sigma$. For $n_i \in \mathbb{Z}_{\geqslant 0}[I_{F_i}]$, we define the indeterminate by

$$\mathrm{Sym}^{n_1}(\mathbb{C}_p)\otimes_{\mathbb{C}_p}\mathrm{Sym}^{n_2}(\mathbb{C}_p) = \bigotimes_{\sigma\in I}\mathbb{C}_p[X_1^\sigma, Y_1^\sigma]_{n_{1\sigma}}\otimes_{\mathbb{C}_p}\mathbb{C}_p[X_2^{\overline{\sigma}}, Y_2^{\overline{\sigma}}]_{n_{2\overline{\sigma}}}\otimes_{\mathbb{C}_p}\mathbb{C}_p[X_2^{\overline{\sigma}\varsigma}, Y_2^{\overline{\sigma}\varsigma}]_{n_{2\overline{\sigma}\varsigma}}$$

It has natural $\mathrm{GL}_2(E\otimes_F\mathbb{C}_p) \cong \prod_{\sigma\in I}\mathrm{GL}_2(\mathbb{C}_p)^3$-action.

Lemma 3.3.4. For any $s \geqslant 0$, we have

$$\Theta_\Phi = \sum_{b\in\mathbb{A}_{F,f}^\times B^\times\backslash\widehat{B}^\times/\Sigma_0(\mathbf{p}^s)}\left(\mathrm{Nrd}_{B/F}(b)^{-\frac{1}{2}}, 1\right)\int_{[\{1\}\times\mathbf{p}^s\mathcal{O}_{F_p}]\times[\mathcal{O}_{\widehat{F_2}_p}\times\{1\}]}\delta(x)\,d(\Phi(b))(x),$$

PROOF. It's proved in the same way of the proof of Lemma 2.4.9 □

For each $\mathfrak{p} \mid p$ splitting completely in F_2, we define

$$\xi_{\mathfrak{p}} := \left(2^{-1}, -2^{-1}\right) \in \mathcal{O}_{F_{2\mathfrak{p}}} = \mathcal{O}_{F_{\mathfrak{p}}}^2.$$

We note that we have the following description of the restriction of δ on $[\{1\} \times \mathbf{p}^s\mathcal{O}_{F_{\mathbf{p}}}] \times [\mathcal{O}_{\widehat{F_{2\mathbf{p}}}} \times \{1\}] \cap \prod_{\mathfrak{p}\mid p} V_{\mathfrak{p}}^E$:

$$\begin{aligned}\delta(z_1, z_2) =& \chi\left(1, \frac{z_2 - z_2^{\varsigma}}{2\xi(1 - z_2^{\varsigma}z_1)}\right)^{-1} \\ &\times \left[\frac{\left\langle \frac{2\xi(1-z_2^{\varsigma}z_1)(1-z_2z_1)}{(z_2-z_2^{\varsigma})}, \frac{(z_2-z_2^{\varsigma})(1-z_2z_1)}{2\xi(1-z_2^{\varsigma}z_1)} \right\rangle^{-\frac{1}{2}}}{\left(\frac{2\xi(1-z_2^{\varsigma}z_1)(1-z_2z_1)}{(z_2-z_2^{\varsigma})}, \frac{(z_2-z_2^{\varsigma})(1-z_2z_1)}{2\xi(1-z_2^{\varsigma}z_1)}\right)}, \right]\end{aligned}$$

where

$$\xi := (\xi_{\mathfrak{p}})_{\mathfrak{p}\mid p} \in \mathcal{O}_{E_p}.$$

Theorem 3.3.5. Assume Φ_2 are ordinary and

$$(\varpi_{\mathfrak{p}})\Phi_2 = a_{2,\mathfrak{p}}\Phi_2$$

for some $a_{2,\mathfrak{p}} \in \mathbb{I}^{\times}$. Let $P \in \mathscr{X}(\mathbb{I})$ such that $P|_{\mathbf{G}_E} = P_{k_1,w_1,\omega_1,\omega_1'} \times P_{k_2,w_2,\omega_2,\omega_2'}$ with $2w_i - k_i = \alpha_i \underline{t}_i$ for $i = 1, 2$ and $\alpha_1 \in 2\mathbb{Z}$. For any $s \geqslant s(\psi_1), \sum_{\mathfrak{p}\mid p} e_{\mathfrak{p}}(F_2/F)^{-1}s(\psi_2)_{\mathfrak{p}}\mathfrak{p}$ ($e_{\mathfrak{p}}(F_2/F)$ is the ramification index at $\mathfrak{p}$) , we have

$$\begin{aligned}&P(\Theta_{\Phi}) \\ &= \frac{\#\left(\mathcal{O}_{F_{\mathfrak{p}}}/\mathbf{p}^s\mathcal{O}_{F_{\mathfrak{p}}}\right)^{\times} \#\left(\mathcal{O}_{F_{\mathfrak{p}}}/\mathbf{p}^s\mathcal{O}_{F_{\mathfrak{p}}}\right)}{(-1)^{k_1}\mathrm{vol}(\Sigma_0(\mathbf{p}^s)) \prod_{\mathfrak{p}\mid p} a_{2,\mathfrak{p}}^{s_{\mathfrak{p}}}(P)^u} \\ &\int_{\mathbb{A}_F^{\times}B^{\times}\backslash\widehat{B}^{\times}} \left\langle \Delta_k^{\xi}, \widehat{\mathrm{Sp}}_P^{(s)}(\Phi_1)^u(x) \otimes \mathrm{Sp}_P(\Phi_2)^u(x\begin{pmatrix} \varpi_p^s & \xi \\ 0 & 1 \end{pmatrix}) \right\rangle_{k*-2\underline{t}*} (\omega_1\omega_2)^{-1/2}(\mathrm{Nrd}_{B/F}(x))dx\end{aligned}$$

where the symbol $(\cdot)^u$ is that explained in Remark 2.1.4 and

$$\xi = (\xi_{\mathfrak{p}})_{\mathfrak{p}|p}$$

$$\text{with } \xi_{\mathfrak{p}} := \begin{cases} \text{the } \xi_{\mathfrak{p}} \text{ defined in 3.1.2} & \text{if } \mathfrak{p} \text{ does not split in } F_2 \\ (2^{-1}, -2^{-1}) & \text{if } \mathfrak{p} \text{ splits completely in } F_2 \end{cases}$$

$$\Delta_k^{\xi} := \prod_{\sigma\in I} (2\xi_{\mathfrak{p}(\sigma)})^{1-k^*_{1,\sigma}} (X_2^{\overline{\sigma}} Y_2^{\overline{\sigma}\varsigma} - X_2^{\overline{\sigma}\varsigma} Y_2^{\overline{\sigma}})^{k^*_{1\,\sigma}-1}$$
$$\times (X_2^{\overline{\sigma}\varsigma} Y_1^{\sigma} - X_1^{\sigma} Y_2^{\overline{\sigma}\varsigma})^{k^*_{2\,\overline{\sigma}}-1} (X_2^{\overline{\sigma}} Y_1^{\sigma} - X_1^{\sigma} Y_2^{\overline{\sigma}})^{k^*_{2\,\overline{\sigma}\varsigma}-1}$$

$$\text{with } \begin{cases} k^*_{1\,\sigma} := \dfrac{k_{1\sigma} + k_{2\overline{\sigma}} + k_{2\overline{\sigma}\varsigma}}{2} - k_{1\sigma}, \\ k^*_{2\,\overline{\sigma}} := \dfrac{k_{1\sigma} + k_{2\overline{\sigma}} + k_{2\overline{\sigma}\varsigma}}{2} - k_{2\overline{\sigma}}, \\ k^*_{2\,\overline{\sigma}\varsigma} := \dfrac{k_{1\sigma} + k_{2\overline{\sigma}} + k_{2\overline{\sigma}\varsigma}}{2} - k_{2\overline{\sigma}\varsigma}, \\ \mathfrak{p}(\sigma) \text{ is a prime such that } \sigma \text{ factor through } F_{\mathfrak{p}(\sigma)}, \end{cases}$$

$$(\omega_1\omega_2)^{1/2} := \left(\omega_1\omega_2\tau_F^{-(\alpha_1+2\alpha_2)}\right)^{1/2} \tau_F^{\alpha_1+2\alpha_2},$$
$$a_{i,\mathfrak{p}}(P)^u := \varpi_{\mathfrak{p}}^{t_F - w_i} |\varpi_{\mathfrak{p}}|_{\mathbb{A}_F}^{[2w_i-k_i]/2} a_{i,\mathfrak{p}}(P).$$

PROOF. Set elements of $\mathbb{Z}[I]$ as follows:

$$h_2 := \sum_{\sigma\in I} k_{2\,\overline{\sigma}}\sigma,$$
$$v_2 := \sum_{\sigma\in I} w_{2\,\overline{\sigma}}\sigma,$$
$$h_3 := \sum_{\sigma\in I} k_{2\,\overline{\sigma}\varsigma}\sigma.$$
$$v_3 := \sum_{\sigma\in I} w_{2\,\overline{\sigma}\varsigma}\sigma.$$

We denote by $F'(x,y)$ the function on $B(\mathbb{A}_{F_2}) \times B(\mathbb{A}_{F_1})$

$$F'(x,y) := \left\langle \Delta_k^{\xi}, \widehat{\mathrm{Sp}}_P^{(s)}(\Phi_1)^u(x) \otimes \mathrm{Sp}_P(\Phi_2)^u(y) \right\rangle.$$

Since for any $(z_1, z_2) \in \mathcal{O}_{F_{1,p}} \times \mathcal{O}_{F_{2,p}}$ such that $z_1 \in \mathbf{p}^s\mathcal{O}_{F_{1,p}}$ and $z_2 - z_2$

$$P\left(\left\langle \frac{2\xi(1-z_2^{\varsigma}z_1)(1-z_2z_1)}{(z_2-z_2^{\varsigma})}, \frac{(z_2-z_2^{\varsigma})(1-z_2z_1)}{2\xi(1-z_2^{\varsigma}z_1)} \right\rangle^{-\frac{1}{2}}, 1\right)^2$$
$$= \omega_1\omega_2^{-1}\left((2\xi)^{-1}(z_2-z_2^{\varsigma})\right)\chi\left(1,\ (2\xi)^{-1}(z_2-z_2^{\varsigma})\right)^2$$
$$\times \epsilon_{\mathrm{cyc},F_1}^{2[2w_1^*-k_1^*]}\left((2\xi)^{-1}(z_2-z_2^{\varsigma})\right)\epsilon_{\mathrm{cyc},F_2}^{2[2v_2^*-h_2^*]}(1-z_2^{\varsigma}z_1)\epsilon_{\mathrm{cyc},F_2}^{2[2v_3^*-h_3^*]}(1-z_2z_1),$$

we have

$$P\left(\left\langle \frac{2\xi(1-z_2^{\varsigma}z_1)(1-z_2z_1)}{(z_2-z_2^{\varsigma})}, \frac{(z_2-z_2^{\varsigma})(1-z_2z_1)}{2\xi(1-z_2^{\varsigma}z_1)} \right\rangle^{-\frac{1}{2}}, 1\right)$$
$$= (\omega_1\omega_2^{-1})^{1/2}\left((2\xi)^{-1}(z_2-z_2^{\varsigma})\right)\chi\left(1,\ (2\xi)^{-1}(z_2-z_2^{\varsigma})\right)$$
$$\times \epsilon_{\mathrm{cyc},F_1}^{[2w_1^*-k_1^*]}\left((2\xi)^{-1}(z_2-z_2^{\varsigma})\right)\epsilon_{\mathrm{cyc},F_2}^{[2v_2^*-h_2^*]}(1-z_2^{\varsigma}z_1)\epsilon_{\mathrm{cyc},F_2}^{[2v_3^*-h_3^*]}(1-z_2z_1),$$

where

$$(\omega_1\omega_2^{-1})^{1/2} := \left(\omega_1\omega_2\tau_F^{-(\alpha_1-2\alpha_2)}\right)^{1/2}\tau_F^{\alpha_1-2\alpha_2}$$

Thus we have

$$\begin{aligned}P(\Theta_\Phi) &= \frac{(-1)^{k_1}}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}|p}a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u}\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)}\sum_{\substack{c\in\mathcal{O}_{F_2\,p}/\mathbf{p}^s\mathcal{O}_{F_2\,p}\\ c-c^S\in(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p})^\times}}\\
&\times(\omega_1\omega_2^{-1})^{1/2}\omega_1'^{-1}\omega_2\omega_2'^{-1}\left((2\xi)^{-1}(c-c^S)\right)\\
&\times F'(x,x\begin{pmatrix}\varpi_p^s & c\\ 0 & 1\end{pmatrix})(\omega_1\omega_2)^{-1/2}(\mathrm{Nrd}_{B/F}(x))dx\\
&= \frac{(-1)^{k_1}}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}|p}a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u}\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)}\sum_{\substack{c_1,c_2\in\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}\\ c_2\in\in(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p})^\times}}\\
&\times(\omega_1\omega_2^{-1})^{1/2}\omega_1^{-1}\omega_1'\omega_2'\left((2\xi)^{-1}(c_2)\right)\\
&\times F'(x,x\begin{pmatrix}\varpi_p^s & c_1+c_2\xi\\ 0 & 1\end{pmatrix})(\omega_1\omega_2)^{-1/2}(\mathrm{Nrd}_{B/F}(x))dx\end{aligned}$$

By changing variable, we have □

$$\begin{aligned}&= \frac{(-1)^{k_1}\#\left(\mathcal{O}_{F_\mathfrak{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathfrak{p}}\right)^\times\#\left(\mathcal{O}_{F_\mathfrak{p}}/\mathbf{p}^s\mathcal{O}_{F_\mathfrak{p}}\right)}{\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}|p}a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u}\\
&\times\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)}F'(x,x\begin{pmatrix}\varpi_p^s & \xi\\ 0 & 1\end{pmatrix})(\omega_1\omega_2)^{-1/2}(\mathrm{Nrd}_{B/F}(x))dx.\end{aligned}$$

Thus we have the formula.

Corollary 3.3.6. Assume Φ_1,Φ_2 are ordinary and

$$T_0(\varpi_\mathfrak{p})\Phi_i = a_{i,\mathfrak{p}}\Phi_i$$

for some $a_i\in\mathbb{I}^\times$. For any $P\in\mathscr{X}(\mathbb{I})$ such that $P|_{\mathbf{G}_E} = P_{k_1,w_1,\omega_1,\omega_1'}\times P_{k_2,w_2,\omega_2,\omega_2'}$ with $2w_i-k_i=\alpha_i\underline{t}_i$ for $i=1,2$. Then we have

$$\begin{aligned}&P(\Theta_\Phi)\\
&= \frac{\#\left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}\right)^\times\#\left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}\right)}{(-1)^{k_1}\mathrm{vol}(\Sigma_0(\mathbf{p}^s))\prod_{\mathfrak{p}|p}a_{1,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u a_{2,\mathfrak{p}}^{s_\mathfrak{p}}(P)^u}\int_{\mathbb{A}_F^\times B^\times\backslash B^\times(\mathbb{A}_F)}\\
&\left\langle\Delta_k^\xi,\mathrm{Sp}_P(\Phi_1)^u(x\tau_{-\varpi^s})\otimes\mathrm{Sp}_P(\Phi_2)^u(x\begin{pmatrix}\varpi_p^s & \xi\\ 0 & 1\end{pmatrix})\right\rangle_{k^*-2\underline{t}^*}(\omega_1\omega_2)^{-1/2}(\mathrm{Nrd}_{B/F}(x))dx,\end{aligned}$$

where the notations are as in Theorem 3.3.5

CHAPTER 4

A review of $\mathbb{I}$-adic forms on GL_2 over totally real fields

4.1. Hilbert modular forms and q-expansions

4.1.1. Definitions of modular forms. We fix non zero ideals $\mathfrak{n}, \mathfrak{n}' \subset \mathcal{O}_F$ such that $\mathfrak{n} + \mathfrak{n}' = 1$ (we will assume that $\mathfrak{n}$ is prime to p and $\mathfrak{n}' \mid p^\gamma$ for sufficiently large γ from Section 4.2 below). Recall $I := \{\sigma\colon F \hookrightarrow \mathbb{C}_p \cong \mathbb{C} : \text{embeddings of fields}\}$. Let

$$\mathrm{GL}_2^+(\mathbb{R}) := \left\{ g \in \mathrm{GL}_2(\mathbb{R}) \middle| \det(g) > 0 \right\}$$

and let

$$\mathfrak{H} := \left\{ z \in \mathbb{C} \middle| \mathrm{Im}(z) > 0 \right\}$$

be the upper half plane and we identify $\mathfrak{H}$ with $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times SO_2(\mathbb{R})$ by

$$\mathrm{GL}_2^+(\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a\sqrt{-1}+b}{c\sqrt{-1}+d} \in \mathfrak{H}.$$

For $\delta = \left(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \right)_{\sigma \in I} \in \mathrm{GL}_2^+(\mathbb{R})^I$ and $z = (z_\sigma)_{\sigma \in I} \in \mathfrak{H}^I$, we define

$$j(\delta, z) := (c_\sigma z_\sigma + d_\sigma)_{\sigma \in I} \in \mathbb{C}^I \cong F \otimes_{\mathbb{Q}} \mathbb{C}.$$

For any fractional ideals $\mathfrak{a}, \mathfrak{b}$. we define subgroup $\Gamma_1(\mathfrak{n}, \mathfrak{n}'; \mathfrak{a}) \subset \mathrm{GL}_2(F)$, which is discrete in $\mathrm{GL}_2^+(\mathbb{R})^I$, by

$$\Gamma_1(\mathfrak{n}, \mathfrak{n}'; \mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_F & \mathfrak{a}^{-1} \\ \mathfrak{n}\mathfrak{n}'\mathfrak{a} & \mathcal{O}_F \end{pmatrix} \middle| \begin{array}{l} ad - bc \in \mathcal{O}_{F,+}^\times \\ d \equiv 1 \mod \mathfrak{n} \\ a \equiv d \equiv 1 \mod \mathfrak{n}' \end{array} \right\}.$$

Let $k, w \in \mathbb{Z}[I]$. For each $\mathbb{C}$-valued function f on $\mathfrak{H}^I$ and for $\gamma \in \mathrm{GL}_2^+(\mathbb{R})^I$, we define

$$f|_{k,w}\gamma(z) := j(\gamma, z)^{-k} \det(\gamma)^w f(\gamma(z)).$$

Definition 4.1.1. Let $k, w \in \mathbb{Z}[I]$ such that $k > 0$ and $2w - k \in \mathbb{Z}\underline{t}$. Let $\mathfrak{a}$ be a nonzero fraction ideal of $\mathcal{O}_F$. We define the space of classical Hilbert modular forms by

$$\left\{ \begin{array}{l} f \ : \mathfrak{H}^I \longrightarrow \mathbb{C} \\ \quad : \text{holomorphic} \end{array} \middle| \begin{array}{ll} 1) & f|_{k,w}\gamma = f \text{ for } \gamma \in \Gamma_1(\mathfrak{n}, \mathfrak{n}'; \mathfrak{a}) \\ 2) & \text{For any } \alpha \in \mathrm{GL}_2(F) \cap \mathrm{GL}_2^+(\mathbb{R})^I, f|_{k,w}\alpha \text{ has} \\ & \text{the following type of Fourier expansion:} \\ & f|_{k,w}\alpha = a(0, f|_{k,w}\alpha) + \displaystyle\sum_{\xi \in F_+^\times} a(\xi, f|_{k,w}\alpha)\mathbf{e}_F(\xi z), \\ & \text{where } \mathbf{e}_F(\xi z) = \exp\left(2\pi\sqrt{-1}\displaystyle\sum_{\sigma \in I} \xi^\sigma z_\sigma\right) \end{array} \right\}.$$

We denote the space by $\mathscr{M}_{k,w}(\Gamma(\mathfrak{n}, \mathfrak{a}))$.

Remark 4.1.2. Let $f \in \mathscr{M}_{k,w}(\Gamma_1(\mathfrak{n}, \mathfrak{n}'; \mathfrak{a}))$. For any $\alpha \in \mathrm{GL}_2(F) \cap \mathrm{GL}_2^+(\mathbb{R})^I$, there exists a nonzero ideal $\mathfrak{n}'$ and a fractional ideal $\mathfrak{a}'$,$f|_{k,w} \in \mathscr{M}_{k,w}(\Gamma_1(\mathfrak{n}, \mathfrak{n}'; \mathfrak{a}'))$. For $\epsilon \in \mathcal{O}_{F,*}^\times$ such that $\epsilon \equiv 1 \mod \mathfrak{n}'$, we have

$$f|_{k,w}\alpha(\epsilon z) = \epsilon^{-w} f|_{k,w}\alpha(z).$$

In particular, unless $w \in \mathbb{Z}\underline{t}$, we have

$$a(0, f|_{k,w}\alpha) = 0.$$

For a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$, let

$$\mathrm{Cl}_F^+(\mathfrak{a}) := \frac{\{\text{the group of fractional ideals prime to } \mathfrak{a}\}}{\{x \in F_+^\times \,|\, |x-1|_{F_\mathfrak{q}} < 1 \text{ for } \mathfrak{q}|\mathfrak{a}\}}.$$

For $k, w \in \mathbb{Z}[I]$, a complex valued function f on $\mathrm{GL}_2(\mathbb{A}_F)$ and $g = (g_f, g_\infty) \in \mathrm{GL}_2(\mathbb{A}_{F,f}) \times \mathrm{GL}_2^+(\mathbb{R})^I$, we define

$$f|_{k,w}g(x) := j\left(g_\infty, (\sqrt{-1})_\sigma\right)^{-k} \det(g_\infty)^w f(xg^{-1}).$$

For each nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$, we define several open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_{F,f})$ as follows:

$$K_0(\mathfrak{a}) := \left\{ u \in \mathrm{GL}_2(\widehat{\mathcal{O}_F}) \,\middle|\, u_\mathfrak{q} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \mathfrak{a}\mathrm{M}_2(\mathcal{O}_{F_\mathfrak{q}}) \text{ for } \mathfrak{q} \neq 0 \right\},$$

$$K_1(\mathfrak{a}) := \left\{ u \in K_0(\mathfrak{a}) \,\middle|\, u_\mathfrak{q} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{a}\mathrm{M}_2(\mathcal{O}_{F_\mathfrak{q}}) \text{ for } \mathfrak{q} \neq 0 \right\},$$

$$K(\mathfrak{a}) := \left\{ u \in K_0(\mathfrak{a}) \,\middle|\, u_\mathfrak{q} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{a}\mathrm{M}_2(\mathcal{O}_{F_\mathfrak{q}}) \text{ for } \mathfrak{q} \neq 0 \right\}.$$

For $\mathfrak{b} \subset \mathcal{O}_F$ be a ideal prime such that $\mathfrak{a} + \mathfrak{b} = (1)$, we define

$$K_1(\mathfrak{a}, \mathfrak{b}) := K_1(\mathfrak{a}) \cap K(\mathfrak{b}).$$

Definition 4.1.3. Let $k, w \in \mathbb{Z}[I]$ such that $k \geqslant 0$ and $2w - k = \alpha\underline{t}$ for some $\alpha \in \mathbb{Z}$. Let $U \subset \mathrm{GL}_2(\mathbb{A}_{F,f})$ be a nonempty open compact subgroup. We define the space of Hilbert modular forms weight (k, w), level U denote by $\mathscr{M}_{k,w}(U)$ as follows:

$$\left\{ \begin{array}{l} f \ : \mathrm{GL}_2(\mathbb{A}_F) \longrightarrow \mathbb{C} \\ \quad : \text{smooth} \end{array} \,\middle|\, \begin{array}{ll} 1) & f(ag) = f(g) \text{ for } a \in \mathrm{GL}_2(F) \\ 2) & f|_{k,w}u = f \text{ for } u \in U \times (\mathbb{R}^\times SO_2(\mathbb{R}))^I \\ 3) & \dfrac{\partial f_x(z)}{\partial \overline{z}} = 0 \text{ for } x \in \mathrm{GL}_2(\mathbb{A}_{F,f}) \\ 4) & \text{When } F = \mathbb{Q}, \text{ there exists } C > 0 \text{ such that} \\ & |f_x(z)| < C \text{ for all } x \in \mathrm{GL}_2(\mathbb{A}_{F,f}) \end{array} \right\},$$

where f_x is define as a well-defined function on $\mathfrak{H}^I$

$$f_x((z)_{\sigma \in I}) = j\left(g_\infty, (\sqrt{-1})_\sigma\right)^k \det(g_\infty)^{-w} f(xg_\infty)$$

with $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})^I$ such that $z = g_\infty(\sqrt{-1})_\sigma$. We also define the space of cusp forms of weight (k, w) and level U by

$$\mathscr{S}_{k,w}(U) := \left\{ f \in \mathscr{M}_{k,w}(U) \,\middle|\, \int_{\mathbb{A}_F/F} f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) da = 0 \text{ for } g \in \mathrm{GL}_2(\mathbb{A}_F) \right\}.$$

Remark 4.1.4. Let $\mathfrak{n}'$ be a nonzero ideal such that $\mathfrak{n} + \mathfrak{n}' = (1)$. By strong approximation theorem, we have a decomposition

$$\mathrm{GL}_2(\mathbb{A}_F) = \bigsqcup_{i=1}^{\#\mathrm{Cl}_F^+(\mathfrak{n}')} \mathrm{GL}_2(F) t_i K_1(\mathfrak{n}, \mathfrak{n}') \, \mathrm{GL}_2^+(\mathbb{R})^I,$$

where $t_i = \begin{pmatrix} a_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ for an $a_i \in \mathbb{A}_{F,f}$. Then we have an isomorphism

$$\mathscr{M}_{k,w}(K_1(\mathfrak{n})) \cong \bigoplus_{i=1}^{\#\mathrm{Cl}_F^+(\mathfrak{n}')} \mathscr{M}_{k,w}\left(\Gamma_1(\mathfrak{n}, \mathfrak{n}'; a_i \mathscr{O}_F)\right); f \mapsto (f_{t_i})_{i=1}^{\#\mathrm{Cl}_F^+(\mathfrak{n}')}.$$

Through the isomorphism, $\mathscr{S}_{k,w}(K(\mathfrak{n}))$ correspond to a space such that

$$a(0, f_{t_i}|_{k,w}\alpha) = 0$$

for any $i = 1, \ldots, \#\mathrm{Cl}_F^+(\mathfrak{n})$ and $\alpha \in \mathrm{GL}_2(F) \cap \mathrm{GL}_2^+(\mathbb{R})^I$. In particular, by Remark 4.1.2, unless $w \in \mathbb{Z}\underline{t}$. , we have

$$\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}')) = \mathscr{S}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}')).$$

We introduce the notion of nearly Hilbert holomorphic modular forms including the usual Hilbert modular forms.

Definition 4.1.5. Let $k, w, m \in \mathbb{Z}[I]$ such that $k, m \geqslant 0$ and $2w - k = \alpha \underline{t}$ for some $\alpha \in \mathbb{Z}$. Let $U \subset \mathrm{GL}_2(\mathbb{A}_{F,f})$ a open compact subgroup. We define the space of nearly holomorphic Hilbert modular forms" of weight (k, w), order $\leqslant m$, and level $\mathfrak{n}$ denote by $\mathscr{N}_{k,w,m}(U)$ as the space consisting of smooth function

$$f\colon \mathrm{GL}_2(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

satisfying

(1) $f(ag) = f(g)$ for $a \in \mathrm{GL}_2(F)$
(2) $f|_{k,w}u = f$ for $u \in U \times (\mathbb{R}^\times SO_2(\mathbb{R}))^I$
(3) For $x \in \mathrm{GL}_2(\mathbb{A}_{F,f})$, a function f_x on $\mathfrak{H}^I$ defined below has the following type of Fourier expansion

$$f_x(z) = a(0, f_x)\left((4\pi y)^{-1}\right) + \sum_{\xi \in L(x)} a(\xi, f_x)\left((4\pi y)^{-1}\right) \mathbf{e}_F(\xi z),$$

where $z = x + y\sqrt{-1}$ $(x \in \mathbb{R}^I, y \in \mathbb{R}^I_{>0})$, $L(x)$ is a lattice of F depending on x, $a(\xi, f_x)(Y) \in \mathbb{C}[\{Y_\sigma\}_{\sigma \in I}]$ is $[F : \mathbb{Q}]$-variable polynomial such that the degree in Y_σ is less than m_σ and

$$\mathbf{e}_F(\xi z) = \exp\left(2\pi\sqrt{-1}\sum_{\sigma \in I} \xi^\sigma z_\sigma\right).$$

The function f_x is define as a well-defined function on $\mathfrak{H}^I$ by

$$f_x((z)_{\sigma \in I}) = j\left(g_\infty, (\sqrt{-1})_\sigma\right)^k \det(g_\infty)^{-w} f(x g_\infty)$$

with $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})^I$ such that $z = g_\infty(\sqrt{-1})_\sigma$.

We define an action of $\mathrm{Cl}_F^+(\mathfrak{n}\mathfrak{n}')$ called the diamond.

Definition 4.1.6. Let k, w, α, m be as in Definition 4.1.5. For a class $[\mathfrak{a}] \in \mathrm{Cl}_F^+(\mathfrak{n}\mathfrak{n}')$ and for $f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}'))$, we define

$$f|\langle \mathfrak{a} \rangle_{k,w} := \begin{cases} |a^{-1}|_{\mathbb{A}_{F,f}}^{\alpha} f|_{k,w} a^{-1} & \text{if } \mathfrak{a} \text{ is prime to } \mathfrak{n}\mathfrak{n}' \\ 0 & \text{otherwise,} \end{cases}$$

where $a \in \mathbb{A}_{F,f}$ is an element such that $a_{\mathfrak{n}\mathfrak{n}'} = 1$ and

$$\mathfrak{a} = a\mathcal{O}_F.$$

Proposition 4.1.7. The both of the spaces of Hilbert modular forms and cusp forms are invariant under the diamond operators: for a nonzero prime ideal $\mathfrak{q}$ being prime to $\mathfrak{n}$,

$$\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'))|\langle \mathfrak{q} \rangle_{k,w} \subset \mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}')),$$
$$\mathscr{S}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'))|\langle \mathfrak{q} \rangle_{k,w} \subset \mathscr{S}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}')).$$

Let $C^\infty(\mathrm{GL}_2(\mathbb{A}_F))$ be the space of smooth function on $\mathrm{GL}_2(\mathbb{A}_F)$. The Lie algebra $\mathfrak{gl}_2(\mathbb{R})^I \otimes \mathbb{C}$ is acting on $C^\infty(\mathrm{GL}_2(\mathbb{A}_F))$. For each $\sigma \in I$, we define $R_\sigma \in \mathfrak{gl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ by

$$R_\sigma := -\frac{1}{8\pi} \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix}$$

For $r \in \mathbb{Z}_{\geqslant 0}[I]$, we define

$$R^r := (R_\sigma^{r_\sigma})_{\sigma \in I} \in \mathfrak{gl}_2(\mathbb{R})^I \otimes \mathbb{C}.$$

Regarding nearly holomorphic Hilbert modular forms, the following are well known:

Proposition 4.1.8. Let $f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}'))$ be a nearly holomorphic Hilbert modular form. For $r \in \mathbb{Z}_{\geqslant 0}[I]$, we have

$$R^r f \in \mathscr{N}_{k+2r,w+r,m+r}(K_1(\mathfrak{n}, \mathfrak{n}')).$$

Moreover, if $k \not> 2m$, we have

$$\mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}')) = \{0\}$$

and if $k > 2m$, we have the following isomorphism

$$\begin{array}{ccc} \bigoplus_{m \geqslant r \geqslant 0} \mathscr{M}_{k-2r,w-r}(K_1(\mathfrak{n}, \mathfrak{n}')) & \cong & \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}')) \\ \Psi & & \Psi \\ (f_r)_r & \mapsto & \sum_{m \geqslant r \geqslant 0} R^r f_r. \end{array}$$

Remark 4.1.9. Let $x \in \mathrm{GL}_2(\mathbb{A}_{F,f})$ and $\sigma \in I$, the differential operator R_σ is described as

$$(R_\sigma f)_x(z) = \frac{1}{2\pi\sqrt{-1}} \left(\frac{k_\sigma}{z - \overline{z}} + \frac{\partial}{\partial z_\sigma} \right) f_x(z).$$

Definition 4.1.10. Let $f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}'))$ with $k > 2m$. By the second assertion of Proposition 4.1.8, f can be described as

$$f = \sum_{m \geqslant r \geqslant 0} R^r f_r,$$

where $f_r \in \mathscr{M}_{k-2r,w-r}(K_1(\mathfrak{n}, \mathfrak{n}'))$. We define the holomorphic projection of f by

$$\mathcal{H}(f) := f_0.$$

We define the two type of operators $T(y)$ and $T(a,b)$ acting on $\mathscr{N}_{k,w,m}(K_1(\mathfrak{n},\mathfrak{n}'))$.

Definition 4.1.11. Let k,w,m and $\mathfrak{n}$ be as in Definition 4.1.5. Let $f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n},\mathfrak{n}'))$. For $a,b \in \mathbb{A}_{F,f}^\times \times \mathbb{A}_{F,f}^\times$, we define

$$f|T(a,b)(g) := \frac{1}{\mathrm{vol}(K_1(\mathfrak{n},\mathfrak{n}'))}\int_{\mathrm{GL}_2(\mathbb{A}_{F,f})} f(gx^{-1})\mathbf{1}_{K_1(\mathfrak{n},\mathfrak{n}')\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & b^{-1}\end{smallmatrix}\right)K_1(\mathfrak{n},\mathfrak{n}')}(x)dx$$
$$= \frac{1}{\mathrm{vol}(K_1(\mathfrak{n},\mathfrak{n}'))}\int_{\mathrm{GL}_2(\mathbb{A}_{F,f})} f(gx)\mathbf{1}_{K_1(\mathfrak{n},\mathfrak{n}')\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right)K_1(\mathfrak{n},\mathfrak{n}')}(x)dx.$$

For each element $x_\mathfrak{q} \in \mathcal{O}_{F_\mathfrak{q}} \cap F_\mathfrak{q}^\times$ for a nonzero prime ideal $\mathfrak{q}$, we define a well-defined operator as follows:

$$T(x_\mathfrak{q}) := \begin{cases} T(x_\mathfrak{q},1) & \text{if } \mathfrak{q}|\mathfrak{n}\mathfrak{n}', \\ \displaystyle\sum_{\substack{i\geqslant j\geqslant 0\\ i+j=\mathrm{ord}_\mathfrak{q}(x)}} T(\pi_\mathfrak{q}^i,\pi_\mathfrak{q}^j). & \text{if } \mathfrak{q}\nmid \mathfrak{n}\mathfrak{n}', \end{cases}$$
$$= T(x_\mathfrak{q},1) + \langle\mathfrak{q}\rangle_{k,w}T(x_\mathfrak{q}\pi_\mathfrak{q}^{-1}),$$

where $\pi_\mathfrak{q}$ is a prime element of $\mathcal{O}_{F_\mathfrak{q}}$. Moreover, for $x \in \widehat{\mathcal{O}_F} \cap \mathbb{A}_{F,f}^\times$, we define

$$T(x) := \prod_\mathfrak{q} T(x_\mathfrak{q}).$$

Note that for almost all $\mathfrak{q}$, $T(x_\mathfrak{q}) = 1$ and the operator $T(x)$ above is well-defined.

Proposition 4.1.12. Let k,w and $\mathfrak{n}$ be as in Definition 4.1.5. Let $\pi_\mathfrak{q}$ be a prime element of $\mathcal{O}_{F_\mathfrak{q}}$. For $r>1$, we have

$$T(\pi_\mathfrak{q}^r) = T(\pi_\mathfrak{q})T(\pi_\mathfrak{q}^{r-1}) - |\pi_\mathfrak{q}|_{\mathbb{A}_F}^{\alpha-1}\langle\mathfrak{q}\rangle_{k,w}T(\pi_\mathfrak{q}^{r-2})$$

Proof. When $\mathfrak{q}|\mathfrak{n}\mathfrak{n}'$, we can deduce the statement easily from the following explicit decomposition:

$$K_1(\mathfrak{n},\mathfrak{n}')\begin{pmatrix}\pi_\mathfrak{q}^r & 0\\ 0 & 1\end{pmatrix}K_1(\mathfrak{n},\mathfrak{n}') = \bigsqcup_{u\in\mathcal{O}_{F_\mathfrak{q}}/\pi_\mathfrak{q}^r\mathcal{O}_{F_\mathfrak{q}}}\begin{pmatrix}\pi_\mathfrak{q}^r & u\\ 0 & 1\end{pmatrix}K_1(\mathfrak{n},\mathfrak{n}').$$

Let $\mathfrak{q}\nmid\mathfrak{n}\mathfrak{n}'$ and put $K := K_1(\mathfrak{n},\mathfrak{n}')_\mathfrak{q} = \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{q}})$. Let f be a modular form and put

$$\phi_1(g) := \mathbf{1}_{K\left(\begin{smallmatrix}\pi_\mathfrak{q} & 0\\ 0 & 1\end{smallmatrix}\right)K}(g)$$
$$\phi_2(g) := \mathbf{1}_{\left\{\substack{|\det(g)|=|\pi_\mathfrak{q}|^r\\ g\in\mathrm{M}_2(\mathcal{O}_{F_\mathfrak{q}})}\right\}}(g)$$

By definition,

$$f|T(\pi_\mathfrak{q})T(\pi_\mathfrak{q}^{m-1})(g) = \frac{1}{\mathrm{vol}(K_1(\mathfrak{n},\mathfrak{n}')_\mathfrak{q})}\int_{\mathrm{GL}_2(F_\mathfrak{q})}\int_{\mathrm{GL}_2(F_\mathfrak{q})} f(gx^{-1}y^{-1})\phi_2(x^{-1})\phi_1(y^{-1})dxdy$$
$$= \frac{1}{\mathrm{vol}(K_1(\mathfrak{n},\mathfrak{n}')_\mathfrak{q})}\int_{\mathrm{GL}_2(F_\mathfrak{q})} f(gy)\int_{\mathrm{GL}_2(F_\mathfrak{q})}\phi_2(x)\phi_1(x^{-1}y)dxdy$$

Let

$$\phi_3(g) := \int_{\mathrm{GL}_2(F_\mathfrak{q})}\phi_2(x)\phi_1(x^{-1}g)dx.$$

Since ϕ_3 is left and right invariant under the action of K, we can describe ϕ_3 as

$$\phi_3 = \sum_{i \geqslant -\frac{r+1}{2}} a_i \mathbf{1}_{K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K}.$$

Since

$$a_i = \frac{1}{\mathrm{vol}\left(K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K\right)} \int_{\mathrm{GL}_2(F_\mathfrak{q})} \phi_3(g)\mathbf{1}_{K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K}(g)dg$$

and

$$\begin{aligned}
\int_{\mathrm{GL}_2(F_\mathfrak{q})} \phi_3(g)\mathbf{1}_{K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K}(g)dg &= \int\int \phi_2(x)\phi_1(g)\mathbf{1}_{K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K}(xg)dgdx \\
&= \mathrm{vol}\left(K\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)K\right)\int \phi_1(g)\mathbf{1}_{K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K}(\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}g) \\
&= \mathrm{vol}\left(K\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)K\right) \\
&\qquad \times \left(\mathrm{M}_2(\mathcal{O}_{F_\mathfrak{q}}) \cap \left(\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K\right) \\
&= \begin{cases} (|\pi_\mathfrak{q}|_{\mathbb{A}_F}^{-1}+1)\mathrm{vol}\left(K\left(\begin{smallmatrix} \pi^{i+r+1} & 0 \\ 0 & \pi^{-i} \end{smallmatrix}\right)K\right) & \text{if } 0 > i \geqslant -\frac{r+1}{2} \\ (|\pi_\mathfrak{q}|_{\mathbb{A}_F}^{-1}+1)|\pi_\mathfrak{q}|_{\mathbb{A}_F}^{-m+1} & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases}
\end{aligned}$$

Thus

$$a_i = \begin{cases} |\pi_\mathfrak{q}|_{\mathbb{A}_F}^{-1}+1 & \text{if } 0 > i \geqslant -\frac{r+1}{2} \\ 1 & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases}$$

we have the proposition. □

4.1.2. Fourier expansions and the notion of coefficients of modular forms. Let $\mathcal{D}_{F/\mathbb{Q}}$ be the different of $F/\mathbb{Q}$ and we fix an element $\mathbf{d} \in \mathbb{A}_{F,f}^\times$ such that

$$d\mathcal{O}_F = \mathcal{D}_{F/\mathbb{Q}}.$$

We define

$$K_F(\mathfrak{n}\mathfrak{n}') := \mathbb{A}_F^\times \cap K_1(\mathfrak{n}, \mathfrak{n}').$$

Proposition 4.1.13. Let k, w, m and $\mathfrak{n}$ are as in Definition 4.1.5. For $f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}'))$, we have the following form of Fourier expansion:

$$f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{cases} |y|_{\mathbb{A}_F} \cdot \left[\begin{array}{l} \mathbf{a}_0(y_f, f)((4\pi y_\infty)^{-1})|y|_{\mathbb{A}_F}^{\beta-1} \\ \quad + \sum_{\xi \in F_+^\times} \mathbf{a}(\xi y_f, f)((4\pi y_\infty)^{-1})(\xi y_\infty)^{w-\underline{t}}\mathbf{e}_F(\sqrt{-1}\xi y_\infty)\mathbf{e}_F(\xi x) \end{array}\right] & \text{if } w = \beta\underline{t} \\ |y|_{\mathbb{A}_F} \sum_{\xi \in F_+^\times} \mathbf{a}(\xi y_f, f)((4\pi y_\infty)(\xi y_\infty)^{w-\underline{t}}\mathbf{e}_F(\sqrt{-1}\xi y_\infty)\mathbf{e}_F(\xi x) & \text{if } w \notin \mathbb{Z}\underline{t}, \end{cases}$$

where $x \in \mathbb{A}_F, y \in \mathbb{A}_F^\times$ with $y_\infty \in \mathbb{R}_{>0}^I$, $\mathbf{e}_F : \mathbb{A}_F/F \longrightarrow \mathbb{C}^\times$ is a unique continuous group homomorphism such that

$$\mathbf{e}_F(z) = \prod_{\sigma\in I} e^{2\pi\sqrt{-1}z_\sigma}$$

for $z = (z_\sigma)_{\sigma\in I} \in F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^I \subset \mathbb{A}_F$ and

$$\mathbf{a}_0(\cdot, f)(Y) \colon \mathbb{A}_{F,f}/F_+^\times K_F(\mathfrak{n}) \longrightarrow \mathbb{C}\left[\{Y_\sigma\}_{\sigma\in I}\right]$$
$$\mathbf{a}(\cdot, f)(Y) \colon \mathbb{A}_{F,f}^\times/K_F(\mathfrak{n}) \longrightarrow \mathbb{C}\left[\{Y_\sigma\}_{\sigma\in I}\right]$$

are $\mathbb{C}\left[\{Y_\sigma\}_{\sigma\in I}\right]$-valued continuous functions whose supports are

$$\left\{x \in \mathbb{A}_F^\times \middle| |x_\mathfrak{q}|_\mathfrak{q} \leqslant |d^{-1}|_\mathfrak{q} \text{ for all prime } \mathfrak{q}\right\}$$

and the degree for Y_σ is less than m_σ.

Proof. It follows from Proposition 4.1.8 and [**Hi86**, Proposition 4.1]. □

We introduce the notion of normalized Hilbert modular forms as follows:

Definition 4.1.14. We call a Hilbert modular form $f \in \mathscr{M}_{k,w}\left(K_1(\mathfrak{n}, \mathbf{p}^s)\right)$ is normalized (at d^{-1}) if

$$\mathbf{a}(d^{-1}, f) = 1.$$

We note that the notion of normalized is depend on the choice of d_p.

For each $n \in \mathbb{Z}[I]$, we define a field by

$$F^n := \mathbb{Q}(\{x^n\}_{x\in F}).$$

We note that F^n is the same as a number field fixed by $\left\{\varsigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \middle| \sum_{\sigma\in I} n_\sigma(\sigma\circ\tau) = n\right\}$. For any fractional ideal $\mathfrak{a} \subset \mathcal{O}_F$, we define a fractional ideal of $\mathcal{O}_{F^n}$ by

$$\mathfrak{a}^n := \langle\{a^n\}_{a\in\mathfrak{a}}\rangle.$$

Definition 4.1.15. Let k, w, m and $\mathfrak{n}$ be as in Definition 4.1.5. For any subring $R \subset \mathbb{C}$ containing $\mathcal{O}_{F^w}$, we define nearly holomorphic modular forms of R-coefficient by

$$\begin{aligned}&\mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}'); R)\\ &:= \left\{ f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}')) \middle| \begin{array}{rcl} \mathbf{a}_0(x, f)(Y) & \in & R[Y] \\ \mathbf{a}(x, f)(Y) & \in & (xd\mathcal{O}_F)^{\underline{t}-w}\, R[Y] \end{array} \text{ for } x \in \mathbb{A}_{F,f}^\times \right\}.\end{aligned}$$

We note that the notion of the coefficient for *modular forms* is compatible with the abstract scaler extensions as follows:

Theorem 4.1.16. Let $k, w, \mathfrak{n}$ be as in Definition 4.1.3. For any subring $R \subset \mathbb{C}$ containing $\mathcal{O}_{F^w}$, we have

$$\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'); \mathcal{O}_{F^w}) \otimes_{\mathcal{O}_{F^w}} R \cong \mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'); R)$$

Proof. The theorem is deduced from the duality theorem ([**Hi86**, Theorem 5.1]). □

The actions of operators in Definition 4.1.11 are described as follows:

Proposition 4.1.17. Let $f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathfrak{n}'))$ with $2w - k = \alpha \underline{t}$. Let $\pi_{\mathfrak{q}}$ be a prime element of $\mathcal{O}_{F_{\mathfrak{q}}}$. Then we have

$$\mathbf{a}_0(y, f|T(\pi_{\mathfrak{q}})) = \mathbf{a}_0(\pi_{\mathfrak{q}} y, f)|\pi_{\mathfrak{q}}|_{\mathbb{A}_F}^{\beta-1}\mathbf{1}_{\{|yd|_{\mathbb{A}_F} \leqslant 1\}}(y) + \mathbf{a}_0(\pi_{\mathfrak{q}}^{-1} y, f|\langle \mathfrak{q} \rangle_{k,w})|\pi_{\mathfrak{q}}|_{\mathbb{A}_F}^{\alpha-\beta} \quad (w = \beta \underline{t}),$$
$$\mathbf{a}(y, f|T(\pi_{\mathfrak{q}})) = \mathbf{a}(\pi_{\mathfrak{q}} y, f)\mathbf{1}_{\{|yd|_{\mathbb{A}_F} \leqslant 1\}}(y) + \mathbf{a}(\pi_{\mathfrak{q}}^{-1} y, f|\langle \mathfrak{q} \rangle_{k,w})|\pi_{\mathfrak{q}}|_{\mathbb{A}_F}^{\alpha-1}.$$

For $a, b \in \left(\widehat{\mathcal{O}_F}\right)_{\mathfrak{n}'}^{\times}$,

$$\mathbf{a}_0(y, f|T(a, b)) = \mathbf{a}_0(yab^{-1}, f|_{k,w} b^{-1}),$$
$$\mathbf{a}(y, f|T(a, b)) = \mathbf{a}(yab^{-1}, f|_{k,w} b^{-1}).$$

Theorem 4.1.18. Let $k, w, \mathfrak{n}$ and $\mathfrak{n}$ be as in Definition 4.1.3. For any nonzero prime ideal $\mathfrak{q} \subset \mathcal{O}_F$, we have

$$\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'); \mathcal{O}_{F^w})|\langle \mathfrak{q} \rangle_{k,w} \subset \mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'); \mathcal{O}_{F^w})$$
$$\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'); \mathcal{O}_{F^w})|T(\pi_{\mathfrak{q}}^r) \subset \mathfrak{q}^{r(\underline{t}-w)} \mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'); \mathcal{O}_{F^w}),$$

where $\pi_{\mathfrak{q}}$ is a prime element of $\mathcal{O}_{F_{\mathfrak{q}}}$.

Proof. [**Hi91**, Theorem 2.2, (ii)]. □

4.2. The theory of p-adic modular forms

From now on, we assume that nonzero ideal $\mathfrak{n}$ is prime to p. We fix a finite flat $\mathbb{Z}_p$-algebra $\mathcal{O} \subset \mathbb{C}_p$ containing all the conjugation of $\mathcal{O}_F$ and fix a uniformizer π. We usually use the symbol $s = \sum_{\mathfrak{p}|p} s_{\mathfrak{p}} \mathfrak{p}$ for the element of semigroup $\bigoplus_{\mathfrak{p}|p} \mathbb{Z}_{\geqslant 0} \mathfrak{p}$ and define the ideal $\mathbf{p}^s \subset \mathcal{O}_F$ by

$$\mathbf{p}^s := \prod_{\mathfrak{p}|p} \mathfrak{p}^{s_{\mathfrak{p}}}.$$

4.2.1. The universal Hecke rings. For any ring $R \subset \mathbb{C}_p$ containing $\mathcal{O}_{F^w}$ (we will mainly consider R as $\mathcal{O}$), we define

$$\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathbf{p}^s); R) := \mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathbf{p}^s); \mathcal{O}_{F^w}) \otimes_{\mathcal{O}_{F^w}} R,$$
$$\mathscr{S}_{k,w}(K_1(\mathfrak{n}, \mathbf{p}^s); R) := \mathscr{S}_{k,w}(K_1(\mathfrak{n}, \mathbf{p}^s); \mathcal{O}_{F^w}) \otimes_{\mathcal{O}_{F^w}} R.$$

Definition 4.2.1. Let k, w as in Definition 4.1.3. For $x \in \widehat{\mathcal{O}_F} \cap \mathbb{A}_{F,f}^{\times}$, we define an endomorphism of $\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathfrak{n}'), \mathbb{C}_p)$ by

$$T_0(x) := x_p^{w-\underline{t}} T(x).$$

We note that by Proposition 4.1.18, $T_0(x)$ is also defined as an endomorphism of the space of cusp forms.

The operator above determines the endomorphisms of $\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathbf{p}^s), \mathcal{O}[\pi^{-1}])$, but by Theorem 4.1.18, it's actually endomorphisms of $\mathscr{M}_{k,w}(K_1(\mathfrak{n}, \mathbf{p}^s), \mathcal{O})$. For any ring $R \subset \mathbb{C}_p$

containing all the conjugation of $\mathcal{O}_F$, we define rings called R-coefficient Hecke rings by

$$\begin{aligned}\mathbf{H}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);R) &:= R\left[\{T_0(x)\}_{x\in\mathbb{A}_{F,f}}, \{T(a,b)\}_{a,b\in(\widehat{\mathcal{O}_F})^\times_{\mathfrak{n}p}}\right] \\ &\subset \mathrm{End}_R\left(\mathscr{M}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);R)\right) \\ \mathbf{h}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);R) &:= R\left[\{T_0(x)\}_{x\in\mathbb{A}_{F,f}}, \{T(a,b)\}_{a,b\in(\widehat{\mathcal{O}_F})^\times_{\mathfrak{n}p}}\right] \\ &\subset \mathrm{End}_R\left(\mathscr{S}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);R)\right)\end{aligned}$$

Let

$$\begin{aligned}\mathrm{Cl}_F^+(\mathfrak{n}p^\infty) &:= \varprojlim_s \mathrm{Cl}_F^+(\mathfrak{n}\mathbf{p}^s) \\ &\cong \mathbb{A}_{F,f}^\times/\overline{F_+^\times K_F(\mathfrak{n})^{(p)}}, \qquad (4.2.1)\end{aligned}$$

where the isomorphism (4.2.1) is induced from

$$\begin{array}{ccc}\mathrm{Cl}_F^+(\mathfrak{n}\mathbf{p}^s) & \xrightarrow{\cong} & \mathbb{A}_F^\times/F^\times(\mathbb{A}_{F,f}^\times\cap K_1(\mathfrak{n},\mathbf{p}^s))\mathbb{R}_{>0}^I \\ \Psi & & \Psi \\ \mathfrak{a} & \longmapsto & [a^{-1}] \text{ such that } a_{p\mathfrak{n}}=1 \text{ and } a\mathcal{O}_F=\mathfrak{a},\end{array} \qquad (4.2.2)$$

We define

$$\mathbf{G} := \mathrm{Cl}_F^+(\mathfrak{n}p^\infty)\times\mathcal{O}_{F_p}^\times$$

By Proposition 4.1.12, for each nonzero prime ideal $\mathfrak{q}$ being prime to $p\mathfrak{n}$, the operator $|\pi_\mathfrak{q}|^\alpha\langle\mathfrak{q}\rangle_{k,w} = \epsilon_{F,cyc}(\pi_\mathfrak{q})^{-\alpha}\langle\mathfrak{q}\rangle_{k,w}$ ($\pi_\mathfrak{q}$ is a prime element of $\mathcal{O}_{F_\mathfrak{q}}$) is an element of each Hecke ring. For any fractional ideal $\mathfrak{a}$ being prime to $\mathfrak{n}p$, we define a element of a Hecke ring by

$$\langle\mathfrak{a}\rangle := \epsilon_{cyc,F}(\mathfrak{a})^\alpha\langle\mathfrak{a}\rangle_{k,w},$$

where we regard $\epsilon_{cyc,F}$ as a continuous character of $\mathrm{Cl}_F^+(\mathfrak{n}p^\infty)$ via the isomorphism (4.2.1). We also define the action of $x\in\mathcal{O}_{F_p}^\times$ on each Hecke ring by $T(x,1)$. By the correspondence $\mathbf{G}\ni(z,a)\mapsto T(a,1)\langle z\rangle$, Hecke rings has a $\mathbf{G}$-action and also $\mathscr{M}_{k,w}\left(K_1(\mathfrak{n},\mathbf{p}^s);\mathcal{O}[\omega]\right)$ has a $\mathbf{G}$-action. We denote a quotient of $\mathbf{G}$ by

$$\mathbf{G}_s := \mathrm{Cl}_F^+(\mathfrak{n}\mathbf{p}^s)\times\left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}\right)^\times.$$

For any characters,

$$\begin{aligned}\omega\colon\ &\mathrm{Cl}_F^+(\mathfrak{n}\mathbf{p}^s)\longrightarrow\mathbb{C}^\times, \\ \omega'\colon\ &\left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}\right)^\times\longrightarrow\mathbb{C}^\times.\end{aligned}$$

we define

$$\begin{aligned}&\mathscr{M}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';\mathcal{O}[\omega,\omega']) \\ &\quad := \left\{ f\in\mathscr{M}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);\mathcal{O}[\omega,\omega']) \;\middle|\; \begin{array}{l}(\mathfrak{z},a)f \\ =\omega(\mathfrak{z})\omega'(a)\epsilon_{cyc,F}(\mathfrak{z})^\alpha f \text{ for } (\mathfrak{z},a)\in\mathbf{G}\end{array}\right\}, \\ &\mathscr{S}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';\mathcal{O}[\omega,\omega']) \\ &\quad := \left\{ f\in\mathscr{S}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);\mathcal{O}[\omega,\omega']) \;\middle|\; \begin{array}{l}(\mathfrak{z},a)f \\ =\omega(\mathfrak{z})\omega'(a)\epsilon_{cyc,F}(\mathfrak{z})^\alpha f \text{ for } (\mathfrak{z},a)\in\mathbf{G}\end{array}\right\},\end{aligned}$$

and also define

$$\begin{aligned}\mathbf{H}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';R[\omega]) &:= R[\omega,\omega']\left[\{T_0(x)\}_{x\in\mathbb{A}_{F,f}},\{T(a,b)\}_{a,b\in(\widehat{\mathcal{O}_F})^\times_{\mathfrak{n}p}}\right]\\ &\subset \mathrm{End}_R\left(\mathscr{M}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega;R[\omega])\right),\\ \mathbf{h}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';R[\omega,\omega']) &:= R[\omega,\omega']\left[\{T_0(x)\}_{x\in\mathbb{A}_{F,f}},\{T(a,b)\}_{a,b\in(\widehat{\mathcal{O}_F})^\times_{\mathfrak{n}p}}\right]\\ &\subset \mathrm{End}_R\left(\mathscr{S}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';R[\omega,\omega'])\right).\end{aligned}$$

It is well known that the paring

$$\langle f,h\rangle := a_p(d^{-1},f|h)$$

for $f\in\mathscr{S}_{k,w}\left(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';R[\omega,\omega']\right)$ and $h\in\mathbf{h}_{k,w}\left(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';R[\omega,\omega']\right)$ is a perfect pairing. Let

$$\begin{aligned}\mathscr{M}_{k,w}(K_1(\mathfrak{n},p^\infty);\mathcal{O}) &:= \bigcup_s \mathscr{M}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);\mathcal{O}),\\ \mathscr{S}_{k,w}(K_1(\mathfrak{n},p^\infty);\mathcal{O}) &:= \bigcup_s \mathscr{S}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);\mathcal{O}).\end{aligned}$$

We define rings acting on the each space above by

$$\begin{aligned}\mathbf{H}_{k,w}(K_1(\mathfrak{n},p^\infty);\mathcal{O}) &:= \varprojlim_s \mathbf{H}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);\mathcal{O}),\\ \mathbf{h}_{k,w}(K_1(\mathfrak{n},p^\infty);\mathcal{O}) &:= \varprojlim_s \mathbf{h}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s);\mathcal{O}).\end{aligned}$$

Then both Hecke rings above has the continuous action of $(z,a)\in\mathbf{G}$ via $T(a,1)\langle z\rangle$. For the universal Hecke ring for cusp forms, we have the following theorem by Hida

Theorem 4.2.2 ([Hi89-1, Theorem 2.3]). For $k,w\in\mathbb{Z}[I]$ satisfying the condition in Definition 4.1.3, there exists a canonical isomorphism

$$\mathbf{h}_{k,w}(K_1(\mathfrak{n},p^\infty);\mathcal{O}) \cong \mathbf{h}_{2\underline{t},\underline{t}}(K_1(\mathfrak{n},p^\infty);\mathfrak{O})$$

such that $T_0(y)$ and $T(a,b)$ of the left hand side correspond to $T_0(y)$ and $T(a,b)$ of the right hand side.

Definition 4.2.3. We define the universal Hecke ring for cusp forms by

$$\mathbf{h}(\mathfrak{n};\mathcal{O}) := \mathbf{h}_{2\underline{t},\underline{t}}(K_1(\mathfrak{n},p^\infty);\mathcal{O}).$$

4.2.2. The space of p-adic modular forms. Let

$$\mathfrak{I} := \mathrm{Cl}_F^+(\mathfrak{n}p^\infty)\sqcup \mathbb{A}_{F,f}^\times/K_F(\mathfrak{n})^{(p)}.$$

For any subring $R\subset\mathbb{C}_p$, we define $\mathcal{C}_b^0(\mathfrak{I},R)$ by

$$\mathcal{C}_b^0(\mathfrak{I},R) := \left\{\begin{array}{cc} \phi & :\mathfrak{I}\longrightarrow\mathbb{C}_p \\ & :\ \text{continuous}\end{array}\middle| \begin{array}{l}\phi(\mathfrak{I})\subset R\\ |\phi(y)|_p \text{ is bounded for } y\end{array}\right\}.$$

On $\mathcal{C}_b^0(\mathfrak{I},\mathbb{C}_p)$, we define the supremum norm $||\cdot||_p$:

$$||\phi||_p := \sup_{y\in\mathfrak{I}}\left\{|\phi(y)|_p \middle| y\in\mathfrak{I}\right\}.$$

Definition 4.2.4. Let $s = \sum_{\mathfrak{p}\mid p} s_{\mathfrak{p}}\mathfrak{p}$ and $k, w, m \in \mathbb{Z}[I]$ such that $k \geqslant 2\underline{t}$, $2w - k \in \mathbb{Z}$ and $m \geqslant 0$. Let $f \in \mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathbf{p}^s); \overline{\mathbb{Q}})$. We define a continuous function (depending on the choice of $d \in \mathbb{A}_{F,f}$)

$$\mathbf{a}_p(\cdot, f)\colon \mathfrak{I} \longrightarrow \mathbb{C}_p\left[\{Y_\sigma\}_{\sigma\in I}\right]$$

by

$$\mathbf{a}_p(y, f) := \begin{cases} \mathbf{a}_0(y, f)\epsilon_{cyc,F}^{\beta-1}(dy) & \text{if} \quad y \in \mathrm{Cl}_F^+(\mathfrak{n}p^\infty) \text{ and } w = \beta\underline{t} \\ 0 & \text{if} \quad y \in \mathrm{Cl}_F^+(\mathfrak{n}p^\infty) \text{ and } w \notin \mathbb{Z}\underline{t} \\ \mathbf{a}(y, f)(dy)_p^{w-t} & \text{if} \quad y \in \mathbb{A}_{F,f}^\times/K_F(\mathfrak{n})^{(p)}. \end{cases}$$

By the correspondence, $f \mapsto \mathbf{a}_p(\cdot, f)(y_p^{-1})$, we can embed $\mathscr{N}_{k,w,m}(K_1(\mathfrak{n}, \mathbf{p}^s); \overline{\mathbb{Q}})$ into $\mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p)$ for any k, w, m and $\mathfrak{n}$ of Definition 4.1.5. In particular, we can embed $\mathscr{M}_{k,w}(K_1(\mathfrak{n}, p^\infty))$ into $\mathcal{C}_b^0(\mathfrak{I}; \mathcal{O})$. Let $\overline{\mathscr{S}_{k,w}\left(K_1(\mathfrak{n}, p^\infty); \overline{\mathbb{Q}}\right)}$ be the closure of $\mathscr{S}_{k,w}(K_1(\mathfrak{n}, p^\infty); \mathbb{Q})$ in $\mathcal{C}_b^0(\mathfrak{I}; \mathbb{C}_p)$. Then we have the following remarkable theorem:

Theorem 4.2.5. The space $\overline{\mathscr{S}_{k,w}\left(K_1(\mathfrak{n}, p^\infty); \overline{\mathbb{Q}}\right)}$ is independent of k, w if $k \geqslant 2\underline{t}$.

Definition 4.2.6. We define the space of p-adic modular forms by

$$\overline{\mathscr{M}}(\mathfrak{n}) := \text{the closure of } \sum_{\substack{k,w \\ k-2\underline{t}\geqslant 0 \\ 2w-k\in\mathbb{Z}\underline{t}}} \mathscr{M}_{k,w}\left(K_1(\mathfrak{n}, p^\infty); \overline{\mathbb{Q}}\right) \text{ in } \mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p),$$

and the space of p-adic cusp forms by

$$\begin{aligned} \overline{\mathscr{S}}(\mathfrak{n}) &:= \text{the closure of } \sum_{\substack{k,w \\ k-2\underline{t}\geqslant 0 \\ 2w-k\in\mathbb{Z}\underline{t}}} \mathscr{S}_{k,w}\left(K_1(\mathfrak{n}, p^\infty); \overline{\mathbb{Q}}\right) \text{ in } \mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p) \\ &\underset{\text{Theorem 4.2.5}}{=} \overline{\mathscr{S}_{2\underline{t},\underline{t}}(\mathfrak{n}p^\infty; \overline{\mathbb{Q}})}. \end{aligned}$$

For any subring $R \subset \mathbb{C}_p$ containing all the conjugation of $\mathcal{O}_F$, we define

$$\begin{aligned} \overline{\mathscr{M}}(\mathfrak{n}; R) &:= \overline{\mathscr{M}}(\mathfrak{n}) \cap \mathcal{C}_b^0(\mathfrak{I}; R), \\ \overline{\mathscr{S}}(\mathfrak{n}; R) &:= \overline{\mathscr{S}}(\mathfrak{n}) \cap \mathcal{C}_b^0(\mathfrak{I}; R). \end{aligned}$$

Between the universal Hecke ring and p-adic cusp forms, there exists the following duality:

Theorem 4.2.7 ([Hi91, Theorem 3.1]). The pairing

$$\mathbf{h}(\mathfrak{n}; \mathcal{O}) \otimes \overline{\mathscr{S}}(\mathfrak{n}; \mathcal{O}) \longrightarrow \mathcal{O}; \quad T \otimes f \mapsto \mathbf{a}_p(d^{-1}, f|T)$$

is a perfect pairing, namely, we have the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}}(\mathbf{h}(\mathfrak{n}; \mathcal{O}), \mathcal{O}) &\cong \overline{\mathscr{S}}(\mathfrak{n}; \mathcal{O}), \\ \mathrm{Hom}_{\mathcal{O}}(\overline{\mathscr{S}}(\mathfrak{n}; \mathcal{O}), \mathcal{O}) &\cong \mathbf{h}(\mathfrak{n}; \mathcal{O}). \end{aligned}$$

Remark 4.2.8. There exists a duality theorem for p-adic modular forms (See [**Hi91**, Theorem 3.1]).

4.2.3. Several operations for p-adic modular forms. We fix a character

$$(\omega, \omega') \colon \mathbf{G} \longrightarrow \mathbb{C}_p^\times$$

of finite image. In this section, we summarize several continuous operation defined on $\mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p)$, which fix spaces of modular forms of character (ω, ω').

4.2.3.1. *Hecke operators.* Let $a(y) \in \mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p)$. For $\mathfrak{q}$ being prime to $\mathfrak{n}p$, we define

$$\begin{aligned} a|T_0(\pi_\mathfrak{q})(y) &:= a(\pi_\mathfrak{q} y)\mathbf{1}_{\left\{|yd|_{\mathbb{A}_F} \leqslant 1\right\}}(y) + \epsilon_{\mathrm{cyc}}(\mathfrak{q})^{\alpha-1}\omega(\mathfrak{q})a(\pi_\mathfrak{q}^{-1}y), \\ T(u_\mathfrak{q}) &:= 1, \end{aligned}$$

where $u_\mathfrak{q} \in \mathcal{O}_{F_\mathfrak{q}}^\times$, $\pi_\mathfrak{q}$ is a prime element of $\mathcal{O}_{F_\mathfrak{q}}$ and $(\pi_\mathfrak{q}, 1)$ is the element of $\mathbf{G}$. We note that $T_0(\pi_\mathfrak{q})$ is independent of the choice of $\pi_\mathfrak{q}$. For any $r > 0$, and $u \in \mathcal{O}_{F_\mathfrak{q}}^\times$, we define $T_0(\pi_\mathfrak{q}^r u)$ inductively by

$$T_0(\pi_\mathfrak{q}^r) := T_0(\pi_\mathfrak{q})T_0(\pi_\mathfrak{q}^{r-1}u) - \epsilon_{cyc,F}(\mathfrak{q})^{\alpha-1}\omega(\mathfrak{q})T_0(\pi_\mathfrak{q}^{r-2}u).$$

For $\mathfrak{q} \mid p\mathfrak{n}$ and $0 \neq x_\mathfrak{q} \in \mathcal{O}_{F_\mathfrak{q}}$, we define

$$a|T_0(x_\mathfrak{q})(y) := a(x_\mathfrak{q} y)\mathbf{1}_{\left\{|yd|_{\mathbb{A}_F} \leqslant 1\right\}}(y).$$

For $x \in \widehat{\mathcal{O}_F} \cap \mathbb{A}_{F,f}^\times$, we define

$$T_0(x) := \prod_\mathfrak{q} T_0(x_\mathfrak{q}).$$

For $a, b \in \widehat{\mathcal{O}_{F}}_p$, we define

$$a|T(a,b)(y) = \omega'(ab^{-1})a(yab^{-1}).$$

Proposition 4.2.9. For $f \in \mathscr{M}_{k,w}\left(K_1(\mathfrak{n}, p^s), \omega, \omega'; \overline{\mathbb{Q}}\right)$. We have

$$\begin{aligned} \mathbf{a}_p(y,f)|T_0(x) &= \mathbf{a}_p(y, f|T_0(x)) \\ \mathbf{a}_p(y,f)|T(a,b) &= \mathbf{a}_p(y, f|T(a,b)) \end{aligned}$$

PROOF. It follows from Proposition 4.1.17. □

4.2.3.2. *Twisted p-depletions.* Let $\eta' \colon \mathcal{O}_{F_p}^\times \longrightarrow \mathbb{C}_p^\times$ be a character of finite image and let $\eta'_\mathfrak{p} := \eta'|_{\mathcal{O}_{F_\mathfrak{p}}}$. We denote the conductor of $\eta'_\mathfrak{p}$ by $\mathfrak{p}^{c(\eta')} \geqslant 0$. We define a twisted p-depression associated with $a(y) \in \mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p)$ denoted by $\theta_{\eta'}^{(p)}a$ by

$$\theta_{\eta'}^{(p)}a(y) := \begin{cases} \eta'(dy_p)a(y) & \text{if } |y_p d_p|_{\mathbb{A}_F} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.2.10. For $f \in \mathscr{M}_{k,w}\left(K_1(\mathfrak{n}, p^s), \omega, \omega'; \overline{\mathbb{Q}}\right)$, we have

$$\begin{aligned} \theta_{\eta'}^{(p)}\mathbf{a}(\cdot, f) &= \mathbf{a}_p(\cdot, \widetilde{\theta}_{\eta'}^{(p)}f) \\ &\in \mathscr{M}_{k,w}\left(K_1(\mathfrak{n}, \mathbf{p}^{\max\{s, c(\eta')\}}), \omega, \omega'\eta'; \overline{\mathbb{Q}}\right) \end{aligned}$$

where $c(\eta') := \sum_{\mathfrak{p}\,|\,p} c(\eta'_{\mathfrak{p}})\mathfrak{p}$. and $\widetilde{\theta}^{(p)}_{\eta'}$ is defined as follows: Let

$$\widetilde{\theta}^{(\mathfrak{p})}_{\eta'_{\mathfrak{p}}} f = \begin{cases} f - \begin{pmatrix} \varpi_{\mathfrak{p}}^{-1} & 0 \\ 0 & 1 \end{pmatrix} (f|T_0(\varpi_{\mathfrak{p}})) & \text{if } c(\eta'_{\mathfrak{p}}) = 0, \\ \dfrac{1}{\mathfrak{g}(\eta'_{\mathfrak{p}})} \displaystyle\sum_{u \in \left(\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}^{c(\eta')}\mathcal{O}_{F_{\mathfrak{p}}}\right)^{\times}} {\eta'}^{-1}_{\mathfrak{p}}(u) \begin{pmatrix} 1 & u\varpi_{\mathfrak{p}}^{-c(\eta'_{\mathfrak{p}})} \\ 0 & 1 \end{pmatrix} f & \text{if } c(\eta'_{\mathfrak{p}}) > 0, \end{cases}$$

where

$$\mathfrak{g}(\eta'_{\mathfrak{p}}) := \sum_{u \in \left(\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}^{c(\eta')}\mathcal{O}_{F_{\mathfrak{p}}}\right)^{\times}} {\eta'}^{-1}_{\mathfrak{p}}(u)\mathbf{e}_F(u\varpi_{\mathfrak{p}}^{-c(\eta'_{\mathfrak{p}})} d_{\mathfrak{p}}^{-1}).$$

Define

$$\widetilde{\theta}^{(p)}_{\eta'} := \prod_{\mathfrak{p}\,|\,p} \widetilde{\theta}^{(\mathfrak{p})}_{\eta'_{\mathfrak{p}}}$$

Proof. By direct computation, we have

$$\theta^{(p)}_{\eta'}\mathbf{a}_p(\cdot, f) = \mathbf{a}_p(\cdot, \widetilde{\theta}^{(p)}_{\eta'} f).$$

□

4.2.3.3. *Central twists.* Let $\eta\colon \mathrm{Cl}_F^+(\mathfrak{n}p^{\infty}) \to \mathbb{C}_p^{\times}$ be a continuous character of finite image. For $a(y) \in \mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p)$, we define a twist $(a \otimes \eta)$ of $a(y)$ by η by

$$(a \otimes \eta|\cdot|^{\beta}_{\mathbb{A}_F})(y) := \eta(y_f d)\epsilon_{cyc,F}(yd)^{-\beta}a(y).$$

Here, we regard η as a continous group homomorphism on $\mathbb{A}_F^{\times}/F^{\times}$ via (4.2.1).

Proposition 4.2.11. For $f \in \mathscr{M}_{k,w}\left(K_1(\mathfrak{n}, p^s), \omega, \omega'; \overline{\mathbb{Q}}\right)$, we have

$$\begin{aligned} \mathbf{a}_p(\cdot, f) \otimes \eta|\cdot|^{\beta}_{\mathbb{A}_F} &= \eta(d)|d|^{\beta}_{\mathbb{A}_F}\mathbf{a}_p(\cdot, f \otimes \eta|\cdot|^{\beta}_{\mathbb{A}_F}) \\ &\in \mathscr{M}_{k,w+\beta\underline{t}}\left(K_1(\mathfrak{n}, \mathbf{p}^s), \omega\eta^2, \omega'\eta|^{-1}_{\mathcal{O}^{\times}_{F_p}}; \overline{\mathbb{Q}}\right), \end{aligned}$$

where $\mathbf{p}^c(\eta) \subset \mathcal{O}_F$ is the conductor of η at primes dividing p.

4.2.3.4. *Differential operators.* Let $r \in \mathbb{Z}_{\geqslant 0}[I]$ with $r \neq 0$. For $a(y) \in \mathcal{C}_b^0(\mathfrak{I}, \mathbb{I})$, we define

$$D^r a(y) = \begin{cases} 0 & \text{if } y \in \mathrm{Cl}_F^+(\mathfrak{n}p^{\infty}), \\ a(y)(dy)_p^r & \text{if } y \in \mathbb{A}_{F,f}^{\times}/K_F(\mathfrak{n})^{(p)}. \end{cases}$$

The operator preserves the space of p-adic modular forms as explained below: let k, w, m as in Definition 4.1.5. For $f \in \mathscr{N}_{k,w,m}\left(K_1(\mathfrak{n}, \mathbf{p}^s); \overline{\mathbb{Q}}\right)$, we define an element of $\mathcal{C}_b^0(\mathfrak{I}, \mathbb{C}_p)$ by

$$\mathbf{a}_p(y, c(f)) := \mathbf{a}_p(y, f)(0).$$

Then we have

Theorem 4.2.12. For $f \in \mathscr{N}_{k,w,m}\left(K_1(\mathfrak{n}, \mathbf{p}^s); \overline{\mathbb{Q}}\right)$ with $k > 2m$, the function $\mathbf{a}_p(\cdot, c(f))$ is an element of $\overline{\mathscr{M}}(\mathfrak{n})$. In particular, for any $f \in \mathscr{M}_{k,w}\left(K_1(\mathfrak{n}, \mathbf{p}^s); \overline{\mathbb{Q}}\right)$ and $r \in \mathbb{Z}_{\geqslant 0}[I]$ with $r \neq 0$, the function

$$\mathbf{a}_p(\cdot, c(R^r f)) = D^r\mathbf{a}(\cdot, f)$$

is also a p-adic modular form.

Proof. See [**Hi91**, p.369-370]. □

4.2.4. The restriction map. We introduce a notion of restriction between Hilbert modular forms of different base field. The notion is important to construct p-adic L-function. In this section, we need to distinguish the base field F for each notion of Hilbert modular forms for GL_2/F, for example, the set of embeddings from F into $\mathbb{C}_p$, the space of modular forms and so on. Thus, for each symbol X relating to the field F, we denote by X_F, for example,

$$I_F := \{\sigma\colon F \hookrightarrow \mathbb{C}_p : \text{ field embedding}\},$$
$$\mathscr{M}_{k,w}(U)_F := \begin{pmatrix} \text{the space of Hilbert modular forms associated with } \mathrm{GL}_2/F \\ \text{of weight } k, w \in \mathbb{Z}[I_F] \text{ and level U.} \end{pmatrix},$$
$$\mathfrak{I}_F := \mathrm{Cl}_F^+(\mathfrak{n}p^\infty) \sqcup \mathbb{A}_{F,f}^\times/K_F(\mathfrak{n})^{(p)},$$
$$d_F : \text{ the element of } \mathbb{A}_{F,f}^\times \text{ such that } d_F\mathcal{O}_F = \mathcal{D}_{F/\mathbb{Q}}.$$

Let E/F be a extension of totally real fields. We assume that the fixed ring $\mathcal{O}$ is containing all the conjugation of $\mathcal{O}_E$. Let $N \subset \mathcal{O}_E$ and $\mathfrak{n} \subset \mathcal{O}_F$ be nonzero ideals such that $\mathfrak{n} = \mathcal{O}_F \cap N$ and both of them are prime to p.

Definition 4.2.13. Let $a \in \mathcal{C}_b^0(\mathfrak{I}_E, \mathbb{C}_p)$ such that there exists $C > 0$ such that $a(y) = 0$ if $y \in \mathbb{A}_{F,f}^\times/K_F(\mathfrak{n})^\times$ and $|y|_{\mathbb{A}_{F,f}} > C$. We define $\mathrm{Res}_{E/F}\, a \in \mathcal{C}_b^0(\mathfrak{I}_F, \mathbb{C}_p)$ by

$$\mathrm{Res}_{E/F}\, a(y) = \begin{cases} \epsilon_{cyc,F}^{1-[E:F]}(yd_F)a(y) & \text{if } y \in \mathrm{Cl}_F^+(\mathfrak{n}p^\infty) \\ \epsilon_{cyc,F}^{1-[E:F]}(yd_F) \displaystyle\sum_{\substack{\eta \in E_+ \\ \mathrm{Tr}_{E/F}(\eta)=1}} a(\eta y) & \text{if } y \in \mathbb{A}_{F,f}^\times/K_F(\mathfrak{n})^{(p)} \end{cases}$$

Note that under the condition for support of the function $a(y)$ in the definition above, the sum $\displaystyle\sum_{\substack{\eta \in E_+ \\ \mathrm{Tr}_{E/F}(\eta)=1}} a(\eta y)$ is finite sum.

We have the following lemma:

Lemma 4.2.14. Let $f \in \mathscr{M}_{k,w}(K(N)_E; \mathbb{C}_p)_E$ be a Hilbert modular form of weight $k, w \in \mathbb{Z}[I_{/E}]$ and level N. We have

$$\mathrm{Res}_{E/F}\, \mathbf{a}_p(\cdot, f) = |d_F|_{\mathbb{A}_F}^{[E:F]-1}(d_{E/F})_p^{w-t_E}\mathbf{a}_p(\cdot, f|_{\mathrm{GL}_2(\mathbb{A}_F)}),$$

where $d_{E/F} = d_E d_F^{-1}$

Proof. It follows by direct computations by definition. □

Corollary 4.2.15. For any $f \in \overline{\mathscr{M}}(N)$, we have $\mathrm{Res}_{E/F}\, f \in \overline{\mathscr{M}}(\mathfrak{n})$.

4.3. Hida's ordinary idempotents and the control theorems

The notations are as in the previous subsection.

Definition 4.3.1. Let $\mathfrak{p} \mid p$ be a prime ideal. We define the $\mathfrak{p}$-ordinary idempotent $e_\mathfrak{p} \in \mathbf{h}(\mathfrak{n}; \mathcal{O})$ by

$$e_\mathfrak{p} := \lim_{n\to\infty} T_0(\varpi_\mathfrak{p})^{n!},$$

where $\varpi_\mathfrak{p}$ is a prime element of $\mathcal{O}_{F_\mathfrak{p}}$. We also define

$$e := \prod_{\mathfrak{p} \mid p} e_\mathfrak{p}.$$

It is well-known that $e_{\mathfrak{p}}$ exists and is independent of the choice of $\varpi_{\mathfrak{p}}$ and satisfying

$$e_{\mathfrak{p}}^2 = e_{\mathfrak{p}}.$$

Definition 4.3.2. For any complete ring $R \subset \mathbb{C}_p$ containing $\mathcal{O}$, We define the nearly ordinary part of the universal Hecke ring and p-adic cusp forms by

$$\mathbf{h}^{n,ord}(\mathfrak{n}; R) := e_p \mathbf{h}(\mathfrak{n}; R)$$
$$\overline{\mathscr{S}}^{n,ord}(\mathfrak{n}; R) := \overline{\mathscr{S}}(\mathfrak{n}; R)|e_p$$

The ordinary part of universal Hecke ring is also a continuous $\mathcal{O}[[\mathbf{G}]]$-module. Let

$$\begin{aligned} \mathscr{X} &:= \mathrm{Hom}_{\mathrm{cont}}(\mathbf{G}, \mathbb{C}^\times) \\ &\cong \mathrm{Hom}_{\mathcal{O}\text{-cont}}\left(\mathcal{O}[[\mathbf{G}]], \mathbb{C}_p\right). \end{aligned}$$

For each point $P \in \mathscr{X}$, we denote the kernel of the induced $\mathcal{O}$-algebra homomorphism $P \colon \mathcal{O}[[\mathbf{G}]] \to \mathbb{C}$ by

$$P_{\mathcal{O}} := \mathrm{Ker}(P) \subset \mathcal{O}[[\mathbf{G}]].$$

For $k, w \in \mathbb{Z}[I]$ such that $k - 2\underline{t} \geqslant 0$ and a finite order character $\omega \times \omega' : \mathbf{G} \longrightarrow \mathbb{C}_p^\times$, we define $P_{k,w,\omega,\omega'} \in \mathscr{X}$ by

$$P_{k,w,\omega,\omega'}(\mathfrak{z}, a) = \omega(\mathfrak{z})\omega'(a)\epsilon_{F,cyc}(\mathfrak{z})^{[2w-k]} a^{\underline{t}-w}.$$

We define the set of arithmetic points of $\mathscr{X}$ by

$$\mathscr{X}^{\mathrm{arith}} := \left\{ P_{k,w,\omega,\omega'} \;\middle|\; \begin{array}{l} k, w \in \mathbb{Z}[I] \text{ such that } k - 2\underline{t} \geqslant 0,\ 2w - k = \alpha \underline{t} \\ \omega \times \omega' \colon \mathbf{G} \longrightarrow \mathbb{C}_p^\times : \text{finite order image} \end{array} \right\}$$

Theorem 4.3.3. The nearly ordinary part of $\mathbf{h}(\mathfrak{n}; \mathcal{O})$ is finite over $\mathcal{O}[[\mathbf{G}]]$ and for any $P = P_{k,w,\omega,\omega'} \in \mathscr{X}^{\mathrm{arith}}$, we have

$$\mathbf{h}^{n,ord}(\mathfrak{n}; \mathcal{O}) \otimes_{\mathcal{O}[[\mathbf{G}]]} \kappa(P_{\mathcal{O}}) \cong e\mathbf{h}_{k,w}\left(K_1(\mathfrak{n}, \mathbf{p}^{s(\omega,\omega')}), \omega; \mathrm{Frac}(\mathcal{O})[\omega, \omega']\right),$$
$$\overline{\mathscr{S}}^{n,ord}(\mathfrak{n}; \mathcal{O}[\omega, \omega'])[P_{\mathcal{O}[\omega,\omega']}] = \mathscr{S}_{k,w}\left(K_1(\mathfrak{n}, \mathbf{p}^{s(\omega,\omega')}), \omega, \omega'; \mathcal{O}[\omega, \omega']\right)\Big| e,$$

where $\kappa(P_{\mathfrak{O}})$ is the residue field of the point $P_{\mathcal{O}}$ and

$$s(\omega, \omega') := \inf \left\{ s \in \bigoplus_{\mathfrak{p}|p} \mathbb{Z}_{\geqslant 1}\mathfrak{p} \;\middle|\; \omega \times \omega' \text{ factors through } \mathbf{G}_s \text{ for } s > 0 \right\}$$

4.4. The theory of $\mathbb{I}$-adic forms

4.4.1. The definition of nearly ordinary $\mathbb{I}$-adic forms. Recall

$$\begin{aligned} \mathbf{G}_s &:= \mathrm{Cl}_F^+(\mathfrak{n}\mathbf{p}^s) \times \left(\mathcal{O}_{F_p}/\mathbf{p}^s\mathcal{O}_{F_p}\right)^\times, \\ \mathbf{G} &:= \varprojlim_s \mathbf{G}_s \\ &= \mathrm{Cl}_F^+(\mathfrak{n}p^\infty) \times \mathcal{O}_{F_p}^\times. \end{aligned}$$

We fix an $\mathbb{I}$ noetherian complete semi-local continuous $\mathcal{O}[[\mathbf{G}]]$-algebra. We remark that $\mathbb{I}$ is not necessarily finite over $\mathcal{O}[[\mathbf{G}]]$. If $\mathbb{I}$ is domain, we denote by $\mathbb{L}$ the fraction field of $\mathbb{I}$. For any $\mathcal{O}$-algebra homomorphism $P \colon \mathbb{I} \to \mathbb{C}_p$, we denote by $P|_{\mathbf{G}}$ the composition

$\mathbf{G} \to \mathbb{I}^\times \xrightarrow{P} \mathbb{C}_p^\times$. When $P|_{\mathbf{G}} = P_{k,w,\omega,\omega'}$, we denote k, w, ω and ω' by k_P, w_P, ω_P, and ω'_P, respectively. We define

$$\mathscr{X}(\mathbb{I}) := \{P\colon \mathbb{I} \longrightarrow \mathbb{C}_p : \text{ continuous } \mathcal{O}\text{-algebra homomorphism}\}$$

We assume that there exists a countable subset $\mathscr{X}^{\mathrm{arith}}(\mathbb{I}) \subset \mathscr{X}(\mathbb{I})$ satisfying

$$\bigcap_{P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})} \mathrm{Ker}(P) = 0, \tag{4.4.1}$$

$$P|_{\mathbf{G}} \in \mathscr{X}^{\mathrm{arith}} \text{ for any } P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I}), \tag{4.4.2}$$

$$P(\mathbb{I}) \text{ is finite over } \mathbb{Z}_p \text{ for any } P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I}). \tag{4.4.3}$$

Note that by (4.4.3), for any maximal ideal $\mathfrak{m} \subset \mathbb{I}$, $\mathbb{I}/\mathfrak{m}$ is a finite field and in particular, $\mathbb{I}$ is a compact ring.

Remark 4.4.1. When $\mathbb{I}$ is finite over $\mathcal{O}[[\mathbf{G}]]$, we can take $\mathscr{X}^{\mathrm{arith}}(\mathbb{I})$ as the set

$$\left\{P \in \mathscr{X}(\mathbb{I}) \middle| P|_{\mathbf{G}} \in \mathscr{X}^{\mathrm{arith}}\right\} \tag{4.4.4}$$

by Lemma 2.2.16.

We define a topology on $\mathbb{I}^{\mathfrak{I}}$ by the weak topology associated with maps

$$||P_*||_p\colon \mathbb{I}^{\mathfrak{I}} \longrightarrow \mathbb{C}_p; F \mapsto ||P \circ F||_p$$

for $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$. Equivalently, the topology is same as that comes from the norm $|\cdot|_w$ defined by

$$|F|_w := \sum_i 2^{-i} ||P_i \circ F||_p.$$

Here we give an order along positive integers on $\mathscr{X}(\mathbb{I})^{\mathrm{arith}}$ and denote it by $\{P_i\}_{i=1}^{\infty}$. It is acutually a norm by the assumption above.

Proposition 4.4.2. The topology on $\mathbb{I}^{\mathfrak{I}}$ determined by the norm $|\cdot|_w$ above is compolete.

Proof. Denote $\mathscr{X}^{airth}(\mathbb{I})$ by $\{P_i\}_{i>0}$. Let $\{\phi^r = (\phi^r_x)_{x \in \mathfrak{I}}\}_{r=1}^{\infty} \subset \mathbb{I}^{\mathfrak{I}}$ be a Cauchy sequence for the norm $|\cdot|_w$. Since the natural morphism

$$\mathbb{I} \longrightarrow \varprojlim_n \mathbb{I}/\cap_{i=1}^n P_i,$$

has a dense image and rings of both sides are compact, it is isomorphism. For each $x \in \mathfrak{I}$ and $i > 0$, there exists $\phi_x \in \mathbb{I}$ such that $\lim_{r\to\infty} P_i(\phi_x^r - \phi_x) = 0$. Thus $|(\phi^r_x)_{x\in\mathfrak{I}} - (\phi_x)_{x\in\mathfrak{I}}|_w \to 0$ $(r \to \infty)$. □

Definition 4.4.3. We define the space of $\mathbb{I}$-adic form by

$$\mathscr{M}(\mathfrak{n}; \mathbb{I}) := \left\{\mathscr{F} \in \mathbb{I}^{\mathfrak{I}} \middle| \; P \circ \mathscr{F} \in \overline{\mathscr{M}}(\mathfrak{n}) \text{ for } P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I}) \;\right\}.$$

We define the space of nearly-ordinary $\mathbb{I}$-adic forms by

$$\mathscr{S}^{n,ord}(\mathfrak{n}; \mathbb{I}) := \left\{\mathscr{F} \in \mathbb{I}^{\mathfrak{I}} \;\middle|\; \begin{array}{l} P \circ \mathscr{F} \in \mathscr{S}^{n,ord}_{k_P,w_P}(\mathfrak{n}\mathbf{p}^\gamma, \omega_P, \omega'_P; P(\mathbb{I}))|e_p \\ \text{for some } \gamma \geqslant s(\omega_P) \text{ for } P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I}). \end{array}\right\}.$$

For ordinary $\mathbb{I}$-adic modular form $\mathscr{F}$ and $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$, we denote by $\mathfrak{F}_P$ the classical modular form corresponding to $P \circ \mathscr{F}$:

$$\mathbf{a}_p(\cdot, \mathscr{F}_P) = P \circ \mathscr{F}.$$

Remark 4.4.4. Since for each $F \in \mathscr{S}^{n,ord}(\mathfrak{n};\mathbb{I})$ and $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$, $P \circ F$ is actually in $\mathscr{S}_{k_P,w_P}(\mathfrak{n}p^{s(\omega_P)}, \omega_P; P(\mathbb{I}))$ (independent of γ !) by Theorem 4.3.3. Thus we have $\mathscr{S}^{n,ord}(\mathfrak{n};\mathbb{I})$ is closed subspace.

Remark 4.4.5. Let $F \in \mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n};\mathbb{I})$. If $\mathbb{I}$ is finite over $\mathcal{O}[[\mathbf{G}]]$, for any $P \in \mathscr{X}(\mathbb{I})$, we have $P \circ F \in \overline{\mathscr{S}}(\mathfrak{n}; P(\mathbb{I}))$

For $\mathbb{I}$-adic form, we also define the notion of normalized:

Definition 4.4.6. Let $\mathscr{F} \in \mathscr{M}(\mathfrak{n};\mathbb{I})$. We call $\mathscr{F}$ is normalized (at d^{-1}) if

$$\mathscr{F}(d^{-1}) = 1.$$

Theorem 4.4.7. Assume that $\mathbb{I}$ is domain. The space $\mathscr{S}^{n,ord}(\mathfrak{n};\mathbb{I})$ is torsion free finitely generated $\mathbb{I}$-module.

Proof. We only prove that $\mathscr{S}^{n,ord}(\mathfrak{n};\mathbb{I})$ is finitely generated. Let $\mathfrak{m}_{\mathbb{I}}$ be the maximal ideal of $\mathbb{I}$ and let $r := \dim_{\mathbb{I}/\mathfrak{m}_{\mathbb{I}}} \mathbf{h}^{n,\mathrm{ord}}(\mathfrak{n};\mathcal{O}) \otimes_{\mathcal{O}[[\mathbf{G}]]} \mathbb{I}/\mathfrak{m}_{\mathbb{I}}$. By the duality, for any $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$we have

$$\mathrm{rank}_{P(\mathbb{I})} \left(e_p \mathscr{S}_{k_P,w_P}(K_1(\mathfrak{n}, \mathbf{p}^{s(\omega_P,\omega'_P)}), \omega_P, \omega'_P; P(\mathbb{I}) \right) \leqslant r.$$

Let $f_1, \ldots, f_t \in \mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n};\mathbb{I})$ be elements which are linearly independent over $\mathbb{L}$. There exists $y_1, \ldots, y_t \in \mathfrak{I}$ such that $\Delta := \det((f_i(y_j))_{i,j}) \in \mathbb{L}^{\times}$. We write

$$\Delta = a/b$$

for some $a, b \in \mathbb{I}$. By (4.4.1), there exists $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$ not containing $ab \neq 0$. Thus $P \circ f_1, \ldots P \circ f_t \in e_p\mathscr{S}_{k,w}(K_1(\mathfrak{n}, \mathbf{p}^s(\omega)) \otimes_{\mathcal{O}} \mathrm{Frac}(\mathcal{O})$ are linearly indepenedent. Thus we have $t \leqslant r$. We can take a finite basis $f_1, \ldots, f_t$ of $\mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n};\mathbb{I}) \otimes_{\mathbb{I}} \mathbb{L}$ with $t \leqslant r$. We define Δ as above. Since

$$\mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n};\mathbb{I}) \subset \Delta^{-1}(\mathbb{I}f_1 + \cdots + \mathbb{I}f_t),$$

and $\mathbb{I}$ is noetherian, we have the theorem. □

Theorem 4.4.8. If $\mathbb{I}$ is sufficiently large integrally closed domain, for any $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$, we have

$$\mathscr{S}^{n,ord}(\mathfrak{n};\mathbb{I}) \xrightarrow{P_*} e_p\mathscr{S}^{n,ord}_{k_P,w_P}(\mathfrak{n}p^{s(\omega_P)}, \omega_P, \omega'_P; P(\mathbb{I})) \tag{4.4.5}$$

is surjective.

Proof. Let $f \in e_p\mathscr{S}^{n,ord}_{k_P,w_P}(\mathfrak{n}p^{s(\omega_P)}, \omega_P, \omega'_P; P(\mathbb{I}))$. By [**?**, Corollary 2.2], the local vector at p of F comes from the one dimentional eigen space of $T(\varpi)$, thus it suffice to prove that for any vector $f \in e_p\mathscr{S}^{n,ord}_{k_P,w_P}(\mathfrak{n}p^{s(\omega_P)}, \omega_P, \omega'_P; P(\mathbb{I}))$ which is new out side p, there exists an element $\mathscr{F} \in \mathscr{S}^{n,ord}(\mathfrak{n};\mathbb{I})$ such that

$$P_*(\mathscr{F}) = f.$$

It follows from the same argument preceding [**Wi88**, Theorem 1.4.1]. □

4.4.2. Hecke operators on $\mathbb{I}^{\mathfrak{I}}$. We define the following action on $\mathbb{I}^{\mathfrak{I}}$ as follows: let $F \in \mathbb{I}^{\mathfrak{I}}$. For $\mathfrak{q}$ being prime to $\mathfrak{n}p$, we define

$$\begin{aligned} F|T_0(\pi_{\mathfrak{q}})(y) &:= F(\pi_{\mathfrak{q}} y)\mathbf{1}_{\{|yd|_{\mathbb{A}_F} \leqslant 1\}}(y) + \epsilon_{\mathrm{cyc}}(\mathfrak{q})^{-1}([\mathfrak{q}], 1)F(\pi_{\mathfrak{q}}^{-1} y), \\ T(u_{\mathfrak{q}}) &:= 1, \end{aligned}$$

where $u_{\mathfrak{q}} \in \mathcal{O}_{F_{\mathfrak{q}}}^{\times}$, $\pi_{\mathfrak{q}}$ is a prime element of $\mathcal{O}_{F_{\mathfrak{q}}}$ and $(\pi_{\mathfrak{q}}, 1)$ is the element of $\mathbf{G}$. We note that $T_0(\pi_{\mathfrak{q}})$ is independent of the choice of $\pi_{\mathfrak{q}}$. For any $r > 0$, and $u \in \mathcal{O}_{F_{\mathfrak{q}}}^{\times}$, we define $T_0(\pi_{\mathfrak{q}}^r u)$ inductively by

$$T_0(\pi_{\mathfrak{q}}^r) := T_0(\pi_{\mathfrak{q}})T_0(\pi_{\mathfrak{q}}^{r-1}u) - \epsilon_{\mathrm{cyc}}(\mathfrak{q})^{-1}([\mathfrak{q}], 1)T_0(\pi_{\mathfrak{q}}^{r-2}u).$$

For $\mathfrak{q} \mid p\mathfrak{n}$ and $0 \neq x_{\mathfrak{q}} \in \mathcal{O}_{F_{\mathfrak{q}}}$, we define

$$F|T_0(x_{\mathfrak{q}})(y) := F(x_{\mathfrak{q}} y)\mathbf{1}_{\{|yd|_{\mathbb{A}_F} \leqslant 1\}}(y).$$

For $x \in \widehat{\mathcal{O}_F} \cap \mathbb{A}_{F,f}^{\times}$, we define

$$T_0(x) := \prod_{\mathfrak{q}} T_0(x_{\mathfrak{q}}).$$

For $a, b \in \widehat{\mathcal{O}_F}_{p\mathfrak{n}}^{\times}$, we define

$$F|T(a,b)(y) = (b,1)F(yab^{-1}).$$

Proposition 4.4.9. Let $F \in \mathscr{M}(\mathfrak{n}; \mathbb{I})$. For any $P \in \mathscr{X}(\mathbb{I})$, we have

$$\begin{aligned} P \circ (F|T_0(x)) &= (P \circ F)|T_0(x), \\ P \circ (F|T(a,b)) &= (P \circ F)|T(a,b). \end{aligned}$$

PROOF. It follows from Proposition 4.1.17. □

We give topologies defined by the operator norm to $\mathrm{End}_{\mathbb{I}}(\mathscr{M}(\mathfrak{n}; \mathbb{I}))$ and $\mathrm{End}_{\mathbb{I}}\left(\mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n}; \mathbb{I})\right)$. We define $\mathbb{I}$-adic Hecke algebra by

$$\begin{aligned} \mathbf{H}(\mathfrak{n}; \mathbb{I}) &:= \overline{\mathbb{I}\left[\{T_0(x)\}_{x \in \mathbb{A}_{F,f}^{\times} \cap \widehat{\mathcal{O}_F}}, \{T(a,b)\}_{a,b \in \widehat{\mathcal{O}_F}_{p\mathfrak{n}}^{\times}}\right]} \subset \mathrm{End}_{\mathbb{I}}(\mathscr{M}(\mathfrak{n}; \mathbb{I})), \\ \mathbf{h}^{n,ord}(\mathfrak{n}; \mathbb{I}) &:= \overline{\mathbb{I}\left[\{T_0(x)\}_{x \in \mathbb{A}_{F,f}^{\times} \cap \widehat{\mathcal{O}_F}}, \{T(a,b)\}_{a,b \in \widehat{\mathcal{O}_F}_{p\mathfrak{n}}^{\times}}\right]} \subset \mathrm{End}_{\mathbb{I}}\left(\mathscr{S}^{n,ord}(\mathfrak{n}; \mathbb{I})\right). \end{aligned}$$

Then we have the following corollary:

Corollary 4.4.10. The ordinary idempotent

$$e_{\mathfrak{p}} := \lim_{n \to \infty} T_0(\varpi_{\mathfrak{p}})^{n!}$$

is exists for each $\mathfrak{p} \mid p$. Put $e_p := \prod_{\mathfrak{p} \mid p} e_{\mathfrak{p}}$, then we have

$$P \circ e_p = e_p \circ P$$

for any $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$

The relation between the universal Hecke algebra and $\mathbb{I}$-adic Hecke algebra is as follws:

Theorem 4.4.11. Let $\mathbb{I}$ be a sufficiently large integrally closed domain as in Theorem 4.4.8. There exists a canonical surjection

$$j\colon \mathbf{h}^{n,\mathrm{ord}}(\mathfrak{n}; \mathcal{O}) \otimes_{\mathcal{O}[[\mathbf{G}]]} \mathbb{I} \longrightarrow \mathbf{h}^{n,\mathrm{ord}}(\mathfrak{n}; \mathbb{I})$$

such that $T_0(x)$ and $T(a,b)$ of each side correspond to each other.

PROOF. By Lemma 2.2.16, Theorem 4.4.8 and Proposition 4.4.9, there are a natural injection

$$\mathbf{h}^{n,\mathrm{ord}}(\mathfrak{n};\mathbb{I}) \hookrightarrow \prod_{P\in\mathscr{X}^{\mathrm{arith}}(\mathbb{I})} e_p h_{k_P,w_P}(K_1(\mathfrak{n},\mathbf{p}^{s(\omega_P)}),\omega_P;P(\mathbb{I})]),$$

and a natural homomorphism

$$\mathbf{h}^{n,\mathrm{ord}}(\mathfrak{n};\mathcal{O})\otimes_{\mathcal{O}[[\mathbf{G}]]}\mathbb{I} \longrightarrow \prod_{P\in\mathscr{X}^{\mathrm{arith}}(\mathbb{I})} e_p h_{2\underline{t},\underline{t}}(K_1(\mathfrak{n},\mathbf{p}^{s(\omega)}),\omega_P;P(\mathbb{I})).$$

The images of the homomorphisms above are the same. Thus we have the Theorem. □

By Theorem 4.4.11, we have a paring between nearly ordinary $\mathbb{I}$-adic cusp forms and universal Hecke algebra as for $F\in\mathscr{S}^{n,ord}(\mathfrak{n};\mathbb{I})$ and $T\in \mathrm{h}^{n,\mathrm{ord}}(\mathfrak{n};\mathcal{O})$,

$$\langle F,T\rangle := F|j(T)(d^{-1})\in\mathbb{I}. \tag{4.4.6}$$

Clearly, for any $\mathscr{X}^{\mathrm{arith}}(\mathbb{I})$, we have

$$P(\langle F,T\rangle) = \langle P\circ F, T \bmod P\rangle \in P(\mathbb{I}).$$

By duarlity for usual modular form, the paring induce the follwing isomorphism:

$$\mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n};\mathbb{I}) \cong \mathrm{Hom}_{\mathbb{I}}\left(\mathbf{h}^{n,\mathrm{ord}}(\mathfrak{n};\mathcal{O})\otimes_{\mathcal{O}[[\mathbf{G}]]}\mathbb{I},\mathbb{I}\right). \tag{4.4.7}$$

4.4.3. Trace operators. Let $\mathfrak{m}$ be a divisor of $\mathfrak{n}$. For $f\in\mathscr{M}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega')$, the natural homomorphism

$$\mathrm{Tr}_{\mathfrak{n}/\mathfrak{m}}f(g) := \frac{1}{[K_1(\mathfrak{m},\mathbf{p}^s):K_1(\mathfrak{n},\mathbf{p}^s)]}\sum_{u\in K_1(\mathfrak{m},\mathbf{p}^s)/K_1(\mathfrak{n},\mathbf{p}^s)} f(gu).$$

preserve the integrality, namely, let $A := [K_1(\mathfrak{m},\mathbf{p}^s):K_1(\mathfrak{n},\mathbf{p}^s)]$, we get the trace operator

$$\mathrm{Tr}_{\mathfrak{n}/\mathfrak{m}}\colon \mathscr{M}_{k,w}(K_1(\mathfrak{n},\mathbf{p}^s),\omega,\omega';\mathcal{O}) \longrightarrow A^{-1}\mathscr{M}_{k,w}(K_1(\mathfrak{m},\mathbf{p}^s),\omega,\omega';\mathcal{O}).$$

Since the trace operator commutes with $T_0(\varpi_{\mathfrak{p}})$ for $\mathfrak{p}\mid p$ and preserves the space of cuspforms, we have

$$\mathrm{Tr}_{\mathfrak{n}/\mathfrak{m}}\colon \overline{\mathscr{S}}^{n,\mathrm{ord}}(\mathfrak{n};\mathcal{O}) \longrightarrow A^{-1}\overline{\mathscr{S}}^{n,\mathrm{ord}}(\mathfrak{m};\mathcal{O}).$$

By Theorem 4.2.7, we have

$$\mathrm{Tr}^*_{\mathfrak{n}/\mathfrak{m}}\colon A^{-1}\mathbf{h}^{n,\mathrm{ord}}(\mathfrak{m};\mathcal{O}) \longrightarrow \mathbf{h}^{n,ord}(\mathfrak{n};\mathcal{O}).$$

By taking $\mathrm{Hom}_{\mathbb{I}}((\cdot)\otimes_{\mathcal{O}[[\mathbf{G}]]}\mathbb{I},\mathbb{I})$ and (4.4.7), we finally obtain the $\mathbb{I}$-adic trace operator:

$$\mathrm{Tr}^{\mathbb{I}}_{\mathfrak{n}/\mathfrak{m}}\colon \mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n};\mathbb{I}) \longrightarrow A^{-1}\mathscr{S}^{n,\mathrm{ord}}(\mathfrak{m};\mathbb{I}). \tag{4.4.8}$$

For any $P\in\mathscr{X}^{\mathrm{arith}}(\mathbb{I})$, we have

$$P\circ\mathrm{Tr}_{\mathfrak{m}/\mathfrak{n}} = \mathrm{Tr}_{\mathfrak{m}/\mathfrak{n}}\circ P.$$

4.4.4. The restriction maps for $\mathbb{I}$-adic forms. We define the $\mathbb{I}$-adic version of the restriction map defined in Section 4.2.4. In this section, we need to distinguish the base field F for each notion of Hilbert modular forms for GL_2/F, for example, the set of embeddings from F into $\mathbb{C}_p$, the space of modular forms and so on. Thus, for each symbol X relating to the field F, we denote by X_F, for example,

$$\begin{aligned} I_F &:= \{\sigma\colon F \hookrightarrow \mathbb{C}_p : \text{ field embedding}\}, \\ \mathscr{M}(\mathfrak{n};\mathbb{I})_F &:= \begin{pmatrix} \text{the space of } \mathbb{I}\text{-adic modular forms associated with} \\ \text{Hilbert modular forms over } \mathrm{GL}_2/F. \end{pmatrix}, \\ \mathfrak{I}_F &:= \mathrm{Cl}_F^+(\mathfrak{n}p^\infty) \sqcup \mathbb{A}_{F,f}^\times/K_F(\mathfrak{n})^{(p)}, \\ d_F &: \text{ the element of } \mathbb{A}_{F,f}^\times \text{ such that } d_F\mathcal{O}_F = \mathcal{D}_{F/\mathbb{Q}}. \end{aligned}$$

Let E/F be a extension of totally real fields. We assume that the fixed ring $\mathcal{O}$ is containing all the conjugation of $\mathcal{O}_E$. Let $N \subset \mathcal{O}_E$ and $\mathfrak{n} \subset \mathcal{O}_F$ be nonzero ideals such that $\mathfrak{n} = N \cap \mathcal{O}_F$ and both of them are prime to p.

Definition 4.4.12. Let $\mathscr{G} \in \mathscr{M}(N;\mathbb{I}_E)_E$. We define a element $\mathrm{Res}_{E/F}\mathscr{G} \in \mathscr{M}(\mathfrak{n};\mathbb{I}_E)_F$ by

$$\mathrm{Res}_{F_2/F_1}\mathscr{G}(y) := \epsilon_{cyc,F}^{1-[E:F]}(yd_F) \sum_{\substack{\eta\in(F_2)_+ \\ \mathrm{Tr}_{E/F}(\eta)=1}} \mathscr{G}(\eta y).$$

Note that under the support of $\mathscr{G}$ is bounded , the sum $\sum_{\substack{\eta\in E_+ \\ \mathrm{Tr}_{E/F}(\eta)=1}} \mathscr{G}(\eta y)$ is finite sum.

Then we have the following proposition:

Proposition 4.4.13. For $\mathscr{G} \in \mathscr{M}(N;\mathbb{I}_E)_E$ and $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I}_E)$, we have the following formula:

$$\begin{aligned} P \circ \mathrm{Res}_{E/F}(\mathscr{G}) &= \mathrm{Res}_{E/F}(P \circ \mathscr{G}) \\ &\in \mathscr{M}_{k_P|_F, w_P|_F}\left(\mathfrak{n}, \omega_P|_{\mathbb{A}_F^\times/F^\times}, \omega'_P|_{\mathcal{O}_{F_p}^\times}\right). \end{aligned}$$

PROOF. It follows from direct computations. □

4.5. Deformations of differential operators

In this section, we construct a homomorphism Θ which is important for constructing p-adic L-functions.

Definition 4.5.1. Let $\mathscr{G} \in \mathscr{M}(\mathfrak{n}_2;\mathbb{I}_2)_{F_2}$. We define $\Theta(\mathscr{G}) \in \overline{\mathscr{M}}(\mathfrak{n}_2;\mathbb{I}_1\otimes\mathbb{I}_2)_{F_2}$ ($\Theta(\mathscr{G})$ is actually the element by Theorem 4.2.12) by

$$\Theta(\mathscr{G})(y) := \begin{cases} \left(\left(\langle yy_p^{-1}\rangle^{\frac{1}{2}}, d_py_p\right), \left(\langle yy_p^{-1}\rangle^{-\frac{1}{2}}, (d_py_p)^{-1}\right)\right)\mathscr{G}(y) & \text{if } |y_pd_p|_{\mathbb{A}_F} = 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $\left(\left(\langle yy_p^{-1}\rangle^{\frac{1}{2}}, d_py_p\right), \left(\langle yy_p^{-1}\rangle^{-\frac{1}{2}}, (d_py_p)^{-1}\right)\right)$ is an element of $\mathbf{G}_{F_2}\times\mathbf{G}_{F_2} \subset (\mathbb{I}_1\otimes\mathbb{I}_2)^\times$ and we regards $\mathrm{Cl}_F^+(\mathfrak{n}p^\infty)$ as a quotient of $\mathbb{A}_F^\times$ via (4.2.1)

Then we have the following interpolation formula for $(\mathrm{Res}_{F_2/F_1}\Theta(\mathscr{G}))|e_p$ as follows:

Theorem 4.5.2. Let $\mathscr{G} \in \mathscr{M}(\mathfrak{n}_2; \mathbb{I}_2)_{F_2}$, For any arithmetic $P \otimes Q \in \mathscr{X}(\mathbb{I}_1 \widehat{\otimes}_{\mathcal{O}} \mathbb{I}_2)$ with $w_P - w_Q \geqslant 0$, we have

$$P \circ (\operatorname{Res}_{F_2/F_1} \Theta(\mathscr{G}))|e_p$$
$$= \operatorname{Res}_{F_2/F_1} \left(D^{w_P - w_Q} \theta^{(p)}_{(\omega_P \omega_Q^{-1})^{-\frac{1}{2}} \omega'_P \omega'_Q{}^{-1}} (P \circ \mathscr{G}) \right) \Big| e_p \otimes \left(\omega_P \omega_Q^{-1}\right)\Big|^{\frac{1}{2}}_{\mathbb{A}_{F_1}/F_1^\times} |\cdot|^{\alpha_P - \alpha_Q}_{\mathbb{A}_{F_1}}$$

Here, $\alpha_P = [2w_P - k_P]$, $\alpha_Q = [2w_Q - k_Q]$ and we define

$$(\omega_1 \omega_2^{-1})^{1/2} := (\omega_1 \omega_2^{-1} \tau_{F_2}^{\alpha_Q - \alpha_P})^{1/2} \tau_{F_2}^{\alpha_P - \alpha_Q}.$$

For the notations, see Section 4.2.3. Since we assume p is odd now, $(\omega_P^{-1} \omega_Q)^{\frac{1}{2}}$ is well-defined. Moreover, let g_Q be a nearly ordinary cusp form obtained by specialization of $\mathscr{G}$ at Q. Then there exists a nearly ordinary cusp form

$$h_{P,Q} := \mathcal{H} \left(\left(R^{w_P - w_Q} \widetilde{\theta}^{(p)}_{(\omega_P \omega_Q^{-1})^{-\frac{1}{2}} \omega'_P \omega'_Q{}^{-1}} g_Q \right) \Big|_{\mathrm{GL}_2(\mathbb{A}_{F_1})} \Big| e_p \right) \otimes (\omega_P \omega_Q^{-1})^{\frac{1}{2}} \Big|_{\mathbb{A}_{F_1}/F_1^\times} |\cdot|^{\alpha_P - \alpha_Q}_{\mathbb{A}_{F_1}}$$

of

$$\text{weight } \left(k_Q|_{F_1} + 2w_P|_{F_1} - 2w_Q|_{F_1}, w_P|_{F_1} + (\alpha_P - \alpha_Q)/2\right),$$
$$\text{character } (\omega_P, \omega'_P),$$

such that

$$\left((\operatorname{Res}_{F_2/F_1} \Theta(\mathscr{G}))|e_p\right)_P = |d_{F_1}|_{\mathbb{A}_{F,f}} (\omega_P \omega_Q^{-1})^{\frac{1}{2}} (d_{F_1}) (d_{F_2/F_1})_p^{w_Q - t_{F_2}} a(\cdot, h_{P,Q}),$$

where $d_{F_2/F_1} = d_{F_2} d_{F_1}^{-1}$.

Proof. It follows by direct computation by using Lemma 4.2.14 and [**Hi91**, Proposition 7.3]. ☐

Corollary 4.5.3. For any $\mathscr{G} \in \mathscr{S}(\mathfrak{n}_2; \mathbb{I}_2)_{F_2}$, The element $(\operatorname{Res}_{F_2/F_1} \Theta(\mathscr{G}))|e_p$ is an element of $\mathscr{S}^{n,\mathrm{ord}}(\mathfrak{n}_2 \cap \mathcal{O}_{F_1}; \mathbb{I}_2)_{F_1}$.

CHAPTER 5

Integral formulas for computing local period integrals

5.1. An integral formula for triple local integrals and Rankin-Selberg local integrals

In this section, F_2/F_1 denotes a quadratic extension of finite extension fields over $\mathbb{Q}_p$ for a prime number p. We fix a generator $\varpi_{F_i} \in \mathcal{O}_{F_i}$ of the maximal ideal $\mathcal{O}_{F_i}$. When F_2 is an unramified field extension over F_1, we take $\varpi_{F_2} = \varpi_{F_1}$. We put $q_i := \#\mathcal{O}_{F_i}/\varpi_{F_i}\mathcal{O}_{F_i}$. For $x \in F_2$, we denote by $\overline{x}$ the image of x under the non-trivial automorphism of F_2 over F_1.

We fix a non-trivial additive character of F_1

$$\psi\colon F_1 \longrightarrow \mathbb{C}^\times.$$

For $\xi \in F_2^\times$ with $\mathrm{tr}_{F_2/F_1}(\xi) = 0$, we define an additive character of F_2

$$\psi_\xi : F_2 \longrightarrow \mathbb{C}^\times;\ x \mapsto \psi(\mathrm{tr}_{F_2/F_1}(x\xi)),$$

which is trivial on F_1. Note that the correspondence

$$\xi \mapsto [x \mapsto \psi(\mathrm{tr}_{F_2/F_1}(x\xi))]$$

gives a bijection between the set of elements of F_2 with trace-zero and that of additive characters of F_2 which are trivial on F_1. A generator of the conductor of ψ_ξ is given by $\varpi^{c(\psi)}\xi^{-1}\mathcal{D}_{F_2/F_1}^{-1}$, where $c(\psi)$ is the exponent of the conductor of ψ, and $\mathcal{D}_{F_2/F_1}$ is a generator of the different ideal of F_2/F_1. Note that the conductor of ψ_ξ has a form of $\varpi_{F_1}^r\mathcal{O}_{F_2}$ for some integer r

Let π_2 be an irreducible admissible representation of $\mathrm{GL}_2(F_2)$ with central character ω_2. For $\xi \in F_2^\times$ with $\mathrm{tr}_{F_2/F_1}(\xi) = 0$, let $\mathscr{W}(\pi_2, \psi_\xi)$ be the Whittaker model of π_2 associated with ψ_ξ. For any non-archimedean local field L and irreducible admissible representation π of $\mathrm{GL}_2(L)$, we define $\lambda(\pi) \in \mathbb{R}_{\geqslant 0}$ by

$$\lambda(\pi) := \begin{cases} 0 & \text{if } \pi \text{ is temperd,} \\ \max\{\,|\mathrm{Re}(\lambda_1)|,\ |\mathrm{Re}(\lambda_2)|\,\} & \text{if } \pi \cong \pi(\chi_1|\cdot|_L^{\lambda_1}, \chi_2|\cdot|_L^{\lambda_2}) \text{ is a principal series,} \end{cases}$$

where χ_1, χ_2 are unitary characters. For any quasi-character $\eta : L^\times \to \mathbb{C}^\times$, we define

$$\lambda(\eta) = \frac{\log|\eta(\pi_L)|_L}{\log(q_L)},$$

where π_L is a uniformizer of $\mathcal{O}_L$ and q_L is the order of the residue field of L.

Fix $\xi \in F_2^\times$ with $\mathrm{tr}_{F_2/F_1}(\xi) = 0$. For Whittaker functions $W \in \mathscr{W}(\pi_2, \psi_\xi)$ and $\widetilde{W} \in \mathscr{W}(\pi_2^\vee, \psi_\xi)$, we define

$$\langle W, \widetilde{W}\rangle := \int_{F_2^\times} W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \widetilde{W}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} d_{F_2}^\times a,$$

$$\Phi_{W,\widetilde{W}}(g) := \langle gW, \widetilde{W}\rangle.$$

Let $G := \mathrm{PGL}_2(F_1)$ and $K := \mathrm{PGL}_2(\mathcal{O}_{F_1})$. Let $\mu, \nu : F_1^\times \to \mathbb{C}^\times$ be quasi-characters such that $(\omega_2|_{F_1^\times})\mu\nu$ is trivial on $F_1^\times$. For $f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$ and $\widetilde{f} \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu^{-1} \boxtimes \nu^{-1})$, we define

$$\langle f, \widetilde{f} \rangle_0 := \int_K f(k)\widetilde{f}(k)\, dk,$$
$$\Phi_{f,\widetilde{f}}(g) := \langle gf, \widetilde{f} \rangle_0.$$

Here, the dk is the invariant measure on G satisfying $\mathrm{vol}(K, dk) = 1$, and $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$ is the induction normalized by the modulus character of $B(F_1) \subset \mathrm{GL}_1(F_1)$, which is the subgroup of upper triangular matrices.

For $W \in \mathscr{W}(\pi_2, \psi_\xi)$ and $f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$, we define

$$\Psi\bigl(W, f\bigr) := \int_{N\backslash G} W(g) f(g)\, dg,$$
$$\widetilde{\Psi}\bigl(W, f\bigr) := \int_{N\backslash G} W(\eta g) f(g)\, dg,$$

where $\eta := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and $N := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ is the subgroup composed of unipotent upper triangular matrices.

For $W \in \mathscr{W}(\pi_2, \psi_\xi)$, $\widetilde{W} \in \mathscr{W}(\pi_2^\vee, \psi_\xi)$, $f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$, and $\widetilde{f} \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu^{-1} \boxtimes \nu^{-1})$, we put

$$I\bigl(W \boxtimes f, \widetilde{W} \boxtimes \widetilde{f}\bigr) := \int_G \Phi_{W,\widetilde{W}}(g) \Phi_{f,\widetilde{f}}(g)\, dg.$$

Theorem 5.1.1. Assume

$$\Lambda := \max\{\, \lambda(\mu),\, \lambda(\nu)\,\} + 2\lambda(\pi_2) < \frac{1}{2}.$$

Then the integrals $I\bigl(W \boxtimes f, \widetilde{W} \boxtimes \widetilde{f}\bigr)$, $\Psi\bigl(W, f\bigr)$, and $\Psi\bigl(\widetilde{W}, \widetilde{f}\bigr)$ converge absolutely, and we have the following equality:

$$I\bigl(W \boxtimes f, \widetilde{W} \boxtimes \widetilde{f}\bigr) = |\xi \mathcal{D}_{F_2/F_1}|_{F_2}^{-1/2} \frac{\zeta_{F_2}(1)}{\zeta_{F_1}(1)}\, \Psi\bigl(W, f\bigr)\, \widetilde{\Psi}\bigl(\widetilde{W}, \widetilde{f}\bigr).$$

Proof. The absolute convergence of the above integrals is a consequence of the assumption $\Lambda < 1/2$ and [**Bu97**, Chapter 4, Proposition 4.7.2, Theorem 4.7.2 and Theorem 4.7.3].

We put $q := q_1$. We may assume ψ has a conductor $\mathcal{O}_{F_1}$. There exists $\xi \in F_2^\times$ with $\mathrm{tr}_{F_2/F_1}(\xi) = 0$ and

$$\psi(b) = \psi_\xi(a + b\theta)$$

for any $a, b \in F_1$, where $\theta \in \mathcal{O}_{F_2}$ is an element with $\mathcal{O}_{F_2} = \mathcal{O}_{F_1}[\theta]$. We note that the conductor of ψ_ξ is $\mathcal{O}_{F_2}$ and $|\xi \mathcal{D}_{F_2/F_1}|_{F_2} = 1$. We put

$$K_n := \bigcup_{i=0}^{n} K \begin{pmatrix} \varpi_{F_1}^i & 0 \\ 0 & 1 \end{pmatrix} K.$$

We note that

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in K_n \Leftrightarrow q^{-n} \leqslant |y|_{F_1} \leqslant q^n \text{ and } |x|_{F_1}^2 \leqslant q^n |y|_{F_1}.$$

Let $\varphi_n := \mathbf{1}_{K_n}$ be the characteristic function of K_n and let $\chi_n := \mathbf{1}_{\varpi_{F_1}^n \mathcal{O}_{F_1}}$ be the characteristic function of $\varpi_{F_1}^n \mathcal{O}_{F_1}$. We put

$$I_n := \int_G \Phi_{W,\widetilde{W}}(g)\Phi_{f,\widetilde{f}}(g)\varphi_{2n}(g)\,dg.$$

At first, we prove when F_2/F_1 is unramified. Formally, we have

$$\begin{aligned}
I_n &= \int_G \int_K f(kg)\widetilde{f}(k)\left\langle \pi_2(g)W,\, \widetilde{W}\right\rangle \varphi_{2n}(g)\,dk\,dg \\
&= \int_K \int_K \int_{F_1^\times} \int_{F_1} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k'\right)\widetilde{f}(k) \\
&\quad \times \left\langle \pi_2\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k'\right)W,\ \pi_2(k)\widetilde{W}\right\rangle \varphi_{2n}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} d_{F_1}x\,\frac{d_{F_1}^\times y}{|y|_{F_1}}\,dk'\,dk \\
&= \int_K \int_K f(k')\widetilde{f}(k) \int_{q^{-2n}\leqslant |y|_{F_1}\leqslant q^{2n}} \mu(y)|y|_{F_1}^{-1/2} \\
&\quad \times \int_{F_2^\times} W\left(\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix} k'\right)\widetilde{W}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} k\right) \\
&\quad \times \int_{F_1} \psi_\xi(ax)\,\chi_{-n+[\mathrm{ord}_{F_1}(y)/2]}(x)\,d_{F_1}x\,d_{F_2}^\times a\,d_{F_1}^\times y\,dk'\,dk.
\end{aligned}$$

Here, $[r]$ is the smallest integer with $[r]\geqslant r$. We put

$$\begin{aligned}
A(k',k) := &\int_{q^{-2n}\leqslant |y|_{F_1}\leqslant q^{2n}} \mu(y)|y|_{F_1}^{-1/2} \int_{F_2^\times} W\left(\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix} k'\right)\widetilde{W}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} k\right) \\
&\times q^{n-[\mathrm{ord}_{F_1}(y)/2]}\,\chi_{n-[\mathrm{ord}_{F_1}(y)/2]}\left(\frac{a-\overline{a}}{\theta-\overline{\theta}}\right) d_{F_2}^\times a\,d_{F_1}^\times y\,dk'\,dk.
\end{aligned}$$

Then we have

$$\begin{aligned}
I_n &= \int_K \int_K f(k')\widetilde{f}(k)A(k',k)\,dk'\,dk, \\
A(k',k) &= \sum_{m\in\mathbb{Z}} \int_{q^{-2n-m}\leqslant |y|_{F_1}\leqslant q^{2n-m}} \mu(\varpi_{F_1}^{-m}y)|\varpi_{F_1}^{-m}y|_{F_1}^{-1/2} \\
&\quad \times \int_{\mathcal{O}_{F_2}^\times} W\left(\begin{pmatrix} uy & 0 \\ 0 & 1 \end{pmatrix} k'\right)\widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m u & 0 \\ 0 & 1 \end{pmatrix} k\right) \\
&\quad \times q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\chi_{n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\left(\frac{u-\overline{u}}{\theta-\overline{\theta}}\right) d_{F_2}^\times u\,d_{F_1}^\times y.
\end{aligned}$$

Here ord_{F_1} is the additive valuation such that $\mathrm{ord}_{F_1}(\varpi_{F_1}) = 1$. Now, we focus on the integration

$$\begin{aligned}
J^{m,n} := &\int_{q^{-2n-m}\leqslant |y|_{F_1}\leqslant q^{2n-m}} \mu(y)|y|_{F_1}^{-1/2} \int_{\mathcal{O}_E^\times} W\left(\begin{pmatrix} uy & 0 \\ 0 & 1 \end{pmatrix} k'\right)\widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m u & 0 \\ 0 & 1 \end{pmatrix} k\right) \\
&\times q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\chi_{n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\left(\frac{u-\overline{u}}{\theta-\overline{\theta}}\right) d_{F_2}^\times u\,d_{F_1}^\times y.
\end{aligned}$$

To confirm the commutation of the sum and integrations above, we prove the uniformly convergence, namely, we claim that there exists a positive constant $C>0$ independent of

k, and k' such that

$$\begin{aligned}&\int_{q^{-2n-m}\leqslant|y|_{F_1}\leqslant q^{2n-m}}|\varpi_{F_1}^{-m}y|_{F_1}^{\lambda(\pi_1)-1/2}\int_{\mathcal{O}_E^\times}\left|W\left(\begin{pmatrix}uy&0\\0&1\end{pmatrix}k'\right)\widetilde{W}\left(\begin{pmatrix}-\varpi_{F_1}^m u&0\\0&1\end{pmatrix}k\right)\right|\\&\times q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\chi_{n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\left(\frac{u-\overline{u}}{\theta-\overline{\theta}}\right)d_{F_2}^\times u\,d_{F_1}^\times y\\&\leqslant Cmq^{-m(2\lambda(\pi_2)+\lambda(\pi_1)+\delta-1/2)}\end{aligned}$$

for any sufficiently small $\delta>0$, where the constant C depends on δ.

Let us prove the claim. We take a sufficiently small open compact normal subgroup $H\lhd \mathrm{GL}_2(\mathcal{O}_{F_2})$ such that

$$\pi_2(h)W=W,\quad \pi_2(h)\widetilde{W}=\widetilde{W}\quad\text{for any } h\in H.$$

Since

$$\begin{aligned}|W|&\leqslant\sum_{\sigma\in\mathrm{GL}_2(\mathcal{O}_{F_2})/H}|\pi_2(\sigma)W|,\\|\widetilde{W}|&\leqslant\sum_{\sigma\in\mathrm{GL}_2(\mathcal{O}_{F_2})/H}|\pi_2(\sigma)\widetilde{W}|,\end{aligned}$$

we may assume that $|W|$ and $|\widetilde{W}|$ are $\mathrm{GL}_2(\mathcal{O}_{F_2})$-invariant. Thus there exists $N>0$ and $C_1>0$ independent of m,n such that, for any $k\in\mathrm{GL}_2(\mathcal{O}_{F_2})$ and $z\in E^\times$, we have

$$\left|W\left(\begin{pmatrix}z&0\\0&1\end{pmatrix}k\right)\right|,\ \left|\widetilde{W}\left(\begin{pmatrix}z&0\\0&1\end{pmatrix}k\right)\right|\leqslant C_1|z|_{F_2}^{-\lambda(\pi_2)+\delta+1/2}\mathbf{1}_{|z|\leqslant q^N}\tag{5.1.1}$$

for any sufficiently small $\delta>0$. Here we use the result written in [**Bu97**, Chapter 4, Proposition 4.7.2, Theorem 4.7.2 and 4.7.3].

We divide the integration as

$$\begin{aligned}\int_{q^{-2n-m}\leqslant|y|_{F_1}\leqslant q^{2n-m}}&=\int_{\substack{q^{-2n-m}\leqslant|y|_{F_1}\leqslant q^{2n-m}\\ n-m\leqslant[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}}+\int_{\substack{q^{-2n-m}\leqslant|y|_{F_1}\leqslant q^{2n-m}\\ n-m>[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}}\\&=I_1^{m,n}+I_2^{m,n}.\end{aligned}$$

Here, we denote by $I_1^{m,n}$ the first integration and by $I_2^{m,n}$ the second integration. For $I_1^{m,n}$, by (5.1.1), there exists $C_2>0$ which is independent of m,n such that

$$\begin{aligned}I_1^{m,n}&\leqslant C_2mq^{(2\lambda(\pi_2)+\lambda(\pi_1)+\delta-3/2)m}\\&\quad\times\int_{\substack{q^{-2n-m}\leqslant|y|_{F_1}\leqslant q^N\\ n-m\leqslant[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}}|y|_{F_1}^{-2\lambda(\pi_2)+\lambda(\pi_1)+1/2}|\mathrm{ord}_{F_1}(y)|\,q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\,d_{F_1}^\times y\\&\leqslant\left(C_2\int_{|y|_{F_1}\leqslant\min\{q^N,q^{-2n+m+1}\}}|y|_{F_1}^{-2\lambda(\pi_2)+\lambda(\pi_1)+1/2}|\mathrm{ord}_{F_1}(y)|\,d_{F_1}^\times y\right)mq^{m(2\lambda(\pi_2)+\lambda(\pi_1)+\delta-1/2)}\end{aligned}$$

We note that by this formula, we have

$$I_1^{m,n}\to 0\qquad\text{if } n\to\infty.\tag{5.1.2}$$

For $I_2^{m,n}$, we have

$$
\begin{aligned}
I_2 = & \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{2n-m} \\ n-m > [\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}} |\varpi_{F_1}^{-m} y|_{F_1}^{-1/2+\lambda(\pi_1)} \int_{\mathcal{O}_E^\times} \left| W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k' \right) \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m & 0 \\ 0 & 1 \end{pmatrix} k \right) \right| \\
& \times q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]} \chi_{n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]} \left(\frac{u-\overline{u}}{\theta-\overline{\theta}} \right) d_{F_2}^\times u \, d_{F_1}^\times y \\
= & \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{2n-m} \\ n-m > [\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}} |\varpi_{F_1}^{-m} y|_{F_1}^{-1/2+\lambda(\pi_1)} \\
& \times \int_{\mathcal{O}_F^\times \times \varpi_{F_1}^{n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]} \mathcal{O}_F} d_{F_1}u_1 \, d_{F_1}u_2 \left| W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k' \right) \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m & 0 \\ 0 & 1 \end{pmatrix} k \right) \right| \\
& \times q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]} d_{F_1}^\times y, \\
= & \left(1+\frac{1}{q}\right)^{-1} q^m \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{2n-m} \\ n-m > [\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}} |\varpi_{F_1}^{-m} y|_{F_1}^{-1/2+\lambda(\pi_1)} \\
& \times \left| W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k' \right) \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m & 0 \\ 0 & 1 \end{pmatrix} k \right) \right| d_{F_1}^\times y.
\end{aligned}
$$

By using (5.1.1), we have

$$
\begin{aligned}
I_2^{m,n} \leqslant \Bigg(& \left(1+\frac{1}{q}\right)^{-1} C_1 \int_{|y|_{F_1} \leqslant q^N} \mathrm{ord}_{F_1}(y) \, |y|^{-2\lambda(\pi_2)+\lambda(\pi_1)+1/2} |y| \, d_{F_1}^\times y \Bigg) \\
& \times m q^{m(2\lambda(\pi_2)+\lambda(\pi_1)+\delta-1/2)}
\end{aligned}
$$

for any sufficiently small $\delta > 0$. We have proved the claim.

As above, we divide the integration $J^{m,n}$ as

$$
\begin{aligned}
J^{m,n} &= \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{2n-m} \\ n-m \leqslant [\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}} + \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} R \leqslant q^{2n-m} \\ n-m > [\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}} \\
&= J_1^{m,n} + J_2^{m,n},
\end{aligned}
$$

where we denote by $J_1^{m,n}$ the first integration and by $J_2^{m,n}$ the second integration.

To prove the formula of Theorem 5.1.1, it suffices to prove that

$$
\begin{aligned}
& J_1^{m,n} \underset{n\to\infty}{\longrightarrow} 0 \\
& J_2^{m,n} \underset{n\to\infty}{\longrightarrow} \frac{\zeta_{F_2}(1)}{\zeta_F(1)} \int_{F^\times} \mu(y)|y|^{-1/2} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k' \right) d_F^\times y, \\
& \qquad\qquad \times \int_{\mathcal{O}_F^\times} |\varpi_{F_1}|^{-m/2} \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m u & 0 \\ 0 & 1 \end{pmatrix} k \right) d_F^\times u.
\end{aligned}
$$

By (5.1.2) and $|J_1^{m,n}| \leqslant I_1^{m,n}$, we immediately have the first assertion. For the second assertion, by the same calculation as $I_2^{m,n}$, we have

$$\begin{aligned}
&J_2^{m,n}\\
&= \zeta_{F_2}(1) \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{2n-m} \\ n-m > [\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}} |\varpi_{F_1}^{-m} y|_{F_1}^{-1/2}\\
&\times \int_{\mathcal{O}_F^\times} \int_{\varpi_{F_1}^{n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\mathcal{O}_F} W\left(\begin{pmatrix} y(u_1+u_2\theta) & 0 \\ 0 & 1 \end{pmatrix} k'\right) \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m(u_1+u_2\theta) & 0 \\ 0 & 1 \end{pmatrix} k\right)\\
&\times q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\, d_{F_1}u_2\, d_{F_1}u_1\, d_{F_1}^\times y.
\end{aligned}$$

Fix a sufficiently large $M > 0$ such that $gW = W$ and $g\widetilde{W} = \widetilde{W}$ for any $g \in \begin{pmatrix} 1+\varpi_{F_1}^M\mathcal{O}_E & 0 \\ 0 & 1 \end{pmatrix}$.

We divide $J_2^{m,n}$ as

$$\begin{aligned}
J_2^{m,n} &= \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{2n-m} \\ n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2] > M}} + \int_{\substack{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{2n-m} \\ M \geqslant n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2] > 0}}\\
&= (J_2^{m,n})' + (J_2^{m,n})''.
\end{aligned}$$

Here we denote by $(J_2^{m,n})'$ (resp. $(J_2^{m,n})''$) the first (resp. second) integration. For $(J_2^{m,n})'$, we have

$$\begin{aligned}
(J_2^{m,n})' =& \zeta_{F_2}(1) \int_{q^{-2n+m+2M} \leqslant |y|_{F_1} \leqslant q^{2n-m}} \mu(y)|y|_{F_1}^{-1/2}\\
&\times \int_{\mathcal{O}_F^\times} W\left(\begin{pmatrix} yu_1 & 0 \\ 0 & 1 \end{pmatrix} k'\right) d_{F_1}^\times y\, |\varpi_{F_1}|^{-m/2}\, \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m u_1 & 0 \\ 0 & 1 \end{pmatrix} k\right) d_{F_1}u_1\\
=& \frac{\zeta_{F_2}(1)}{\zeta_{F_1}(1)} \int_{q^{-2n+m+2M} \leqslant |y|_{F_1} \leqslant q^{2n-m}} \mu(y)|y|_{F_1}^{-1/2}\, W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k'\right) d_{F_1}^\times y\\
&\times \int_{\mathcal{O}_F^\times} |\varpi_{F_1}|_{F_1}^{-m/2}\, \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^m u_1 & 0 \\ 0 & 1 \end{pmatrix} k\right) d_{F_1}^\times u_1.
\end{aligned}$$

On the other hand, $(J_2^{m,n})''$ is estimated as follows (in the same manner as $I_2^{m,n}$)

$$\begin{aligned}
&|(J_2^{m,n})''|\\
&\leqslant \left(\left(1+\frac{1}{q}\right)^{-1} C_1 \int_{q^{-2n-m} \leqslant |y|_{F_1} \leqslant q^{-2n+m+2M+1}} \mathrm{ord}_{F_1}(y)\, |y|^{-2\lambda(\pi_1)+\lambda(\pi_1)+1/2}\, |y|\, d_{F_1}^\times y\right)\\
&\quad \times mq^{m(2\lambda(\pi_2)+\lambda(\pi_1)+\delta-1/2)}
\end{aligned}$$

Hence we have $(J_2^{m,n})'' \to 0$ $(n \to \infty)$. Therefore, The proof of Theorem 5.1.1 is complete when F_2/F_1 is unramified.

Finally, we shall prove Theorem 5.1.1 when F_2/F_1 is ramified. We take θ as ϖ_{F_2}. By the similar calculation as above, we have

$$A(k',k) = \sum_{m=0}^{\infty} J^{m,n,0} + \sum_{m=0}^{\infty} J^{m,n,1}. \tag{5.1.3}$$

where for $\varepsilon = 0, 1$, we put

$$J^{m,n,\varepsilon} := \int_{q^{-2n-m}\leqslant |y|_{F_1}\leqslant q^{2n-m}} \mu(y)|y|_{F_1}^{-1/2} \int_{\mathcal{O}_E^\times} W\left(\begin{pmatrix} u\varpi_{F_2}^{\varepsilon} y & 0 \\ 0 & 1 \end{pmatrix} k'\right) \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^{m}\varpi_{F_2}^{\varepsilon} u & 0 \\ 0 & 1 \end{pmatrix} k\right)$$
$$\times\, q^{n-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\, \chi_{n-m-[\mathrm{ord}_{F_1}(y\varpi_{F_1}^{-m})/2]}\left(\frac{\varpi_{F_2}^{\varepsilon} u - \overline{\varpi_{F_2}^{\varepsilon} u}}{\varpi_{F_2} - \overline{\varpi_{F_2}}}\right) d_{F_2}^\times u\, d_{F_1}^\times y.$$

We can prove there exists $C_3 > 0$ which is independent of m, n such that

$$J^{m,n,\varepsilon} < C_3 m q^{2\lambda(\pi_2)+\lambda(\pi_1)+\delta-1/2}$$

for any sufficiently small $\delta > 0$ in the same manner as before. Hence, the equality (5.1.3) makes sense. Since

$$\frac{\varpi_{F_2} u - \overline{\varpi_{F_2} u}}{\varpi_{F_2} - \overline{\varpi_{F_2}}}$$

is a unit, by the similar calculation as $J_1^{m,n}$, we have

$$J^{m,n,1} \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

For $J^{m,n,0}$, we can also apply the same method for the estimation of $J^{m,n}$ (actually, the calculation is slightly simpler since $a + b\varpi_{F_2} \in \mathcal{O}_{F_2}^\times$ if and only if $a \in \mathcal{O}_{F_1}^\times$), we have

$$J^{m,n,0} \underset{n\to\infty}{\longrightarrow} \frac{\zeta_{F_2}(1)}{\zeta_{F_1}(1)} \times \int_{F^\times} \mu(y)|y|_{F_1}^{-1/2} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k'\right) d_F^\times y$$
$$\int_{\mathcal{O}_F^\times} |\varpi_{F_1}|_{F_1}^{-m/2} \widetilde{W}\left(\begin{pmatrix} -\varpi_{F_1}^{m} u & 0 \\ 0 & 1 \end{pmatrix} k\right) d_F^\times u.$$

The proof of Theorem 5.1.1 is complete. □

5.2. Asai L-functions and its functional equation.

5.2.1. Asai L-functions and Rankin-Selberg integrals. The notations are the same as in the previous section. In this subsection, we define an Asai L-function using Rankin-Selberg integrals. Let π_2 be an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(F_2)$ with central character ω_2. As in previous section, we fix an element $\xi \in F_2^\times$ with $\mathrm{tr}_{F_2/F_1}(\xi) = 0$.

Let $\mathfrak{S}(F_1^2)$ be the space of Bruhat-Schwartz functions on F_1^2. For any $\Phi \in \mathfrak{S}(F_1^2)$ and $W \in \mathscr{W}(\pi_2, \psi_\xi)$, we define a function on $s \in \mathbb{C}$ by

$$Z(s, W, \Phi) := \int_{N(F_1)\backslash \mathrm{GL}_2(F_1)} W(g)\, \Phi((0,1)g)\, |\det(g)|_{F_1}^s\, dg,$$

where we normalize the invariant measure so that $\mathrm{vol}(\mathrm{GL}_2(\mathcal{O}_{F_1}), dg) = 1$. We note that $Z(s, W, \Phi)$ converges absolutely for sufficiently large $\mathrm{Re}(s)$, and is analytically continued to the whole complex plane as a meromorphic function. Moreover, it is an element of $\mathbb{C}[q_1^s, q_1^{-s}]$. The $\mathbb{C}$-vector space generated by $Z(s, W, \Phi)$'s for $W \in \mathscr{W}(\pi_2, \psi_\xi)$ and $\Phi \in \mathfrak{S}(F_1^2)$ is actually an ideal of $\mathbb{C}[q_1^s, q_1^{-s}]$. There exists $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$ such that this ideal is generated by $P(q_1^{-s})^{-1}$ (see [**Kab04**, p.801] or [**Fl93**, Appendix, Theorem]).

We define the *Asai L-function* by

$$L_{\mathrm{RS}}(s, \mathrm{As}\pi_2) := \frac{1}{P(q_1^{-s})}.$$

This function satisfies the following functional equation (see [**Kab04**, Theorem 3] or [**Fl93**, Appendix, Theorem]):

$$\frac{Z(1-s, W\otimes\omega_2^{-1}, \widehat{\Phi})}{L_{\mathrm{RS}}(1-s, \mathrm{As}\,\pi_2^\vee)} = \omega_2(-1)\varepsilon(s, \mathrm{As}\,\pi_2, \psi, \xi)\frac{Z(s, W, \Phi)}{L_{\mathrm{RS}}(s, \mathrm{As}\,\pi_2)}, \tag{5.2.1}$$

where there exists $c \in \mathbb{C}^\times$ and $m \in \mathbb{Z}$ depending only on π_2, ψ, and ξ such that

$$\varepsilon_{\mathrm{RS}}(s, \mathrm{As}\,\pi_2, \psi, \xi) := c q_1^{-ms},$$

and

$$\widehat{\Phi}(x, y) := \int_{F_1\times F_1} \Phi(u, v)\psi(uy - vx)\, du\, dv.$$

Here $du\, dv$ is the *self-dual* measure associated with $F_1 \times F_1 \to \mathbb{C}; (x, y) \mapsto \psi(x + y)$. We note that for any $a \in F_1^\times$, we have

$$\varepsilon_{\mathrm{RS}}(s, \mathrm{As}\,\pi_2, \psi^a, \xi) = \omega_2^2(a)\, |a|_{F_1}^{4s-2}\, \varepsilon_{\mathrm{RS}}(s, \mathrm{As}\,\pi_2, \psi, \xi),$$
$$\varepsilon_{\mathrm{RS}}(s, \mathrm{As}\,\pi_2, \psi, a\xi) = \omega_2(a)\, |a|_{F_1}^{2s-1}\, \varepsilon_{\mathrm{RS}}(s, \mathrm{As}\,\pi_2, \psi, \xi),$$

where $\psi^a(x) := \psi(ax)$.

There are other definitions of the Asai L-function. By applying the Langlands-Shahidi method ([**Sha90**]) to $\mathrm{U}(2, 2)$, we have another L-function whose inverse is an element of $\mathbb{C}[q_1^s, q_1^{-s}]$. We denote it by

$$L_{\mathrm{LS}}(s, \mathrm{As}\,\pi_2).$$

Moreover, let ρ_2 be the representation of the Weil-Deligne group of F_2 corresponding to π_2 via the local Langlands correspondence . We define the L-function

$$L_{\mathrm{Gal}}(s, \mathrm{As}\,\pi_2)$$

as the L-function for the multiplicative induction of ρ_2 (see [**Pra92**, Section 7]).

It is known that L_{RS}, L_{LS}, and L_{Gal} are the same by [**Hen10**, Section 1.5, Théorèm], [**Mat09**, Theorem 1.3], and [**AR05**, Theorem 1.6] (see also [**?**, Theorem 4.2] and the paragraph following it). Therefore, we denote

$$L(s, \mathrm{As}\,\pi_2) := L_{\mathrm{RS}}(s, \mathrm{As}\,\pi_2) = L_{\mathrm{LS}}(s, \mathrm{As}\,\pi_2) = L_{\mathrm{Gal}}(s, \mathrm{As}\,\pi_2).$$

5.2.2. Intertwining operators and functional equations. We discuss the relation between intertwining operators and functional equations. Put

$$\gamma(s, \mathrm{As}\,\pi_2, \psi, \xi) := \varepsilon_{\mathrm{RS}}(1/2, \mathrm{As}\,\pi_2, \psi, \xi)\, \frac{L(1-s, \mathrm{As}\,\pi_2^\vee)}{L(s, \mathrm{As}\,\pi_2)}.$$

For any quasi-character $\mu_0 : F_1^\times \longrightarrow \mathbb{C}^\times$, $g \in \mathrm{GL}_2(F_1)$, and $\Phi \in \mathfrak{S}(F_1^2)$, we define an element of

$$\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(|\cdot|_{F_1}^{s-1/2} \boxtimes \mu_0|\cdot|_{F_1}^{-s+1/2})$$

by

$$z\big(s,\, \mu_0,\, \Phi\big)(g) := |\det(g)|_{F_1}^s \int_{F_1^\times} (g\Phi)(0, t)\, \mu_0^{-1}(t)\, |t|_{F_1}^{2s}\, d_{F_1}^\times t.$$

By a direct computation, we have

$$\begin{aligned} Z\big(s,\, W,\, \Phi\big) &= \int_{N(F_1)\backslash \mathrm{PGL}_2(F_1)} W(g)\, z(s, \omega_2^{-1}, \Phi)(g)\, dg \\ &= \Psi\big(W,\, z\big(s,\, \omega_2^{-1},\, \Phi\big)\big). \end{aligned}$$

For quasi-characters $\mu, \nu : F_1^\times \to \mathbb{C}^\times$ with $\mu \neq \nu$, and for any element

$$h \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu),$$

we define

$$Mh \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\nu \boxtimes \mu)$$

by the analytic continuation of the following integral

$$Mh(g) := \int_{F_1} h\Big(\tau_1 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\Big)dx,$$

where $\tau_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; see [**Bu97**, Section 4.5]). Hence we have

$$M\colon \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu) \longrightarrow \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\nu \boxtimes \mu).$$

When $\mu\nu^{-1} \neq |\cdot|^{\pm 1}$, it becomes an isomorphism. When $\mu\nu^{-1} = |\cdot|$, it is a zero-homomorphism. When $\mu\nu^{-1} = |\cdot|^{-1}$, it induces an isomorphism from the irreducible quotient $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$ to the irreducible subspace of $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\nu \boxtimes \mu)$.

Lemma 5.2.1. Let $\mu, \nu\colon F_1^\times \longrightarrow \mathbb{C}^\times$ be quasi-characters such that $\mu \neq \nu$ and $(\omega_2|_{F_1^\times})\mu\nu$ is trivial on $F_1^\times$. We assume that

$$\lambda(\pi_2) + \max\{\lambda(\mu), \lambda(\nu)\} < 1/2.$$

For $W \in \mathscr{W}(\pi_2, \psi_\xi)$ and $f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$, we have

$$\widetilde{\Psi}\big(W,\, Mf\big) = \mu(-1)\frac{\varepsilon(0, \mathbf{1}, \psi)\, \gamma(1/2, \mathrm{As}\pi_2 \otimes \mu, \psi, \xi)}{\gamma(0, \mu\nu^{-1}, \psi)}\Psi\big(W,\, f\big),$$

where Ψ and $\widetilde{\Psi}$ are as in Section 5.1 and $\tau_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. Let

$$\mathfrak{S}(F_1^2)_0 := \big\{\, \Phi \in \mathfrak{S}(F_1^2) \;\big|\; \Phi(0,0) = 0 \,\big\}.$$

For any $\Phi \in \mathfrak{S}(F_1^2)_0$, the function $s \mapsto z(s, \mu, \Phi)$ is an entire function. The $\mathrm{GL}_2(F_1)$-invariant homomorphism

$$\mathfrak{S}(F_1^2)_0 \ni \Phi \mapsto z\big(1/2,\, \nu\mu^{-1},\, \Phi\big) \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mathbf{1} \boxtimes \nu\mu^{-1})$$

is surjective since this map is non-zero and the right hand side $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mathbf{1} \boxtimes \nu\mu^{-1})$ is an irreducible representation of $\mathrm{GL}_2(F_1)$. We take an element $\Phi \in \mathfrak{S}(F_1^2)_0$ satisfying

$$z\big(1/2,\, \nu\mu^{-1},\, \Phi\big) = f \otimes \mu^{-1}.$$

On the other hand, by [**Jac72**, the proof of Theorem 14.7], for $\Phi \in \mathfrak{S}(F_1^2)$, we have

$$z\big(1 - s,\, \mu\nu^{-1},\, \widehat{\Phi}\big) \otimes \nu\mu^{-1} = \nu\mu^{-1}(-1)\frac{\gamma(2s - 1, \mu\nu^{-1}, \psi)}{\varepsilon(0, \mathbf{1}, \psi)}Mz\big(s,\, \nu\mu^{-1},\, \Phi\big).$$

(Note that the measure used to define the intertwining operator satisfies $\mathrm{vol}(\mathcal{O}_F, dx) = 1$. But in [**Jac72**], the self-dual measure associated with ψ is used for it. Thus there is a difference between the above formula and that in [**Jac72**]). By the above formula, we have

$$\widetilde{\Psi}\big(W,\, Mf\big) = \frac{\varepsilon(0, \mathbf{1}, \psi)}{\gamma(0, \mu\nu^{-1}, \psi)}Z\big(1/2,\, W \otimes \nu,\, \widehat{\Phi}\big).$$

By combining it with the functional equation (5.2.1), the assertion follows. □

5.3. A relation between pairings on different models

The notations are the same as in the previous section. We denote $F = F_1$. We define

$$W_f(g) := \varepsilon(0, \nu^{-1}, \psi)^{-1} \lim_{N \to \infty} \int_{\varpi_F^{-N}\mathcal{O}_F} f\left(\tau_1 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi^{-1}(x)\, dx$$

for $f \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\mu \boxtimes \nu)$ (for this integral, we do not need the condition $\mu \neq \nu$). The limit exists and gives an element of the space of Whittaker functions $\mathscr{W}(\pi, \psi)$, where π is the unique irreducible infinite dimensional subquotient of $\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu \boxtimes \nu)$. For $\mu \neq \nu$, we define a $\mathrm{GL}_2(F)$-invariant pairing

$$\mathfrak{j}_\pi \colon \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\mu \boxtimes \nu) \times \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\nu^{-1} \boxtimes \mu^{-1}) \longrightarrow \mathbb{C}$$

by

$$\mathfrak{j}_\pi(f, h) := \int_K f(k) M h(k)\, dk,$$

where $K = \mathrm{PGL}_2(\mathcal{O}_F)$ and dk is normalized so that $\mathrm{vol}(K, dk) = 1$. The pairing $\mathfrak{j}_\pi$ induces an invariant pairing

$$\mathfrak{j}_\pi : \pi \times \pi^\vee \to \mathbb{C}$$

and we denote it by the same symbol $\mathfrak{j}_\pi$. Therefore, there exists a constant $C \in \mathbb{C}$ such that

$$C \cdot \mathfrak{j}_\pi(f, h) = \mathfrak{i}_\pi(W_f, W_h).$$

The constant C can be determined explicitly as follows:

Proposition 5.3.1. Assume $\mu \neq \nu$. Then the constant C is described as

$$C = \mu(-1)\, \mathcal{E}(\pi, \mathrm{Ad}),$$

where

$$\begin{aligned} \mathcal{E}(\pi, \mathrm{Ad}) &:= \frac{\varepsilon(0, \pi \otimes \nu^{-1}, \psi)}{\varepsilon(0, \mu, \psi)\, \varepsilon(0, \nu^{-1}, \psi)\, L(1, \pi^\vee \otimes \nu)} L(1, \mathrm{Ad}\pi)\, \left.\frac{\zeta_F(t)}{L(t, \pi \otimes \nu^{-1})}\right|_{t=0}, \\ &= \begin{cases} \dfrac{\varepsilon(0, \mathbf{1}, \psi)\, \varepsilon(0, \mu\nu^{-1}, \psi)}{\varepsilon(0, \mu, \psi)\, \varepsilon(0, \nu^{-1}, \psi)\, L(1, \mu\nu^{-1})\, L(0, \mu\nu^{-1})} & \text{if } \pi \cong \pi(\mu, \nu), \\[2ex] \dfrac{\varepsilon(0, \mathrm{St} \otimes |\cdot|_F^{-1/2}, \psi)}{\varepsilon(0, \nu|\cdot|_F^{-1}, \psi)\, \varepsilon(0, \nu^{-1}, \psi)} & \text{if } \pi \cong \mathrm{St} \otimes \nu|\cdot|_F^{-1/2}. \end{cases} \end{aligned}$$

Proof. Let

$$I_m := \left\{ x \in \mathrm{GL}_2(\mathcal{O}_F) \;\middle|\; x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \varpi_F^m \mathcal{O}_F \right\}.$$

We take a unique element

$$f \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\mu \boxtimes \nu)$$

characterized by

$$\begin{aligned} f(\tau_1) &= 1, \\ nf &= f \quad \text{for } n \in N(\mathcal{O}_F), \\ f\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right) &= 0 \quad \text{for } u \in \varpi_F \mathcal{O}_F. \end{aligned}$$

The uniqueness follows from [**Hi89-2**, Corollary 2.2]. By the uniqueness, for sufficiently large $m \in \mathbb{Z}_{>0}$, and for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_m$, we have

$$uf = \mu(d)\,\nu(a)\,f.$$

Thus we have

$$\begin{aligned}
&\mathrm{vol}(\overline{I_m}, dk)^{-1}\,\mathfrak{j}_\pi\big(\tau_{\varpi_F^m} f,\ Mf\otimes\mu^{-1}\nu^{-1}\big)\\
&= \sum_{u\in\mathcal{O}_F/\varpi_F^m\mathcal{O}_F} \tau_{\varpi_F^m} f\Big(\tau_1\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\Big)\, Mf\otimes(\mu\nu)^{-1}\Big(\tau_1\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\Big)\\
&+ \sum_{u\in\varpi_F\mathcal{O}_F/\varpi_F^m\mathcal{O}_F} \tau_{\varpi_F^m} f\Big(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\Big)\, Mf\otimes(\mu\nu)^{-1}\Big(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\Big).
\end{aligned}$$

Here, $\tau_{\varpi_F^m} := \begin{pmatrix} 0 & 1 \\ -\varpi_F^m & 0 \end{pmatrix}$ and we denote the image of I_m in K by $\overline{I_m}$. By using the well-known formula for matrices:

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} -x^{-1} & -1 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix},$$

all terms but $\tau_{\varpi_F^m} f(1)$ are zero. Since

$$\begin{aligned}
Mf\otimes(\mu\nu)^{-1}(1) &= 1,\\
\mathrm{vol}(\overline{I_m}, dk) &= |\varpi_F^m|\,\frac{\zeta_F(2)}{\zeta_F(1)},
\end{aligned}$$

We have

$$\mathfrak{j}_\pi\big(\tau_{\varpi_F^m} f,\ Mf\otimes\mu^{-1}\nu^{-1}\big) = \nu|\cdot|^{1/2}(\varpi_F^m)\,\frac{\zeta_F(2)}{\zeta_F(1)}.$$

On the other hand, note that

$$W_f\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big) = \varepsilon(0,\nu^{-1},\psi)^{-1}\,\nu|\cdot|^{1/2}(a)\,\mathbf{1}_{\varpi_F^{c(\psi)}\mathcal{O}_F}(a),$$

where $c(\psi)$ is the exponent of the conductor of ψ. Thus for sufficiently large $m \in \mathbb{Z}$, we have

$$\begin{aligned}
&\frac{\zeta_F(1)\,L(1,\mathrm{Ad}\pi)}{\zeta_F(2)}\,\mathfrak{i}_\pi\big(\tau_{\varpi_F^m} W_f,\ W_{f\otimes\mu^{-1}\nu^{-1}}\big)\\
&= \varepsilon(0,\mu^{-1},\psi)^{-1}\int_{F^\times} \tau_1 W_f\Big(\begin{pmatrix} a & 0 \\ 0 & \varpi_F^m \end{pmatrix}\Big)\,\nu|\cdot|^{1/2}(-a)\,\mathbf{1}_{\varpi_F^{c(\psi)}\mathcal{O}_F}(a)\,\mu\nu(-a)^{-1}\,d^\times a\\
&= \varepsilon(0,\mu^{-1},\psi)^{-1}\,\mu(-1)\,\nu|\cdot|_F^{1/2}(\varpi_F^m)\int_{F^\times} \tau_1 W_f\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)\,\mu|\cdot|_F^{1/2}(a)\,d^\times a.
\end{aligned}$$

The last equality follows since the support of $\tau_1 W_f\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)$ is bounded and m is sufficiently large. By the functional equation for an irreducible admissible representation of $\mathrm{GL}_2(F)$, it is equal to

$$\begin{aligned}&\varepsilon(0,\mu^{-1},\psi)^{-1}\,\mu(-1)\,\nu|\cdot|_F^{1/2}(\varpi_F^m)\\&\quad\times\left(\gamma(1-t,\pi\otimes\nu^{-1},\psi)\int_{F^\times}W_f\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)\nu^{-1}|\cdot|_F^{1/2-t}(a)\,d^\times a\right)\Bigg|_{t=1}\\&=\mu(-1)\,\nu|\cdot|_F^{1/2}(\varpi_F^m)\,L(1,\mathrm{Ad}\pi)\,\mathcal{E}(\pi,\mathrm{Ad}).\end{aligned}$$

Thus we have the desired formula. □

5.4. Epsilon-factors of Asai L-functions

We rewrite Theorem 5.1.1:

Theorem 5.4.1. Let π_2 be an irreducible admissible representation over $\mathrm{GL}_2(F_2)$ with central character ω_2 and fix $\xi \in F_2^\times$ with $\mathrm{tr}_{F_2/F_1}(\xi) = 0$. Let $\mu,\nu : F_1^\times \longrightarrow \mathbb{C}^\times$ be a quasi-character such that $(\omega_2|_{F_1^\times})\mu\nu = 1$. We assume that

$$\max\{\lambda(\mu),\lambda(\nu)\} + \lambda(\pi_2) < \frac{1}{2}.$$

For any $W \in \mathscr{W}(\pi_2,\psi_\xi)$, $\widetilde{W} \in \mathscr{W}(\pi_2^\vee,\psi_\xi)$, $f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu\boxtimes\nu)$, and $\widetilde{f} \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\nu^{-1}\boxtimes\mu^{-1})$, we have

$$\begin{aligned}&\mathcal{I}_{\pi_2\boxtimes\mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu\boxtimes\nu)}\big(W\boxtimes W_f,\,\widetilde{W}\boxtimes W_{\widetilde{f}}\big)\\&=|\xi\mathcal{D}_{F_2/F_1}|_{F_2}^{-1/2}\,\frac{|\varpi_{F_1}|_{F_1}^{(c(\mu)+c(\nu))/2}\,\varepsilon_{\mathrm{RS}}(1/2,\mathrm{As}\pi_2\otimes\mu,\psi,\xi)}{\varepsilon(1/2,\mu,\psi)\,\varepsilon(1/2,\nu^{-1},\psi)\,L(1/2,\mathrm{As}\pi_2\otimes\mu)^2}\,\Psi\big(W,\,f\big)\,\Psi\big(\widetilde{W},\,\widetilde{f}\big).\end{aligned}$$

Proof. When $\mu \neq \nu$, it follows from the formal calculations by applying Proposition 5.3.1, Theorem 5.1.1, and Corollary 5.2.1. By analytic continuation, the formula holds even when $\mu = \nu$. □

Corollary 5.4.2. The notations are the same as in Theorem 5.4.1. If $\mu = \nu$ (hence $\omega_2|_{F_1^\times}\mu^2 = \mathbf{1}$), we have

$$\varepsilon_{\mathrm{RS}}\big(1/2,\,\mathrm{As}\pi_2\otimes\mu,\,\psi,\xi\big) = 1.$$

Proof. We take a unique element

$$f \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mu\boxtimes\mu)$$

characterized by

$$\begin{aligned}f(\tau_1)&=1,\\nf&=f\quad\text{for } n\in N(\mathcal{O}_{F_1}),\\f\left(\begin{pmatrix}1&0\\u&1\end{pmatrix}\right)&=0\quad\text{for } u\in\varpi_{F_1}\mathcal{O}_{F_1}.\end{aligned}$$

By a direct computation, we have

$$\begin{aligned}\mathfrak{j}_\pi(f,f)&=\frac{\zeta_F(2)}{\zeta_{F_1}(1)},\\\mathfrak{i}_\pi\big(W_f,\,W_f\big)&=\frac{\zeta_{F_1}(2)}{\zeta_{F_1}(1)^3}.\end{aligned}$$

Thus for $h, \widetilde{h} \in \mathrm{Ind}_{B(F_1)}^{\mathrm{GL}_2(F_1)}(\mathbf{1} \boxtimes \mathbf{1})$, we have

$$\mathfrak{i}_{\pi_1}\big(W_h,\, W_{\widetilde{h}}\big) = \zeta_F(1)^{-2} \int_K h(k)\widetilde{h}(k)\, dk.$$

By applying Theorem 5.1.1 and Corollary 5.2.1, we have

$$\begin{aligned} &\mathcal{I}_{\pi_2 \boxtimes \pi_1}\big(W \boxtimes W_f,\, \widetilde{W} \boxtimes W_{\widetilde{f}}\big) \\ &= |\xi \mathcal{D}_{F_2/F_1}|_{F_2}^{-1/2}\, |\varpi|^{(c(\mu)+c(\nu))/2}\, L(1/2, \mathrm{As}\pi_2 \otimes \mu)^{-2}\, \Psi\big(W,\, f\big)\, \Psi\big(\widetilde{W},\, \widetilde{f}\big). \end{aligned}$$

Therefore by comparing this with the formula of the theorem above, we have the corollary. □

Remark 5.4.3. Assume π_2 is a distinguished discrete series representation. By combining Corollary 5.4.2 with the result of Anandavardhanan [**Ana08**, Theorem 1.1], we have

$$\omega_2^{-1}(\xi)\, |\xi|_{F_2}^{-s+1/2}\, \varepsilon(1/2, \chi_{F_2/F_1}, \psi)\, \varepsilon_{\mathrm{RS}}(s, \mathrm{As}\pi_2, \psi, \xi) = \varepsilon(s, \mathrm{As}\pi_2, \psi).$$

where χ_{F_2/F_1} is the quadratic character associated with the extension F_2/F_1 and the ε-factor of the right hand side is defined from the Langlands-Shahidi method. We expect that the above equality holds for any generic representation π_2.

CHAPTER 6

Constructions of p-adic L-functions for twisted triple products

In this section, we assume that p is odd. Let $F/\mathbb{Q}$ be a real quadratic extension with the discriminant $D \in \mathbb{Z}_{>0}$. Let $I_F = \{\sigma, \rho\}$ be the set of embeddings from F to $\mathbb{C}$. We assume that p is not split and denote by $\mathfrak{p}$ the prime ideal of $\mathcal{O}_F$ above p.

For convenience, we give a table of local L-functions for a irreducible admissible representation π over $\mathrm{GL}_2(F)$ for a local field F below. Let q be the order of residue field of F for each non-archimedean place v.

(6.0.1)

	π	$L(s,\pi)$	$L(s,\mathrm{Ad}\pi)$
$v<\infty$	$\pi(\mu,\nu)$	$L(s,\mu)L(s,\nu)$	$\zeta_F(s)L(s,\mu\nu^{-1})L(s,\nu\mu^{-1})$
	$\mathrm{St}\otimes\chi$	$L(s+1/2,\chi)$	$\zeta_F(s+1)$
$v\mid\infty$	σ_{k-1}	$\Gamma_{\mathbb{C}}(s+(k-1)/2)$	$\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+k-1)$

Here, let $\eta\colon F^\times \longrightarrow \mathbb{C}$ be a continuous group homomorphism if F is non-archimedean , we define

$$\begin{aligned}
L(s,\eta) &:= \begin{cases} (1-\eta(\varpi)q^{-s})^{-1} & \text{if } \eta \text{ is unramified,} \\ 1 & \text{otherwise,} \end{cases} \\
\Gamma_{\mathbb{R}}(s) &:= \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \\
\Gamma_{\mathbb{C}}(s) &:= 2(2\pi)^{-s}\Gamma(s) \\
\zeta_F(s) &:= \begin{cases} (1-q^{-s})^{-1} & \text{if } F \text{ is non-archimedean,} \\ \Gamma_{\mathbb{R}}(s) & \text{if } F=\mathbb{R}, \\ \Gamma_{\mathbb{C}}(s) & \text{if } F=\mathbb{C}. \end{cases}
\end{aligned}$$

6.1. The review of Ichino's formula

Let F be a totally real fields and E a étale cubic algebra over F such that the image of any F-algebra homomorphism $E \to \mathbb{C}$ is contained in $\mathbb{R}$. Let D be a (not necessarily definite) quaternion algebra over F of discriminant $\mathfrak{n}^-$ and let $D_E := D \otimes_F E$. Let $\Pi \cong \otimes'_v \Pi_v$ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}(\mathbb{A}_E)$ with central character trivial on $\mathbb{A}_F^\times$. We suppose that there exists an irreducible unitary automorphic representation Π^D of $D^\times(\mathbb{A}_E)$ associated with Π by the Jacquet-Langlands correspondence. We define the element of

$$I \in \mathrm{Hom}_{D^\times(\mathbb{A}_F)\times D^\times(\mathbb{A}_F)}\big(\Pi^D \boxtimes (\Pi^D)^\vee, \mathbb{C}\big)$$

by

$$I(\phi\boxtimes\phi^\vee) := \int_{\mathbb{A}_F^\times D^\times(F)\backslash D^\times(\mathbb{A}_F)} \int_{\mathbb{A}_F^\times D^\times(F)\backslash D^\times(\mathbb{A}_F)} \phi(x)\phi^\vee(y)\, dxdy,$$

for $\phi \in \Pi^D$ and $\phi' \in (\pi^D)^\vee$, where dx and dy are the Tamagawa measure on $\mathbb{A}_F^\times \backslash D(\mathbb{A}_F)$. We define

$$\mathcal{B} \in \mathrm{Hom}_{D^\times(\mathbb{A}_E)\times D^\times(\mathbb{A}_E)}\big(\Pi^D \boxtimes (\Pi^D)^\vee, \mathbb{C}\big)$$

be an invariant pairing by

$$\mathcal{B}(\phi,\phi') := \int_{\mathbb{A}_E^\times D(E)\backslash D(\mathbb{A}_E)} \phi(x)\phi'(x)dx,$$

for $\phi \in \Pi^D$ and $\phi' \in (\pi^D)^\vee$, where the measure dx is the Tamawagawa measure. For each place v of F, we fix an element

$$\mathcal{B}_v \in \mathrm{Hom}_{D^\times(\mathbb{A}_{F_v})\times D^\times(\mathbb{A}_{F_v})}\big(\Pi_v^D \boxtimes (\Pi_v^D)^\vee, \mathbb{C}\big)$$

and assume for any $\otimes'_v \phi_v \in \otimes'_v \Pi_v^D$ and $\otimes'_v \phi'_v \in \otimes'_v (\Pi_v^D)^\vee$,

$$\mathcal{B}_v(\phi_v, \phi'_v) = 1$$

for almost all v. Then there exists $C_1 \in \mathbb{C}^\times$ such that

$$\mathcal{B} = C_1 \prod_v \mathcal{B}_v.$$

For $\otimes'_v \phi_v \in \otimes'_v \Pi_v^D$ and $\otimes'_v \phi'_v \in \otimes'_v (\Pi_v^D)^\vee$,

$$\mathcal{I}_{\Pi_v^D}(\phi) := \frac{\zeta_{F_v}(2)}{\zeta_{E_v}(2)} \frac{L(1,\mathrm{Ad}\Pi_v)}{L(1/2,\Pi_v)} \int_{F_v^\times \backslash D^\times(F_v)} \mathcal{B}_v(\Pi_v^D(g)\phi_v, \phi'_v)\, d_v x,$$

Here, let $G := \mathrm{Res}_{E/F}\,\mathrm{GL}_2$ and let $\widehat{G}$ be the dual group. $L(s,\mathrm{Ad}\Pi_v)$'s and $L(s,\Pi_v)$'s are defined by representations $\mathbb{C}^{\otimes 3}$ and $\mathrm{Lie}(\widehat{G})/\mathrm{Lie}(Z(\widehat{G}))$ respectively of ${}^L G = \mathrm{GL}_2(\mathbb{C})^3 \rtimes \mathrm{Gal}(\overline{F}/F)$, where $\mathrm{Gal}(\overline{F}/F)$ acts on it as $\mathscr{S}_3$ through the permutation of $\mathrm{Spec}(E \times_F \overline{F})$. The measures $d_v x$ are invariant measures defined as follows:

- In the case F_v is nonarchimedean and $D(F_v)$ is a division algebra, let R_v is the max order of $D(F_v)$ and define $d_v x$ such that
$$\mathrm{vol}(\mathcal{O}_{F_v}^\times \backslash R_v^\times, d_v x) = 1.$$
- In the case F_v is nonarchimedean and $D(F_v)$ is a matrix algebra, we define $d_v x$ satisfying
$$\mathrm{vol}(\mathcal{O}_{F_v}^\times \backslash D(\mathcal{O}_{F_v}), d_v x) = 1.$$
- In the case $F_v = \mathbb{R}$ and $D(F_v)$ is a division algebra, we define $d_v x$ such that
$$\mathrm{vol}(\mathbb{R}^\times \backslash D(\mathbb{R}), d_v x/d^\times t) = 1,$$
where $d^\times t$ is an invariant measure on $\mathbb{R}$ defined by $d^\times t = dt/|t|_{\mathbb{R}}$ (Here, dt is a invariant measure on $\mathbb{R}$ with $\mathrm{vol}([0,1], dt) = 1$).
- In the case $F_v = \mathbb{R}$ and $D(F_v)$ is a matrix algebra, we define
$$d_v x = \frac{1}{2\pi} \frac{dxdy}{|y|_{\mathbb{R}}^2} d\theta$$
for the coordinate
$$x = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \ (a \in \mathbb{R}, y \in \mathbb{R}^\times, \theta \in [0, 2\pi)).$$
Here, dx and dy are a invariant measure on $\mathbb{R}$ such that the volume of $[0,1]$ is one.

We remark that by [**IP**, Lemma 6.1], we have

$$dx = C_{D/F}\prod_v{}' d_v x,$$

where

$$C_{D/F} := (4\pi^2)^r (2\pi)^{[F:\mathbb{Q}]-r} |d_F|^2_{\mathbb{A}_{F,f}} \zeta_F^{(\infty)}(2)^{-1} \prod_{\substack{v<\infty \\ D(F_v):\ \text{division}}} (q_v - 1)^{-1},$$

where $d_F \in \mathbb{A}_{F,f}$ such that $d_F\mathcal{O}_F$ is equal to the absolute different ideal $\mathcal{D}_{F/\mathbb{Q}}$, r is the number of the infinite places v such that $D(F_v)$ are division algebras. Ichino's formula [**Ich08**] is as follows:

Theorem 6.1.1. For $\phi = \otimes'_v \phi_v \in \Pi^D$ and $\phi' := \otimes'_v \phi'_v \in (\Pi^D)^\vee$ such that

$$\mathcal{B}(\phi, \phi') \neq 0,$$

we have

$$\frac{I(\phi,\phi')}{\mathcal{B}(\phi,\phi')} = \frac{C_{D/F}}{2^c} \cdot \frac{\zeta_E(2)}{\zeta_F(2)} \cdot \frac{L(1/2,\Pi)}{L(1,\mathrm{Ad}\Pi)} \cdot \prod_v \frac{\mathcal{I}_{\Pi_v^D}(\phi_v,\phi'_v)}{\mathcal{B}_v(\phi_v,\phi'_v)},$$

where c is the number of connected components of $\mathrm{Spec}(E)$.

6.2. Constructions of p-adic L-functions for unbalanced twisted triple products

6.2.1. Setting. Let $f_0 \in \mathscr{S}_{k,w}(K_1(p^{s(\omega_1)}, \omega_1))_{\mathbb{Q}}$ be normalized Hilbert cuspidal eigenform over $\mathbb{Q}$ and let $g_0 \in \mathscr{S}_{h,v}(K_1(\mathfrak{p}^{s(\omega_2)}), \omega_2)/F$ be normalized at $\sqrt{D}^{-1}$. Both of them are ordinary at each place dividing p and new at each place dividing outside p. We assume the following conditions:

$$k_1 \geqslant h_\sigma + h_\rho,$$
$$\omega_1 = \omega_2|_{\mathrm{Cl}^+_{F_1}(\mathfrak{n}_1 p^\infty)},$$

where H is the subgroup of $\mathrm{Cl}^+_F(p^\infty) \subset \mathbf{G}_F$ such that

$$\mathrm{Cl}^+_F(p^\infty) = \mathrm{Cl}^+_F(p^\infty)(p) \times H$$

is a decomposition into p-sylow group and the subgroup H composing of elements of order being prime to p. Let $\mathbb{I}_1$ and $\mathbb{I}_2$ be a sufficiently large integrally closed domain. Suppose that both of them are finite over a component of the normalizer of $\mathcal{O}[[\mathbf{G}_F]]$ and H acts on both of $\mathbb{I}_1$ and $\mathbb{I}_2$ via χ. We denote by $\mathbb{K}_i$ the fraction field of $\mathbb{I}_i$. We define $\mathscr{X}^{\mathrm{arith}}(\mathbb{I}_i)$ as in (4.4.4).

6.2.2. The p-adic interpolation of Petersson inner products. Let

$$\lambda_1 \colon \mathbf{h}^{n,\mathrm{ord}}(1;\mathcal{O}) \longrightarrow \mathbb{I}_1$$

be a Hida family on the f_0 and we denote by $\mathscr{F} \in \mathscr{S}^{n,ord}((1);\mathbb{I}_1)$ the lift of f_1 defined by λ_1. We note that for each $P \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I}_1)$, the weight of the specialization $P \circ \mathscr{F}$ is $(k_P|_{\mathbb{Q}} - 2\underline{t}_{\mathbb{Q}}, w_P|_{\mathbb{Q}} - \underline{t}_{\mathbb{Q}})$ and its character is the restriction of (ω_P, ω'_P) to $\mathbf{G}_{\mathbb{Q}}$. Since $\mathbf{h}^{n,\mathrm{ord}}((1);\mathcal{O})$ has no nilpotent element, we have a decomposition as a $\mathbb{K}_1$-algebra

$$\mathbf{h}^{n,\mathrm{ord}}((1);\mathcal{O}) \otimes_{\mathbb{I}_1} \mathbb{K}_1 \cong \mathbb{K}_1 \times \mathbb{B},$$

where the projection $\mathbf{h}^{n,\mathrm{ord}}((1);\mathcal{O}) \to \mathbb{K}_1$ is identical to λ_1. We denote by $\mathbf{1}_{\lambda_1} \in \mathbf{h}^{n,\mathrm{ord}}((1);\mathcal{O})\otimes_{\mathbb{I}_1} \mathbb{K}_1$ the idempotent corresponding to $(1,0)$ of the right hand side above. Let $\mathscr{G} \in \mathscr{S}^{n,\mathrm{ord}}((1);\mathbb{I}_2)$ be a lift of g_0, namely, $P(\mathscr{G}) = g_0$.

Definition 6.2.1. We define the square root of p-adic L-function in $\mathbb{K}_1\hat{\otimes}_{\mathcal{O}}\mathbb{I}_2$ as

$$\mathscr{L}_p(\mathscr{F}\otimes\mathscr{G}) := \left\langle 1_{\lambda_1}, \operatorname{Res}_{F/\mathbb{Q}}(\Theta(\mathscr{G}))\middle| e_p\right\rangle,$$

where the pairing is the one defined in (4.4.6).

For any nonzero ideal $a \in \mathbb{Z}$, let

$$\tau_{a,q} := \begin{pmatrix} 0 & 1 \\ -q^{\operatorname{ord}_q(a)} & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_q)$$

$$\tau_\infty := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

and define

$$\tau_a := \tau_\infty \prod_q \tau_{a,q} \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}).$$

For $f, g \in \mathscr{S}_{k,w}(K_1(p^s), \omega_1)$, we define

$$\langle f, g\rangle_{p^s} := \int_{\mathrm{PGL}_2(\mathbb{A}_{F_1})} (\tau_{p^s} f_1)(x) f_2(x)\omega_1(\det(x))^{-1}|\det(x)|_{\mathbb{A}_{F_1}}^{-[2w-k]}dx,$$

where dx is the Tamagawa measure. We have the following interpolation formula by Theorem :

Theorem 6.2.2. For $P\otimes Q \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I}_1\hat{\otimes}_{\mathcal{O}}\mathbb{I}_2)$ satisfying the following condition

$$2w_P - 2w_Q = k_P - k_Q - t_F \geqslant 0$$

and the denominator of $\mathscr{L}_p(\mathscr{F}\otimes\mathscr{G})$ is not contained in the kernel of P. Then for suffiently large s, we have

$$(P\otimes Q)(\mathscr{L}_p(\mathscr{F}\otimes\mathscr{G})) = \sqrt{D}_{\mathfrak{p}}^{w_Q - t_{F_2}} \frac{\langle \mathscr{F}_P, h_{P,Q}\rangle_{p^s}}{\langle \mathscr{F}_P, \mathscr{F}_P\rangle_{p^s}},$$

where $h_{P,Q}$ is defined in Theorem 4.5.2.

Proof. It follows immediately from Theorem 4.5.2. □

6.2.3. Ichino's formula for GL_2. We rewrite Ichino's formula ([**Ich08**]) for GL_2 by means of Waldspurger's formula [**Wal85**, Proposition 6] for our setting. For convenience, we often denote $\mathbb{Q}$ and F by F_1 and F_2. For $P_i \in \mathscr{X}^{\mathrm{arith}}(\mathbb{I})$. We put

$$f_1(x) := \mathscr{F}_{P_1}(x)|\det(x)|_{\mathbb{A}_{\mathbb{Q}}}^{-\frac{[2w_{P_1}-k_{P_1}]}{2}}$$

$$f_2(x) := \mathscr{G}_{P_2}(x)|\det(x)|_{\mathbb{A}_F}^{-\frac{[2w_{P_2}-k_{P_2}]}{2}}$$

which have unitary central characters ω_{P_1} and ω_{P_2} respectively. Let $\psi_1 := \mathbf{e}$ be an additive character on $\mathbb{A}_F/F$ and let $\xi := \sqrt{D}^{-1}$ and define

$$\psi_2(x) := \mathbf{e}_F(\xi x).$$

Let π_i be an irreducible automorphic representation generated by f_i^u. We fix an isomorphism

$$\pi_i \cong {\bigotimes_v}' \pi_{i,v}, \tag{6.2.1}$$

where for each place v_i of F_i we fix an isomorphism between π_{i,v_i} and its Whittaker model $\mathscr{W}(\pi_{i,v_i}, \psi_{i,v_i})$ and for unramified non-archimedean place v_i, the isomorphisms above are determined by spherical vector $W^0 \in \mathscr{W}(\pi_{i,v_i}, \mathbf{e}_{F_{v_i}})$ such that $W^0(1) = 1$. In addition, we assume that the isomorphism above satisfies for any $\phi_i \in \pi_i$ corresponding to $\otimes_{v_i}\phi_{i,v_i} \in \bigotimes'_{v_i} \pi_{i,v}$,

$$W_{\phi_i}(g) = \prod_{v_i} \phi_{i,v}(g_v),$$

where

$$W_{\phi_i}(g) := \int_{\mathbb{A}_{F_i}/F_i} \phi_i\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi_i(-x)dx,$$

where dx is the self dual measure associated with ω_i Here, we take dg as the Tamagawa measure on $\mathrm{PGL}_2(\mathbb{A}_{F_i})$. For each non-archimedean (resp. archimedean) place v, we fix a $\mathrm{GL}_2(F_{i,v})$ (resp. $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2))$)-invariant pairing

$$\mathfrak{i}_{\pi_{i,v}}(W, W') := \frac{\zeta_{F_{i,v}}(2)}{\zeta_{F_{i,v}}(1) L(1, \mathrm{Ad}\pi_{i,v})} \langle W, W' \rangle_v.$$

Here, we define

$$\langle W, W' \rangle_v := \int_{F_{i,v}^\times} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) W'(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}) d^\times a,$$

where $d^\times a$ is an invariant measure on $F_i^\times$ such that $\mathrm{vol}(\mathcal{O}_{F_i}^\times, da^\times) = 1$. Suppose $\pi_i^\vee$ holds the same assumption as π_i. By the formula proved in [**Wal85**, Proposition 6], we have

$$\mathcal{B}_{\pi_i} = \frac{2|d_{F_i}|_{\mathbb{A}_{F_i}}^{1/2} L(1, \mathrm{Ad}\pi_i)}{\zeta_{F_i}(2)} \prod_v \mathfrak{i}_{\pi_{i,v_i}} \tag{6.2.2}$$

where $d_{F_i} \in \mathbb{A}_{F_i}$ is an element such that $d_{F_i}\mathcal{O}_{F_i}$ is equal to the different ideal of $F_i/\mathbb{Q}$. Note that in [**Wal85**], they use the self dual measure to define the measure $d^\times a$ for the paring of the Whittaker model. Let $E := \mathbb{Q} \times F$. and let

$$\Pi := \pi_1 \boxtimes \pi_2$$

We chose the paring $\mathcal{B}$ as follows:

$$\mathcal{B}_{\Pi_{v_1}} := \mathfrak{i}_{\pi_{1,v_1}} \otimes \mathfrak{i}_{\pi_{2,v_1}} : \Pi_v \otimes \Pi_v^\vee \longrightarrow \mathbb{C}$$

By Ichino's formula, we have

Theorem 6.2.3. Let $\phi \in \Pi$ and $\phi' \in \Pi^\vee$ be elements corresponding to $\otimes'_{v_1}\phi_{v_1} \in \bigotimes'_{v_1} \Pi_{v_1}$ and $\otimes'_{v_1}\phi'_{v_1} \in \bigotimes'_{v_1} \Pi_{v_1}^\vee$, respectively. Then we have

$$I_\Pi(\phi \otimes \phi') = \frac{L(1/2, \Pi)}{D^{1/2}\zeta_{F_1}(2)^2} \prod_{v_1} \mathcal{I}_{\Pi_{v_1}}(\phi_{v_1} \otimes \phi'_{v_1}).$$

We determine the local test vectors $\pi_{1,v}$ (resp. $\pi_{2,w}$) corresponding to f_1 (resp. f_2). For an archimedian place v (resp. w ($w \in I_F$)), W_v (resp. W'_w) is an element of the discrete series representations $\sigma_{k_{P_1}-1}$ (resp. $\sigma_{h_{P_2,w}-1}$ of the positive lowest weight k (resp. $h_{P_{2,w}}$) and we take $W_v \in \sigma_{k_{P_1}-1}$ (resp. $W'_w \in \sigma_{h_{P_{2,w}}-1}$) as a unique element satisfying

$$W_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = |a|_{\mathbb{R}}^{k/2} e^{-2\pi|a|_{\mathbb{R}}} e^{\sqrt{-1}k\theta},$$

$$(\text{resp. } W'_w\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \left(|a|_{\mathbb{R}}/\sqrt{D}\right)^{h_w/2} e^{-2\pi|a|_{\mathbb{R}}/\sqrt{D}} e^{\sqrt{-1}h_w\theta}\ ).$$

For non-archimedean v (resp. w) being prime to p, let W_v (resp. W'_w) be a spherical vector of $\mathscr{W}(\pi_{1,v}, \psi_{1,v})$ (resp. $\mathscr{W}(\pi_{2,w}, \psi_{2,w})$) such that

$$W_v(1) = 1 \qquad (\text{resp.} \quad W'_w(1) = 1)$$

For $v = p$ (resp. $w = \mathfrak{p}$), we define the *ordinary vector* $\phi_{1,p}$ (resp. $\phi_{2,\mathfrak{p}}$) as follows: when $\pi_{1,p}$ (resp. $\pi_{2,\mathfrak{p}}$ is principal series, we take a unique element

$$f_p \in \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mu_1 \boxtimes \nu_1) \qquad (\text{resp.} \quad g_{\mathfrak{p}} \in \mathrm{Ind}_{B(F_{\mathfrak{p}})}^{\mathrm{GL}_2(F_{\mathfrak{p}})}(\mu_2 \boxtimes \nu_2))$$

characterized by

$$\begin{aligned} f_p(\tau_1) &= 1, \\ n f_p &= f_p \quad \text{for } n \in N(\mathbb{Z}_p), \\ f_p\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right) &= 0 \quad \text{for } u \in p\mathbb{Z}_p \end{aligned}$$

$$\left(\text{resp.} \quad \begin{aligned} g_{\mathfrak{p}}(\tau_1) &= 1, \\ n g_{\mathfrak{p}} &= g_{\mathfrak{p}} \quad \text{for } n \in N(\mathcal{O}_F), \\ g_{\mathfrak{p}}\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right) &= 0 \quad \text{for } u \in \varpi_F \mathcal{O}_F. \end{aligned} \right),$$

and define

$$W_p := \varepsilon(0, \nu_1^{-1}, \psi_p) W_{f_p} \qquad (\text{resp.} \quad W'_{\mathfrak{p}} := \varepsilon(0, \nu_2^{-1}, \psi_{F,\mathfrak{p}}) W_{g_{\mathfrak{p}}}).$$

When the local representation is special, we define the vectors by replacing μ_i by $\nu_i |\cdot|^{-1}$.

Remark 6.2.4. Let $a(f_i, p)$ be the p-th Fourier coefficient of f_i, then we have

$$\begin{aligned} a(f_1, p) &= \nu_1(p)\, p^{(k-1)/2}, \\ a(f_2, \mathfrak{p}) &= \nu_2(\varpi_{F_{\mathfrak{p}}})\, |\varpi_{F_{\mathfrak{p}}}|_p^{-(h_\sigma + h_\rho - 2)/2}. \end{aligned}$$

and the ordinary condition for f (resp. g) is equivalent to $|\nu_1(p)|_p = p^{(k-1)/2}$ (resp. $|\nu_2(p)|_p = p^{(h_\sigma + h_\rho - 2)/2}$).

When $\pi_{1,p}$ is spherical, there exists a elliptic modular form $\widetilde{f}$ of level 1 such that

$$f = \widetilde{f} - \mu_1(p)\, p^{-1/2}\, \widetilde{f}.$$

Thus we define in general

$$\widetilde{f} = \begin{cases} \widetilde{f}\text{: defined as above} & \text{if } \pi_{f,p} \text{ is spherical}, \\ f & \text{otherwise}. \end{cases}$$

For each place v of $\mathbb{Q}$, we denote $\phi_{2,v} := \boxtimes_{w|v} W'_w$. We assume that

$$\begin{aligned} \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} f\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x\right) dt &= \prod_v W_v(x_v), \\ \int_{\mathbb{A}_F/F} g\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} y\right) dt &= \prod_w W'_w(y_w). \end{aligned}$$

6.2.4. Main results (interpolation formulas for unbalanced case).

Theorem 6.2.5. Let $P \otimes Q \in \mathscr{X}(\mathbb{I})$ be an element such that $P|_{\mathbf{G}} = P_{k_P, w_P, \omega_1, \mathbf{1}}$ and $Q|_{\mathbf{G}} = P_{k_P - 2r\sigma - \underline{t}_F, w_P - r\sigma, \omega_2, \mathbf{1}}$ for some $r \geqslant 0$, we have

$$\begin{aligned}
&(P \otimes Q)(\mathscr{L}_p\left(\mathscr{F} \otimes \mathscr{G})^2\right) \\
&= 2^{r+4} \sqrt{D_{\mathbf{p}}}^{2w_P|_{\mathbb{Q}} - 2\underline{t}_F - r\sigma} a(\mathscr{F}_P, p)^{2c(\omega_1)} \varepsilon_{\mathrm{RS}}(1/2, \mathrm{As}\pi_{\mathscr{G}_{Q,p}} \otimes \mu_{\mathscr{F}_{P,p}}, \psi, \sqrt{D}^{-1}) \\
&\quad \times \left(\frac{L(1, \mu_{\mathscr{F}_P} \nu_{\mathscr{F}_P}^{-1}) L(0, \mu_{\mathscr{F}_P} \nu_{\mathscr{F}_P}^{-1})}{L(1/2, \mathrm{As}\pi_{\mathscr{G}_{Q,p}} \otimes \mu_{\mathscr{F}_{P,p}})} \right)^2 \cdot \frac{L(1/2, \pi_{\mathscr{F}_P} \otimes \pi_{\mathscr{G}_Q} \otimes \sqrt{\omega_P \omega_Q}^{-1})}{D \cdot \Omega(P)^2},
\end{aligned}$$

where $c(\omega_1)$ is the exponent of p of conductor of ω_1 and $\pi_{\mathscr{F}_P}$ and $\pi_{\mathscr{G}_Q}$ is a unitary cuspidal automorphic representation associated with $\mathscr{F}_P$ and $\mathscr{G}_Q$ respectively. We suppose that $\pi_{\mathscr{F}_P, p}$ is the irreducible subquotient of $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mu_{\mathscr{F}_P}, \nu_{\mathscr{F}_p})$. $\Omega(P) \in \mathbb{C}^\times$ is a complex number defined by

$$\Omega(P) := 2^{k_P|_{\mathbb{Q}}} p^{c(\omega_1)((k_P|_{\mathbb{Q}})/2 - 1)} \varepsilon(1/2, \pi_{\mathscr{F}_P}) \left(\widetilde{\mathscr{F}_P}, \widetilde{\mathscr{F}_P}\right)_{\Gamma_0(p^{c(\omega_1)})},$$

where we define

$$\Gamma_0(p^{c(\omega_1)}) := \left\{ x \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p^{c(\omega_1)} \right\},$$

$$\left(\widetilde{\mathscr{F}_P}, \widetilde{\mathscr{F}_P}\right)_{\Gamma_0(p^{c(\omega_1)})} := \int_{\Gamma_0(p^{c(\omega_1)}) \backslash \mathfrak{H}} \left| \widetilde{\mathscr{F}_P}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) \right|^2 \frac{dx\, dy}{y^2}.$$

Proof. Put

$$\omega := (\omega_P \omega_Q)^{1/2}$$

Let

$$\Pi := \pi_1 \boxtimes \pi_2 \otimes \omega^{-1}$$

and we apply Theorem 6.2.3 to the vector of $\Pi \boxtimes \Pi^\vee$:

$$\left(f \boxtimes R_\sigma^r \theta^{(\mathfrak{p})}_{(\omega_P \omega_Q^{-1})^{-1/2}} g \otimes \omega^{-1} \right) \boxtimes \left(f \otimes \omega_P^{-1} \boxtimes R_\sigma^r \theta^{(\mathfrak{p})}_{(\omega_P \omega_Q^{-1})^{-1/2}} g \otimes \omega^{-1} \omega_P \right)$$

and we have

$$\begin{aligned}
&\left\langle f, R_\sigma^r \theta^{(\mathfrak{p})}_{(\omega_P \omega_Q^{-1})^{-1/2}} g|_{\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})} \otimes \omega^{-1} \right\rangle_m^2 \\
&= L(1/2, \pi_f \otimes \pi_g) \frac{D^{-1/2}}{\zeta_F(2)^2} \\
&\quad \times \mathcal{I}_{\Pi_p}\left(\tau_p^m W_p \boxtimes \theta^{(\mathfrak{p})}_{(\omega_P \omega_Q^{-1})^{-1/2}} W'_p \otimes \omega_p^{-1},\ \tau_p^m W_p \otimes \omega_{P,p}^{-1} \boxtimes \theta^{(\mathfrak{p})}_{(\omega_P \omega_Q^{-1})^{-1/2}} W'_p \otimes \omega_p^{-1} \omega_{P,p} \right) \\
&\quad \times \mathcal{I}_{\Pi_\infty}\left(W_p \boxtimes R_{\sigma_1}^r W'_\infty \otimes \omega_\infty^{-1},\ W_\infty \otimes \omega_{P,\infty} \boxtimes R_{\sigma_1}^r W'_\infty \otimes \omega_{P,\infty} \omega_\infty^{-1} \right).
\end{aligned}$$

Here we use [**Ich08**, Lemmga 2.2]. By [**CC16**, Proposition 3.11], we have

$$\mathcal{I}_{\Pi_\infty}\left(W_p \boxtimes R_{\sigma_1}^r W'_\infty \otimes \omega_\infty^{-1},\ W_\infty \otimes \omega_{P,\infty} \boxtimes R_{\sigma_1}^r W'_\infty \otimes \omega_{P,\infty} \omega_\infty^{-1} \right) = 2^{4-r-k-h_\sigma-h_\tau}.$$

We note that the Whittaker functions we use at the archimedean places are slightly different from those of [**CC16**, (3.3)]. On the other hand, by Theorem 5.4.1, we have

$$\begin{aligned}&\mathcal{I}_{\Pi_p}\big(\tau_p^m W_p \boxtimes \theta^{(\mathfrak{p})}_{(\omega_P\omega_Q^{-1})^{-1/2}} W'_p \otimes \omega_p^{-1},\ \tau_p^m W_p \otimes \omega_{P,p}^{-1} \boxtimes \theta^{(\mathfrak{p})}_{(\omega_P\omega_Q^{-1})^{-1/2}} W'_p \otimes \omega_p^{-1}\omega_{P,p}\big)\\ &= \frac{\varepsilon_{\mathrm{RS}}(1/2, \mathrm{As}\pi_2 \otimes \mu_1, \psi_\xi)}{L(1/2, \mathrm{As}\pi_2 \otimes \mu_1)^2}\, \Psi\big(\theta^{(\mathfrak{p})}_{(\omega_P\omega_Q^{-1})^{-1/2}} W'_{\mathfrak{p}} \otimes \omega^{-1},\ \tau_{p^m} f_p\big)^2.\end{aligned}$$

Let m be a sufficiently large m. Then we have

$$\begin{aligned}&\Psi\big(\theta^{(\mathfrak{p})}_{(\omega_P\omega_Q^{-1})^{-1/2}} W'_{\mathfrak{p}} \otimes \omega^{-1},\ \tau_{p^m} f_p\big)\\ &= \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p^\times} \theta^{(\mathfrak{p})}_{(\omega_P\omega_Q^{-1})^{-1/2}} W'_{\mathfrak{p}}\left(\begin{pmatrix} y & 0 \\ x & 1\end{pmatrix}\right) \omega^{-1}(y)\, f\left(\begin{pmatrix} y & 0 \\ x & 1\end{pmatrix}\tau_{p^m}\right) \frac{d^\times y}{|y|}\, dx\\ &= \nu_1(p^m)\, p^{-\frac{m}{2}}\, \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Q}_p^\times} \theta^{(\mathfrak{p})}_{(\omega_P\omega_Q^{-1})^{-1/2}} W'_p\left(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right) \omega^{-1}(y)\, \mu_1(y)\, |y|^{-1/2}\, d^\times y.\\ &= \nu_1(p^m)\, p^{-\frac{m}{2}}\, \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Z}_p^\times} d^\times y\\ &= \nu_1(p^m)\, p^{-\frac{m}{2}}\, \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)}.\end{aligned}$$

On the other hand, by means of [**Wal85**, Proposition 6], we have

$$\begin{aligned}\big\langle \mathscr{F}_P,\ \mathscr{F}_P\big\rangle_m &= \frac{2L(1, \mathrm{Ad}\pi_{\mathscr{F}_P})}{\zeta_{\mathbb{Q}}(2)} \omega_1(-1) \cdot \nu_1(p^m)\, p^{-\frac{m}{2}}\\ &\quad \times \frac{\varepsilon(1/2, \pi_{\mathscr{F}_P})\, p^{(k_P c(\omega_1))/2} a(\mathscr{F}_P, p)^{-c(\omega_1)}}{L(1, \mu_{\mathscr{F}_P}\nu_{\mathscr{F}_P}^{-1}) L(0, \mu_{\mathscr{F}_P}\nu_{\mathscr{F}_P}^{-1})} \cdot \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \cdot 2^{-k_P},\\ \big(\widetilde{\mathscr{F}_P},\ \widetilde{\mathscr{F}_P}\big)_{\Gamma_0(p^{c(\omega_1)})} &= C^{-1}_{\mathrm{GL}_2/\mathbb{Q}}\, p^{c(\omega_1)} \cdot \frac{2L(1, \mathrm{Ad}\pi_{\mathscr{F}_P})}{\zeta_{\mathbb{Q}}(2)} \cdot 2^{-k_P}.\end{aligned}$$

By putting them together, we have the theorem. □

Remark 6.2.6. Let A be $\mathcal{O}[\{a(\mathscr{F}_P, n)\}_n] \subset \mathbb{C}_p$. There exists a number called congruence number $\eta(P\circ\lambda_1) \in \overline{\mathbb{Q}}_p^\times$ and Hida's canonical period $\Omega(+, P\circ\lambda_1, A)\Omega(-, P\circ\lambda_1, A) \in \mathbb{C}^\times$ such that $\Omega/\Omega(+, P \circ \lambda_1, A)\Omega(-, P \circ \lambda_1, A) \in \overline{\mathbb{Q}}$ and we have (see [**Hi16**, Theorem 5.7]),

$$|\eta(P \circ \lambda_1)|_p = \left|\frac{\Omega(P)}{\Omega(+, P \circ \lambda_1, A)\Omega(-, P \circ \lambda_1, A)}\right|_p,$$

where $\lambda_1 \in \mathrm{Hom}_{\mathbb{I}}(\mathrm{h}((1), \mathbb{I}), \mathbb{I})$ associated with $\mathscr{F}$ and note the formula deduced from [**Wal85**, Proposition 6]:

$$L(1, \mathrm{Ad}\pi_{\mathscr{F}_P}) = 2^{k_P|_{\mathbb{Q}}} p^{-s(\omega_1,\mathbf{1})} (\widetilde{\mathscr{F}}_{P1}, \widetilde{\mathscr{F}}_{P1})_{\Gamma_0(p^{s(\omega_1,\mathbf{1})})}.$$

6.3. Constructions of p-adic L-functions for balanced twisted triple products

6.3.1. Main results (interpolation formulas for the balanced case). For notations, see the beginning of Chapter 3, Theorem 3.3.5 and its corollary. Note that ,although it's abuse of ntation, we use F as a quadratic extension over $\mathbb{Q}$ here and let $F_2 = F$ and $F_1 = \mathbb{Q}$. Let $B_F := B \otimes_{\mathbb{Q}} F$. We denote by N^- the discriminant of B. We assume χ_1

and χ_2 are trivial and Φ_1 and Φ_2 be nonzero Hecke eigen forms of $S(K_0^B(1);\mathscr{D}(\mathbb{I}_1))$ and $S(K_0^{B_F}(1);\mathscr{D}(\mathbb{I}_2))$ respectively. We denote by Π_P the representation generated by

$$\phi := \langle \mathrm{Sp}_P(\Phi)^u \otimes \mathrm{Sp}_P(\Psi)^u, \Delta_{\underline{k}}^{\xi}\rangle_{k^*-2\underline{t}^*}.$$

Then we have an isomorphism

$$\Pi_P \cong \bigotimes_{v<\infty}{}' \Pi_{P,v} \otimes (L_{k_1}(\mathbb{C}) \boxtimes L_{k_2}(\mathbb{C})),$$

where

$$L_{k_i}(\mathbb{C}) := \mathrm{Sym}_{k_i-2\underline{t}_{F_i}}(\mathbb{C}) \otimes \det{}^{1-k_i/2}.$$

We fix additive characters

$$\psi := \mathbf{e}_p$$
$$\psi_{(2\xi)^{-1}}(x) := \psi(\mathrm{tr}_{F/\mathbb{Q}}(x/\sqrt{D}))$$

and $\Pi_{P,p} = \pi_{1,P} \boxtimes \pi_{2,P}$, which $\pi_{i,P}$ is a irreducible subquotient of $\mathrm{Ind}_{B(F_i)}^{\mathrm{GL}(F_i)}(\mu_{i,P} \boxtimes \nu_{i,P})$ such that.

$$|\nu_{i,P}(p)|_p > |\mu_{i,P}(p)|_p.$$

The ϕ is corresponding to an element $\otimes_v \phi_v$ as follows: At $v \neq p, \infty$, ϕ_v is a spherical vector, at $v = \infty$, $\phi_v = \Delta_{\underline{k}}^{\xi}$ defined in Theorem 3.3.5 and at $v = p$, $\phi_v = W_p \boxtimes W'_{\mathfrak{p}} \in \mathscr{W}(\pi_{1,P}, \psi) \boxtimes \mathscr{W}(\pi_{2,P}, \psi_{(2\xi)^{-1}})$ determined in the same manner in the preceding part of Remark 6.2.4. We have the following theorem

Theorem 6.3.1. Let $P \in \mathscr{X}(\mathbb{I})$ such that $P|_{\mathbf{G}_E} = P_{k_1,w_1,\boldsymbol{\omega}_1} \times P_{k_2,w_2,\boldsymbol{\omega}_2}$ with $\boldsymbol{\omega}_i = (\omega_i, \mathbf{1})$ and ω_i factoring through $\mathrm{Cl}_F^+(p^s)$ and $k_1 < k_{2,\sigma} + k_{2,\rho}$, $k_{2,\sigma} < k_1 + k_{2,\rho}$ and $k_{2,\rho} < k_1 + k_{2,\sigma}$ hold. We have

$$\begin{aligned}
&P(\mathscr{L}_p(\Phi_1 \otimes \Phi_2)) \\
=&D_{\mathfrak{p}}^{-k_1^*-1} \cdot \prod_{q|N^-} e_q(F/\mathbb{Q}) \cdot \frac{\left[K_0^B(1) : K_1^B(\mathfrak{n}_1 \cap \mathfrak{n}_2)\right]^2}{\left[K_0^B(1) : K_1^B(\mathfrak{n}_1)\right]\left[K_0^{B_F}(1) : K_1^{B_F}(\mathfrak{n}_2)\right]} \\
&\times \frac{\mathcal{E}_p(\Pi_P)}{\mathcal{E}(\pi_{1,P}, \mathrm{Ad})\mathcal{E}(\pi_{2,P}, \mathrm{Ad})} \cdot \left(\frac{L(1/2, \mu_{1,P}\nu_{2,P})}{L(1/2, \mathrm{As}\pi_{2,P} \otimes \mu_{1,P})L(1/2, \mu_{1,P}^{-1}\nu_{2,P}^{-1})}\right)^2 \cdot \frac{L(1/2, \Pi_P)}{L(1, \mathrm{Ad}\Pi_P)}.
\end{aligned}$$

Here, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are defined in the beginning of Chapter 3, $e_q(F/\mathbb{Q})$ is the ramified index of $F/\mathbb{Q}$ at q,

$$\mathcal{E}_p(\Pi_P) := \frac{\varepsilon_{\mathrm{RS}}(1/2, \mathrm{As}\pi_2 \otimes \mu_{1,P}, \psi_{\sqrt{D}^{-1}})\varepsilon(1/2, \mu_{1,P}^{-1}\nu_{2,P}^{-1}, \psi^{-1})}{\varepsilon(1/2, \mu_{1,P}\nu_{2,P}, \psi)}$$

and $\mathcal{E}(\pi_{i,P}, \mathrm{Ad})$ is that defined in Proposition 5.3.1.

Proof. We denote by $c(\omega_i)$ the conductor of ω_i. Put

$$u_s := \begin{pmatrix} \pi_{F_{\mathfrak{p}}}^s & \xi \\ 0 & 1 \end{pmatrix},$$
$$\omega := \omega_1\omega_2,$$
$$a_{1,P} := \mu_{1,P}(p)p^{1/2},$$
$$a_{2,P} := \nu_{2,P}(\varpi_F)q^{1/2},$$

where q is the order of residue field of $F_{\mathfrak{p}}$. By Proposition 4.4, Proposition 4.9 and Corollary 5.2 of [**CC16**], we have

$$\begin{aligned}
&P(\mathscr{L}_p(\Phi_1 \otimes \Phi_2)) \\
=&\frac{(2\xi)^{1-k_1^*} C_{B/\mathbb{Q}}}{2^4\pi^2} \cdot \frac{\zeta_E(2)}{\zeta_{\mathbb{Q}}(2)} \cdot \frac{L(1/2, \Pi_P)}{L(1, \mathrm{Ad}\Pi)} \prod_{q|N^-} e_q(F/\mathbb{Q}) \\
&\times \frac{\left[K_0^B(1) : K_1^B(\mathfrak{n}_1 \cap \mathfrak{n}_2)\right]^2}{\left[K_0^B(1) : K_1^B(\mathfrak{n}_1)\right]\left[K_0^{B_F}(1) : K_1^{B_F}(\mathfrak{n}_2)\right]} \\
&\times \frac{p^{-6s}(1-p^{-2})^2}{C_{B/\mathbb{Q}}^2 a_{1,P}^{2s} a_{2,P}^{2e_p(F/\mathbb{Q})s}} \frac{a_{1,P}^{c(\omega_1)} a_{2,P}^{c(\omega_2)} C_{B/\mathbb{Q}} C_{B_F/F}}{p^{c(\omega_1)}(1+p^{-1}) q^{c(\omega_2)}(1+q^{-1})} \\
&\times \frac{\mathcal{I}_{\Pi_{P,p}}\left(\tau_{-p^s} W_p \boxtimes u_s W'_{\mathfrak{p}} \otimes \omega^{1/2}, \tau_{-p^s} W_p \otimes \omega_1^{-1} \boxtimes u_s W'_{\mathfrak{p}} \otimes (\omega_1\omega_2)^{-1/2}\omega_1\right)}{\mathfrak{i}_{\pi_{1,P}}(\tau_{-p^{s(\omega_1)}} W_p, W_p \otimes \omega_1^{-1}) \mathfrak{i}_{\pi_{2,P}}(\tau_{-\varpi_{F_{\mathfrak{p}}}^{c(\omega_2)}} W'_p, W'_p \otimes \omega_2^{-1})}.
\end{aligned}$$

by Theorem 5.4.1, we have

$$\begin{aligned}
&\mathcal{I}_{\Pi_{P,p}}\left(\tau_{-p^s} W_p \boxtimes u_s W'_{\mathfrak{p}} \otimes \omega^{-1}, \tau_{-p^s} W_p \otimes \omega_1^{-1} \boxtimes u_s W'_{\mathfrak{p}} \otimes \omega^{-1}\omega_1\right) \\
&= \frac{\varepsilon_{\mathrm{RS}}(1/2, \mathrm{As}\pi_2 \otimes \mu_{1,P}, \psi_{2\xi}^{-1})}{L(1/2, \mathrm{As}\pi_2 \otimes \mu_{1,P})^2} \Psi\left(u_s W'_{\mathfrak{p}} \otimes \omega^{-1}, \tau_{-p^s} f_p\right)^2.
\end{aligned}$$

We proceed the computation of Ψ:

$$\begin{aligned}
&\Psi\left(u_s W'_{\mathfrak{p}} \otimes \omega^{-1}, \tau_{-p^s} f_p\right) \\
&= \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^\times} W'_p\left(\begin{pmatrix} y & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} p^s & \xi \\ 0 & 1 \end{pmatrix}\right) f\left(\begin{pmatrix} y & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ p^s & 0 \end{pmatrix}\right) |y|_{\mathbb{Q}_p}^{-1} \, d^\times y dx \\
&= \nu(-p^s) p^{-s/2} \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^\times} W'_p\left(\begin{pmatrix} y & 0 \\ p^s x & 1 \end{pmatrix} \begin{pmatrix} p^s & \xi \\ 0 & 1 \end{pmatrix}\right) \mu(y) |y|_{\mathbb{Q}_p}^{-1/2} \mathbf{1}_{\mathbb{Z}_p}(x) \, d^\times y dx.
\end{aligned}$$

Since

$$\begin{pmatrix} y & 0 \\ p^s x & 1 \end{pmatrix} \begin{pmatrix} p^s & \xi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y p^s & y\xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - p^s x\xi & -x\xi^2 \\ p^{2s} x & 1 + p^s x\xi \end{pmatrix},$$

for sufficiently large s, we have

$$= \nu(-p^s) p^{-s/2} \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Q}_p^\times} W'_p\left(\begin{pmatrix} y p^s & y\xi - xyp^s\xi^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^s & \xi \\ 0 & 1 \end{pmatrix}\right) \mu(y) |y|_{\mathbb{Q}_p}^{-1/2} \, d^\times y.$$

Since $\psi_{(2\xi)^{-1}}(\mathbb{Q}_p) = \{1\}$, we have

$$\begin{aligned}
&= \mu_{1,P}(-p^s) p^{-s/2} \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Q}_p^\times} \nu_{2,P}(y) |y|_{F_{\mathfrak{p}}}^{1/2} \psi(p^{-s} y) \mu_{1,P}(p^{-s} y) |p^{-s} y|_{\mathbb{Q}_p}^{-1/2} \mathbf{1}_{\mathbb{Z}_p}(y) \, d^\times y. \\
&= \mu_{1,P}(-p^s) \mu_{1,P}(p^{-s}) p^{-s} \frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)} \int_{\mathbb{Q}_p^\times} \nu_{2,P} \mu_{1,P}(y) \psi(p^{-s} y) \mathbf{1}_{\mathbb{Z}_p}(y) |y|_{\mathbb{Q}_p}^{1/2} \, d^\times y.
\end{aligned}$$

By using the functional equation, we have

$$= \mu_{1,P}(-p^s)\mu_{1,P}(p^{-s})p^{-s}\gamma(1/2, \nu_{2,P}^{-1}\mu_{1,P}^{-1}, \psi^{-1})\frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)}\int_{\mathbb{Q}_p^\times} \nu_{2,P}^{-1}\mu_{1,P}^{-1}(y)\mathbf{1}_{-p^{-s}+\mathbb{Z}_p}(y)|y|_{\mathbb{Q}_p}^{1/2}\,d^\times y$$

$$= \mu_{1,P}(-p^s)\nu_{2,P}(p^s)p^{-s/2}\gamma(1/2, \nu_{2,P}^{-1}\mu_{1,P}^{-1}, \psi^{-1})\frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)}\mathrm{vol}(1+p^s\mathbb{Z}_p, d^\times y)$$

$$= \mu_{1,P}(-1)\zeta_{\mathbb{Q}_p}(2)(\mu_{1,P}(p)p^{1/2})^s(\nu_{2,P}(p)p)^s\gamma(1/2, \nu_{2,P}^{-1}\mu_{1,P}^{-1}, \psi^{-1})p^{-3s}.$$

On the other hand, by direct computation, we have

$$\mathfrak{i}_{\pi_{1,P}}(\tau_{-p^{c(\omega_1)}}W_p, W_p\otimes\omega_1^{-1}) = a_{1,P}^{c(\omega_1)}p^{-c(\omega_1)}\frac{\zeta_{\mathbb{Q}_p}(2)}{\zeta_{\mathbb{Q}_p}(1)}\mathcal{E}(\pi_{1,P}, \mathrm{Ad})$$

$$\mathfrak{i}_{\pi_{2,P}}(\tau_{-q^{c(\omega_2)}}W'_{\mathfrak{p}}, W'_{\mathfrak{p}}\otimes\omega_2^{-1}) = a_{2,P}^{c(\omega_2)}p^{-c(\omega_2)}\frac{\zeta_{F_{\mathfrak{p}}}(2)}{\zeta_{F_{\mathfrak{p}}}(1)}\mathcal{E}(\pi_{2,P}, \mathrm{Ad}),$$

where $\mathcal{E}(\cdot, \mathrm{Ad})$ is defined in Proposition 5.3.1. □

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