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Kyoto University
AN ENHANCEMENT OF THE ZAGIER CONJECTURE

NOBUO SATO

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1. Introduction

The enhanced zeta value was introduced in Zagier-Gangl’s article [18] as a $\mathbb{C}/(2\pi i)^m\mathbb{Q}$-valued lift (more precisely, with a bounded denominator) of the partial zeta value $\zeta(m, A) \in \mathbb{R}$ attached to an ideal class $A$ of an imaginary quadratic number field. Together with the enhanced polylogarithms, they formulated a conjecture in terms of the Bloch groups of the Hilbert class field of the imaginary quadratic field, which refines the Zagier conjecture for partial zeta values. Despite its presumable importance in relation with the regulators of $K$-groups or Bloch groups, its nature as an $L$-value is not clear from the definition. Our main goal of this article is to show that the enhanced zeta value is a partial derivative of a certain two-variable analog of the partial zeta function, namely the Shintani $L$-function for imaginary quadratic fields.

Zagier and Gangl considered the enhanced zeta value as a lift of a partial zeta value at a positive argument, while we consider them as a lift of the first derivative of a partial zeta function at a negative argument. These points of views do not make any difference, as long as we consider the partial zeta value for an ideal class, since the values of a partial zeta function at positive integers are equal to the first derivative of another partial zeta function at negative integers under the functional equation, up to a simple factor of $\pi$ and the discriminant of $F$. But when one comes to think of partial zeta values of a ray class, the two viewpoints make difference, since, for a general ray class, a partial zeta value at a positive integer is only equal to a linear combination of the first derivative of partial zeta functions at a negative integer with not necessarily rational coefficients. As we shall show in Section 5, the first derivative of the partial zeta function of a general ray class at a negative integer has a natural decomposition into partial derivatives of a certain two-variable zeta function. These partial derivatives give well-defined $\mathbb{C}/2\pi i\mathbb{Q}$-valued invariants which are complex-conjugate of each other and we shall define our ray class invariant by these partial derivatives. We shall show in Section 6, our ray class invariant is equal to Zagier-Gangl’s enhanced zeta value for a ray class up to a simple factor of $\pi$. Thus, if we consider a lift of partial zeta values at positive integers, it should be a linear combination of our ray class invariants with non-rational coefficients which can no longer be defined as a value in $\mathbb{C}/(2\pi i)^m\mathbb{Q}$. Therefore, the enhanced zeta value should be regarded as a lift of the first derivative of a partial zeta function at a negative integer if we wish to define it for a general ray class.
In Section 2, we first recall the history and explain the motivation for the study of zeta values, from Euler’s discoveries to Dedekind’s class number formula which gives a description for the residue at $s = 1$ of the Dedekind zeta function as a determinant of logarithms of algebraic numbers. At the end, we give a refined formulation of the Stark conjecture due to Rubin [10], which gives a vast generalization of the class number formula. For Stark’s original conjecture, see [12].

In Section 3, we review the classical polylogarithms and hyperbolic volume formulas in relation with the Zagier conjecture. While the Stark conjecture concerns with zeta values at $s = 1$, the Zagier conjecture generalizes the class number formula to zeta values at general positive integers. The Humbert volume formula [5] gives an expression for the volume of a certain hyperbolic orbifold in terms of a Dedekind zeta value at $s = 2$. Together with Lobachevsky’s volume calculation for hyperbolic tetrahedra [6], this leads to an expression for $\zeta_F(2)$ as a linear combination of special values of the Bloch-Wigner dilogarithm $D(z)$ at certain algebraic arguments. The Zagier conjecture is a higher analog of this formula. In the place of the real-valued logarithm $\log |z|$ that appears in the expression for zeta values at $s = 1$, and real-valued dilogarithm $D(z)$ for zeta values at $s = 2$, the Zagier conjecture uses a real-valued $s$-logarithm for $s \geq 2$. Also, the unit group that occurs in the description for zeta values at $s = 1$ is replaced by higher Bloch groups, whose “concrete” description is yet to be obtained for general $s$.

In Section 4, we explain an enhancement of the Zagier conjecture given by Zagier and Gangl [18]. On one hand, we construct the enhanced polylogarithms. Since the regulator map $K_{2m-1}(\mathbb{C}) \to \mathbb{R}$ is known to have a natural lift to $\mathbb{C}/(2\pi i)^m\mathbb{Q}$, such an enhancement is natural under the compatibility of regulators of $K$-groups and polylogarithm maps of Bloch groups given by Beilinson and Deligne. On the other hand, we construct the enhanced zeta value $I_m(\mathcal{A}) \in \mathbb{C}/(2\pi i)^m\mathbb{Q}$ of the partial zeta value $\zeta(k,\mathcal{A}) \in \mathbb{R}$ of an ideal class $\mathcal{A}$ of an imaginary quadratic field.

In Section 5, we introduce the Shintani L-function of two variables associated to a lattice in $\mathbb{C}$. By modifying an integral representation of the Shintani L-function, we give a meromorphic continuation of the Shintani L-function to consider the partial derivatives of the Shintani L-function. Then, we construct $\mathbb{C}/2\pi i\mathbb{Q}$-valued invariants $\Lambda_1(-k,\mathcal{A})$ and $\Lambda_2(-k,\mathcal{A}) = \Lambda_1(-k,\mathcal{A})$ of a ray class $\mathcal{A}$ of an imaginary quadratic field using partial derivatives of the Shintani L-function at the negative integer $(s_1, s_2) = (-k, -k)$. From the construction, it gives a natural lift of the first derivative of the partial zeta function $\zeta(s, \mathcal{A})$ at the negative integer $s = -k$. Due to the singularity of the Shintani L-function at negative integers, its partial derivative depends on the direction of approach to the point and the analysis becomes a little complicated. Note that, in the construction of our ray class invariant, we fix a “CM”-lattice i.e. a fractional ideal of an imaginary quadratic field, however, to show the equality between our enhanced zeta value and Zagier-Gangl’s one, we let the lattice deform continuously to consider a differentiation with respect to the deformation parameter.

In Section 6, we give a Fourier expansion of the restriction of the Shintani L-function $L(s_1, s_2)$ to $L(s, 0)$, using a one-dimensional integral representation for $L(s, 0)$. By showing that the Maass operator acts on Shintani L-values as a shifting operator, we associate the first partial derivative at $(s_1, s_2) = (-k - k)$ of the Shintani L-function with the first partial derivative at $(s_1, s_2) = (-2k, 0)$ of the
Shintani L-function. Together with several propositions, we shall prove our main theorem

**Theorem.** Let $\mathcal{A} \in \mathrm{Cl}(F)$ be an ideal class of an imaginary quadratic field $F$. Then, for a positive integer $m$, we have

$$I_{m+1}(\mathcal{A}^{-1}) = \frac{(2\pi i)^m}{m!} \Lambda_1 (-m, \mathcal{A}),$$

as an element of $\mathbb{C}/(2\pi i)^{m+1}\mathbb{Q}$.

Zagier-Gangl's enhanced zeta value $I_{m+1}(\mathcal{A})$ is defined for an ideal class $\mathcal{A}$ of an imaginary quadratic field and our ray class invariant $\Lambda_1 (-m, \mathcal{A})$ is defined for a ray class $\mathcal{A}$ of an imaginary quadratic field. Therefore, our main theorem above shows that our ray class invariant gives a generalization to a ray class of the Zagier-Gangl's enhanced zeta value. In Section 7, we expand Zagier-Gangl's enhanced conjecture to a general ray class. We also give several numerical examples that are not included in Zagier-Gangl's enhanced conjecture to verify the correctness of our conjecture.

**General notations.**

- We denote by $\mathbb{Z}(m)$ and $\mathbb{Q}(m)$, the subgroups $(2\pi i)^m\mathbb{Z}$ and $(2\pi i)^m\mathbb{Q}$ of $\mathbb{C}$ respectively.
- We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ and fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$.
- We denote by $\Re(z)$ and $\Im(z)$, the real part and the imaginary part of $z$ respectively.
- We denote by $M \otimes N$ and $M \wedge N$ a tensor product and wedge product over $\mathbb{Z}$ of two modules $M$ and $N$ respectively.
- For a subset $X$ of $\mathbb{R}$, we denote by $X_{\geq a}$ the subset $\{ x \in X | x \geq a \}$ of $X$ (the notations $X_{\leq a}$, $X_{> a}$ and $X_{< a}$ are also used in a similar manner).
- For a set $X$, we denote by $\#X$ the number of the elements of $X$.
- For a set $X$, we denote by $1_X$ the characteristic function of $X$.
- For a set $X$, we denote by $\mathbb{Z}[X]$ the free $\mathbb{Z}$-module formally generated by the elements of $X$.
- For a set $X$ and an abelian group $A$, we denote by $\text{Map}(X, A)$ the abelian group formed by the maps from $X$ to $A$.
- For a number field $F$, we denote by $\mathcal{O}_F$ the ring of integers of $F$ and $\mu_F$ the subgroup of $\mathcal{O}_F^\times$ formed by the roots of unity in $F$, $\text{Cl}(F)$ the ideal class group of $F$.
- We denote the number of elements of $\mu_F$ by $w_F$ and those of $\text{Cl}(F)$ by $h_F$.
- We denote by $S_F$ the set of all places of $F$ and by $\Sigma_F$ the set of all embeddings of $F$ into $\mathbb{C}$.
- We denote by $r_1$ and $r_2$ the numbers of real and complex places of $F$ respectively.
- For an element $x$ of $F$, we denote by $N(x) \in \mathbb{Q}$ the field norm of $x$ i.e.

$$N(x) := \prod_{\rho \in \Sigma_F} \rho(x),$$

- For a non-zero fractional ideal $\mathfrak{a}$ of $F$, we denote by $N(\mathfrak{a}) \in \mathbb{Q}^\times$ the norm of $\mathfrak{a}$, i.e.

$$N(\mathfrak{a}) := \frac{\#((\mathfrak{a} + \mathcal{O}_F)/\mathfrak{a})}{\#((\mathfrak{a} + \mathcal{O}_F)/\mathcal{O}_F)}.$$
Note that, for an element \( x \) of \( F \), we have \( N(xO_F) = |N(x)| \).
- For each place \( v \) of \( F \), we denote by \( F_v \) the local field at \( v \).
- For each finite place \( v \) of \( F \), we denote by \( k_v \) the residue field of \( F_v \).
- We denote by \( | |_v \) the \( v \)-adic absolute value defined by
  \[
  |z|_v := \begin{cases} 
  q_v^{-\text{ord}_v z} & \text{if } v \text{ is finite} \\
  |z| & \text{if } v \text{ is real} \\
  z\overline{v} & \text{if } v \text{ is complex,}
  \end{cases}
  \]
  where \( q_v = \#k_v \).
- For each imaginary quadratic field \( F \), we fix a complex embedding \( \rho : F \hookrightarrow \mathbb{C} \) and regard \( F \) as a subfield of \( \mathbb{C} \).

2. Zeta values and Regulators

2.1. The Riemann zeta function. Special values of zeta and L-functions are one of the most important research objects in the study of algebraic number theory. The history of the study of the zeta and the L-function goes back to the era of Euler, who discovered the celebrated equality

\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.
\]

After the discovery of this equality, Euler showed, in general, that the value

\[
1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots
\]

is equal to a rational multiple of \( \pi^k \) for an even positive integer \( k \). Moreover, he calculated the “value” of the divergent series

\[
1 + 2^{k-1} + 3^{k-1} + 4^{k-1} + \cdots,
\]

and stated that the “value” is a rational number for a positive integer \( k \). He also discovered the following duality relating the above two types of series.

\[
1 + 2^{k-1} + 3^{k-1} + \cdots = \frac{2(k-1)! \cos \left( \frac{k\pi}{2} \right)}{(2\pi)^k} \left( 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots \right).
\]

Although Euler’s calculation was not rigorous from the modern point of view, his result was not a total nonsense. After the work of Euler, Riemann gave a rigorous proof for Euler’s result. He defined the Riemann zeta function by the Dirichlet series

\[
\zeta(s) := \sum_{m=1}^{\infty} \frac{1}{m^s},
\]

which absolutely converges for \( \Re s > 1 \). He then gave an integral representation

\[
\zeta(s) = \frac{\pi^{\frac{s}{2}}}{2\Gamma\left(\frac{s}{2}\right)} \int_0^\infty (\vartheta(i\tau) - 1) \tau^{s-1} d\tau,
\]

where

\[
\vartheta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 \tau}
\]
is the Jacobi theta function. Using this integral representation, together with the
modular transformation formula
\[
\vartheta \left( -\frac{1}{\tau} \right) = \sqrt{\frac{-\pi}{i}} \vartheta (\tau)
\]
for the Jacobi theta function, Riemann proved the functional equation
\[
\tilde{\zeta}(s) = \tilde{\zeta}(1 - s)
\]
of \(\zeta(s)\), where
\[
\tilde{\zeta}(s) := \pi^{-s} \frac{\Gamma \left( \frac{s}{2} \right)}{\Gamma(s)} \zeta(s)
\]
is the complete zeta function. This functional equation is a rigorous restatement of
Euler’s duality which now makes sense for general \(s\).

2.2. Dirichlet L-function. The Riemann zeta function has several curious as-
pects. In particular, the Euler product expression
\[
\zeta(s) = \prod_{p \text{ prime}} \left( 1 - p^{-s} \right)^{-1},
\]
which is a restatement of the unique factorization of integers, is of fundamental
importance, when we consider the relation to number theory. It is somewhat a
miracle that such a product over the set of primes has an analytic continuation to
the entire complex plane (except for a simple pole at \(s = 1\)), since generally it is
not easy to achieve both properties at the same time.

After the work of Riemann, several analogous functions has been studied. Let
\(N\) be a positive integer. A homomorphism
\[
\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times
\]
is called a Dirichlet character of conductor \(N\), if \(\chi\) does not factor through \((\mathbb{Z}/N')\mathbb{Z})^\times
with \(N' < N\). For an integer \(m\) such that \((m, N) = 1\), we use the same notation
\(\chi(m)\) for \(\chi(m \mod N)\). The Dirichlet L-function is defined by
\[
L(s, \chi) := \sum_{(m, N) = 1, m > 0} \frac{\chi(m)}{m^s}
\]
for a Dirichlet character \(\chi\) of conductor \(N\). A Dirichlet character \(\chi\) is said to be
odd if \(\chi(-1) = -1\) and even if \(\chi(-1) = 1\). From the definition, the Riemann zeta
function is a special case of the Dirichlet L-functions with \(N = 1\). As can easily be
seen, Dirichlet L-functions, too, admits the Euler product expression
\[
L(s, \chi) = \prod_{p \nmid N} \left( 1 - \chi(p)p^{-s} \right)^{-1}
\]
which generalizes that of the Riemann zeta function. There is also a closely related
function to Dirichlet L-functions called the partial zeta functions. The partial zeta
functions are defined by
\[
\zeta(s, a) = \sum_{m \equiv a, m > 0} \frac{1}{m^s}
\]
for \(a \in (\mathbb{Z}/N\mathbb{Z})^\times\). The partial zeta functions has no Euler product expression in
general, but related to the Dirichlet L-function by
\[
L(s, \chi) = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)\zeta(s, a).
\]
The partial zeta functions and the Dirichlet L-function have several analogous special values, for example, it is known that

\[
\begin{cases}
L(1 - 2k, \chi) \in \mathbb{Q}^x \\
L(-2k, \chi) \in N^{-\frac{1}{2}}\pi^{2k}\mathbb{Q}^x
\end{cases}
\]

for \(k \in \mathbb{Z}_{\geq 1}\) and an even character \(\chi\).

On the other hand, there appear a new type of special values. For example, let \(\chi_5\) be a Dirichlet character of conductor 5 such that \(\chi_5(2) = -1\). Since 2 is a generator of the cyclic group \((\mathbb{Z}/5\mathbb{Z})^\times \simeq \mathbb{Z}/4\mathbb{Z}\), this determines \(\chi_5\). The resulting Dirichlet L-function is then

\[
L(s, \chi_5) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s} + \cdots
\]

\[
= \prod_{p=\pm 1(5)} (1 - p^{-s}) \prod_{p=\pm 2(5)} (1 + p^{-s}),
\]

where \(p \equiv a(N)\) means that the product is over the set \(\{x : \text{prime s.t } x \equiv a \mod N\}\).

Using the Taylor series for \(\log\) i.e.

\[-\log(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n}\]

which is valid for \(\{z \in \mathbb{C} \mid |z| \leq 1\} \setminus \{1\}\), one obtains

\[
L(1, \chi_5) = -\frac{\sum_{a \in (\mathbb{Z}/5\mathbb{Z})^\times} \chi_5(a) \log \left(1 - e^{2\pi i a/5}\right)}{\sum_{a \in (\mathbb{Z}/5\mathbb{Z})^\times} \chi_5(a)e^{2\pi i a/5}}
\]

\[
= \frac{2}{\sqrt{5}} \log \varepsilon
\]

where \(\varepsilon := \frac{1+i\sqrt{5}}{2}\) is the golden ratio.

2.3. Zeta functions for number fields. Let \(F\) be a number field, \(\mathcal{O}_F\) the ring of integers of \(F\). The definition of the Riemann zeta function as a sum of \((-s)\)-th power of positive integers does not simply extend to the definition of zeta functions for general number fields. As is well-known, the unique factorization of integers into prime numbers fails if we consider a general number field. In stead, the unique factorization of ideals into prime ideals holds, which leads to the right definition

\[
\zeta_F(s) : = \sum_{a \subseteq \mathcal{O}_F} \frac{1}{N(a)^s}
\]

\[
= \prod_p (1 - N(p)^{-s})^{-1}
\]

of the Dedekind zeta function. Here, the sum is over all non-zero ideals in \(\mathcal{O}_F\) and the product runs over all non-zero prime ideals in \(\mathcal{O}_F\). The Dedekind zeta function admits a holomorphic continuation to \(\mathbb{C}\) except for a simple pole at \(s = 1\).

Now consider the case where \(F = \mathbb{Q}(\sqrt{5})\). Then, from the quadratic reciprocity law, a (rational) prime \(p\) with \(p \equiv \pm 1 \mod 5\) splits in \(\mathbb{Q}(\sqrt{5})\), a prime \(p\) with
\( p \equiv \pm 2 \mod 5 \) stays prime in \( \mathbb{Q}(\sqrt{5}) \), and \( p = 5 \) is the only prime that ramifies in \( \mathbb{Q}(\sqrt{5}) \). Therefore, we have
\[
\zeta_{\mathbb{Q}(\sqrt{5})}(s) = (1 - 5^{-s})^{-1} \prod_{p \equiv \pm 1 \pmod{5}} (1 - p^{-s})^{-2} \times \prod_{p \equiv \pm 2 \pmod{5}} (1 - p^{-2s})^{-1} = \zeta(s)L(s, \chi_5),
\]
where \( \chi_5 \) is the Dirichlet character defined in the preceding section. Thus
\[
\text{Res}_{s=1} \left( \zeta_{\mathbb{Q}(\sqrt{5})}(s) \right) = \frac{2}{\sqrt{5}} \log \varepsilon.
\]

The point of this formula is that \( \varepsilon = \frac{1 + \sqrt{5}}{2} \) is the fundamental unit of \( \mathbb{Q}(\sqrt{5}) \). In fact, if we define the regulator map
\[
\lambda_F : \mathcal{O}_F^* \to \left\{ \sum_v x_v v \in \bigoplus_v \mathbb{R} \mid \sum_v x_v = 0 \right\} \simeq \mathbb{R}^{r_1 + r_2 - 1},
\]
where \( v \) runs over all infinite places of \( F \), by
\[
\lambda_F (\varepsilon) := -\sum_v \log |\varepsilon|_v v,
\]
then the following theorem holds in general.

**Theorem 2.1** (Dedekind’s analytic class number formula). For a number field \( F \), one has
\[
\text{Res}_{s=1} \left( \zeta_F(s) \right) = \frac{2^{r_1}(2\pi)^{r_2}h_FR_F}{w_F |D_F|^{1/2}},
\]
where \( D_F \) is the discriminant of \( F \), \( w_F \) the number of roots of unity in \( F \), \( h_F \) the class number of \( F \) and \( R_F \) the covolume of \( \lambda_F(\mathcal{O}_F^*) \).

**Remark 2.2.** Note that the \( R_F \) is expressed as the absolute value of the determinant
\[
\det (\log |\varepsilon_i|_v)_{1 \leq i \leq r_1 + r_2 - 1, v \neq w}
\]
where \( \varepsilon_1, \ldots, \varepsilon_{r_1 + r_2 - 1} \) is any basis of \( \mathcal{O}_F^*/\mu_F \) and the excluded \( w \) is any infinite place of \( F \). By the application of the functional equation of the Dedekind zeta function, the analytic class number formula has another equivalent form
\[
\zeta_F(s) = -\frac{h_FR_F}{w_F} s^{r_1 + r_2 - 1} + O(s^{r_1 + r_2}),
\]
which looks simpler.

It is natural to ask what comes in the place of Dirichlet L-function for a general number field. Let \( S_F \) be the set of all places of \( F \) and
\[
m : S_F \to \mathbb{Z}_{\geq 0}
\]
be the map such that \( m(p) = 0 \) except for a finite number of places, \( m(p) \in \{0, 1\} \) if \( p \) is real and \( m(p) = 0 \) if \( p \) is complex. Such an \( m \) is called a modulus of \( F \). The restriction of \( m \) to finite places is naturally identified with an integral ideal \( m_f \) of \( F \), and the restriction of \( m \) to infinite places is determined by the real places such that \( m(p) = 1 \). Thus, traditionally one identifies a modulus \( m \) with an expression
\[
m = \prod_p p^{m(p)} = m_f \prod_{p, \text{real}, m(p)=1} p.
\]
For a modulus \( m \), we define a subgroup

\[
F_m^\times := \{ x \in F^\times \mid x \equiv 1 \mod m, x_p > 0 \text{ if } m(p) = 1 \}
\]

of \( F^\times \). Here, \( x_p \) denotes the image of \( x \) under the real embedding corresponding to \( p \). For a finite set \( S \) of places of \( F \), let \( I_S \) be the subgroup of the ideal group of \( F \) which is generated by prime ideals of \( F \) not in \( S \) and \( S(m) \) the set of places such that \( m(p) > 0 \). We denote by \( I_m \) for \( I_{S(m)} \) if there is no risk of confusion. Define \( i : F_m^\times \to I_m \) by

\[
i(x) = xO_F.
\]

Then the ray class group \( Cl_m(F) \) associated to a modulus \( m \) is defined by

\[
Cl_m(F) := \frac{I_m}{i(F_m^\times)}.
\]

Using the ray class group, we define the Hecke L-function by the Dirichlet series

\[
L(s, \chi) := \sum_{a \in I_m, a \subset O_F} \frac{\chi(a)}{N(a)^s},
\]

where the sum is over all integral ideals in \( I_m \) and \( \chi \) is a (generalized) Dirichlet character \( \chi : I_m \to Cl_m(F) \to \mathbb{C}^\times \).

Again, the minimal \( m \) such that \( \chi \) factors through \( Cl_m(F) \) is called the conductor of \( \chi \).

As an example, let \( F = \mathbb{Q} \). Since \( \mathbb{Z} \) is a PID, \( m_f \) is identified with its generator \( N \mathbb{Z} \). There is only one real place \( \infty \) and no complex places. Therefore, one has

\[
\mathbb{Q}^\times_{\infty} = \{ x \in \mathbb{Q}^\times \mid x \equiv 1 \mod N \},
\]

\[
I_N = \left\{ (x) \mid x = \frac{a}{b} \text{ with } (a, N) = (b, N) = 1 \right\}
\]

and

\[
\mathbb{Q}^\times_{N, \infty} = \{ x \in \mathbb{Q}^\times_{\geq 0} \mid x \equiv 1 \mod N \},
\]

\[
I_{N, \infty} = I_N,
\]

thus \( Cl_N(\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times / \{ \pm 1 \} \) and \( Cl_{N, \infty}(\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \) respectively. Therefore, for \( m = N \infty \), \( L(s, \chi) \) coincides with a Dirichlet L-function of conductor \( d/N \) and , for \( m = N \), \( L(s, \chi) \) coincides with a Dirichlet L-function of conductor \( d/N \) whose character is even. We define the partial zeta function by

\[
\zeta(s, \mathcal{A}) := \sum_{a \in \mathcal{A}} \frac{1}{N(a)^s},
\]

for \( \mathcal{A} \in Cl_m(F) \), where the sum runs over all integral ideals in \( \mathcal{A} \). Thus, the partial zeta function is related to the Hecke L-function by

\[
L(s, \chi) = \sum_{\mathcal{A} \in Cl_m(F)} \chi(\mathcal{A})\zeta(s, \mathcal{A}).
\]

From the class field theory, the partial zeta function can be defined in terms of abelian extension of \( F \). Let \( H/F \) be an abelian extension. From the class field theory, there exists a modulus \( m \) and a surjection

\[
(2.3) \quad \text{rec} : Cl_m(F) \to \text{Gal}(H/F).
\]
The conductor of $H/F$ is a minimal $\mathfrak{m}$ for which such a natural surjection exists. If $\mathfrak{m}$ is the conductor of $H/F$, then (2.3) does not factor through $\text{Cl}_\mathfrak{m}(F)$ for all $\mathfrak{m}' < \mathfrak{m}$. We say that $H$ is a ray class field of conductor $\mathfrak{m}$ if (2.3) is an isomorphism and $\mathfrak{m}$ is the conductor of $H/F$. We define a partial zeta function for $\sigma \in \text{Gal}(H/F)$ by

$$\zeta(s, \sigma) := \sum_{\mathcal{A} \in \text{Cl}_\mathfrak{m}(F) \atop \text{rec}(\mathcal{A}) = \sigma} \zeta(s, \mathcal{A})$$

where $\mathfrak{m}$ is a conductor of $H/F$. For $F = \mathbb{Q}$, the ray class field of modulus $N \cdot \infty$ is $\mathbb{Q}(\zeta_N)$ and that of modulus $N$ is $\mathbb{Q}(\zeta_N + \zeta_N^{-1})$, which is the maximal totally real subfield of $\mathbb{Q}(\zeta_N)$. The partial zeta function in this case is given by

$$\sum_{m \equiv a(N)} \frac{1}{m^s}, \sum_{m \equiv \pm a(N)} \frac{1}{m^s}$$

for the the element $\sigma_a : \zeta_N \mapsto \zeta_N^a$ of $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ and $\text{Gal}(\mathbb{Q}(\zeta_N + \zeta_N^{-1})/\mathbb{Q})$, respectively.

Motivated by the analytic class number formula, Stark formulated the Stark conjecture which generalizes the analytic class number formula to partial zeta functions (more generally, to Artin $L$-functions). Here, we give a refined formulation of the Stark conjecture due to Rubin [10]. First, let us define several notations. Fix a finite abelian extension $H$ of $F$, a finite set $S, T$ of places of $F$ such that

$$\begin{cases} S \supset S_\infty \cup S_{\text{ram}} \\ T \cap S = \emptyset. \end{cases}$$

where $S_\infty$ and $S_{\text{ram}}$ denotes the set of infinite places of $F$ and the set of all the places ramified in $H/F$, respectively. Now set $G = \text{Gal}(H/F)$. Following Rubin, we define

- $S_H := \{\text{places of } H \text{ lying above } S\}, T_H := \{\text{places of } H \text{ lying above } T\}.$
- $\mathcal{O}_S := \{x \in H | |x|_v \leq 1 \text{ for all } v \notin S_H\}.$
- $\mathcal{O}_S^{\leq} := \{x \in \mathcal{O}_S | x \equiv 1 \text{ mod } v \text{ for all } v \in T_H\}.$
- $X_S := \ker(\bigoplus_{v \in S_H} \mathbb{Z}_v \to \mathbb{Z}), \text{ with } v \mapsto 1 \text{ for } v \in S_H.$
- $\lambda_{S,T} : \mathcal{O}_S^{\leq} \to \mathbb{R} \otimes X_S \text{ with } x \mapsto -\sum_v \log |x|_v.$

Here, we assume that $\mathcal{O}_S^{\leq}$ is torsion-free. Define the $S$-modified partial zeta function $\zeta_S(s, \sigma)$ by

$$\zeta_S(s, \sigma) := \sum_a N(a)^{-s},$$

where $a$ runs over integral ideals in $\text{rec}^{-1}(\sigma)$ which is coprime to $S$, and define the Stickelberger functions

$$\theta_S(s) := \sum_{\sigma \in G} \zeta_S(s, \sigma^{-1}) \sigma \in \mathbb{C}[G],$$

$$\theta_{S,T}(s) := \theta_S(s) \cdot \prod_{v \in T} (1 - \text{Frob}_v^{-1} \cdot N(v)^{1-s}) \in \mathbb{C}[G].$$

Here, we denoted by $\text{Frob}_v \in G$ the Frobenius element of $v$. Take any $V \subsetneq S$ such that each $v \in V$ splits completely in the extension $H/F$. We choose any labeling $v_0, v_1, \ldots, v_r$ of the elements of $S$ so that $V = \{v_1, \ldots, v_r\}$. We also fix $w_i \in S_H$
such that $w_i | v_i$ for $i \in \{0, \ldots, n\}$. Then it is known that $s^{-r} \theta_{S,T}(s)$ is holomorphic at $s = 0$ (see [10, Section 2.1]) and thus the value

$$\theta_{S,T}^{(r)} := \lim_{s \to 0} s^{-r} \theta_{S,T}(s)$$

is well-defined. Since $O_{S,T}^\times$ and $X_S$ are $G$-modules, we can define

$$\begin{cases}
  O_{S,T}^{(r)} := \bigwedge_{\mathbb{Z}[G]} O_{S,T}^\times \\
  X_S^{(r)} := \bigwedge_{\mathbb{Z}[G]} X_S.
\end{cases}$$

Then $\lambda_{S,T}$ induces the isomorphism

$$\det \lambda_{S,T} : \mathbb{R} \otimes O_{S,T}^{(r)} \xrightarrow{\sim} \mathbb{R} \otimes X_S^{(r)}.$$

The Rubin-Stark element $V_{S,T} \in \mathbb{R} \otimes O_{S,T}^{(r)}$ is then defined by

$$V_{S,T} := (\det \lambda_{S,T})^{-1} \left( \theta_{S,T}^{(r)} \cdot (w_1 - w_0) \wedge \cdots \wedge (w_r - w_0) \right).$$

Define the Rubin lattice $\bigcap_{\mathbb{Z}[G]} O_{S,T}^\times \subset \mathbb{Q} \otimes O_{S,T}^{(r)}$ by

$$\bigcap_{\mathbb{Z}[G]} O_{S,T}^\times := \left\{ \varepsilon \in \mathbb{Q} \otimes O_{S,T}^{(r)} \mid \Phi(\varepsilon) \in \mathbb{Z}[G], \forall \Phi \in \mathcal{O}_{S,T}^{(r)\ast} \right\}$$

where $\mathcal{O}_{S,T}^{(r)\ast} := \bigwedge_{\mathbb{Z}[G]} \text{Hom}_{\mathbb{Z}[G]} \left( \mathcal{O}_{S,T}^\times, \mathbb{Z}[G] \right)$ and $\Phi(\varepsilon)$ is the natural paring defined by

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r) := \det (\varphi_i(\varepsilon_j))_{1 \leq i, j \leq r}$$

for $\Phi = \varphi_1 \wedge \cdots \wedge \varphi_r$ and $\varepsilon = \varepsilon_1 \wedge \cdots \wedge \varepsilon_r$. Rubin conjectured

**Conjecture 2.4** (Rubin). Let the notations be as above. Then

$$V_{S,T} \in \bigcap_{\mathbb{Z}[G]} O_{S,T}^\times.$$

### 3. Polylogarithms and hyperbolic volumes

**3.1. The classical polylogarithms.** In the Zagier conjecture, polylogarithms play an essential role. Therefore, we first review the classical polylogarithms. The $k$-logarithm $\text{Li}_k(z)$ is defined, most naively, by the power series

$$\text{Li}_k(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

for $|z| < 1$. In particular,

$$\text{Li}_1(z) = - \log (1 - z),$$

where the branch of log is taken to be the principal value.

By differentiation, one obtains the inductive relation

$$z \frac{d}{dz} \text{Li}_k(z) = \text{Li}_{k-1}(z),$$

for $k \geq 2$, and

$$(1 - z) \frac{d}{dz} \text{Li}_1(z) = 1,$$
which leads to the linear differential equation
\[
\frac{d}{dz}(1 - z) \frac{d}{dz} \left( \frac{z d}{dz} \right)^{k-1} \text{Li}_k(z) = 0.
\]

Note that \(\text{Li}_1(z) = -\log(1 - z)\) can be analytically continued to \(\mathbb{C} \setminus \mathbb{R}_{\geq 1}\). Hence

\[
\text{Li}_k(z) = \int_0^z \text{Li}_{k-1}(z) \frac{dz}{z}
\]

is also analytically continued to \(\mathbb{C} \setminus \mathbb{R}_{\geq 1}\).

**Proposition 3.2** (Distribution relation). For each positive integer \(n\), we have

\[
\text{Li}_k(z^n) = n^{k-1} \sum_{\omega^n = 1} \text{Li}_k(\omega z), \quad |z| < 1
\]

**Proof.** Since
\[
\sum_{\omega^n = 1} \omega^r = \begin{cases} n & \text{if } n \mid r, \\ 0 & \text{if } n \nmid r, \end{cases}
\]
it follows that
\[
\sum_{\omega^n = 1} \text{Li}_k(\omega z) = \sum_{\omega^n = 1} \sum_{m=1}^{\infty} \frac{(\omega z)^m}{m^k} = n \sum_{m=1}^{\infty} \frac{z^n m^k}{(nm)^k} = n^{1-k} \text{Li}_k(z^n).
\]

**Proposition 3.3** (Reflection formula). For \(z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}\), we have

\[
\text{Li}_k(z) + (-1)^k \text{Li}_k(z^{-1}) = - \frac{(2\pi i)^k}{k!} B_k \left( \frac{\log z}{2\pi i} \right),
\]

where \(B_k(t)\) is the \(k\)-th Bernoulli polynomial. Here, the branch of \(\log\) is taken so that

\[
0 < \Im(\log z) < 2\pi.
\]

**Proof.** If \(k = 1\), then we have
\[
\text{Li}_1(z) - \text{Li}_1(z^{-1}) = -\log(1 - z) + \log(1 - z^{-1}) = -\log z + \pi i = -2\pi i B_1 \left( \frac{\log z}{2\pi i} \right).
\]

For \(k > 1\), we proceed by induction. Assume that the proposition holds for \(k\). Put

\[
f_k(z) := \text{Li}_k(z) + (-1)^k \text{Li}_k(z^{-1}) + \frac{(2\pi i)^k}{k!} B_k \left( \frac{\log z}{2\pi i} \right)
\]

for \(k \in \mathbb{Z}_{\geq 1}\). Then
\[
z \frac{d}{dz} f_{k+1}(z) = \text{Li}_k(z) + (-1)^k \text{Li}_k(z^{-1}) + \frac{(2\pi i)^k}{(k+1)!} B_{k+1} \left( \frac{\log z}{2\pi i} \right).
\]
Since $B_{k+1}'(t) = (k+1)B_k(t)$, we have $f_{k+1}'(z) = 0$. It follows that $f_{k+1}(z)$ is a constant. Since
\[
\lim_{z \to 1} f_{k+1}(z) = \zeta(k+1) + (-1)^{k+1}\zeta(k+1) + \frac{(2\pi i)^{k+1}}{(k+1)!} B_{k+1} = 0,
\]
we have $f_{k+1}(z) = 0$. \hfill \Box

3.2. The dilogarithm function. If we focus our attention to the dilogarithm, much more is known. For example, we have

**Proposition 3.4** (Reflection formula).

The dilogarithm function $\text{Li}_2(z)$ satisfies the following functional equations.

1. $\text{Li}_2(z) + \text{Li}_2(z^{-1}) = -\frac{\pi^2}{6} - \frac{1}{2} \left(\log(-z)\right)^2$ for $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$.
2. $\text{Li}_2(z) + \text{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z) \log(1-z)$ for $z \in \mathbb{C} \setminus (\mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 1})$.

**Proof.** (1) is a special case of the previous proposition. For (2), by integration by parts, we have
\[
\text{Li}_2(z) = -\int_0^z \frac{\log(1-z)}{z} \, dz = -\log z \cdot \log(1-z) - \int_0^z \frac{\log z}{1-z} \, dz = -\log z \cdot \log(1-z) - (\text{Li}_2(1-z) - \text{Li}_2(1)).
\]
Since $\text{Li}_2(1) = \pi^2/6$, we obtain (2). \hfill \Box

Either of (1) or (2) yields the following corollary.

**Corollary 3.5.** The upper and lower limits to the real axis
\[
\text{Li}_2(z_\pm) := \lim_{\varepsilon \to +0} \text{Li}_2(z \pm i\varepsilon) \quad (z \in \mathbb{R}_{\geq 1})
\]
of $\text{Li}_2(z)$ are related by
\[
\text{Li}_2(z_+) - \text{Li}_2(z_-) = 2\pi i \log z.
\]

**Proof.** Since $1 - z \in \mathbb{R}_{<0}$, we have
\[
\lim_{\varepsilon \to +0} \text{Li}_2(1 - (z + i\varepsilon)) = \lim_{\varepsilon \to +0} \text{Li}_2(1 - (z - i\varepsilon)).
\]
Thus, by formula (2) of Proposition 3.4,
\[
(3.6) \quad \text{Li}_2(z_+) - \text{Li}_2(z_-) = \lim_{\varepsilon \to +0} (\text{Li}_2(z + i\varepsilon) - \text{Li}_2(z - i\varepsilon)) = \lim_{\varepsilon \to +0} (\log(z + i\varepsilon) \cdot \log(1 - z + i\varepsilon) - \log(z + i\varepsilon) \cdot \log(1 - z - i\varepsilon)).
\]
Since
\[
\lim_{\varepsilon \to +0} \log(z \pm i\varepsilon) = \log z, \quad \lim_{\varepsilon \to +0} \log(1 - z \pm i\varepsilon) = \log(z - 1) \pm \pi i,
\]
the right-hand side of the equality (3.6) equals
\[
2\pi i \log z.
\]
\hfill \Box
By reflection formulas, we see that $\text{Li}_2(z)$, $-\text{Li}_2(z^{-1})$ and $-\text{Li}_2(1-z)$ are equal up to a few terms which are products of $\log z$, $\log(1-z)$ and $2\pi i$. The reflections $\sigma_1 : z \mapsto z^{-1}$ and $\sigma_2 : z \mapsto 1-z$ generate six values
\begin{equation}
(3.7)
\end{equation}

\begin{align*}
S_z := \left\{ z, \frac{1}{1-z}, \frac{1-z}{z}, \frac{1}{z}, 1-z, \frac{z}{z-1} \right\}
\end{align*}
of $S_3$-symmetry, thus the values

$\text{Li}_2(z), \text{Li}_2\left(\frac{1}{1-z}\right), \text{Li}_2\left(\frac{z-1}{z}\right), -\text{Li}_2\left(\frac{1}{z}\right), -\text{Li}_2(1-z), -\text{Li}_2\left(\frac{z}{z-1}\right)$

are equal up to a few terms which are products of $\log z$, $\log(1-z)$ and $2\pi i$. From this symmetry, together with the distribution relation

$\text{Li}_2(z^2) = 2 \left( \text{Li}_2(z) + \text{Li}_2(-z) \right),$

one can obtain the following values of $\text{Li}_2(z)$ in terms of $2\pi i$ and log of some algebraic numbers.

\begin{align*}
\text{Li}_2(0) &= 0, \\
\text{Li}_2(1) &= \frac{\pi^2}{6}, \\
\text{Li}_2(-1) &= \frac{-\pi^2}{12}, \\
\text{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2, \\
\text{Li}_2(2^\pm) &= \frac{\pi^2}{4} \pm \pi i \log 2,
\end{align*}

and

\begin{align*}
\text{Li}_2\left(\varepsilon^{-2}\right) &= \frac{\pi^2}{15} - (\log \varepsilon)^2, \\
\text{Li}_2\left(\varepsilon^{-1}\right) &= \frac{\pi^2}{10} - (\log \varepsilon)^2, \\
\text{Li}_2(\varepsilon_\pm^2) &= \frac{4\pi^2}{15} \pm 2\pi i \log \varepsilon - (\log \varepsilon)^2, \\
\text{Li}_2(-\varepsilon) &= -\frac{\pi^2}{10} + \frac{1}{2}(\log \varepsilon)^2, \\
\text{Li}_2\left(-\varepsilon^{-1}\right) &= -\frac{\pi^2}{15} + \frac{1}{2}(\log \varepsilon)^2, \\
\text{Li}_2(\varepsilon_\pm) &= \frac{7\pi^2}{30} \pm \pi i \log \varepsilon + \frac{1}{2}(\log \varepsilon)^2.
\end{align*}
Then from the equality
\[ S = \{ \epsilon, -\epsilon, \epsilon^{-2}, \epsilon^{-1}, -\epsilon^{-1}, -\epsilon^2 \} , \]
Therefore, we can express \( L_2(\alpha) \) for \( \alpha \in S \) by a sum of the form
\[ \pm L_2(\epsilon^{-1}) + a(2\pi i)^2 + b(2\pi i) \log \epsilon + c(\log \epsilon)^2 \]
with some \( a, b, c \in \mathbb{Q} \). On the other hand, we have
\[ L_2(\epsilon^{-2}) = 2 \left( L_2(\epsilon^{-1}) + L_2(-\epsilon^{-1}) \right) \]
from the distribution relation. Combining these equalities yields the equality of the form
\[ 5L_2(\epsilon^{-1}) = a(2\pi i)^2 + b(\log \epsilon)^2 \quad (a, b \in \mathbb{Q}) \]
from which we obtain the values \( L_2(\alpha) \) for \( \alpha \in S \). Note that, for \( \alpha \in \{ \epsilon, \epsilon^2 \} \) \( = S \cap \mathbb{R}_{\geq 1} \), we have two values i.e. \( L_2(\alpha_{\pm}) \) neither of which are real numbers. Similarly, for \( z = -1 \), the reflections 3.7 generate the three values
\[ S_{-1} = \left\{ -1, 1, \frac{1}{2} \right\} . \]
Again, we can express \( L_2(\alpha) \) for \( \alpha \in S_{-1} \) by a sum of the form
\[ \pm L_2(-1) + a(2\pi i)^2 + b(2\pi i) \log 2 + c(\log 2)^2 \quad (a, b, c \in \mathbb{Q}) . \]
In this case, we have
\[ L_2(1) = 2 \left( L_2(1) + L_2(-1) \right) \]
from the distribution relation (here, we consider \( L_2(1) \) as a limit from the inside of the unit circle). Therefore, we obtain the expressions for \( L_2(\alpha) \) for \( \alpha \in S_{-1} \). Finally, for \( \alpha \in S_1 = \{ 0, 1, (\infty) \} \), the values \( L_2(\alpha) \) are well-known.

Proposition 3.8 (The five-term relation). For \( 0 < x, y < 1 \), we have
\[
L_2(x) + L_2(y) + L_2 \left( \frac{1 - x}{1 - xy} \right) + L_2 \left( 1 - xy \right) + L_2 \left( \frac{1 - y}{1 - xy} \right)
= \frac{\pi^2}{2} - \log x \cdot \log(1 - x) - \log y \cdot \log(1 - y) - \log \left( \frac{1 - x}{1 - xy} \right) \cdot \log \left( \frac{1 - y}{1 - xy} \right) .
\]

Proof. One can show this equality by simply differentiating both sides. To avoid tedious calculations, we make use of the symmetry in the equality. Consider the real variables \( \{ x_i \}_{i \in \mathbb{Z}/5\mathbb{Z}} \) with relations
\[ \{ 1 - x_i = x_{i-2}^2 + 1 \}_{i \in \mathbb{Z}/5\mathbb{Z}} . \]
These variables are called cluster variables associated to the quiver of type \( A_2 \). By setting \( x_0 = x, x_1 = y \) and expressing \( \{ x_i \}_{i \in \mathbb{Z}/5\mathbb{Z}} \) by \( x \) and \( y \), one finds that
\[ x_0 = x, x_1 = y, x_2 = \frac{1 - x}{1 - xy}, x_3 = 1 - xy, x_4 = \frac{1 - y}{1 - xy} \]
is exactly the arguments appearing on the left-hand side of the five-term relation. Then from the equality
\[
\sum_{i \in \mathbb{Z}/5\mathbb{Z}} x_i \otimes x_{i+2} = x_0 \otimes (x_2x_3) + x_1 \otimes (x_3x_4) + x_2 \otimes x_4 = x \otimes (1 - x) + y \otimes (1 - y) + \left( \frac{1 - x}{1 - xy} \right) \otimes \left( \frac{1 - y}{1 - xy} \right)
\]
in $\text{Sym}^2 \left( \mathbb{R}_+^\times \right) := \left( \mathbb{R}_+^\times \otimes \mathbb{R}_+^\times \right) / \langle u \otimes v - v \otimes u \mid u, v \in \mathbb{R}_+^\times \rangle$ (we remind the reader that the tensor product is over $\mathbb{Z}$), we see that

$$\sum_{i \in \mathbb{Z}/\mathbb{S}Z} \log x_i \log x_{i+2} = \log x \log(1-x) + \log y \log(1-y) + \log \left( \frac{1-x}{1-xy} \right) \log \left( \frac{1-y}{1-xy} \right).$$

Thus the five-term relation is equivalent to the equality

$$\sum_{i \in \mathbb{Z}/\mathbb{S}Z} (\text{Li}_2(x_i) + \log x_i \log x_{i+2}) = \frac{\pi^2}{2}.$$

Since

$$d \left\{ \sum_{i \in \mathbb{Z}/\mathbb{S}Z} (\text{Li}_2(x_i) + \log x_i \log x_{i+2}) \right\} = \sum_{i \in \mathbb{Z}/\mathbb{S}Z} \left( \text{Li}_1(x_i) \frac{dx_i}{x_i} + \log x_{i+2} \frac{dx_i}{x_i} + \log x_i \frac{dx_{i+2}}{x_{i+2}} \right)$$

$$= \sum_{i \in \mathbb{Z}/\mathbb{S}Z} (\text{Li}_1(x_i) + \log x_{i+2} + \log x_{i-2}) \frac{dx_i}{x_i}$$

$$= 0,$$

the quantity $\sum_{i \in \mathbb{Z}/\mathbb{S}Z} (\text{Li}_2(x_i) + \log x_i \log x_{i+2})$ is a constant. By letting $x, y \to +0$, we find that the constant is $\pi^2/2$. \[ \square \]

**Remark 3.9.** From the relation

$$\sum_{i \in \mathbb{Z}/\mathbb{S}Z} x_i \otimes x_{i+2} = \frac{1}{2} \sum_{i \in \mathbb{Z}/\mathbb{S}Z} (x_i \otimes x_{i-2} + x_i \otimes x_{i+2})$$

$$= \frac{1}{2} \sum_{i \in \mathbb{Z}/\mathbb{S}Z} x_i \otimes (1-x_i),$$

the five-term relation is also equivalent to

$$\sum_{i \in \mathbb{Z}/\mathbb{S}Z} \left( \text{Li}_2(x_i) + \frac{1}{2} \log x_i \log (1-x_i) \right) = \frac{\pi^2}{2}.$$

Here, the function

$$R(x) := \text{Li}_2(x) + \frac{1}{2} \log x \log (1-x) \quad (0 < x < 1)$$

is called the Rogers dilogarithm which can be extended to a real analytic function on $\mathbb{P}^1(\mathbb{R}) \setminus \{0, 1, \infty\}$ by

$$R(x) := \begin{cases} R \left( \frac{1}{1-x} \right) - \frac{\pi^2}{6} & (x < 0) \\ -R \left( \frac{1}{x} \right) + \frac{\pi^2}{6} & (1 < x) \end{cases}.$$ 

Then the Rogers dilogarithm defines a continuous function on $\mathbb{R}$ as shown in the following graph.
The behavior of $R(x)$

Furthermore, if we define the function $R^* : \mathbb{P}^1(\mathbb{R}) \setminus \{0, 1, \infty\} \to \mathbb{R}/2^3\mathbb{Z}(2)$ by

$$R^*(x) := R(x) \mod \frac{1}{2^3}\mathbb{Z}(2)$$

it is continuous on $\mathbb{P}^1(\mathbb{R})$ and satisfies cleaner functional equations, such as

$$R^*(x) + R^*(y) + R^*(\frac{1 - x}{1 - xy}) + R^*(1 - xy) = 0.$$  

We shall explain a curious relation between the Rogers dilogarithm and the enhanced dilogarithm in the end of Section 4.1.

### 3.3. The monodromy of polylogarithms

So far, we settled a branch cut along $[0, +\infty)$, to avoid the complexity of the multivaluedness of the polylogarithms. The monodromy of $\text{Li}_k(z)$ was first computed by Ramakrishnan [8]. In this section, we calculate the monodromy of $\text{Li}_k(z)$ as follows. Consider the linear differential equation

$$d\frac{dz}{dz} (1 - z) d\frac{dz}{dz} (z d\frac{dz}{dz} f(z))^k = 0$$

satisfied by $\text{Li}_k(z)$. By solving the equation 3.10 inductively, one can see that any local solution is given by

$$\text{Li}_k(c; z) := c_k \text{Li}_k(z) + \sum_{j=0}^{k-1} c_j (2\pi i)^{k-j} \frac{(\log z)^j}{j!}$$

with $c := (c_k, c_{k-1}, \ldots, c_0) \in \mathbb{C}^{k+1}$. Let $\gamma_0, \gamma_1$ denote the generator of the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ corresponding to the counter-clockwise loop around 0, 1 respectively. Then we have

**Proposition 3.11** (Ramakrishnan). *Define a $(k + 1)$-dimensional representation

$$\mathcal{M} : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) = \langle \gamma_0, \gamma_1 \rangle \to GL_{k+1}(\mathbb{Q})$$
by
\[ M(\gamma_0) := \exp \begin{bmatrix} 0 & 0 \\ 0 & J_k \end{bmatrix}, \quad M(\gamma_1) := \exp \begin{bmatrix} -J_2 & 0 \\ 0 & 0 \end{bmatrix}, \]
with \( J_n := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathcal{M}_n(\mathbb{Q}). \]

Then we have
\[ \text{Li}_k(c; \gamma z) = \text{Li}_k(c; M(\gamma); z) \]
for \( \gamma \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}). \)

Proof. By the definition of \( \mathcal{M} \), it is sufficient to prove the statement for \( \gamma_0 \) and \( \gamma_1 \). Since \( \text{Li}_k(z) \) is holomorphic around \( z = 0 \), and \( \log(\gamma_0 z) = \log z + 2\pi i \), the monodromy around \( z = 0 \) is just
\[ \text{Li}_k(c; \gamma_0 z) = c_k \text{Li}_k(z) + \sum_{0 \leq j \leq k-1} c_j (2\pi i)^k \frac{(\log z + 2\pi i)^j}{j!} \]
\[ = c_k \text{Li}_k(z) + \sum_{0 \leq j' \leq k-1} \frac{c_j (2\pi i)^k}{(j - j')!} \frac{(\log z)^{j'}}{j'!} \]
\[ = c_k \text{Li}_k(z) + \sum_{0 \leq j' \leq k-1} \left( \sum_{j' \leq j \leq k-1} \frac{c_j}{(j - j')!} \right) \frac{(2\pi i)^k}{j'!} \frac{(\log z)^{j'}}{j'!}. \]

Since \( e^{J_k} = 1 + \frac{1}{\pi} J_k + \frac{1}{\pi^2} J_k^2 + \cdots + \frac{1}{(k-1)!} J_k^{k-1}, \) we have proved \( \text{Li}_k(c; \gamma_0 z) = \text{Li}_k(c; M(\gamma_0); z). \)

For \( \gamma_1 \), we prove the formula by induction on \( k \). In contrast to the \( \gamma_0 \) case, \( \log z \) is holomorphic around \( z = 1 \). On the other hand, \( \text{Li}_k(z) \) has non-trivial monodromy and we are to show the equality
\[ \text{Li}_k(\gamma_1 z) = \text{Li}_k(z) - 2\pi i \frac{(\log z)^{k-1}}{(k-1)!} \]
for \( k \geq 1 \). Put
\[ F_k(z) := \text{Li}_k(\gamma_1 z) - \text{Li}_k(z) + 2\pi i \frac{(\log z)^{k-1}}{(k-1)!} \]
for \( z \in \mathbb{C} \setminus \mathbb{R}_{\geq 1} \). Then obviously \( F_1(z) = 0 \). For \( k \geq 2 \), we have \( z \frac{d}{dz} (F_k(z)) = F_{k-1}(z). \) If we assume \( F_{k-1}(z) = 0, \) \( F_k(z) \) is a constant. To prove that the constant is zero, we shall show \( \lim_{\varepsilon \to 0^+} F_k(\alpha + i\varepsilon) = 0 \) for \( \alpha \in \mathbb{R}_{>1} \) as follows.

Let \( \alpha \in \mathbb{R}_{>1} \). Since
\[ \lim_{\varepsilon \to 0^+} \text{Li}_k(\gamma_1(\alpha + i\varepsilon)) = \lim_{\varepsilon \to 0^+} \text{Li}_k(\alpha - i\varepsilon), \]
we have
\[ \lim_{\varepsilon \to 0^+} F_k(\alpha + i\varepsilon) = \lim_{\varepsilon \to 0^+} \left( \text{Li}_k(\alpha - i\varepsilon) - \text{Li}_k(\alpha + i\varepsilon) + 2\pi i \frac{(\log(\alpha + i\varepsilon))^{k-1}}{(k-1)!} \right). \]
Since

\[ \text{Li}_k(\alpha \pm i \varepsilon) = (-1)^{k-1} \text{Li}_k ((\alpha \pm i \varepsilon)^{-1}) - \frac{(2\pi i)^k}{k!} B_k \left( \frac{\log(\alpha \pm i \varepsilon)}{2\pi i} \right) \]

by Proposition 3.3 and

\[ \lim_{\varepsilon \to +0} \text{Li}_k ((\alpha + i \varepsilon)^{-1}) = \lim_{\varepsilon \to +0} \text{Li}_k ((\alpha - i \varepsilon)^{-1}) \]

(this is because \( \alpha^{-1} \in \mathbb{C} \setminus \mathbb{R}_{\geq 1} \)), we have

\[ \lim_{\varepsilon \to +0} F_k(\alpha + i \varepsilon) \]

\[ = \frac{(2\pi i)^k}{k!} \lim_{\varepsilon \to +0} \left\{ B_k \left( \frac{\log(\alpha + i \varepsilon)}{2\pi i} \right) - B_k \left( \frac{\log(\alpha - i \varepsilon)}{2\pi i} \right) + k \left( \frac{\log(\alpha + i \varepsilon)}{2\pi i} \right)^{k-1} \right\} \]

\[ = \frac{(2\pi i)^k}{k!} \left\{ B_k \left( \frac{\log \alpha}{2\pi i} \right) - B_k \left( \frac{\log \alpha}{2\pi i} + 1 \right) + k \left( \frac{\log \alpha}{2\pi i} \right)^{k-1} \right\}. \]

From the property \( B_k(t+1) - B_k(t) = kt^{k-1} \) of the Bernoulli polynomial,

\[ \lim_{\varepsilon \to +0} F_k(\alpha + i \varepsilon) = 0. \]

Hence we have shown \( F_k(z) = 0 \) for \( k \geq 1 \) by induction, and this proves \( \text{Li}_k(c; \gamma_1 z) = \text{Li}_k(cM(\gamma_1); z) \).

\( \square \)

Remark 3.12. Proposition 3.11 gives a generalization of the equality given in Corollary 3.5. In particular, we see that for \( \gamma \in \pi_1 \left( \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \right) \), there exist \( \nu_0(\gamma), \ldots, \nu_{k-1}(\gamma) \in \mathbb{Z} \) such that

\[ \text{Li}_k(\gamma z) = \text{Li}_k(z) + \sum_{j=0}^{k-1} \nu_j(\gamma) \frac{(2\pi i)^{k-j}}{(k-1-j)!} \frac{(\log z)^j}{j!}. \]

That is to say, if we choose

\[ \{\text{Li}_k(z)\} \cup \left\{ \frac{(2\pi i)^{k-j}}{(k-1-j)!} \frac{(\log z)^j}{j!} \right\}_{0 \leq j \leq k-1} \]

as a basis of a local solution, we get an integral monodromy representation of \( \pi_1 \left( \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \right) \). It is also worth noting that the value

\[ \text{Li}_k(1) := \left( \lim_{\varepsilon \to +1} \text{Li}_k(z) \right) \mod \frac{1}{(k-1)!} \mathbb{Z}(k) \]

is well-defined and is equal to \( \zeta(k) \), from the above expression for \( \text{Li}_k(\gamma z) \). Similarly, if \( \alpha \) is a \( n \)-th root of unity, the value

\[ \text{Li}_k(\alpha) \mod \frac{1}{n^{k-1}(k-1)!} \mathbb{Z}(k) \]

is well-defined.
3.4. **Hyperbolic 3-space \( \mathbb{H}^3 \).** We define the hyperbolic 3-space \( \mathbb{H}^3 \) by

\[
\mathbb{H}^3 := \mathbb{C} \times \mathbb{R}^*_+
\]

\[
= \{(z, r) | z \in \mathbb{C}, r > 0\}.
\]

We consider \( \mathbb{H}^3 \) as a subset of Hamiltonian quaternion \( \mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \) by

\[
\mathbb{H}^3 = \{(x + iy, r) | x + iy + rj\}
\]

We equip \( \mathbb{H}^3 \) with the hyperbolic metric

\[
ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}
\]

which gives rise to the volume element

\[
dV = \frac{dx dy dz}{r^3}.
\]

The “boundary” of \( \mathbb{H}^3 \) is given by \( \mathbb{C} \cup \{\infty\} \), which is identified with the Riemann sphere \( \mathbb{P}^1(\mathbb{C}) \). The geodesic lines with respect to the hyperbolic metric are half-circles or half-line in \( \mathbb{H}^3 \) which are orthogonal to the boundary plane \( \mathbb{C} = \{x + iy\} \subset \mathbb{H} \) in Euclidean sense. The geodesic planes are Euclidean hemispheres or half-planes which are perpendicular to the boundary plane \( \mathbb{C} \) of \( \mathbb{H} \).

For distinct three points \( a, b, c \in \mathbb{H}^3 := \mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C}) \), the geodesic plane containing \( a, b \) and \( c \) is denoted by \( H_{a,b,c} \). For distinct four points \( a, b, c, d \in \mathbb{H}^3 \), the tetrahedron \( \Delta_{a,b,c,d} \) is the set surrounded by \( H_{a,b,c}, H_{a,b,d}, H_{a,c,d} \) and \( H_{b,c,d} \). If \( \{a, b, c, d\} \subset \mathbb{P}^1(\mathbb{C}) \), the tetrahedron \( \Delta_{a,b,c,d} \) is called an ideal tetrahedron.

There is a natural action of \( SL_2(\mathbb{C}) \) on \( \mathbb{H}^3 \) as follows. Namely, the action of \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{C}) \) on \( P \in \mathbb{H}^3 \subset \mathbb{H} \) is given by

\[
P \mapsto M(P) := (aP + b)(cP + d)^{-1},
\]

where the inverse is taken in \( \mathbb{H} \). In terms of \( \mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^*_+, \) this action is given by

\[
M(z + rj) = z' + r'j
\]

with \( \left\{ \begin{array}{l} z' = \frac{az + bj(cz + d) + ur^2}{|c + dj|^2 + |c|^2 r^2} \\ r' = \frac{r}{|cz + dj|^2 + |c|^2 r^2} = \frac{r}{|cP + d|^2} \end{array} \right. \).

Note that this action stabilizes the plane \( \mathbb{C} \times \{0\} \), where the action 3.13 reduces to the usual Möbius transformation.

**Lemma 3.14.** (1) The action of \( SL_2(\mathbb{C}) \) on \( \mathbb{H}^3 \) is transitive.

(2) The stabilizer of \( j \in \mathbb{H}^3 \) is

\[
SU(2) = \left\{ \left( \begin{array}{cc} u & v \\ -\overline{v} & \overline{u} \end{array} \right) \middle| |u|^2 + |v|^2 = 1 \right\}.
\]

(3) The metric \( ds^2 \) is invariant by the action of \( SL_2(\mathbb{C}) \).

**Proof.** (1) follows from

\[
\left( \begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} \sqrt{r} \\ \sqrt{r}^{-1} \end{array} \right) (j) = (z, r).
\]
For (2), observe that
\[ M(j) = j \]
if and only if
\[ aj + b = j(cj + d) = -\tau + \overline{\tau}j. \]
Hence \( a = \overline{\tau}, b = -\tau \) and thus \( M \in SU(2) \).

As for (3), note that \( SL_2(\mathbb{C}) \) is generated by
\[ \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bigg| b \in \mathbb{C} \right\} \text{ and } \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}, \]
whose action is given by
\[ (z, r) \mapsto (z + b, r) \tag{3.15} \]
and
\[ (z, r) \mapsto \left( \frac{-\tau}{|z|^2 + r^2}, \frac{r}{|z|^2 + r^2} \right), \tag{3.16} \]
respectively. It is now easy to see that the metric \( ds^2 \) is invariant under these transformations, hence under any transformation by \( SL_2(\mathbb{C}) \).

3.5. The Lobachevsky function. The Lobachevsky function \( L(\theta) \) is defined by
\[ L(\theta) := -\int_0^\theta \log |2\sin t| \, dt. \]
We shall show that \( L(\theta) \) is related to the dilogarithm function \( Li_2(z) \) as follows.

**Proposition 3.17.** For \( 0 < \theta < \pi \), we have
\[ 2iL(\theta) = Li_2\left(e^{2i\theta}\right) - \frac{\pi^2}{6} + \pi \theta - \theta^2. \]
In particular,
\[ L(\theta) = \frac{1}{2} \Im \left( Li_2\left(e^{2i\theta}\right) \right). \]

**Proof.** Since the latter equality is an immediate consequence of the former, we prove only the former formula. Recall that the dilogarithm function \( Li_2(z) \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_{\geq 1} \) and continuous on the closed unit disc \( \{|z| \leq 1\} \) by setting \( Li_2(1) = \frac{\pi^2}{6} \).

Put
\[ \phi(z) := \frac{-\log(1-z)}{z} = \sum_{n=1}^\infty \frac{z^{n-1}}{n} \quad \text{for } |z| < 1. \]

From the definition, \( \phi(z) \) defines a holomorphic function on \( \mathbb{C} \setminus \mathbb{R}_{\geq 1} \). As is well-known, \( Li_2(z) \) is expressed as
\[ Li_2(z) = \int_0^z \phi(w) \, dw. \]
Let \( \varepsilon \) and \( \theta \) be real numbers such that \( 0 < \varepsilon < \theta < \pi \). Then we have
\[ \int_0^{e^{2i\theta}} \phi(w) \, dw = \int_0^{e^{2i\varepsilon}} \phi(w) \, dw + \int_{e^{2i\varepsilon}}^{e^{2i\theta}} \phi(w) \, dw. \]
By setting \( w = e^{2i\varphi} \), the second term can be modified as
\[
\int_{e^{2i\varphi}} e^{2i\varphi} \phi(w) \, dw = - \int_{e^{2i\varphi}} \log(1 - e^{2i\varphi}) \, 2idt
\]
\[
= - \int_{e^{2i\varphi}} \left( \log(2\sin t) + i \left( t - \frac{\pi}{2} \right) \right) \, 2idt
\]
\[
= [t^2 - \pi t]_{e^{2i\varphi}} - 2i \int_{e^{2i\varphi}} \log(2\sin t) \, dt.
\]
Thus we have
\[
-2i \int_{e^{2i\varphi}} \log(2\sin t) \, dt = \text{Li}_2 \left( e^{2i\varphi} \right) - \text{Li}_2 \left( e^{2i\pi} \right) - [t^2 - \pi t]_{e^{2i\varphi}}.
\]

Since \( \text{Li}_2 \) is continuous on the unit circle, the improper integral
\[
\int_{\varphi=0}^{2\varphi} \log(2\sin t) := \lim_{\varphi \to 0} \int_{\varphi}^{2\varphi} \log(2\sin t) \, dt
\]
exists, and we have
\[
2i \Pi(\varphi) = \text{Li}_2 \left( e^{2i\varphi} \right) - \frac{\pi^2}{6} + \pi \varphi - \varphi^2
\]
as desired.

**Lemma 3.18.** The Lobachevsky function \( \Pi(\varphi) \) satisfies the following relations.

1. \( \Pi(\varphi + \pi) = \Pi(\varphi) \).
2. \( \Pi(-\varphi) = -\Pi(\varphi) \).
3. For each positive integer \( n \), we have
   \[
   \Pi(n\varphi) = n \sum_{j=0}^{n-1} \Pi \left( \varphi + \frac{j\varphi}{n} \right).
   \]

**Proof.** (1) and (2) follow immediately from
\[
\Pi(\varphi) = \frac{1}{2} \Im \left( \text{Li}_2 \left( e^{2i\varphi} \right) \right)
\]
given in Proposition 3.17. (3) follows from the distribution relation
\[
\text{Li}_2 \left( z^n \right) = n \sum_{j=0}^{n-1} \text{Li}_2 \left( e^{2\pi i j} z \right), \text{ for } |z| < 1
\]
which is valid for \(|z| = 1\) by continuity.

**Lemma 3.19** (Lobachevsky). Let \( z_1, z_2, z_3 \in \mathbb{C} \) be distinct three complex numbers. Let \( \alpha, \beta \) and \( \gamma \) be the angles of the Euclidean triangle \( T_{z_1, z_2, z_3} \) with vertices \( z_1, z_2 \) and \( z_3 \). Then the volume of the ideal tetrahedron \( \Delta_{\infty, z_1, z_2, z_3} \) is given by
\[
\text{Vol} \left( \Delta_{\infty, z_1, z_2, z_3} \right) = \Pi(\alpha) + \Pi(\beta) + \Pi(\gamma).
\]

**Proof.** By the action of \( SL_2(\mathbb{C}) \), we may assume that
\[
|z_1| = |z_2| = |z_3| = 1.
\]
Then the geodesic plane \( H_{z_1, z_2, z_3} \) is the upper half of the unit sphere and the ideal tetrahedron \( \Delta_{\infty, z_1, z_2, z_3} \) is as drawn in the following figure.
The hyperbolic tetrahedron $\Delta_{\infty, z_1, z_2, z_3}$

Suppose, for simplicity, we assume the origin 0 is in the interior of the triangle $T_{z_1, z_2, z_3}$. Join the origin to the three midpoints

$$z_{12} := \frac{z_1 + z_2}{2}, \quad z_{13} := \frac{z_1 + z_3}{2}, \quad z_{23} := \frac{z_2 + z_3}{2}$$

of the three sides, and the three vertices $z_1, z_2, z_3$ by line segments, as indicated in the following figure.

Subdivision of the triangle $T_{z_1, z_2, z_3}$

Thus we obtain a subdivision of $T_{z_1, z_2, z_3}$ into six right triangles. Note that each pair of triangles that share a side perpendicular to a side of $T_{z_1, z_2, z_3}$ are reflection
of one another. Consider, say, the triangle $T$ with vertices $0, z_2$, and $z_3$. Since the angle $\alpha$ is a half of the arc joining $z_2$ and $z_3$, the angle of $T$ at the origin is equal to $\alpha$.

Vertical projection of the subdivision of $T_{z_1, z_2, z_3}$ upwards yields a subdivision of $\triangle_{\infty, z_1, z_2, z_3}$ into six tetrahedra. Thus, the calculation of the volume of $\triangle_{\infty, z_1, z_2, z_3}$ reduces to the calculation of the volume of each small piece of six tetrahedra. Let us calculate the volume of the small tetrahedron $\triangle_T$ corresponding to $T$. Since $\triangle_T$ is the region

$$\triangle_T = \left\{ (z, r) \mid z \in T, r \geq \sqrt{1 - |z|^2} \right\} \subset \mathbb{C} \times \mathbb{R}_{\geq 0},$$

its volume is given by

$$\text{Vol} (\triangle_T) = \int \int_{x + iy \in T} \int_{r = \sqrt{1 - x^2 - y^2}}^{\infty} \frac{dx dy dr}{r^3} = \frac{1}{2} \int \int_{x + iy \in T} \frac{dx dy}{1 - x^2 - y^2}.$$

We may assume

$$T = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \cos \alpha, 0 \leq y \leq x \tan \alpha \}.$$ Setting $a = \sqrt{1 - x^2}$ and integrating with respect to $y$, we have

$$\text{Vol} (\triangle_T) = \frac{1}{4} \int_{\cos \alpha}^{\cos \alpha} \log \left( \frac{\alpha \cos \alpha + x \sin \alpha}{\alpha \cos \alpha - x \sin \alpha} \right) \frac{dx}{\alpha}.$$

Hence, if we set $x = \cos \theta$, the integral becomes

$$\frac{1}{4} \int_{0}^{\cos \alpha} \log \left( \frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)} \right) \frac{dx}{\alpha} = \frac{1}{4} \left( \text{Li}(2\alpha) - \text{Li}(0) - \text{Li} \left( \frac{\pi}{2} + \alpha \right) + \text{Li} \left( \frac{\pi}{2} - \alpha \right) \right).$$

By Lemma 3.18, we have

$$\frac{1}{2} \text{Li}(2\alpha) = \text{Li}(\alpha) + \text{Li} \left( \frac{\pi}{2} + \alpha \right)$$

$$= \text{Li}(\alpha) - \text{Li} \left( \frac{\pi}{2} - \alpha \right).$$

Hence $\text{Vol} (\triangle_T) = \frac{1}{2} \text{Li}(\alpha)$. Thus it readily follows that

$$\text{Vol} (\triangle_{\infty, z_1, z_2, z_3}) = 2 \left( \frac{1}{2} \text{Li}(\alpha) + \frac{1}{2} \text{Li}(\beta) + \frac{1}{2} \text{Li}(\gamma) \right)$$

$$= \text{Li}(\alpha) + \text{Li}(\beta) + \text{Li}(\gamma),$$

which completes the proof of Lemma 3.19. □

3.6. The Bloch-Wigner function. The Bloch-Wigner function or the Bloch-Wigner dilogarithm is a real analytic function on $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$ closely related to the Lobachevsky function [17]. It is defined by

$$D(z) := \Im (\text{Li}_2(z) - \log |z| \text{Li}_1(z)).$$
Put $F(z) = \text{Li}_2(z) - \log |z| \text{Li}_1(z)$. From the monodromy of $\text{Li}_2(z)$ and $\text{Li}_1(z)$ given in Section 3.3, we have

$$F(\gamma_0 z) = F(z),$$

$$F(\gamma_1 z) = F(z) + 2\pi \arg z.$$

Hence we have

$$D(\gamma_0 z) = D(z),$$

$$D(\gamma_1 z) = D(z),$$

from which we see that $D(z)$ defines a real analytic function on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Moreover, the functional equations of $\text{Li}_2(z)$ become clean, if we pass to $D(z)$. For example, the $\mathfrak{S}_3$-symmetry of $\text{Li}_2$ yields the $\mathfrak{S}_3$-symmetry

$$D(z) = D\left(\frac{1}{1-z}\right) = D\left(\frac{z-1}{z}\right),$$

$$= -D\left(\frac{1}{z}\right) = -D\left(1-z\right) = -D\left(\frac{z}{z-1}\right)$$

of $D(z)$, and the five term relation of $\text{Li}_2(z)$ yields the five term relation

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0$$

of $D(z)$ (in this case, valid for general $x, y$ such that $x, y, \frac{1-x}{1-xy}$ etc. lie outside $\{0, 1, \infty\}$, thanks to the vanishing of the monodromy).

From the definition of $D(z)$, its restriction to the unit circle is, in fact, the Lobachevsky function i.e.

$$D(e^{2i\theta}) = 2\Pi(\theta).$$

Therefore, $D(z)$ gives a real analytic extension of the $\Pi(\theta)$ to $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. On the other hand,

**Lemma 3.20.** For $z \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, we have the following relation.

$$D(z) = \frac{1}{2} \left\{ D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-z^{-1}}{1-\bar{z}^{-1}}\right) + D\left(\frac{(1-z)^{-1}}{(1-\bar{z})^{-1}}\right) \right\}$$

$$= \mathfrak{J}\left(\arg z\right) + \mathfrak{J}\left(\arg\left(\frac{z-1}{z}\right)\right) + \mathfrak{J}\left(\arg\left(\frac{1}{1-z}\right)\right).$$

**Proof.** The equality between the expression in $D$ and $\mathfrak{J}$ is clear. Hence it is sufficient to show the first equality. If we set $x = z^{-1}, y = \bar{z}^{-1}z$ in the five-term relation of $D$, we have

$$D(z^{-1}) + D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-z^{-1}}{1-\bar{z}^{-1}}\right) + D(1-\bar{z}^{-1}) + D\left(\frac{1-\bar{z}^{-1}z}{1-\bar{z}^{-1}}\right) = 0.$$

Since $D(z^{-1}) = -D(z)$ and $D(1-\bar{z}^{-1}) = D(\bar{z}) = -D(z)$, we have

$$2D(z) = D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-z^{-1}}{1-\bar{z}^{-1}}\right) + D\left(\frac{1-z/\bar{z}}{1-\bar{z}^{-1}}\right).$$

Since

$$\frac{1-\bar{z}^{-1}z}{1-\bar{z}^{-1}} = 1 - \frac{1-z}{1-\bar{z}},$$

we obtain the desired equality. \qed
Theorem 3.21. For \( z \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \) with \( \Re z > 0 \), we have
\[
\operatorname{Vol}(\Delta_{\infty,0,1,z}) = D(z).
\]

Proof. Recall the settings of Lemma 3.19. Assume that \( \Re z > 0 \). Then the three angles of the triangle \( T_{0,1,z} \) is given by
\[
\arg z, \arg \left(\frac{z-1}{z}\right), \arg \left(\frac{1}{1-z}\right).
\]
Therefore, Theorem 3.21 follows from Lemmas 3.19 and 3.20.

Remark 3.22. We define the sign \( \operatorname{sign}(a, b, c, d) \) of the configuration of four points \( a, b, c, d \in \mathbb{C} \times \mathbb{R}_{>0} \) as the sign of determinant \( \det(c-b, d-b, a-b) \) by regarding \( a, b, c, d \in \mathbb{R}^3 \). This definition naturally extends to the case where one of four points is \( \infty \). We define the “signed volume” \( \operatorname{Vol}^* \) of \( \Delta_{a,b,c,d} \) by
\[
\operatorname{Vol}^*(\Delta_{a,b,c,d}) := \operatorname{sign}(a, b, c, d) \cdot \operatorname{Vol}(\Delta_{a,b,c,d}).
\]
Then one can easily verify that the signed volume is invariant under the action of \( SL_2(\mathbb{C}) \) and Theorem 3.21 extends to the equality
\[
\operatorname{Vol}^*(\Delta_{\infty,0,1,z}) = D(z)
\]
for \( z \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \).

From Theorem 3.21, one can obtain the volume formula for general ideal tetrahedron \( \Delta_{z_0,z_1,z_2,z_3} \). Consider the element \( \gamma \) of \( SL_2(\mathbb{C}) \) such that
\[
\gamma z_0 = \infty, \gamma z_1 = 0, \gamma z_2 = 1.
\]
Such a \( \gamma \) is unique and given by
\[
\gamma z = \frac{(z - z_1)(z_2 - z_0)}{(z - z_0)(z_2 - z_1)},
\]
as easily seen. Therefore, if we define the cross-ratio \( r(z_0, z_1, z_2, z_3) \) of distinct four points \( z_0, z_1, z_2, z_3 \in \mathbb{P}^1(\mathbb{C}) \) by
\[
r(z_0, z_1, z_2, z_3) := \frac{(z_3 - z_1)(z_2 - z_0)}{(z_3 - z_0)(z_2 - z_1)},
\]
we have
\[
\operatorname{Vol}^*(\Delta_{z_0,z_1,z_2,z_3}) = \operatorname{Vol}^*(\Delta_{\infty,0,1,r(z_0,z_1,z_2,z_3)})
= D \circ r(z_0, z_1, z_2, z_3).
\]
Now, the meanings as well as the proofs of the \( \mathfrak{S}_3 \)-symmetry and the five-term relation of \( D \) become clear. By permutation of the vertices \( z_0, z_1, z_2, z_3 \), we have
\[
D \circ r(z_0, z_1, z_2, z_3) = \operatorname{sign}(\sigma) \cdot D \circ r(z_{\sigma(0)}, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})
\]
for \( \sigma \in \mathfrak{S}_4 \). Here, the cross-ratio is invariant under the permutations by Klein four-group
\[
\mathcal{V} = \{ \text{id}, (01)(23), (02)(13), (03)(12) \},
\]
hence the above equality yields the \( \mathfrak{S}_3 \)-symmetry. For the five-term relation, we consider the triangulation of distinct five points \( z_0, z_1, z_2, z_3, z_4 \in \mathbb{P}^1(\mathbb{C}) \). This yields the equality
\[
\sum_{i=0}^{4} (-1)^i D \circ r(z_0, \ldots, \hat{z}_i, \ldots, z_4) = 0.
\]
which is equivalent to the five-term relation.

3.7. The Humbert volume formula. Let \( F = \mathbb{Q}(\sqrt{D}) \) be an imaginary quadratic field of discriminant \( D \). By fixing an embedding \( F \to \mathbb{C} \), \( SL_2(O_F) \) is regarded as a discrete subgroup of \( SL_2(\mathbb{C}) \). As we have explained in Section 3.4, \( SL_2(\mathbb{C}) \) acts on the hyperbolic 3-space \( \mathfrak{H}^3 \), hence the quotient

\[
SL_2(O_F) \setminus \mathfrak{H}^3
\]

is a hyperbolic 3-orbifold of finite volume with respect to the volume element induced from that of \( \mathfrak{H}^3 \). Humbert showed that

**Theorem 3.23** (Humbert [5]). Let \( \text{Vol}(M) \) denote the hyperbolic volume of \( M \). Then one has

\[
\text{Vol} \left( SL_2(O_F) \setminus \mathfrak{H}^3 \right) = \frac{|D|^{3/2}}{4\pi^2} \zeta_F(2).
\]

This theorem was extended to a general number field \( F \) as follows. Let \( F \) be a number field with \( r_1 \) real places and \( r_2 \) complex places. Then \( F \otimes_{\mathbb{Q}} \mathbb{R} \) can be identified with \( \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \). Thus \( SL_2(F) \) can be naturally embedded into \( SL_2(\mathbb{R})^{r_1} \times SL_2(\mathbb{C})^{r_2} = SL_2(F \otimes_{\mathbb{Q}} \mathbb{R}) \).

Then \( SL_2(O_F) \) is a discrete subgroup of \( SL_2(\mathbb{R})^{r_1} \times SL_2(\mathbb{C})^{r_2} \), where \( O_F \) denotes the ring of integers of \( F \). There is a natural action of \( SL_2(\mathbb{R})^{r_1} \times SL_2(\mathbb{C})^{r_2} \) on \( \mathfrak{H}^{r_1} \times (\mathfrak{H}^3)^{r_2} \) where \( \mathfrak{H} \) is the usual upper-half plane. Then the following theorem holds.

**Theorem 3.24.** Let \( D_F \) denote the discriminant of \( F \). Then one has

\[
\text{Vol} \left( SL_2(O_F) \setminus \mathfrak{H}^{r_1} \times (\mathfrak{H}^3)^{r_2} \right) = 2 |D_F|^{3/2} \pi^{-r_1}(8\pi^2)^{-r_2} \zeta_F(2).
\]

Here, the upper-half plane \( \mathfrak{H} \) is equipped with the invariant measure \( dV = \frac{dx\,dy}{y} \), and the hyperbolic 3-space \( \mathfrak{H}^3 \) is equipped with the invariant measure \( dV = \frac{dx\,dy\,dr}{r} \).

**Remark 3.25.** In fact, this theorem follows from the Tamagawa number formula for \( SL_2 \). Let us recall the definition of the Tamagawa measure of a connected semi-simple algebraic group \( G \) defined over \( F \).

Let \( \mathbb{A}_F \) be the adele ring of \( F \). Recall that the Tamagawa measure of \( \mathbb{A}_F \) is the Haar measure of \( \mathbb{A}_F \), normalized as \( \text{Vol}(F \setminus \mathbb{A}_F) = 1 \). For each place \( v \) of \( F \), let \( dx_v \) be the Haar measure of \( F_v \) such that

\[
\begin{align*}
\text{Vol}(O_v) &= 1 & \text{if } v \text{ is finite} \\
\text{Vol}(\mathbb{R}/\mathbb{Z}) &= 1 & \text{if } v \text{ is real} \\
\text{Vol}(\mathbb{C}/\mathbb{Z}[i]) &= 2 & \text{if } v \text{ is complex},
\end{align*}
\]

where we denote by \( O_v \) the ring of integers of \( F_v \). Note that \( dx_v = 2\,da\,db \) if \( F_v = \mathbb{C} \), where \( x_v = a + bi \in \mathbb{C}, a, b \in \mathbb{R} \). The Tamagawa measure of \( \mathbb{A}_F \) is then given by

\[
dx = |D_F|^{-1/2} \prod_v dx_v.
\]

Let \( G \) be a connected semi-simple algebraic group defined over \( F \). Then the space of invariant 1-forms on \( G \) is the dual space \( g^* \) of the Lie algebra \( g \) of \( G \). Choose a non-zero element

\[
\omega \in \bigwedge^{\dim G} g^*
\]
defined over $F$. Then $\omega$ together with $dx_v$ gives rise to a Haar measure $|\omega|_v$ on $G(F_v)$. The Tamagawa measure $dg_{\text{Tam}}$ of $G(\mathbb{A}_F)$ is then defined by

$$dg_{\text{Tam}} := |D_F|^{-\frac{1}{2} \dim G} \prod_v |\omega|_v.$$ 

Since $\wedge^{\dim G} \mathfrak{g}^*$ is a 1-dimensional vector space over $F$, the definition of $dg_{\text{Tam}}$ does not depend on the choice of $\omega$. The Tamagawa number $\tau(G)$ is defined by the volume of $G(F) \setminus G(\mathbb{A}_F)$ with respect to $dg_{\text{Tam}}$, i.e.

$$\tau(G) = \int_{G(F) \setminus G(\mathbb{A}_F)} dg_{\text{Tam}}.$$ 

It is a well-known fact that $\tau(SL_2) = 1$ (see, for example, [15, Theorem 3.3.1]).

The Lie algebra $\mathfrak{g}$ of $G = SL_2$ is given by

$$\mathfrak{g} := \text{Lie}(G) = \{ X \in M_2(F) | \text{tr}(X) = 0 \},$$

which has a basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis. Put $\omega := e_1^* \wedge e_2^* \wedge e_3^*$. Then the Tamagawa measure $dg_{\text{Tam}}$ on $G(\mathbb{A}_F)$ is given by

$$dg_{\text{Tam}} = |D_F|^{-\frac{3}{2}} \prod_v |\omega|_v.$$ 

For each place $v$ of $F$, let $K_v$ be the maximal compact subgroup of $G(F_v)$ given by

$$K_v := \begin{cases} SL_2(O_v) & \text{if } v \text{ is finite} \\ SO(2) & \text{if } v \text{ is real} \\ SU(2) & \text{if } v \text{ is complex}. \end{cases}$$

Put $\mathbb{K}_{\text{fin}} := \prod_{v < \infty} K_v$ and $\mathbb{K}_\infty := \prod_{v | \infty} K_v$. By strong approximation theorem, we have

$$G(\mathbb{A}_F) = G(F) \mathbb{K}_{\text{fin}} G(F_\infty) \text{ with } G(F_\infty) : = \prod_{v | \infty} G(F_v).$$

Since $SL_2(\mathcal{O}_F) = SL_2(F) \cap \mathbb{K}_{\text{fin}} G(F_\infty)$, we have

$$SL_2(\mathcal{O}_F) \setminus G(F_\infty) / \mathbb{K}_\infty \simeq SL_2(F) \setminus SL_2(\mathbb{A}_F) / \mathbb{K}_{\text{fin}} \cdot \mathbb{K}_\infty.$$ 

Note that $G(F_\infty) / \mathbb{K}_\infty \simeq \hat{\mathbb{A}}^* \times (\hat{\mathbb{A}}^3)^{r_2}$. Suppose that $v$ is a finite place. Then we have

$$\int_{\mathbb{K}_v} |\omega|_v = q_v^{-3} |SL_2(k_v)| = 1 - q_v^{-2}.$$

Next consider the case $F_v = \mathbb{R}$. Put

$$f_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, f_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
Then \( \text{Lie}(SO(2)) = \mathbb{R} f_3 \). Let \( dk \) be the Haar measure of \( SO(2) \) determined by the dual basis of \( f_3 \). Then we have

\[
\int_{SO(2)} dk = \text{Vol}(S^1) = 2\pi.
\]

Note that

\[
\exp(x f_1) \cdot i = x + i,
\]
\[
\exp(y f_2) \cdot i = e^{yi}.
\]

It follows that the dual basis of \( \{ f_1, f_2 \} \subset \text{Lie}(SL_2(\mathbb{R}))/\text{Lie}(SO(2)) \) gives rise to an invariant measure

\[
dV = \frac{dx \, dy}{y^2}.
\]

Note that \( e_1 \wedge e_2 \wedge e_3 = 2f_1 \wedge f_2 \wedge f_3 \). Hence we have

\[
|\omega|_v = |e_1^* \wedge e_2^* \wedge e_3^*| = \frac{1}{2} |f_1^* \wedge f_2^* \wedge f_3^*| = \frac{1}{2} dk dV.
\]

Now consider the case \( F_v = \mathbb{C} \). We think of \( SL_2(\mathbb{C}) \) as an algebraic group over \( \mathbb{R} \). Then

\[
\{ e_1, ie_1, e_2, ie_2, e_3, ie_3 \} \subset \text{Lie}(SL_2(\mathbb{C})) = \{ X \in M_2(\mathbb{C}) | \text{tr}(X) = 0 \}
\]

is a basis of \( \text{Lie}(SL_2(\mathbb{C})) \) over \( \mathbb{R} \). Let

\[
\{ e_1^*, (ie_1)^*, e_2^*, (ie_2)^*, e_3^*, (ie_3)^* \}
\]

be the dual basis. Since \( dx_v = 2\, da \, db \) for \( x_v = a + bi \in \mathbb{C} \) \( (a, b \in \mathbb{R}) \), we have

\[
|\omega|_v = 8 \left| e_1^* \wedge (ie_1)^* \wedge e_2^* \wedge (ie_2)^* \wedge e_3^* \wedge (ie_3)^* \right|.
\]

Put

\[
f_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},
\]

\[
f_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad f_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.
\]

Then \( \text{Lie}(SU(2)) = \mathbb{R} f_1 \oplus \mathbb{R} f_5 \oplus \mathbb{R} f_6 \). Let \( dk \) be the Haar measure of \( SU(2) \) which comes from the dual basis \( f_1^*, f_5^* \) and \( f_6^* \). Then

\[
\int_{SU(2)} dk = \text{Vol}(S^3) = 2\pi^2.
\]

As in the real case, \( \{ f_1, f_2, f_3 \} \) gives rise to the invariant measure

\[
dV = \frac{dx \, dy \, dr}{r^3}.
\]
Together with the Humbert volume formula, we obtain an expression of $\mathfrak{A}$. Thus, we can express the volume of $M$ on $H_m$ where each $r_i$ can be triangulated into a finite union of direct products (more precisely, decomposed into a polysimplicial complex) placed into a polysimplicial complex).

Putting together, we have
\[
\tau(SL_2) = \left|DF\right|^{-3/2} \left(\prod_{v<\infty} \text{Vol}(K_v)\right) \cdot \left(\frac{\text{Vol}(SO(2))}{2}\right)^r \cdot (4\text{Vol}(SU(2)))^{r_2} \times \text{Vol}
\left(SL_2(\mathcal{O}_F) \setminus \mathfrak{A} \times (\mathfrak{A}^3)^{r_2}\right)
\]

Here, $Z(SL_2) = \{\pm 1\}$ denotes the center of $SL_2$. Since $\tau(SL_2) = 1$, we have
\[
\text{Vol}
\left(SL_2(\mathcal{O}_F) \setminus \mathfrak{A} \times (\mathfrak{A}^3)^{r_2}\right) = 2 \left|DF\right|^{3/2} \cdot \pi^{-r_1} (8\pi^2)^{-r_2} \zeta_F(2)
\]
as desired.

3.8. The Bloch group. By the theorem due to Lobachevsky, the volume of any tetrahedron in $\mathfrak{A}$ is expressible as a finite combination of the volume of ideal tetrahedra. We linearly extend the definition of $D$ : $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \rightarrow \mathbb{R}$ to $D : Z[\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}] \rightarrow \mathbb{R}$ . Thus the volume of any tetrahedron $\Delta$ in $\mathfrak{A}$ is expressible as a value of $D(\xi)$ at an element $\xi \in Z[\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}]$.

It is known that the quotient
\[
M_F := SL_2(\mathcal{O}_F) \setminus \mathfrak{A} \times (\mathfrak{A}^3)^{r_2}
\]
can be triangulated into a finite union of direct products (more precisely, decomposed into a polysimplicial complex)
\[
\bigcup_{i \in J} \prod_{v<\infty} \Delta^{(i)}_v
\]
where each $\Delta^{(i)}_v$ is a triangle in $\mathfrak{A}$ whose area is a rational multiple of $\pi$ if $v$ is a real place, and a tetrahedron in $\mathfrak{A}$ whose volume is $D(\xi^{(i)}_v)$ where $\xi^{(i)}_v$ is a certain element of $Z[\mathbb{Q}[\mathbb{Q}] \setminus \{0, 1\}] \subset Z[\mathbb{C} \setminus \{0, 1\}]$ if $v$ is a complex place (see [16, Section 2]). Thus, we can express the volume of $M_F$ as
\[
\text{Vol}(M_F) = \sum_{i \in J} \prod_{v<\infty} \text{Vol}(\Delta^{(i)}_v).
\]

Together with the Humbert volume formula, we obtain an expression of $\zeta_F(2)$
\[
\zeta_F(2) = \left|DF\right|^{-3/2} \pi^{2(r_1 + r_2)} \sum_{i \in J} \prod_{v \in \mathbb{C}} D(\xi^{(i)}_v),
\]
where each $m_i$ is a rational number and each $\xi^{(i)}_v$ is an element of $Z[\mathbb{Q}[\mathbb{Q}] \setminus \{0, 1\}]$. Now, for simplicity, we consider the case when $r_2 = 1$. Then the above formula says that there exist $\xi_F \in \mathbb{Q}[\mathbb{Q}[\mathbb{Q}] \setminus \{0, 1\}]$ such that
\[
\zeta_F(2) = \left|DF\right|^{-3/2} \pi^{2(r_1 + 1)} D(\xi_F).
\]
From a combinatorial analysis of the triangulation, \( \xi_{F} = \sum i m_{i}[z_{i}] \) is subject to the relation

\[
\sum i m_{i}(z_{i} \wedge (1 - z_{i})) = 0
\]

in \( \mathbb{C}^{\times} \wedge \mathbb{C}^{\times} \) (= the multiplicative wedge product of \( \mathbb{C}^{\times} \) over \( \mathbb{Z} \)). (See [7, Theorem 1.1].) We put

\[
\mathcal{C}(K) := \left\langle \sum_{i=0}^{4} (-1)^{i} [r(z_{0}, \ldots, z_{i}, \ldots, z_{4})] \mid z_{0}, \ldots, z_{4} \in \mathbb{P}^{1}(K) \text{ such that } z_{i} \neq z_{j} \text{ for } i \neq j \right\rangle
\]

(i.e. the \( \mathbb{Z} \)-module generated by five term relations of \( D(z) \)) for a field \( K \). Then it can be shown that \( \xi_{F} \neq \xi'_{F} \in \mathcal{C}(\mathbb{Q}) \) for elements \( \xi_{F} \) and \( \xi'_{F} \) obtained from different triangulations of \( M_{F} \).

Motivated by this observation, Bloch defined the group \( B(K) \) for a field \( K \) as follows. Let

\[
\beta : \mathbb{Z}[K \setminus \{0,1\}] \rightarrow K^{\times} \wedge K^{\times}
\]

be the map defined by

\[
\beta \left( \sum i m_{i}[z_{i}] \right) := \sum i m_{i} z_{i} \wedge (1 - z_{i}),
\]

and the submodule \( \mathcal{A}(K) \subset \mathbb{Z}[K \setminus \{0,1\}] \) by

\[
\mathcal{A}(K) := \ker \beta.
\]

Then the Bloch group \( B(K) \) is defined by

\[
B(K) := \mathcal{A}(K) / \mathcal{C}(K).
\]

Since \( \mathcal{C}(K) \subset \mathcal{A}(K) \), the group is well-defined. From the definition, \( \xi_{F} \) explained above defines an element of \( B(\mathbb{Q}) \) which is independent of the choice of the triangulation of \( M_{F} \).

3.9. Algebraic \( K \)-groups and the Borel regulator. Let \( R \) be a commutative ring with unit. The \( i \)-th algebraic \( K \)-group \( K_{i}(R) \) of \( R \) is defined as the \( i \)-th homotopy group

\[
K_{i}(R) = \pi_{i}(K(R)),
\]

where \( K(R) \) is a certain topological space. We explain a construction of \( K(R) \) based on Quillen’s plus construction (see [14]).

First, let \( GL(R) \) be the direct limit of \( GL_{n}(R) \) by the standard embedding

\[
GL_{n}(R) \hookrightarrow GL_{n+m}(R)
\]

\[
x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.
\]

Here, we equip \( GL(R) \) with the discrete topology. For distinct integers \( i, j \in \{1, \ldots, n\} \) and \( r \in R \), we denote by \( e_{ij}(r) \) the elementary matrix \( (m_{kl}) \in GL_{n}(R) \) defined by

\[
m_{kl} = \begin{cases} 1 & \text{if } k = l \\ r & \text{if } (i, j) = (k, l) \\ 0 & \text{otherwise}, \end{cases}
\]

and denote by \( E_{n}(R) \) the subgroup of \( GL_{n}(R) \) generated by elementary matrices. A group is called perfect, if it is equal to its commutator subgroup. It is known that the direct limit \( E(R) \) of \( E_{n}(R) \) is the perfect radical (= maximal perfect subgroup)
of $GL(R)$. (Moreover, one can show that $E_n(R)$ is a perfect subgroup of $GL_n(R)$ for $n \geq 3$.)

For a topological space $Y$, let us denote by $Y^I$ the space of paths $p: [0, 1] \to Y$ in $Y$ with the compact-open topology. Let $X, Y$ be based topological spaces and $f: X \to Y$ a continuous map. Let $y$ be the base point of $Y$. A homotopy fiber $F(f)$ of $f$ is the set of pairs $\{(x, p) \in X \times Y^I\}$ such that $p(0) = f(x)$ and $p(1) = y$. We equip $F(f)$ with the subspace topology of $X \times Y^I$. Now, let $X$ and $Y$ be based connected CW-complexes. A map $f: X \to Y$ is said to be acyclic if

$$H_i(F(f); \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0. \end{cases}$$

Let us recall the plus construction.

**Definition 3.26.** Let $X$ be a based connected CW-complex and $P$ a perfect normal subgroup of $\pi_1(X)$. The plus construction on $X$ relative to $P$ is a pair $(f, Y)$ where $Y$ is a based connected CW-complex and $f: X \to Y$ is an acyclic map such that $P = \ker(\pi_1(X) \to \pi_1(Y))$.

Let $X$ be a based connected CW-complex and $P$ a perfect normal subgroup of $\pi_1(X)$. Then, due to Quillen’s theorem, there exists a plus construction $(f_P, X_P^+)$ relative to $P$, which is unique up to homotopy equivalence. In particular, we simply write $X^+$ for $X_P^+$ if $P$ is the perfect radical of $\pi_1(X)$. Let $K_0(R)$ denote the Grothendieck group of $R$ and $BGL(R)$ the classifying space of $GL(R)$. Then, $K(R)$ is defined as

$$K(R) := K_0(R) \times BGL(R)^+,$$

where $BGL(R)^+$ is the plus construction of $BGL(R)$ relative to its perfect radical $E(R)$. Note that, since $K_0(R)$ is discrete and $BGL(R)^+$ is connected, the definition is equivalent to

$$K_i(R) = \pi_i(BGL(R)^+)$$

for $i > 0$.

If $R$ is a number field, or its ring of integers, the rank of $K_i(R)$ was calculated by Borel [2]. Let $F$ be a number field with $r_1$ real and $r_2$ complex places. Put $n_+ := r_1 + r_2, n_- := r_2$. For $i > 1$, Borel constructed the Borel regulator map

$$\lambda_{i, F}: K_i(F) \to \mathbb{R}^{n_i}$$

with

$$n_i := \begin{cases} 0 & \text{for even } i \\ n_+ & \text{for } i = 1 \mod 4 \\ n_- & \text{for } i = 3 \mod 4 \end{cases}$$

and proved the following theorem.

**Theorem 3.27** (Borel [2]). (1) The induced map

$$\lambda_{i, F}: K_i(F) \otimes \mathbb{R} \to \mathbb{R}^{n_i}$$

is an isomorphism, and thus the $\mathbb{Z}$-rank of $K_i(F)$ for $i > 1$ is given by

$$\dim_{\mathbb{Q}}(K_i(F) \otimes \mathbb{Q}) = n_i.$$

(2) Let $R_{m,F}$ denote the covolume of the image of $\lambda_{2m-1,F}$. Then we have

$$\lim_{s \to 0} s^{-n} \zeta_F(1 - m + s) = Q \cdot R_{m,F},$$
where $\equiv_{\mathbb{Q}^*}$ means “equal up to non zero rational multiple”.

Note that if we rewrite (2) of Theorem 3.27 in terms of Dedekind zeta values at positive integers, it is equivalent to the equality

$$\zeta_F(m) = \frac{D_F}{\sqrt{2\pi}}(2\pi)^{-m} \mathcal{R}_{m,F}.$$ 

3.10. Relation between the Bloch-Wigner function and the Borel regulator of $K_3(F)$. Let $\sigma_1, \ldots, \sigma_{r_2}$ denote a set of complex embeddings of $F$ none of which is a complex-conjugate of each other. Bloch constructed a map from $B(F)$ to $K_3(F)$, and later Suslin proved that the map is, in fact, an isomorphism up to torsion [13, Theorem 5.2]. Moreover, under this isomorphism the diagram

$$B(F) \xrightarrow{\sim} K_3(F)$$

$$\begin{array}{c}
\lambda_{3,F} \\
\mathbb{R}^{r_2}
\end{array}$$

commutes. Hence, rewriting $\lambda_{3,F}$ in Theorem 3.27 by the Bloch-Wigner function, we have the following theorem [18, Section 1].

**Theorem 3.28.** Let the notations be as above. The Bloch group $B(F)$ is finitely generated of rank $r_2$. Moreover, let $\xi_1, \ldots, \xi_{r_2}$ be a $\mathbb{Q}$-basis of $B(F) \otimes \mathbb{Q}$. Then we have

$$\zeta_F(2) = \frac{D_F}{\sqrt{2\pi}}(2\pi)^{2(r_1+r_2)} \det(D(\sigma_i(\xi_j)))_{1 \leq i,j \leq r_2}.$$ 

This theorem is an analog of the analytic class number formula, where the unit group $O_F^*$ of rank $r_1 + r_2 - 1$ is replaced by the Bloch group $B(F)$ of rank $r_2$, and the real-valued analog $\log |z|$ of logarithm is replaced by a real-valued analog $D(z)$ of dilogarithm.

3.11. The Zagier conjecture. The Zagier conjecture is a generalization of Theorem 3.28 to zeta values at integers $m \geq 3$. Let $\sigma_1, \ldots, \sigma_{r_2}$ denote a set of complex embeddings of $F$ none of which is a complex-conjugate of each other and $\sigma_{r_2+1}, \ldots, \sigma_{r_2+r_1}$ the set of real embeddings. To formulate a conjecture which generalizes Theorem 3.28 to higher zeta values, one needs a higher analog $L_m(z)$ of $D(z)$, and the higher analog $B_m(F)$ of the Bloch group $B(F)$ such that the diagram

$$B_m(F) \xrightarrow{\sim} K_{2m-1}(F)$$

$$\begin{array}{c}
\lambda_{2m-1,F} \\
\mathbb{R}^{n_z}
\end{array}$$

commutes. (Here and hereafter, $\sim$ ignores finite kernel and cokernel.) A conjectural candidate has been proposed by Zagier.

In place of $D(z)$, we use the following $L_m(z)$ which is a real-valued analog of the polylogarithm function.

**Definition 3.29.** For $m \geq 2$, we define the function $L_m(z)$ by

$$L_m(z) := \Re \left( \sum_{k=0}^{m-1} \frac{B_k}{k!} (2 \log |z|)^k \text{Li}_{m-k}(z) \right),$$
where \( \mathfrak{R}_m(z) := \mathfrak{R}(z^{-1}), \) \( B_k \) is the \( k \)-th Bernoulli number, and \( \text{Li}_k(z) \) is the polylogarithm defined by (3.1).

**Remark 3.30.** \( \text{Li}_k(z) \) has an analytic continuation to the universal covering of \( \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \). As was the case \( \mathcal{L}_2(z) = -D(z), \) \( \mathcal{L}_m(z) \) has no monodromy around the branch points \( z = 0, 1, \infty \) (moreover, these functions are continuous at \( z = 0, 1, \infty \)), hence it is a real analytic function on \( \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \). As before, \( \mathcal{L}_m \) linearly extends to a function on \( \mathbb{Z}[\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}] \), and thus it can be regarded as a function on any submodule of \( \mathbb{Z}[\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}] \).

Next we explain how to define the higher Bloch groups \( B_m(F) \). To define \( B_m(F) \), we need two groups, \( A_m(F) \) i.e. the “natural” domain of definition of \( \mathcal{L}_m \), and the subgroup \( C_m(F) \) i.e. the kernel of

\[ \mathcal{L}_m : \mathbb{Z}[\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}] \rightarrow \mathbb{C}, \]

and \( B_m(F) \) should be defined as the quotient of these groups.

Let \( m \geq 2, \) \( F \) be any field. Let

\[ \beta_{m,F} : \mathbb{Z}[F] \rightarrow \text{Sym}^{m-2}(F^\times) \otimes \bigwedge^2 F^\times \]

be the map defined by

\[ \beta_{m,F}([z]) := \begin{cases} \frac{1}{z \otimes \cdots \otimes z \wedge (1 - z)} & z \notin \{0, 1\} \\ 0 & z \in \{0, 1\} \end{cases} \]

and

\[ \iota : \text{Hom}(F^\times, \mathbb{Z}) \times \mathbb{Z}[F \setminus \{0, 1\}] \rightarrow \mathbb{Z}[F \setminus \{0, 1\}] \]

the bilinear map defined by

\[ \iota \left( \phi, \sum n_i[x_i] \right) := \sum n_i \phi(x_i)[x_i]. \]

for \( z \in F \).

**Definition 3.31.** We define \( C_m(F) \) by

\[ C_m(F) := \left\{ \xi(1) - \xi(0) | \xi(t) \in \ker \beta_{m,F(T)} \right\}, \]

where \( F(T) \) denotes the field of rational functions in one variable over \( F \).

We define \( A_m(F) \) by

\[ A_m(F) := \left\{ \xi \in \mathbb{Z}[F \setminus \{0, 1\}] | \iota(\phi, \xi) \in C_{m-1}(F) \text{ for } \forall \phi \in \text{Hom}(F^\times, \mathbb{Z}) \right\}. \]

**Remark 3.32.** We call the groups \( A_m(F) \) and \( C_m(F) \) \( A \)-groups and \( C \)-groups, respectively. If \( F \) is a number field, there is also an inductive definition of \( A \)-groups and \( C \)-groups which is more practical for numerical computation by computers. First we put

\[ A^*_2(F) := A(F), \]

\[ C^*_2(F) := C(F), \]

where \( A(F) \) and \( C(F) \) are the groups defined in Section 3.8. For \( m \geq 3, \) we define the group \( A^*_m(F) \) by

\[ A^*_m(F) := \left\{ \xi \in \mathbb{Z}[F \setminus \{0, 1\}] | \iota(\phi, \xi) \in C^*_{m-1}(F) \text{ for } \forall \phi \in \text{Hom}(F^\times, \mathbb{Z}) \right\}. \]
and a subgroup $C^*_m(F)$ by

$$C^*_m(F) := \{ \xi \in A^*_m(F) | L_m(\sigma(\xi)) = 0 \text{ for } \forall \sigma \in \Sigma_F \}$$

(here, the function $L_m$ is defined in Definition 3.29). One may show that $C_m(F) \subset C^*_m(F)$, but the equality $C_m(F) = C^*_m(F)$ is known only for $m = 2$.

We define the higher Bloch groups $B_m(F)$ for $m \geq 2$ by

$$B_m(F) := A_m(F)/C_m(F).$$

Since $C_m(\mathbb{C}) \subset C_m(\mathbb{C})$, $L_m$ gives a well-defined function on $B_m(\mathbb{C})$.

Using $C$-groups, the definition of $A$-groups can also be expressed as follows. Define $G_m(F)$ by

$$G_m(F) := \mathbb{Z}[F]/C_m(F)$$

and

$$\delta_{m,F} \in \text{Hom} \left( \mathbb{Z}[F], F^\times \otimes G_{m-1}(F) \right)$$

by $\delta_{m,F}([x]) := x \otimes [x]$ for $x \in F^\times$. Then

$$A_m(F) = \ker \delta_{m,F}.$$

Goncharov [4] defined a complex

$$0 \to G_m(F) \xrightarrow{\partial} F^\times \otimes G_{m-1}(F) \xrightarrow{\partial} \wedge^2 F^\times \otimes G_{m-2}(F) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \wedge^{m-2} F^\times \otimes G_2(F) \xrightarrow{\partial} \wedge^m F^\times$$

where the coboundary map $\partial$ is given by

$$\partial (x_1 \wedge \cdots \wedge x_r \otimes [x]) := x_1 \wedge \cdots \wedge x_r \wedge x \otimes [x]$$

except for the last step, and by

$$\partial (x_1 \wedge \cdots \wedge x_{m-2} \otimes [x]) := x_1 \wedge \cdots \wedge x_{m-2} \wedge x \wedge (1 - x)$$

for the last step. From the definition, the cohomology at the first step of (3.33) gives $B_m(F)$, while the rest of the steps of (3.33) are conjectured to be exact for a number field $F$. (This is naturally understood by assuming Goncharov’s conjecture stating that the cohomology at each step of the complex (3.33) corresponds to the Admas filtration of the algebraic $K$-groups.) Using the Bloch group $B_m(F)$ and the real-valued polylogarithm map $L_m$, the Zagier conjecture is formulated as follows.

**Conjecture** (Zagier). Let $m \geq 2$ and the notations be as above. The Bloch group $B_m(F)$ is finitely generated of rank $n_\pm$. Moreover, let $\xi_1, \ldots, \xi_{n_\pm}$ be a $\mathbb{Q}$-basis of $B_m(F) \otimes \mathbb{Q}$ and $\sigma_1, \ldots, \sigma_{n_\pm}$ a set of complex (and real, if $m$ is odd) embeddings of $F$ none of which is a complex-conjugate of each other. Then we have

$$\zeta_F(m) = \mathbb{Q} \cdot \left| D_F \right|^{1/2} (2\pi)^{mn_\pm} \det (L_m(\sigma_i(\xi_j)))_{1 \leq i,j \leq n_\pm}.$$
see the rank of $B_m(F)$. Also, by choosing any $\mathbb{Q}$-basis of the real values, one can compute the determinant and thus check the conjecture.

There is also theoretical support of this conjecture. One of the most significant one is that of Beilinson and Deligne [1]. They constructed a map

$$i_m : B_m(F) \rightarrow K_{2m-1}(F)$$

such that $(\mathcal{L}_m \circ \sigma_i)_{1 \leq i \leq 2n+2} = \lambda_{2m-1,F} \circ i_m$. What is left to prove is the surjectivity of $i_m$, and hence the expressibility of $\zeta_F(m)$ as a determinant of $\mathcal{L}_m \circ \sigma_i$ is generally an open problem. In the case of $m = 3$, Goncharov [4] proved the surjectivity of $i_3$, and the Zagier conjecture is already a theorem. He also discovered an explicit 2-term relation of $\zeta_3$ in 3 variables, which conjecturally spans $\mathcal{C}_3(F)$, just as 5-term relation did in the $\mathcal{C}_2(F)$ case. Goncharov also conjectures that the interpretation as a hyperbolic volume of the $D$-values are generalizable to higher dimensions. He states that the volume of any hyperbolic manifold of dimension $2m$ and $2m - 1$ is expressible as a $\mathcal{L}_m$-value in $\mathbb{Z}[\mathbb{Q} \setminus \{0, 1\}]$.

4. THE ENHANCED ZAGIER CONJECTURE

In this chapter, we will explain an enhancement of the Zagier conjecture. To understand the motivation of the enhancement, let us first go back to the case of Dedekind’s class number formula. To avoid the difficulty of formulation, we restrict ourselves to a number field $F$ with a rank one unit group (i.e. the case $r_1 + r_2 = 2$). We also exclude the case where the fundamental unit is a totally real number (this occurs only when $F$ is totally real or CM), since, in such a case, the enhancement we consider here does not contain any new information. Thus, we consider the cases $r_1 = r_2 = 1$ or $r_1 = 0$, $r_2 = 2$ here. In these cases, the regulator is simply given by

$$\log |\varepsilon|_v \in \mathbb{R}$$

where $\varepsilon$ is the fundamental unit of the number field, and $| \cdot |_v$ is the $v$-adic absolute value associated with a complex place $v$ of $F$. Since this value can be decomposed as

$$\log |\varepsilon|_v = \log \rho (\varepsilon) + \log \overline{\rho} (\varepsilon),$$

where $\rho, \overline{\rho}$ are the conjugate complex embeddings lying over $v$, the value

$$\log \rho (\varepsilon) \in \mathbb{C}/\mathbb{Z}(1),$$

gives a natural lift of the regulator (here, we ignored the effect of roots of unity in $F$). The regulator only contains the information on the absolute value of $\rho (\varepsilon)$, whereas $\log \rho (\varepsilon)$ contains the whole information on the original value $\rho (\varepsilon)$. Therefore if $\rho (\varepsilon)$ is not real, $\log \rho (\varepsilon)$ contains strictly more informations than the regulator.

Such an enhancement is applicable to much wider cases than above. For example, come to think of an abelian extension of a number field with one complex place ($r_2 = 1$) such that all the real places ramify under the extension. Then, the associated Stark conjecture about the partial zeta value is of rank one, and the Stark unit is a non-real number. Here again, the regulator only has the information on the absolute value of the Stark unit, so we can consider the enhanced regulator to be the logarithm of the original value.

Since the regulators of unit groups have enhancement, so should the zeta values. For imaginary quadratic $F$ (i.e. $r_1 = 0, r_2 = 1$), the situation of the Stark conjecture is always of rank one, and the regulator is expressible as logarithm of the absolute value of theta function evaluated at CM-points. Therefore, one may define the
enhanced zeta value as the logarithm of the theta value itself, and, in fact, it gives
the Stark unit. For imaginary cubic $F$ (i.e. $r_1 = 1, r_2 = 1$), Ren-Szech [9] obtained
such an enhancement of zeta values using the method of Shintani decomposition of
the first derivative of a Shintani zeta function associated to a cone (here, we noted
as “associated to a cone” since there is also a different type of Shintani zeta function
which is associated to a prehomogeneous vector space).

For regulators $\lambda_{m,F} : K_{2m-1}(F) \to \mathbb{R}^{n_z}$ of higher $K$-groups, a natural lift
to $(\mathbb{C}/\mathbb{Q}(m))^n_z$ is also known (although the lift to $(\mathbb{C}/\mathbb{Z}(m))^n_z$ is a more subtle
problem). Hence, there should be a corresponding lift of the polylogarithm map
to $(\mathbb{C}/\mathbb{Q}(m))$. For regulators $\lambda_{m,F} : K_{2m-1}(F) \to \mathbb{R}^{n_z}$ of higher $K$-groups, a natural lift
to $(\mathbb{C}/\mathbb{Q}(m))^n_z$ is also known (although the lift to $(\mathbb{C}/\mathbb{Z}(m))^n_z$ is a more subtle
problem). Hence, there should be a corresponding lift of the polylogarithm map
from the higher Bloch groups. We will explain how to construct such a lift, namely
“the enhanced polylogarithms”, in the following section.

4.1. The enhanced polylogarithm. Following Gangl and Zagier’s construction
in [18], we describe the construction of the enhanced polylogarithm

$$\hat{L}_m : \mathcal{B}_m(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}(m).$$

We define the set $\Omega$ by

$$\Omega := \{(x,U) \in \mathbb{C}^\times \times \mathbb{C} \mid U - \log x \in \mathbb{Q}(1)\}$$

and the function $F_m$ on $\Omega$ by

$$F_m(x,U) := \frac{1}{(m-1)!} \int_0^\infty \frac{(t-U)^{m-1}}{x^{-1}e^t - 1} dt$$

where the contour of integration avoids the poles at $t \in \log x + \mathbb{Z}(1)$. Since the
absolute convergence of the integral is clear, and the change of the contour of
integration can only produce the residue in $\mathbb{Q}(m)$, $F_m$ is a well-defined map
from $\Omega$ to $\mathbb{C}/\mathbb{Q}(m)$. The function $F_m(x,U)$ does not appear to be a polylogarithm, but
one can find a more polylogarithm-like expression

$$F_m(x,U) = \sum_{k=0}^{m-1} \frac{(-U)^k}{k!} \text{Li}_{m-k}(x)$$

by restricting to $|x| < 1$, expanding the integrand in the power series of $x$ and
integrating term by term. Now let $\Lambda$ be the set of group homomorphisms $L : \mathbb{C}^\times \to \mathbb{C}$ such that $(x,L(x)) \in \Omega$ for $x \in \mathbb{C}^\times$. Then we define $\hat{L}_m : (\mathbb{C} \setminus \{0,1\}) \times \Lambda \to \mathbb{C}/\mathbb{Q}(m)$ by

$$\hat{L}_m(x,L) := F_m(x,L(x)) + \frac{(-L(x))^{m-1} \text{Li}_{m-1}(1-x)}{m!}$$

for $(x,L) \in (\mathbb{C} \setminus \{0,1\}) \times \Lambda$. By linearly extending the definition of $\hat{L}_m$ to $\mathbb{Z}[\mathbb{C} \setminus \{0,1\}] \times \Lambda$, and restricting it to $\mathcal{A}_m(\mathbb{C}) \times \Lambda$, one has the following theorem.

**Theorem 4.1** (Gangl-Zagier [18]). (1) The restricted function $\hat{L}_m : \mathcal{A}_m(\mathbb{C}) \times \Lambda \to \mathbb{C}/\mathbb{Q}(m)$ does not depend on $L \in \Lambda$, hence gives a well-defined map

$$\hat{L}_m : \mathcal{A}_m(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}(m).$$

(2) $\mathcal{R}_m \circ \hat{L}_m = L_m$ as a function on $\mathcal{A}_m(\mathbb{C})$.

(3) $\hat{L}_m(\xi) = 0$ for $\xi \in \mathcal{C}_m(\mathbb{C})$. 

From this theorem, we see that $\tilde{L}_m$ gives a well-defined map

$$\tilde{L}_m : B_m(\mathbb{C}) := A_m(\mathbb{C})/\mathbb{C}_m(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}(m)$$

which is a lift of the real valued polylogarithm map $L_m : B_m(\mathbb{C}) \to \mathbb{R}$.

In the case of $m = 2$, there is a slightly more complicated but finer definition of $\tilde{L}_2$ such that the target space $\mathbb{C}/\mathbb{Q}(m)$ is lifted to $\mathbb{C}/\frac{1}{2}\mathbb{Z}(2)$. Let $F : (\mathbb{C} - \mathbb{Z}(1)) \to \mathbb{C}/\mathbb{Z}(2)$ be the map defined by

$$F(u) := \int_0^u \frac{t}{1 - e^t} dt.$$ 

Here again, the choice of the contour may change the value of the integral by an element of $\mathbb{Z}(2)$, we obtain a well-define map by setting the target space as $\mathbb{C}/\mathbb{Z}(2)$.

We define $b_{X} : = \{ (u,v) \in \mathbb{C}^2 | e^u + e^v = 1 \}$, $X : = \{ (x,y) \in (\mathbb{C}^\times)^2 | x + y = 1 \} (= \mathbb{C} \setminus \{0,1\})$.

Then there is an obvious map $\exp : \tilde{X} \to X$. For any finitely generated subgroup $A$ of $A_2(\mathbb{C})$, define a finite set $S_A$ by

$$S_A := \bigcup_{\xi \in A} \left\{ |z_i, 1 - z_i| \right\}$$

and $G_A \subset \mathbb{C}^\times$ by the subgroup generated by elements of $S_A$. Put

$$\tilde{G}_A := \exp^{-1}(G_A) \subset \mathbb{C}$$

and

$$X_A := (G_A \times G_A) \cap X,$$
$$\tilde{X}_A := \exp^{-1}(X_A) \subset \tilde{X}.$$

Then we can define $\tilde{\beta} : \mathbb{Z}[\tilde{X}_A] \to \bigwedge^2 \tilde{G}_A$ and $\beta : \mathbb{Z}[X_A] \to \bigwedge^2 G_A$ by

(4.2) $\tilde{\beta} \left( \sum_i m_i [u_i, v_i] \right) := \sum_i m_i (u_i \wedge v_i),$

(4.3) $\beta \left( \sum_i m_i [x_i, y_i] \right) := \sum_i m_i (x_i \wedge y_i).$

(Here, the wedge product in (4.2) is additive since $\tilde{G}_A$ is an additive group and the wedge product in (4.3) is multiplicative since $G_A$ is a multiplicative group.) For $\xi \in A$, take any $\xi_1 \in \mathbb{Z}[\tilde{X}_A]$ such that $e^{\xi_1} = \xi$. Since

$$\tilde{\beta}(\xi_1) \in \ker \left( \exp : \bigwedge^2 \tilde{G}_A \to \bigwedge^2 G_A \right) = \mathbb{Z}(1) \wedge \tilde{G}_A,$$

there exists $\xi_2 \in \tilde{G}_A/\mathbb{Z}(1)$ such that

$$2\pi i \wedge \xi_2 = \tilde{\beta}(\xi_1).$$
This can be summarized in the following commutative diagram.

\[
\begin{align*}
G_A/\mathbb{Z}(1) & \xrightarrow{2\pi i \wedge} \Lambda^2 G_A \\
\mathbb{Z}[\mathbb{X}_A] & \xrightarrow{\beta} \Lambda^2 G_A \\
A & \xrightarrow{\exp} \mathbb{Z}[X_A] \xrightarrow{\beta} \Lambda^2 G_A
\end{align*}
\]

Here, the vertical arrow on the right-side is exact. Now define \(F: \mathbb{Z}[\mathbb{X}_A] \to \mathbb{C}/2^{-1}\mathbb{Z}(2)\) by

\[
F(\eta) := \sum_i m_i \left( F(u_i) - \frac{1}{2} u_i v_i \right)
\]
for \(\eta = \sum_i m_i (u_i, v_i)\), and \(\tilde{D}\) by

\[
\tilde{D}(\xi_1, \xi_2) := F(\xi_1) - \pi i \xi_2.
\]

Then we have the following theorem.

**Theorem 4.4** (Gangl-Zagier [18]). (1) The function \(\tilde{D}\) depends only on \(\xi \in A\), and does not depend on the choice of \(\xi_1, \xi_2\). Hence the map

\[
\tilde{D}: A_2(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}(2)
\]

is well-defined.

(2) \(\Im \circ \tilde{D} = D\) as a function on \(A_2(\mathbb{C})\).

(3) \(\tilde{D}(\xi) = 0\) for \(\xi \in \mathcal{C}_2(\mathbb{C})\).

It is curious to note that, unlike the real-valued dilogarithm \(D\), which is defined on the whole of \(\mathbb{Z}^n \setminus \{0, 1\}\), the enhanced dilogarithm \(\tilde{D}\) is defined only on the smaller domain \(A_2(\mathbb{C})\). The situation is expressed by the following commutative diagram.

\[
\begin{align*}
A_2(\mathbb{C}) & \xrightarrow{\tilde{D}} \mathbb{C}/\mathbb{Z}(2) \\
\cap \quad \Im & \quad \cap \\
\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] & \xrightarrow{D} \mathbb{R}
\end{align*}
\]

This is one main reason why we called \(A_2(\mathbb{C})\) the “natural domain” of definition.

There is another curious fact in comparison with the diagram (4.5). If we restrict the definition of \(\tilde{D}\) to \(A_2(\mathbb{R})\), its imaginary part \(D = \Im \circ \tilde{D}\) vanishes identically. On the other hand, the real part of \(\tilde{D}\) survives and we have a well-defined map \(\tilde{D}: A_2(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}(2)\). Now, let us recall the Rogers dilogarithm

\[
R^*: \mathbb{P}^1(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}(2)
\]

in Remark 3.9. As we have already seen, \(R^*\) is continuous everywhere, real analytic except for \(\{0, 1, \infty\}\). Moreover, if we restrict \(R^*\) to \(A_2(\mathbb{R}) \subset \mathbb{Z}[\mathbb{R} \setminus \{0, 1\}]\), it
coincides with $\tilde{D}$. Again, the situation is expressed by the following commutative diagram.

\[
\begin{array}{ccc}
A_2(\mathbb{R}) & \xrightarrow{\tilde{D}} & \mathbb{R} / \frac{1}{2}\mathbb{Z}(2) \\
\cap & & \cap \\
\mathbb{Z}[\mathbb{R} \setminus \{0, 1\}] & \xrightarrow{R} & \mathbb{R} / \frac{1}{2}\mathbb{Z}(2)
\end{array}
\]

From the diagrams (4.5) and (4.6), $\tilde{D}$ is a lift of the Bloch-Wigner function as well as an extension of Rogers dilogarithm to complex arguments.

4.2. The enhanced zeta values. Let $F$ be an imaginary quadratic field. We constructed the enhanced polylogarithms in the preceding section. To formulate an enhancement of the Zagier conjecture, it is necessary to construct enhanced zeta values also. For Zagier and Gangl’s construction of enhanced zeta values, we first recall the Eichler integral. For the convenience of notation, set

\[
f_{j,k}(a + bi, c + di) = \left( \frac{a \tau + b}{c \tau + d} \right)^k f(a + bi, c + di).
\]

for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R})$ and $f \in \text{Map}(\mathcal{H}, \mathbb{C})$. Then, surprisingly, we have the following proposition.

**Proposition 4.7.** For $k \in \mathbb{Z}_{\geq 0}$, $\gamma \in SL_2(\mathbb{R})$ and a real analytic function $f$ on $\mathcal{H}$,

\[
\partial^k \left( f|_{\gamma,1-k} \right) = \left( \partial^k f \right)|_{\gamma,1+k},
\]

where $\partial$ denotes the differentiation by $\tau$.

**Proof.** Let $\gamma$ be any $2 \times 2$ matrix

\[
\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(\mathbb{C}).
\]

We show a more general equality

\[
\partial^{r+1} \left( f|_{\gamma,s} \right) = \sum_{i=0}^r \left( \begin{array}{cc} -s - i \\ r - i \end{array} \right) \epsilon^{-i}(\det \gamma)^i \left( \frac{\partial^i}{\partial t^i} f \right)|_{\gamma,s+r+i},
\]

valid for $k \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$, by induction. Since $\left( \begin{array}{cc} -1 \\ 0 \end{array} \right) = 1$, the equality (4.8) is trivial for $r = 0$. Assume that the equality holds for $r$. By differentiation, one has

\[
\partial^{r+1} \left( f|_{\gamma,s} \right) = \sum_{i=0}^r \left( \begin{array}{cc} -s - i \\ r - i \end{array} \right) \epsilon^{-i}(\det \gamma)^i \partial \left\{ \left( \frac{\partial^i}{\partial t^i} f \right)|_{\gamma,s+r+i+1} \right\},
\]

Substituting

\[
\partial \left\{ \left( \frac{\partial^i}{\partial t^i} f \right)|_{\gamma,s+r+i+1} \right\}
\]

\[
= (i + 1) \det \gamma \left( \frac{\partial^{i+1}}{(i + 1)!} f \right)|_{\gamma,s+r+i+2} - (s + r + i) \epsilon \left( \frac{\partial^{i}}{\partial t^{i}} f \right)|_{\gamma,s+r+i+1}
\]
in the above equality, the proposition holds for \( r + 1 \). Hence the equality holds for \( r \in \mathbb{Z}_{\geq 0} \). By setting \((r, s) = (k, 1 - k)\) in the equality, Proposition 4.7 follows.

Let \( M_k(\Gamma) \) denote the vector space formed by holomorphic modular forms on \( \mathfrak{H} \) of weight \( k \) attached to the modular group \( \Gamma \subset SL_2(\mathbb{R}) \). Then an Eichler integral \( F(\tau) \) of \( f(\tau) \in M_k(\Gamma) \) is any function with the property

\[
\left( \frac{\partial}{\partial \tau} \right)^{k-1} F(\tau) = f(\tau).
\]

Eichler integrals inherit the automorphy of \( f(\tau) \) as follows.

**Proposition 4.9.** Let \( k \in \mathbb{Z}_{\geq 0} \) and \( F(\tau) \) be an Eichler integral of an element of \( M_{k+1}(\Gamma) \). Then, for \( \gamma \in \Gamma \),

\[
F(\tau)|_{\gamma, 1-k} - F(\tau)
\]

is a polynomial in \( \tau \) of degree at most \( k - 1 \).

**Proof.** From Proposition 4.7,

\[
\partial^k \left( F|_{\gamma, 1-k} - F \right) = (\partial^k F)|_{\gamma, 1+k} - \partial^k F
\]

for \( \gamma \in \Gamma \). Since \( \partial^k F \in M_{k+1}(\Gamma) \), the right-hand side equals to zero. Therefore, it follows that

\[
F|_{\gamma, 1-k} - F
\]

is a polynomial in \( \tau \) of degree at most \( k - 1 \). \(\square\)

Let \( E_{2k}(\tau) \) be the holomorphic Eisenstein series of weight \( 2k \) with \( k \in \mathbb{Z}_{\geq 2} \). It is well known that its Fourier expansion is given by

\[
E_{2k}(\tau) = \zeta(1 - 2k) + 2 \sum_{m=1}^{\infty} m^{2k-1} \frac{e^{2\pi im\tau}}{1 - e^{2\pi im\tau}},
\]

where \( \tau \) is an element of the upper half plane \( \mathfrak{H} \).

Let \( \tilde{E}_{2-2k, P}(\tau) \) be the specific Eichler integral of \((2\pi i)^{2k-1}E_{2k}(\tau)\), defined by

\[
\tilde{E}_{2-2k, P}(\tau) := \zeta(2k - 1) + P + \zeta(1 - 2k) \frac{(2\pi i)^{2k-1}(2\pi i)^{2k-1}}{(2k - 1)!} + 2 \sum_{m=1}^{\infty} m^{1-2k} \frac{e^{2\pi im\tau}}{1 - e^{2\pi im\tau}},
\]

where \( P \) is any polynomial in \( \tau \) of degree at most \( 2k - 2 \), whose coefficients lie in \( \mathbb{Q}(2k - 1) \). It is known that \( \tilde{E}_{2-2k, P}(\tau) \) satisfies the following remarkable property.

**Proposition 4.10.** For \( \gamma \in SL_2(\mathbb{Z}) \),

\[
\tilde{E}_{2-2k, P}(\tau) - \tilde{E}_{2-2k, P}|_{\gamma, (2-2k,0)}(\tau)
\]

is a polynomial in \( \tau \) of degree at most \( 2k - 2 \), whose coefficients lie in \( \mathbb{Q}(2k - 1) \).

In Section 6.1.3, we give a proof of this proposition from the point of view of the theory of Shintani L-function. For \( s \in \mathbb{C} \), let

\[
\partial_s := \partial + \frac{s}{\tau - \bar{\tau}}
\]
be the Maass raising operator. Set
\[ dk := (\tau - \tau)^k \partial_{-2} \partial_{-4} \cdots \partial_{-2k} = \sum_{j=0}^{k} \frac{(k+j)!}{j!(k-j)!}(\tau - \tau)^{k-j} \partial^{k-j}. \]
Then we have the following proposition.

**Proposition 4.11.** For any real analytic function \( f \) on \( \mathfrak{H} \), we have
\[ dk \left( f|_{\gamma,(-2k,0)}(\tau) \right) = (dkf)|_{\gamma,(-k,-k)}(\tau). \]

**Proof.** From the definition of \( \partial_{\alpha} \), it is easy to check that
\[ \partial_{s_1} \left( f|_{\gamma,(s_1,s_2)}(\tau) \right) = (\partial_{s_1}f)|_{\gamma,(s_1+2,s_2)}(\tau) \]
for any real analytic function \( f \) on \( \mathfrak{H} \). (Thus \( \partial_{\alpha} \) sends a modular form of weight \( s \) to a modular form of weight \( s + 2 \).) Using the equality (4.12) \( k \) times, the lemma is proved. \( \square \)

**Proposition 4.13.** For \( l = 0, \ldots, 2k \), we have
\[ dk(l) = 2l!(2k+l)! \frac{k!}{k!} Z[l, \tau]. \]
In particular, for \( \mathcal{P} \in \mathbb{Q}[\tau] \) with \( \deg \mathcal{P} \leq 2k \), we have \( dk \mathcal{P} \in \mathbb{Q}[\tau + \tau, \tau] \).

We shall restate Proposition 4.13 as Corollary 6.14 in Section 6.4 and give a proof there.

Now define \( \mathcal{E}_{k,p}(\tau) \) by
\[ \mathcal{E}_{k,p}(\tau) := dk-1 \tilde{E}_{2-2k,p}(\tau). \]
Then we have the following proposition.

**Proposition 4.14.** For \( \gamma \in SL_2(\mathbb{Z}) \),
\[ \mathcal{E}_{k,p}(\tau) - \mathcal{E}_{k,p}|_{\gamma,(1-k,1-k)}(\tau) \in (2\pi i)^{2k-1} \mathbb{Q}[\tau + \tau, \tau]. \]

**Proof.** From Proposition 4.11,
\[ \mathcal{E}_{k,p}(\tau) - \mathcal{E}_{k,p}|_{\gamma,(1-k,1-k)}(\tau) = dk-1 \left\{ \tilde{E}_{2-2k,p}(\tau) - \tilde{E}_{2-2k,p}|_{\gamma,(2-2k,0)}(\tau) \right\}. \]
Thus Proposition 4.14 follows from Propositions 4.10 and 4.13. \( \square \)

Now, for \( \mathcal{A} \in \text{Cl}(F) \), there is a corresponding class of integral positive definite binary quadratic form \([A]\). For
\[ Q = mX^2 + nXY + lY^2 \quad (m > 0) \]
in \([A]\), we associate \( \lambda_Q \in \mathbb{Z}_{>0} \) and \( \tau_Q \in \mathfrak{H} \) by
\[ \lambda_Q := m, \]
\[ \tau_Q := \frac{n + \sqrt{n^2 - 4ml}}{2m}. \]

**Definition 4.16.** Let \( \mathcal{A} \) be an ideal class of \( F \) and \( Q \) an element of \([A]\). Let \( \lambda_Q, \tau_Q \) be as above. We define the enhanced zeta value \( I_k(\mathcal{A}) \in \mathbb{C}/\mathbb{Q}(k) \) by
\[ I_k(A) := \left( \frac{1}{w_F} \left( \frac{\lambda_Q}{2\pi i} \right)^{k-1} \mathcal{E}_{k,F}(\tau_Q) \right) \mod Q(k). \]

**Proposition 4.17.** \( I_k(A) \) does not depend on the choice of \( Q \in [A] \) and thus it is well-defined.

**Proof.** Let \( Q(X,Y), Q'(X,Y) \) be any elements of \([A]\). Then, by definition, there exists an element

\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \]

such that

\[ Q'(X,Y) = Q((X,Y) \gamma). \]

Then \( \lambda_Q, \tau_Q \) and \( \lambda_{Q'}, \tau_{Q'} \) are related by

\[ \lambda_{Q'} = \lambda_Q(\tau_Q + d)(\tau_{Q'} + d), \quad \tau_{Q'} = \frac{a\tau_Q + b}{c\tau_Q + d}. \]

Since \( \tau_Q + \tau_{Q}\tau_{Q} \in \mathbb{Q} \), Proposition 4.17 follows from Proposition 4.14. \( \square \)

**Proposition 4.18.** The above defined \( I_k(A) \) satisfies

\[ \mathfrak{R} \left( i^{k-1} I_k(A) \right) = |D_F|^{k-1/2} \frac{(k-1)!}{2(2\pi)^k} \zeta_F(k, A). \]

Hence it gives a lift to \( \mathbb{C}/\mathbb{Q}(k) \) of partial zeta values for an ideal class.

**Proof.** We only give a brief sketch of the proof. Observe that the partial zeta function \( \zeta_F(k, A) \) is expressible as a value at a CM point of the real analytic Eisenstein series

\[ E(\tau, k) := \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{y^k}{|m\tau + n|^{2k}} \left( y := \frac{\tau - \pi}{2i} \right). \]

Its Fourier series expansion is well-known and given by

\[ \frac{(k-1)!}{\pi^k} E(\tau, k) \]

\[ = \zeta(2k)y^k + \bar{\zeta}(2k-1)y^{1-k} + 4 \sum_{l=1}^{\infty} \frac{i^{k-\frac{1}{2}}\sigma_{1-2k}(l)}{l \pi K_{k-\frac{1}{2}}(2\pi ly) \cos(2\pi lx)}, \]

where

\[ \hat{\zeta}(s) := \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) \]

is the complete Riemann zeta function, \( \sigma_s(l) := \sum_{d|l} d^s \) is the sum of \( s \)-th power of divisor function, and \( K_{s}(z) \) is the modified Bessel function of the second kind of order \( s \). Using \( \bar{\zeta}(2k) = \zeta(1-2k) \) and the expression

\[ K_{k-\frac{1}{2}}(z) = \left( \frac{\pi}{2z} \right)^{1/2} e^{-z} \sum_{r=0}^{k-1} \frac{(k-1+r)!}{r!(k-1-r)!} (2z)^{-r} \]
for the modified Bessel function of half-integral order, we obtain
\[
\frac{(k - 1)!}{\pi^k} E(\tau, k) = \frac{(k - 1)!}{2(2k - 1)!} \zeta(1 - 2k)(-4\pi y)^k + \frac{(2k - 2)!}{(k - 1)!} \zeta(2k - 1)(4\pi y)^{1-k} \sum_{n=1}^{\infty} \sigma_1 - 2k(n) \left( e^{2\pi i n \tau} + e^{-2\pi i n \tau} \right) \sum_{r=0}^{k-1} \frac{(k - 1 + r)!}{r!(k - 1 - r)!} n^{k-1-r}(4\pi y)^{-r}.
\]
Since \(|D_F|^2 = i\lambda_d(\tau_A - \tau_A)\) and
\[
d_{k-1}(e^{2\pi i n \tau}) = (4\pi y)^{k-1} \sum_{r=0}^{k-1} \frac{(k - 1 + r)!}{r!(k - 1 - r)!} n^{k-1-r}(4\pi y)^{-r} e^{2\pi i n \tau},
\]
Proposition 4.18 follows by the straight-forward application of Corollaries 6.14 and 6.15. (For the proof of these, see Section 6.4).

From Proposition 4.18, we see that \(I_k(A)\) gives an enhancement of the partial zeta value \(\zeta_F(k, A)\).

Remark 4.19. We use slightly different notations from those in [18]. To avoid confusion, we note the following list of comparison.
\[
\begin{align*}
\mathcal{R}_k^ZG(z) &= (-1)^{[k/2]} \mathcal{R}_k(z) \\
\mathcal{L}_k^ZG(z) &= (-1)^{[k/2]} \mathcal{L}_k(z) \\
\mathcal{E}_k^ZG(z) &= \frac{1}{2}(2\pi i(\tau - \tau))^{1-k} \mathcal{E}_k(z) \\
I_k^ZG(A) &= (-1)^{k-1} I_k(A).
\end{align*}
\]
Here, the notations with superscript of ZG (placed on the left hand-side) indicate that they are the ones used in Zagier and Gangl’s article, and the notations without superscript (placed on the right hand-side) indicate that they are used in this article.

5. The Shintani L-function and its first partial derivative

5.1. The definition of the Shintani L-function. Let \(L\) be a lattice in \(C\). Let \(\tilde{C}^\times\) be the universal covering group of \(C^\times\) with the covering homomorphism \(p : \tilde{C}^\times \rightarrow C^\times\). We define \(\log \in \text{Hom}(\tilde{C}^\times, C)\) by the unique homomorphisms such that the diagram
\[
\begin{array}{ccc}
\tilde{C}^\times & \xrightarrow{\log} & C \\
| & \searrow & \downarrow p \\
C^\times & \xrightarrow{\log} & C/\mathbb{Z}(1)
\end{array}
\]
commutes. We thus define the homomorphism \(\tilde{C}^\times \otimes C \rightarrow C^\times : (\alpha, s) \mapsto \alpha^s\) by
\[
\alpha^s := \exp(s \cdot \log\alpha)
\]
and \(\red{\arg} \in \text{Hom}(\tilde{C}^\times, \mathbb{R})\) by \(\red{\arg} := 3 \circ \log\). For \(x \in \tilde{C}^\times\), define \(\tau \in \tilde{C}^\times\) as the unique element such that \(\red{\arg}\tau = -\red{\arg}x\) and \(|p(\tau)| = |p(x)|\). We denote by \(e_\theta \in \tilde{C}^\times\), the unique element such that \(\red{\arg}e_\theta = \theta\) and \(|p(e_\theta)| = 1\), i.e. the unique element such that \((e_\theta)^s = e^{i\theta s}\). \(\mathbb{R}_{>0} (= \text{the identity component of } \mathbb{R}^\times\) is naturally regarded as a subgroup of \(\tilde{C}^\times\) formed by the elements with \(\red{\arg}x = 0\). For a subset \(X\) of \(\tilde{C}^\times\)
and \( z \in \mathbb{C}^x \), we put \( zX = \{ zx | x \in X \} \). Hereafter, we simply write \( x \) for \( p(x) \), if there is no risk of confusion.

Set \( \mathbb{L} := p^{-1}(\mathbb{L} \setminus \{0\}) \). For \( x, y \in \mathbb{L} \), we define a \((2\text{-dimensional})\) cone \( C(x, y) \) by
\[
C(x, y) := \left\{ z \in \mathbb{L}; \min(a \arg x, a \arg y) < a \arg z < \max(a \arg x, a \arg y) \right\},
\]
and for \( x \in \mathbb{L} \), and a \((1\text{-dimensional})\) cone \( C(x) \) by
\[
C(x) := \left\{ z \in \mathbb{L}; a \arg z = a \arg x \right\}.
\]

We denote by \( 1_{C} \) the characteristic function of a cone \( C \). We say \( C \) is admissible, if \( C \) is a \(2\text{-dimensional}\) cone of the form \( C(x, y) \) with \( 0 < |a \arg (x) - a \arg (y)| < \pi \), or if \( C \) is a \(1\text{-dimensional}\) cone. Since \( 1_{C(x, y)} + 1_{C(y, z)} = 1_{C(x, z)} \) for \( x, y, z \in \mathbb{L} \) with \( a \arg x < a \arg y < a \arg z \), a characteristic function for any cone is a sum of characteristic functions of admissible cones. We denote by \( A(\mathbb{L}) \) the \( \mathbb{Z} \)-module generated by the characteristic functions of admissible cones. Note that we have \( e^{C}(x, y) = C(ex, ey) \) and \( e^{C}(x) = C(ex) \) for \( e \in \ker p \). Thus the group \( \ker p = \{ e^{2\pi n} \}_{n \in \mathbb{Z}} \) acts on \( A(\mathbb{L}) \) by \( (ef)(x) = f(e^{-1}x) \) for \( e \in \ker p \).

We say \( \phi \in \text{Map}(\mathbb{L}, \mathbb{Z}) \) is a periodic function on \( \mathbb{L} \), if there exists a sublattice \( \mathbb{L}' \subset \mathbb{L} \) such that \( \phi(x + l) = \phi(x) \) for \( l \in \mathbb{L}' \). For a periodic function \( \phi \) on \( \mathbb{L} \), define \( N_{\phi} \in \mathbb{N} \) as the smallest integer such that \( N_{\phi} \mathbb{L} \subset \mathbb{L}' \). We basically omit \( \phi \) and simply denote \( N \) for \( N_{\phi} \) if there is no risk of confusion.

**Definition 5.1.** For \( s = (s_1, s_2) \in \mathbb{C}^2 \) with \( \Re(s_1 + s_2) > 2 \), a periodic function \( \phi \) on \( \mathbb{L} \) and \( f \in A(\mathbb{L}) \), we define the Shintani \( L \)-function \( L(s, \phi, f) \) by
\[
L(s, \phi, f) := \sum_{x \in \mathbb{L}} f(x) \phi(p(x)) x^{-s_1} y^{-s_2}.
\]

**Remark 5.2.** The series in the Definition 5.1 is absolutely convergent for \( s = (s_1, s_2) \in \mathbb{C}^2 \) with \( \Re(s_1 + s_2) > 2 \).

**5.2. Fundamental domains.** We define \( S \in \text{Hom}(A(\mathbb{L}), \text{Map}(\mathbb{C}^x, \mathbb{Z})) \) by \( S(f) = \sum_{e \in \ker p} ef \).

**Definition 5.3.** We say \( f \in A(\mathbb{L}) \) is a fundamental domain if \( S(f) = 1_{\mathbb{C}^x} \).

**Example 5.4.** We set \( D(x) := C(x) \cup C(x, xe_{2\pi}) \) for \( x \in \mathbb{L} \). Since \( \bigsqcup_{e \in \ker p} D(ex) = \mathbb{C}^x \), \( 1_{D(x)} \) is a fundamental domain. We say \( f \) is a basic fundamental domain if \( f = 1_{D(x)} \) with some \( x \in \mathbb{L} \).

**Proposition 5.5.** If \( f \in A(\mathbb{L}) \) and \( S(f) = 0 \), there exists \( g \in A(\mathbb{L}) \) such that \( f = (1 - e_{2\pi}g) \).

**Proof.** Let \( x \in \mathbb{L} \). We first fix a domain \( D(x) := C(x) \cup C(x, xe_{2\pi}) \). Any element \( f \in A(\mathbb{L}) \) can be expressed as \( f := \sum_{C} m_{C}1_{C} \) with admissible cones \( C \) such that there exist unique \( e_{C} \in \ker p \) with \( e_{C}C \subset D(x) \). For \( f = \sum_{C} m_{C}1_{C} \in A(\mathbb{L}) \), we
set \( f' := \sum_C m_C 1_{eC} \). Then, if \( S(f) = 0 \), we have
\[
S(f') = S \left( f + \sum_C m_C (e_C - 1) 1_C \right) \\
= \sum_C m_C S((e_C - 1) 1_C) \\
= 0.
\]
Since the support of \( f' \) is included in \( D(x) \), \( S(f') = 0 \) implies \( f' = 0 \). Hence we have
\[
f = \sum_C m_C (1 - e_C) 1_C.
\]
Since \( (1 - e) \in (1 - e_{2\pi})\mathbb{Z}[\ker p] \) for \( e \in \ker p \), this completes the proof. \( \square \)

If \( D_1, D_2 \) are two fundamental domains, there exists \( f \in A(\mathbb{L}) \) such that \( D_1 - D_2 = (1 - e_{2\pi})f \) since \( S(D_1 - D_2) = 0 \). For \( f \in A(\mathbb{L}) \) and \( \Re(s_1 + s_2) > 2 \), we have
\[
L(s, \phi, (1 - e_{2\pi})f) = \sum_{x \in \mathbb{L}} f(x) \phi(p(x)) e^{-s_1 \pi^{-s_2}} - \sum_{x \in \mathbb{L}} f(e_{-2\pi}x) \phi(p(x)) e^{-s_1 \pi^{-s_2}} \\
= (1 - e^{2\pi i(s_2 - s_1)}) L(s, \phi, f).
\]
As we will see in Section 5.4.2, \( L(s, \phi, f) \) has a meromorphic continuation to \( s \in \mathbb{C}^2 \) and admits poles only at \( s_1 + s_2 \in \mathbb{Z} \). Thus, if we restrict the variable to \( \{ (s_1, s_2) \in \mathbb{C}^2 \mid s_2 - s_1 \in \mathbb{Z} \} \), we have \( L(s, \phi, (1 - e_{2\pi})f) = 0 \) for \( f \in A(\mathbb{L}) \), by which we see that \( L(s, \phi, D) \) does not depend on the choice of a fundamental domain \( D \).
In particular, if we choose a basic fundamental domain, we see the following fact.

Remark 5.6. For a fundamental domain \( D \), and \( s = (s_1, s_2) \in \mathbb{C}^2 \) with \( s_2 - s_1 \in \mathbb{Z} \), the Shintani L-function \( L(s, \phi, D) \) reduces to the Maass Eisenstein series
\[
\sum_{x \in \mathbb{L}\{0\}} \phi(x) \frac{x^{s_2-s_1}}{|x|^{s_2}}.
\]

5.3. Relation with partial zeta functions. Let \( F \) be an imaginary quadratic field. For a modulus \( m \), let \( \text{Cl}_m(F) \) be the ray class group of modulus \( m \). Let us recall the definition of the partial zeta function.

Definition 5.7. (1) Let \( m \) be a modulus of \( F \). For \( \Re s > 1 \) and a ray class \( \mathcal{A} \in \text{Cl}_m(F) \), we define the partial zeta function \( \zeta(s, \mathcal{A}) \) by
\[
\zeta(s, \mathcal{A}) := \sum_{\mathcal{I} \in \mathcal{A}} \frac{1}{N(\mathcal{I})^s},
\]
where the sum runs over all non-zero integral ideals in \( \mathcal{A} \).

(2) Let \( \Gamma \) be a subgroup of \( \text{Cl}_m(F) \). For \( \mathcal{A}' \in \text{Cl}_m(F)/\Gamma \), we define the partial zeta function \( \zeta(s, \mathcal{A}') \) by
\[
\zeta(s, \mathcal{A}') := \sum_{\mathcal{A} \in \varphi^{-1}(\mathcal{A}')} \zeta(s, \mathcal{A})
\]
where \( \varphi : \text{Cl}_m(F) \to \text{Cl}_m(F)/\Gamma \) is the natural surjection.
Let $H$ be an abelian extension of $F$ with $G := \text{Gal}(H/F)$, and $\mathfrak{m}$ the conductor of $H/F$. From the class field theory, there exists a subgroup $\Gamma$ of the ray class group $\text{Cl}_\mathfrak{m}(F)$ with a canonical isomorphism $\text{rec} : \text{Cl}_\mathfrak{m}(F)/\Gamma \to G$. Then we define $\zeta(s, \eta)$ for $\eta = \sum_{\sigma \in G} m_\sigma \sigma \in \mathbb{Z}[G]$ by

$$
\zeta(s, \eta) := \sum_{\sigma \in G} m_\sigma \zeta(s, \text{rec}^{-1}(\sigma)).
$$

If we choose an ideal $\mathfrak{a} \in \mathcal{A}^{-1}$ for each ray class $\mathcal{A} \in \text{Cl}_\mathfrak{m}(F)$, we have

$$
\sum_{\mathcal{A} \in \mathcal{A}} \frac{1}{N(\mathfrak{a})^s} = N(\mathfrak{a})^s \sum_{\mathcal{A} \in \mathcal{A}} \frac{1}{N(\mathfrak{a})^s},
$$

where $\mathfrak{a}$ is the identity class in $\text{Cl}_\mathfrak{m}(F)$. Note that $\mathfrak{m}$ is identified with an integral ideal, since $F$ has no real places. We denote by $\mathcal{U}(\mathfrak{m})$ the subgroup of $F^\times$ formed by elements congruent to 1 mod $\mathfrak{m}$. Since $w_\mathfrak{m} := |\mathcal{O}_F^\times \cap \mathcal{U}(\mathfrak{m})|$ is finite, the sum is equal to

$$
\frac{1}{w_\mathfrak{m}} \sum_{x \in \mathcal{A} \cap \mathcal{U}(\mathfrak{m})} \frac{1}{|x|^{2s}}.
$$

We set $L = \mathfrak{a}$ and consider the periodic function $1_{\mathfrak{a} \cap \mathcal{U}(\mathfrak{m})} \in \text{Map}(\mathbb{L}, \mathbb{Z})$. Then, for a fundamental domain $D$, we have

$$
L((s, s), 1_{\mathfrak{a} \cap \mathcal{U}(\mathfrak{m})}, D) = \sum_{x \in \mathcal{A} \cap \mathcal{U}(\mathfrak{m})} \frac{1}{|x|^{2s}}.
$$

Thus we have

$$
\zeta(s, \mathcal{A}) = \frac{1}{w_\mathfrak{m}} N(\mathfrak{a})^s L((s, s), 1_{\mathfrak{a} \cap \mathcal{U}(\mathfrak{m})}, D)
$$

for $\mathcal{A} \in \text{Cl}_\mathfrak{m}(F)$.

### 5.4. Analytic continuation of the Shintani $L$-function

As mentioned in the introduction, our ray class invariants $\Lambda_i(-k, \mathcal{A})$ shall be defined by using the partial derivative $\lim_{s \to k} \frac{\partial}{\partial \sigma} (s, \mathcal{A}, \phi, f)$ for $i \in \{1, 2\}$ and $k \in \mathbb{Z}_{\geq 0}$. Therefore, we study the behaviour of the Shintani $L$-function at $(s_1, s_2) \in \mathbb{Z}_{\geq 0}^2$. For this purpose, we give an analytic continuation of $L((s_1, s_2), \phi, f)$ for general $f \in A(\mathbb{L})$ in this section.

#### 5.4.1. An integral representation for $L((s_1, s_2), \phi, 1_C)$

For an admissible cone $C$, there exists (not unique) $\theta \in \mathbb{R}$ such that

$$
\epsilon_\theta \mathcal{C} \subset H
$$

where $\mathcal{C}$ is the closure of $C$ in $\mathbb{C}^\times$ and

$$
H := \left\{ z \in \mathbb{C}^\times \mid -\frac{\pi}{2} < \text{arg} z < \frac{\pi}{2} \right\}.
$$

(Such $\theta$ can be chosen, e.g. as $-\frac{1}{2} (\text{arg} x + \text{arg} y)$ for a 2-dimensional cone $C(x, y)$.)

Since

$$
z^{-s} = \frac{1}{\Gamma(s)} \int_{\mathbb{R}^+} e^{-zt} t^{s-1} dt
$$
for $z \in \mathbb{H}$, one has
\[
 z^{-s_1} \overline{z}^{-s_2} = e^{	heta_1 (e_\theta z)^{-s_1}} \cdot e^{	heta_2 (e_\theta \overline{z})^{-s_2}} = \frac{e^{	heta_1}}{\Gamma(s_1)} \int_{\mathbb{R}_+} e^{-e_\theta z u_1^{s_1-1}} du_1 \cdot \frac{e^{	heta_2}}{\Gamma(s_2)} \int_{\mathbb{R}_+} e^{-e_\theta \overline{z} u_2^{s_2-1}} du_2
\]
for $z \in C$. Changing variables by $t_1 = e_\theta u_1$, $t_2 = e_\theta u_2$, one has $u_1^{s_1-1} du_1 = e^{-s_1 t_1^{s_1-1}} dt_1$, $u_2^{s_2-1} du_2 = e^{-s_2 t_2^{s_2-1}} dt_2$ and thus
\[
(5.8) \quad z^{-s_1} \overline{z}^{-s_2} = \frac{1}{\Gamma(s_1)} \int_{e_\theta \mathbb{R}_+} e^{-z t_1^{s_1-1}} dt_1 \cdot \frac{1}{\Gamma(s_2)} \int_{e_\theta \mathbb{R}_+} e^{-\overline{z} t_2^{s_2-1}} dt_2.
\]
Define
\[
 F((t_1, t_2), \phi, 1_C) := \sum_{z \in \mathbb{L}} 1_C(z) \phi(z) e^{-z t_1 - \overline{z} t_2}
\]
for an admissible cone $C$. Then, from (5.8), it follows that
\[
 L(s, \phi, 1_C) = \sum_{z \in \mathbb{L}} 1_C(z) \phi(z) z^{-s_1} \overline{z}^{-s_2} = \frac{1}{\prod_{i=1}^2 \Gamma(s_i)} \sum_{z \in \mathbb{L}} 1_C(z) \phi(z) \cdot \int_{e_\theta \mathbb{R}_+ \times e_\theta \mathbb{R}_+} e^{-z t_1 - \overline{z} t_2} t_1^{s_1-1} dt_1 t_2^{s_2-1} dt_2 = \frac{1}{\prod_{i=1}^2 \Gamma(s_i)} \int_{e_\theta \mathbb{R}_+ \times e_\theta \mathbb{R}_+} F((t_1, t_2), \phi, 1_C) t_1^{s_1-1} dt_1 t_2^{s_2-1} dt_2
\]
for $\Re s_1, \Re s_2 > 1$.

Now, for $x, y \in \mathbb{L} \setminus \{0\}$, set
\[
(5.9) \quad F((t_1, t_2), \phi, (x, y)) := \frac{\sum_{\alpha \in [N_\alpha; x, y]} \phi(\alpha) e^{-\alpha t_1 - \overline{\alpha} t_2}}{(1 - e^{-N_\alpha (xt_1 + \overline{\alpha} t_2)}) (1 - e^{-N_\alpha (yt_1 + \overline{\alpha} t_2)})},
\]
\[
(5.10) \quad F((t_1, t_2), \phi, x) := \frac{\sum_{\alpha \in [N_\alpha; x]} \phi(\alpha) e^{-\alpha t_1 - \overline{\alpha} t_2}}{(1 - e^{-N_\alpha (xt_1 + \overline{\alpha} t_2)})},
\]
where we put
\[
[N; x, y] := \{m x + ny | m, n \in \mathbb{Q} \text{ such that } 0 < m, n \leq N\},
\]
\[
[N; x] := \{mx | m \in \mathbb{Q} \text{ such that } 0 < m \leq N\}
\]
for $x, y \in \mathbb{L}$ and $N \in \mathbb{Z}_{>0}$. Here, we put $\phi(z) = 0$ for $z \notin \mathbb{L}$, and so the sums in the numerators of (5.9) i.e.
\[
\sum_{\alpha \in [N_\alpha; x, y]} \phi(\alpha) e^{-\alpha t_1 - \overline{\alpha} t_2}, \quad \sum_{\alpha \in [N_\alpha; x]} \phi(\alpha) e^{-\alpha t_1 - \overline{\alpha} t_2}
\]
are finite sums. Since $\phi(\alpha + N \omega) = \phi(\alpha)$ for $\omega \in \mathbb{L}$,
\[
 F((t_1, t_2), \phi, (x, y))
 = \sum_{k,l=0}^{\infty} \sum_{m,n \in \mathbb{Q} \cap \mathbb{L}, 0 < m < N} \phi((m + k N_\alpha) x + (n + l N_\alpha) y) e^{-(m + k N_\alpha)(xt_1 + \overline{\alpha} t_2) + (n + l N_\alpha)(yt_1 + \overline{\alpha} t_2)}
\]
for \((t_1, t_2) \in \mathbb{C}^2\) such that \(\Re(x_1 + \overline{y_2}) > 0\) and \(\Re(y_1 + \overline{x_2}) > 0\). Thus, for a two dimensional admissible cone \(C(x, y)\),

\[
F((t_1, t_2), \phi, 1_{C(x, y)}) = \sum_{m,n \in \mathbb{Q}_{>0}} \phi(mp(x) + np(y))e^{-mp(x) + np(y)t_1 - (mp(x) + np(y))t_2} = F((t_1, t_2), \phi, (p(x), p(y))),
\]

is a meromorphic function of \((t_1, t_2)\) on \(\mathbb{C}^2\). Similarly, for a one dimensional cone \(C(x)\), we have

\[
F((t_1, t_2), \phi, 1_{C(x)}) = F((t_1, t_2), \phi, p(x)).
\]

We extend the definition of \(F\) by

\[
F((t_1, t_2), \phi, f) := \sum_i m_i F((t_1, t_2), \phi, 1_{C_i})
\]

for \(f = \sum_i m_i 1_{C_i} \in A(\mathbb{L})\) where \(C_i\)'s are admissible cones. This definition does not depend on the choice of the expression of \(f\).

One can check the following important property of \(F((t_1, t_2), \phi, (x, y)), F((t_1, t_2), \phi, x)\) by direct calculation.

**Proposition 5.13.** For \(x, y \in \mathbb{L} \setminus \{0\}\), \(F\) satisfies the relations

\[
F((t_1, t_2), \phi, (x, y)) = F((t_1, t_2), \phi, (y, x)),
\]

and

\[
F((t_1, t_2), \phi, x) + F((t_1, t_2), \phi, -x) = -F((t_1, t_2), \phi, 0).
\]

**5.4.2. Analytic continuation of the Shintani \(L\)-function.** The meromorphic continuation of \(L(s, \phi, f)\) to entire \(\mathbb{C}^2\) is essentially given by Shintani in [11]. Since \(f \in A(\mathbb{L})\) is a sum of admissible cones, it is sufficient to consider the case \(f = 1_C\) with an admissible cone \(C\).

For \(a, b\) with \(\arg a = \arg b\), we denote by \(I(a, b)\) the line that starts from \(a\) and ends at \(b\), and for \(a, b\) with \(|a| = |b|\), we denote by \(I(a, b)\) the arc that starts from \(a\) and ends at \(b\). We thus define a contour

\[
I(a, b; \theta, \psi) := I(e_{\theta}b, e_{\theta}a) + I(e_{\theta}a, e_{\psi}a) + I(e_{\psi}a, e_{\psi}b)
\]

and a closed contour

\[
\gamma(a, b; \theta, \psi) := I(e_{\theta}b, e_{\theta}a) + I(e_{\theta}a, e_{\psi}a) + I(e_{\psi}a, e_{\psi}b) + I(e_{\psi}b, e_{\theta}b)
\]

for \(a, b \in \mathbb{R}_{>0}\) and \(\theta, \psi \in \mathbb{R}\) (see the following figures).
The contour $I(a, b; \theta, \psi)$

The closed contour $\gamma(a, b; \theta, \psi)$

Here, the addition of paths means the concatenation of the paths. For simplicity, we use notations $J_1$, $J_2$, $J_3$, and $J_4$ for $I_1$, $I_2$, $I_3$, and $I_4$, respectively, where $a$ is a sufficiently small fixed number and $b$ is a sufficiently large fixed number suitably chosen for the situation. Moreover, we use the notations $I_1$, $I_2$, $I_3$, and $I_4$ for the particular contours $I_1$, $I_2$, $I_3$, and $I_4$, respectively.

For finite number of contours $I_1, I_2, \ldots, I_m$, we denote the sum

$$c_1 \int_{I_1} f(z)dz + c_2 \int_{I_2} f(z)dz + \cdots + c_m \int_{I_m} f(z)dz$$

by

$$\int_{c_1 I_1 + c_2 I_2 + \cdots + c_m I_m} f(z)dz$$

for simplicity.

Let $C$ be an admissible cone and $\theta \in \mathbb{R}$ an element with $e_{\theta}C \subset \mathbb{H}$. Following Shintani, we divide the domain of integration into two pieces

$$\Delta_1 = \{(e_{\theta}t_1, e_{-\theta}t_2); 0 < t_2 < t_1\},$$

and

$$\Delta_2 = \{(e_{\theta}t_1, e_{-\theta}t_2); 0 < t_1 < t_2\},$$

Then we have,

$$\Gamma(s_1)\Gamma(s_2)L((s_1, s_2), \phi, 1_C)$$

$$= \int_{\Delta_1} \mathcal{F}((t_1, t_2), \phi, 1_C)t_1^{s_1-1}dt_1t_2^{s_2-1}dt_2 + \int_{\Delta_2} \mathcal{F}((t_1, t_2), \phi, 1_C)t_1^{s_1-1}dt_1t_2^{s_2-1}dt_2.$$
If we change the variables by $y = t_1, u = t_2/t_1$ in the first integral, we have
\[
\int_{\Delta_1} F(t_1, t_2, \phi, 1_C) t_1^{s_1-1} t_2^{s_2-1} dt_1 dt_2
\]
\[
= \int_{e^{\varphi R}} y^{s_1+s_2-1} dy \int_{e^{-\varphi R}(0,1]} F((y, yu), \phi, 1_C) u^{s_2-1} du
\]
\[
= \frac{1}{(e^{2\pi i (s_1+s_2)} - 1)} \frac{1}{(e^{2\pi i s_2} - 1)} \int_{I(e, \infty: i \theta)} y^{s_1+s_2-1} dy \int_{I(e, 1: -2\theta)} F((yu, y), \phi, 1_C) u^{s_2-1} du.
\]

Here, we take sufficiently small $\varepsilon > 0$. Similarly, by changing the variables by $u = t_1/t_2, y = t_2$ in the second integral, we get
\[
\int_{\Delta_2} F((t_1, t_2, \phi, 1_C) t_1^{s_1-1} t_2^{s_2-1} dt_1 dt_2
\]
\[
= \frac{1}{(e^{2\pi i (s_1+s_2)} - 1)} \frac{1}{(e^{2\pi i s_2} - 1)} \int_{I(e, \infty: -i \theta)} y^{s_1+s_2-1} dy \int_{I(e, 1: 2\theta)} F((yu, y), \phi, 1_C) u^{s_1-1} du.
\]

For $f \in A(L)$, we set
\[
Z((s_1, s_2), \phi, f) := \left( e^{2\pi i (s_1+s_2)} - 1 \right) L((s_1, s_2), \phi, f).
\]

Using the functional equation of the gamma function
\[
\Gamma(s) \Gamma(1 - s) = \frac{2\pi i}{e^{\pi i s} - e^{-\pi i s}},
\]
we have the following proposition.

**Proposition 5.14.** For an admissible cone $C$ and $\theta \in \mathbb{R}$ with $e_\theta C \subset H$, $Z((s_1, s_2), \phi, 1_C)$ admits an integral representation
\[
Z((s_1, s_2), \phi, 1_C)
\]
\[
= e^{-\pi i (s_1+s_2)} \frac{1}{(2\pi i)^2} \Gamma(1 - s_1) \Gamma(1 - s_2)
\]
\[
\times \left\{ (e^{2\pi i s_1} - 1) \int_{I(e, \infty: i \theta)} y^{s_1+s_2-1} dy \int_{I(e, 1: -2\theta)} F((0, y), \phi, 1_C) u^{s_2-1} du
\]
\[
+ (e^{2\pi i s_2} - 1) \int_{I(e, \infty: -i \theta)} y^{s_1+s_2-1} dy \int_{I(e, 1: 2\theta)} F((yu, y), \phi, 1_C) u^{s_1-1} du \right\}.
\]

**Remark 5.15.** Proposition 5.14 gives a holomorphic continuation of $Z((s_1, s_2), \phi, f)$ to $(s_1, s_2) \in \mathbb{C}^2 \setminus ((\mathbb{Z}_{>0} \times \mathbb{C}) \cup (\mathbb{C} \times \mathbb{Z}_{>0}))$ for $f \in A(L)$. Since $L((s_1, s_2), \phi, f)$ is a meromorphic function in more than one variables, $L((s_1, s_2), \phi, f)$ cannot have isolated poles. Therefore, it is evident from Definition 5.1 that $L((s_1, s_2), \phi, f)$ cannot have poles either along $\mathbb{Z}_{>0} \times \mathbb{C}$ or $\mathbb{C} \times \mathbb{Z}_{>0}$, since it cannot have poles in the domain $\{ (s_1, s_2) \in \mathbb{C}^2 | \Re(s_1 + s_2) > 2 \}$. Thus $Z((s_1, s_2), \phi, f)$ is an entire function on $\mathbb{C}^2$ and $L((s_1, s_2), \phi, f)$ is a meromorphic function on $\mathbb{C}^2$ with possible poles along $s_1 + s_2 \in \mathbb{Z}_{\leq 2}$.

**5.5. The first Taylor coefficients of $Z((s_1, s_2), \phi, f)$ at $(s_1, s_2) \in \mathbb{Z}_{\leq 0}^2$.** For $f \in A(L)$, $F((t_1, t_2), \phi, f)$ admits a Laurent series expansion in $(y, u) = (t_1, t_2 t_1^{-1})$. 

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for sufficiently small \( |y|, |u| \). Thus we define \( b_1((k_1, k_2), \phi, f) \) by “the coefficient of \( t_1^{k_1} t_2^{k_2} \)” of the Laurent expansion of \( F((t_1, t_2), \phi, f) \) in \( y, u \). More precisely, if

\[
F((y, u), \phi, f) = \sum_{i_1 \in \mathbb{Z}_{\geq -2}, i_2 \in \mathbb{Z}_{\geq 0}} c_1((i_1, i_2), \phi, f)y^{i_1}u^{i_2},
\]

we define \( b_1((k_1, k_2), \phi, f) := c_1((k_1 + k_2, k_2), \phi, f) \),

\[
t_1^{k_1} t_2^{k_2} = t_1^{k_1} (\frac{t_2}{t_1})^{k_2} = y^{k_1 + k_2} u^{k_2},
\]

Similarly, we define \( b_2((k_1, k_2), \phi, f) \) by the coefficient of \( t_1^{k_1} t_2^{k_2} \) of the Laurent expansion of \( F((t_1, t_2), \phi, f) \) in \( (t_1 t_2^{-1}, t_2) \) valid for sufficiently small \( |t_1 t_2^{-1}|, |t_2| \). Then we have the following proposition.

**Proposition 5.16.** For \( f \in A(\mathbb{L}) \), the Taylor expansion of \( Z((s_1, s_2), \phi, f) \) is of the form

\[
Z((s_1, s_2), \phi, f) = (-1)^{s_1 + k_2} (2\pi i)^k_1 k_2! \sum_{i=1}^{2} b_i((k_1, k_2), \phi, f) (s_i + k_i) + \sum_{l_1 + l_2 \geq 2} c_{ij}((k_1, k_2), \phi, f)(s_1 + k_1)^{l_1}(s_2 + k_2)^{l_2}
\]

with some \( c_{ij} \)'s at \( (s_1, s_2) = (-k_1, -k_2) \) for \( (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \) and \( f \in A(\mathbb{L}) \).

**Proof.** It suffices to show the proposition for \( f = 1_C \) with an admissible cone \( C \). Let \( C \) be an admissible cone, and \( \theta \in \mathbb{R} \) an element with \( c_\theta C \subset \mathbb{H} \). Let us recall the analytic continuation of \( Z((s_1, s_2), \phi, 1_C) \)

\[
Z((s_1, s_2), \phi, 1_C) = e^{-\pi i(s_1 + s_2)} (2\pi i)^2 \Gamma(1 - s_1) \Gamma(1 - s_2)
\]

\[
\times \left\{ (e^{2\pi i s_1} - 1) \times \int_{I(\epsilon, \infty; \theta)} y^{s_1 + s_2 - 1} dy \int_{I(\epsilon, 1; -\theta)} F((y, u), \phi, 1_C) u^{s_2 - 1} du + (e^{2\pi i s_2} - 1) \times \int_{I(\epsilon, \infty; -\theta)} y^{s_1 + s_2 - 1} dy \int_{I(\epsilon, 1; \theta)} F((y, u), \phi, 1_C) u^{s_1 - 1} du \right\}
\]

given in Proposition 5.14. For \( (s_1, s_2) = (-k_1, -k_2) \) with \( (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \), the integrals on the right-hand side reduce to

\[
\int_{C_{\epsilon}} y^{-(k_1 + k_2 + 1)} dy \int_{C_{\epsilon}} F((y, u), \phi, 1_C) u^{-(k_2 + 1)} du
\]

and

\[
\int_{C_{\epsilon}} y^{-(k_1 + k_2 + 1)} dy \int_{C_{\epsilon}} F((y, u), \phi, 1_C) u^{-(k_1 + 1)} du,
\]

where \( C_{\epsilon} := p(I(\epsilon, c_{2\pi \epsilon})) \) is a circle of radius \( \epsilon \) in \( \mathbb{C}^* \). By evaluating these integrals in terms of the residue at the origin, the proposition is proved. \( \square \)
5.6. Vanishing of the first Taylor coefficients of \( Z((s_1, s_2), \phi, D) \) at \((s_1, s_2) \in \mathbb{Z}_\leq 0^2 \) for a fundamental domain \( D \).

**Proposition 5.17.** For a fundamental domain \( D \), the Taylor expansion of \( Z((s_1, s_2), \phi, D) \) is of the form

\[
Z((s_1, s_2), \phi, D) = \begin{cases} 
\sum_{i+j \geq 2} c_{ij}(s_1 + k_1)^i(s_2 + k_2)^j & \text{for } (k_1, k_2) \in \mathbb{Z}_\leq 0^2 \setminus \{(0, 0)\} \\
-2\pi i \phi(0)(s_1 + s_2) + \sum_{i+j \geq 2} c_{ij}s_1 s_2^j & \text{for } (k_1, k_2) = (0, 0)
\end{cases}
\]

with some \( c_{ij} \)'s.

**Proof.** Let \( t := (t_1, t_2) \). From Proposition 5.16, we have only to observe that \( \mathcal{F}(t, \phi, D) = -\phi(0) \). Since \( D - 1_{D(z)} \in (1 - e_{2\pi})A(\mathbb{L}) \), and \( \mathcal{F}(t, \phi, (1 - e_{2\pi})f) = 0 \) for \( f \in A(\mathbb{L}) \), the proposition follows from \( \mathcal{F}(t, \phi, 1_{D(z)}) = -\phi(0) \). Since

\[
1_{D(z)} = 1_{C(z)} + 1_{C(x,y)} + 1_{C(y)} + 1_{C(\psi, x)} + 1_{C(\psi x, x)} + 1_{C(\psi x, x y)} + 1_{C(\psi y, x)}
\]

for \( y \in \mathbb{L} \cap C(x, e_{2\pi} x) \), we have

\[
\mathcal{F}(t, \phi, 1_{D(z)}) = F(t, \phi, x) + F(t, \phi, (x, y)) + F(t, \phi, y) + F(t, \phi, (y, -x)) + F(t, \phi, (x, -y)) + F(t, \phi, (y, x))
\]

From the Proposition 5.13, the right-hand side equals \(-\phi(0)\). Hence the proposition is proved. \( \square \)

5.7. The first partial derivative \( \frac{\partial L}{\partial s_i} \)((-k_1, -k_2), \phi, D; \lambda) \). In this section, we show the existence of the first partial derivative \( \frac{\partial L}{\partial s_i} \)((-k_1, -k_2), \phi, D; \lambda) \) for a fundamental domain \( D \), and a direction \( \lambda \). For a meromorphic function \( f(s_1, s_2) \) on \( \mathbb{C}^2 \) and a "direction" \( \lambda \in \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \lambda_1 + \lambda_2 = 1 \} \), we set

\[
f(s_1, s_2; \lambda) := \lim_{t \to 0} f(s_1 + t\lambda_1, s_2 + t\lambda_2)
\]

if the limit exists. Let \((k_1, k_2) \in \mathbb{Z}_\leq 0^2 \). From Proposition 5.17, we have

\[
Z((-k_1 - s, -k_2 + s), \phi, D) = O(s^2).
\]

Thus we define \( R_{k_1, k_2}(\phi, D) \) as follows.

**Definition 5.18.** For \((k_1, k_2) \in \mathbb{Z}_\leq 0^2 \), we define \( R_{k_1, k_2}(\phi, D) \) by the coefficient of \( s^2 \) of the Taylor expansion of \( Z((-k_1 - s, -k_2 + s), \phi, D) \).

Then, we have the following two lemmas.

**Lemma 5.19.** For \( i \in \{1, 2\} \), \((k_1, k_2) \in \mathbb{Z}_\leq 0^2 \), a fundamental domain \( D \), and a direction \( \lambda \), \( \frac{\partial L}{\partial s_i} \)((-k_1, -k_2), \phi, D; \lambda) \) exists.

**Lemma 5.20.** For \( i \in \{1, 2\} \), \((k_1, k_2) \in \mathbb{Z}_\leq 0^2 \), a fundamental domain \( D \), and a direction \( \lambda = (\lambda_1, \lambda_2) \), we have

\[
\frac{\partial L}{\partial s_i}((-k_1, -k_2), \phi, D; \lambda) = \frac{\partial L}{\partial s_i}((-k_1, -k_2), \phi, D; e_i) - \frac{(1 - \lambda_i)^2}{2\pi i} R_{k_1, k_2}(\phi, D).
\]

Here, we set \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \).

To prove these lemmas, we first show the following key proposition.
Lemma 5.21. Let $G(s_1, \ldots, s_n)$ be a holomorphic function in a neighborhood of the origin, such that $G(0, \ldots, 0) = 0$. Put $g(s_1, \ldots, s_n) := (s_1 + \cdots + s_n)^{-1}G(s_1, \ldots, s_n)$ and $e_k := (0, \ldots, 1, \ldots, 0) \in \mathbb{C}^n$. Then, $\frac{\partial}{\partial s_k}(te_k)$ is holomorphic at $t = 0$, and for $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ with $\lambda_1 + \cdots + \lambda_n = 0$, the Laurent series expansion of $\frac{\partial}{\partial s_k}(t(e_k + \lambda))$ is of the form

$$\frac{\partial g}{\partial s_k}(t(e_k + \lambda)) = \frac{\partial g}{\partial s_k}(te_k) - t^2G(t\lambda) + O(t).$$

Proof. From the assumption that $G(0) = 0$, we may put

$$g(s_1, \ldots, s_n) = \sum_{1 \leq i \leq n} a_is_i + \sum_{1 \leq i, j \leq n} c_{ij}s_is_j + t^3$$

with some $a_i$'s and $c_{ij}$'s, where $I^m$ denotes the terms with degree $\geq m$. Differentiating with respect to $s_k$, one has

$$\frac{\partial g}{\partial s_k}(s_1, \ldots, s_n)$$

$$= \left(\sum_{1 \leq i \leq n} (c_{ik} + c_{kj})s_i + t^2\right) - \left(\sum_{1 \leq i \leq n} a_is_i + \sum_{1 \leq i, j \leq n} c_{ij}s_is_j + t^3\right)$$

$$= \sum_{1 \leq i \leq n} (a_k - a_i)s_i + \sum_{1 \leq i, j \leq n} (c_{ik} + c_{kj} - c_{ij})s_is_j + t^3.$$ 

Thus

$$\frac{\partial g}{\partial s_k}(te_k) = \frac{c_k t^2 + O(t^3)}{t^2} = c_k + O(t)$$

is holomorphic at $t = 0$.

Now, if $\lambda_1 + \cdots + \lambda_n = 0$, we have

$$\frac{\partial g}{\partial s_k}(te_k + \lambda) - \frac{\partial g}{\partial s_k}(te_k)$$

$$= -t\sum_{1 \leq i \leq n} a_is_i + t^2\left(\sum_{1 \leq i, j \leq n} (c_{ik} + c_{kj} - c_{ij})(\lambda_i + \delta_{ik})(\lambda_j + \delta_{kj}) - \delta_{ik}\delta_{kj}\right) + O(t^3)$$

$$= -t\sum_{1 \leq i \leq n} a_is_i + t^2\sum_{1 \leq i, j \leq n} (c_{ik} + c_{kj} - c_{ij})(\lambda_i\lambda_j + \lambda_i\delta_{kj} + \lambda_j\delta_{ik}) + O(t^3).$$

Since $\sum_{1 \leq i \leq n} (c_{ik} - c_{ij})\lambda_i\delta_{kj} = 0$ and $\sum_{1 \leq i, j \leq n} c_{ij}\lambda_i\lambda_j = \sum_{1 \leq i \leq n} c_{kk}\lambda_i = 0$, we have

$$\sum_{1 \leq i, j \leq n} (c_{ik} + c_{kj} - c_{ij})\lambda_i\delta_{kj} = 0.$$

Similarly, we have

$$\sum_{1 \leq i, j \leq n} (c_{ik} + c_{kj} - c_{ij})\delta_{ik}\lambda_j = 0.$$

Together with $\sum_{1 \leq i, j \leq n} c_{ik}\lambda_i\lambda_j = 0$ and $\sum_{1 \leq i, j \leq n} c_{kj}\lambda_i\lambda_j = 0$, we obtain the equality

$$\frac{\partial g}{\partial s_k}(te_k + \lambda) - \frac{\partial g}{\partial s_k}(te_k) = -t\sum_{1 \leq i \leq n} a_is_i + t^2\sum_{1 \leq i, j \leq n} c_{ij}\lambda_i\lambda_j + O(t^3).$$
Since \( G(t\lambda) = t \sum_{1 \leq i \leq n} a_i \lambda_i + t^2 \sum_{1 \leq i, j \leq n} c_{ij} \lambda_i \lambda_j + O(t^3) \), the proposition is proved. \( \square \)

As a special case of Lemma 5.21, we have the following corollary.

**Corollary 5.22.** Let \( G(s_1, \ldots, s_n), g(s_1, \ldots, s_n) \) be as above. If \( G(t\lambda) \) has second order zero at \( t = 0 \), i.e. \( G(t\lambda) = \sum_{d=2}^{\infty} a_d(\lambda)t^d \), we have

\[
\frac{\partial g}{\partial s_i}(0; e_i + \lambda) = \frac{\partial g}{\partial s_i}(0; e_i) - a_2(\lambda)
\]
for \( i = 1, \ldots, n \).

Now, we prove Lemmas 5.19 and 5.20. Let \((k_1, k_2) \in \mathbb{Z}_{>0}^2 \) and consider the case \( n = 2 \) with \( g(s_1, s_2) = L((s_1 - k_1, s_2 - k_2), \phi, D) \). Then

\[
G(s_1, s_2) = (s_1 + s_2)g(s_1, s_2) = \frac{s_1 + s_2}{e^{2\pi i(s_1 + s_2)} - 1} Z((s_1 - k_1, s_2 - k_2), \phi, D)
\]

Since

\[
\frac{e^{2\pi i(s_1 + s_2)} - 1}{s_1 + s_2} = 2\pi i \sum_{k=0}^{\infty} \frac{(2\pi i(s_1 + s_2))^k}{(k+1)!},
\]

we have

\[
G(t\lambda) = \frac{1}{2\pi i} Z((t\alpha - k_1, -t\alpha - k_2), \phi, D)
\]

for \( \lambda = (\alpha, -\alpha) \). Therefore, we see that

\[
a_2(\lambda) = \frac{\alpha^2}{2\pi i} R_{k_1, k_2}(\phi, D)
\]
in this case. Applying the above corollary, we have

\[
\frac{\partial L}{\partial s_1}((-k_1, -k_2), \phi, D; (1+\alpha, -\alpha)) = \frac{\partial L}{\partial s_1}((-k_1, -k_2), \phi, D; (1, 0)) - \frac{\alpha^2}{2\pi i} R_{k_1, k_2}(\phi, D)
\]
and

\[
\frac{\partial L}{\partial s_2}((-k_1, -k_2), \phi, D; (\alpha, 1-\alpha)) = \frac{\partial L}{\partial s_2}((-k_1, -k_2), \phi, D; (0, 1)) - \frac{\alpha^2}{2\pi i} R_{k_1, k_2}(\phi, D).
\]

Thus the lemmas are proved.

**Remark 5.23.** We can see from Proposition 5.14 that, for \( k_2 \in \mathbb{Z}_{\geq 0} \), the one variable function \( L((s, -k_2), \phi, f) \) is holomorphic at \( s \in \mathbb{C} \setminus \{1, \ldots, k_2 + 2\} \), and the first derivative of \( L((s, -k_2), \phi, f) \) at \( s = -k_1 \in \mathbb{Z}_{\leq 0} \) is equal to

\[
\frac{\partial L}{\partial s_1}((-k_1, -k_2), \phi, f; (1, 0)).
\]
In particular, \( \frac{\partial L}{\partial s_1}((-k_1, -k_2), \phi, f; e_i) \) exists for all \((k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \) and \( f \in A(\mathbb{L}) \).
5.8. Explicit calculation of $Z((-k_1 - s, -k_2 + s), \phi, D)$. For $(k_1, k_2) \in \mathbb{Z}^2_{\geq 0}$, set

$$R_{k_1, k_2}(s, \phi, f) := Z((-k_1 - s, -k_2 + s), \phi, f).$$

In this section, we give an explicit expression for $R_{k_1, k_2}(s, \phi, 1_{D(a)})$. For this purpose, we first give a closed contour integral representation for $R_{k_1, k_2}(s, \phi, f)$.

**Lemma 5.24.** For $(k_1, k_2) \in \mathbb{Z}^2_{\geq 0}$, and an admissible cone $C$, we have

$$R_{k_1, k_2}(s, \phi, 1_C) = (-1)^{k_1 + k_2} \frac{\Gamma(1 + k_1 + s) \Gamma(1 + k_2 - s)}{2\pi i} (e^{-2\pi i s} - 1) \times \int_{(\gamma_{(-2\theta)})} F_{k_1 + k_2}((1, u), \phi, 1_C) u^{s-k_2-1} du.$$

Here, $\theta \in \mathbb{R}$ is any element such that $e^{\theta} \subset H$.

**Proof.** Let $C_\varepsilon := p(I(\varepsilon, r_{2\varepsilon}))$ be a circle-shaped contour of radius $\varepsilon$ in $\mathbb{C} \times$. Put

$$A := \frac{(-1)^{k_1 + k_2} (2\pi i)^2}{\Gamma(1 + k_1 + s) \Gamma(1 + k_2 - s)} R_{k_1, k_2}(s, \phi, 1_C).$$

Then we have from Proposition 5.14,

$$A = \left\{ (e^{-2\pi i s} - 1) \int_{I(\varepsilon, \infty; \theta)} t_1^{-k_1-k_2-1} dt_1 \int_{I(\varepsilon, 1, -2\theta)} F((t_1, ut_1), \phi, 1_C) u^{s-k_2-1} du + (e^{2\pi i s} - 1) \int_{I(\varepsilon, \infty; \theta)} t_1^{-k_1-k_2-1} dt_2 \int_{I(\varepsilon, 1, 2\theta)} F((vt_2, t_2), \phi, 1_C) v^{-s-k_1-1} dv \right\}$$

$$= \left\{ (e^{-2\pi i s} - 1) \int_{C_\varepsilon} t_1^{-k_1-k_2-1} dt_1 \int_{I(\varepsilon, 1, -2\theta)} F((t_1, ut_1), \phi, 1_C) u^{s-k_2-1} du + (e^{2\pi i s} - 1) \int_{C_\varepsilon} t_2^{-k_1-k_2-1} dt_2 \int_{I(\varepsilon, 1, 2\theta)} F((vt_2, t_2), \phi, 1_C) v^{-s-k_1-1} dv \right\}.$$

Define the “degree $i$ homogeneous part” $F_i$ by the $i$-th coefficient of the Laurent expansion

$$F((\lambda t_1, \lambda t_2), \phi, f) = \sum_{i \in \mathbb{Z}^2_{\geq -2}} F_i((t_1, t_2), \phi, f) \lambda^i.$$

Clearly, $F_i((t_1, t_2), \phi, f)$ is a homogeneous rational function of degree $i$ in $(t_1, t_2)$. Since we have

$$F((t_1, ut_1), \phi, f) = \sum_{i \in \mathbb{Z}^2_{\geq -2}} F_i((1, u), \phi, f)t_1^i$$

and

$$F((vt_2, t_2), \phi, f) = \sum_{i \in \mathbb{Z}^2_{\geq -2}} F_i((v, 1), \phi, f)t_2^i,$$

the above expression for $A$ can be rewritten as

$$2\pi i \left\{ (e^{-2\pi i s} - 1) \int_{I(\varepsilon, 1, -2\theta)} F_{k_1+k_2}((1, u), \phi, 1_C) u^{s-k_2-1} du 
\quad + (e^{2\pi i s} - 1) \int_{I(\varepsilon, 1, 2\theta)} F_{k_1+k_2}((v, 1), \phi, 1_C) v^{-s-k_1-1} dv \right\}.$$
by residue calculus. Changing the variable by \( v = e^{2\pi i s} u^{-1} \), we have
\[
(e^{2\pi i s} - 1) \times \int_{I(\varepsilon,1;2\theta)} F_{k_1 + k_2} ((v,1), \phi, 1_C) u^{-s-k_1-1} dv
\]
\[
= (e^{-2\pi i s} - 1) \times \int_{-I(\varepsilon^{-1},1;-2\theta)} F_{k_1 + k_2} ((1,u), \phi, 1_C) u^{s-k_2-1} du.
\]
Since
\[
\int_{I(\varepsilon,1;2\theta)-I(\varepsilon^{-1},1;-2\theta)} f(z) dz = \int_{\gamma(\varepsilon^{-1};-2\theta,2\pi-2\theta)} f(z) dz
\]
for any \( f(z) \), this proves the proposition. \( \square \)

**Remark 5.25.** For \( uv = 1 \), the 1-forms
\[
F_{k_1 + k_2} ((1,u), \phi, 1_C) u^{s-k_2-1} du
\]
and
\[
-F_{k_1 + k_2} ((v,1), \phi, 1_C) v^{-s-k_1-1} dv
\]
are equal, which can be expressed in more symmetric way as
\[
F_{k_1 + k_2} ((1,t_1,t_2), \phi, 1_C) t_1^{-s-k_1} t_2^{-k_2} d log \left( \frac{t_2}{t_1} \right),
\]
by using the homogeneous coordinate \( (t_1 : t_2) \) on the universal covering of \( \mathbb{C}^\times = \mathbb{P}^1 \setminus \{0, \infty\} \).

Using Lemma 5.24, we can show the following proposition.

**Proposition 5.26.** For \( (k_1, k_2) \in \mathbb{Z}_0^2 \), \( x \in \mathbb{L} \) and \( y \in \mathbb{L} \cap C(x,e_{\pi} x) \), we have
\[
R_{k_1,k_2}(s, \phi, 1_D(x)) = (-1)^{k_1 + k_2} \frac{\Gamma(1 + k_1 + s) \Gamma(1 + k_2 - s)}{2\pi i} (e^{-2\pi i s} - 1)(1 - e^{4\pi i s})
\]
\[
\times \int_{\gamma(-2\theta,-2\theta')} F_{k_1 + k_2} ((1,u), \phi, (-x,-y)) u^{s-k_2-1} du.
\]
Here, \( \theta, \theta' \in \mathbb{R} \) are any elements such that \( e_\theta C(x,y) \subset \mathbb{H} \) and \( e_{\theta'} C(y,e_{\pi} x) \subset \mathbb{H} \).

**Proof.** For \( x, y \in \mathbb{L} \), we define \( F_i((t_1,t_2), \phi, (x,y)) \) and \( F_i((t_1,t_2), \phi, x) \) by
\[
F((\lambda t_1, \lambda t_2), \phi, (x,y)) = \sum_{i \in \mathbb{Z}_{-2}} F_i((t_1,t_2), \phi, (x,y)) \lambda^i
\]
\[
F((\lambda t_1, \lambda t_2), \phi, x) = \sum_{i \in \mathbb{Z}_{-1}} F_i((t_1,t_2), \phi, x) \lambda^i.
\]
Clearly, we have
\[
F_i((t_1,t_2), \phi, 1_C(x,y)) = F_i((t_1,t_2), \phi, (p(x), p(y))),
\]
\[
F_i((t_1,t_2), \phi, 1_C(x)) = F_i((t_1,t_2), \phi, p(x))
\]
for admissible cones \( C(x) \) and \( C(x,y) \). Since \( F((t_1,t_2), \phi, (x,y))(xt_1 + \overline{x}t_2)(yt_1 + \overline{y}t_2) \) is homogeneous polynomial of degree \( i + 2 \) in \( t_1, t_2 \). Similarly, \( F_i((t_1,t_2), \phi, x)(xt_1 + \overline{x}t_2) \) is a homogeneous polynomial of degree \( i + 1 \) in \( t_1, t_2 \). Also, it holds that
\[
F_i((t_1,t_2), \phi, (x,y)) + F_i((t_1,t_2), \phi, (-x,y)) = -F_i((t_1,t_2), \phi, y),
\]
\[
F_i((t_1,t_2), \phi, x) + F_i((t_1,t_2), \phi, -x) = -\phi(0) \delta_{i,0},
\]
where $\delta_{i,0}$ is the Kronecker delta. Set
\[
\omega(0) := \phi(0)\delta_{i,k_1+k_2,0} u^{s-k_2-1} du,
\omega(x) := F_{k_1+k_2}((1,u),\phi,x)u^{s-k_2-1} du,
\omega(x,y) := F_{k_1+k_2}((1,u),\phi,(x,y))u^{s-k_2-1} du.
\]
Since
\[
D(x) = (1 + e^x)(1_C(x) + 1_C(x,y) + 1_C(y) + 1_C(y,e,x))
\]
we have, from Lemma 5.24,
\[
\frac{(-1)^{k_1+k_2}2\pi i}{\Gamma(1+k_1+s)\Gamma(1+k_2-s)} R_{k_1,k_2}(s,\phi,1D(x))
\]
\[
= (e^{-2\pi is} - 1) \times \left\{ \int_{\gamma(2\pi - 2\theta)} (\omega(x) + \omega(x,y)) + \int_{\gamma(-2\theta)} (\omega(y) + \omega(-x,y)) \right\}
\]
\[
+ \int_{\gamma(2\pi - 2\theta)} (\omega(-x) + \omega(-x,-y)) + \int_{\gamma(2\pi - 2\theta)} (\omega(-y) + \omega(x,-y)) \right\}.
\]
From the relation $\omega(y) + \omega(x,y) + \omega(-x,y) = 0$, we have
\[
\int_{\gamma(2\pi - 2\theta)} \omega(-x,y) + \int_{\gamma(2\pi - 2\theta)} (\omega(-y) + \omega(x,-y)) = \int_{\gamma(2\pi - 2\theta) - \gamma(2\pi - 2\theta)} \omega(-x,y).
\]
From the relation $\omega(-y) + \omega(x,-y) + \omega(-x,-y) = 0$, we have
\[
\int_{\gamma(2\pi - 2\theta)} \omega(-x,y) + \int_{\gamma(2\pi - 2\theta)} (\omega(-y) + \omega(x,-y)) = \int_{\gamma(2\pi - 2\theta) - \gamma(2\pi - 2\theta)} \omega(-x,-y).
\]
From the relation $\omega(0) + \omega(x) + \omega(-x) = 0$, we have
\[
\int_{\gamma(2\pi - 2\theta)} \omega(x) + \int_{\gamma(2\pi - 2\theta)} (\omega(-x) + \omega(x,y)) = \int_{\gamma(2\pi - 2\theta) - \gamma(2\pi - 2\theta)} \omega(x) - \int_{\gamma(2\pi - 2\theta) - \gamma(2\pi - 2\theta)} \omega(x).
\]
Thus we have
\[
\frac{(-1)^{k_1+k_2}2\pi i}{\Gamma(1+k_1+s)\Gamma(1+k_2-s)} R_{k_1,k_2}(s,\phi,1D(x))
\]
\[
= (e^{-2\pi is} - 1) \times \left\{ \int_{\gamma(2\pi - 2\theta)} \omega(x) + \int_{\gamma(-2\theta)} \omega(x,y) \right\}
\]
\[
+ \int_{\gamma(2\pi - 2\theta) - \gamma(2\pi - 2\theta)} \omega(-x,-y) \right\}.
\]
Since $\int_{\gamma(\theta_1)-\gamma(\theta_2)} f(z) dz = \int_{\gamma(\theta_1,\theta_2)} f(z) dz$ for any $f(z)$, we obtain
\[
\int_{\gamma(-2\theta)-\gamma(2\pi - 2\theta)} (\omega(-x) + \omega(x,y) + \int_{\gamma(2\pi - 2\theta) - \gamma(2\pi - 2\theta)} \omega(-x,-y)) = (1 - e^{2\pi is}) \left\{ \int_{\gamma(-2\theta)} \omega(x) + \int_{\gamma(-2\theta,2\pi - 2\theta)} \omega(x,y) + \int_{\gamma(2\pi - 2\theta,2\pi - 2\theta)} \omega(-x,-y) \right\}.
\]
The poles of $\omega(x)$ are located at

$$u \in \epsilon(1+2\pi) + 2\pi \vec{g}x.$$ 

From the assumption on $\theta, \theta'$, we have

$$-2\theta < \pi + 2\pi \vec{g}x < -2\theta'$$

hence $\omega(x)$ has no poles inside $\gamma(-2\theta', 2\pi - 2\theta)$. Thus we have

$$\int_{\gamma(-2\theta)} \omega(x) = \int_{\gamma(-2\theta', -2\theta')} \omega(x).$$

Again, using $\omega(x) + \omega(x, y) = \omega(-x, -y) + \omega(-y)$ ($= -\omega(x, -y)$), we have

$$\int_{\gamma(-2\theta, -2\theta')} (\omega(x) + \omega(x, y)) = \int_{\gamma(-2\theta, -2\theta')} (\omega(-x, -y) + \omega(-y)).$$

The poles of $\omega(-y)$ are located at

$$u \in \epsilon(1+2\pi) + 2\pi \vec{g}y.$$ 

From the assumption on $\theta, \theta'$, we have

$$-\pi + 2\pi \vec{g}y < -2\theta < -2\theta' < \pi + 2\pi \vec{g}y,$$

hence $\omega(-y)$ has no poles inside $\gamma(-2\theta, -2\theta')$, from which we see that

$$\int_{\gamma(-2\theta, -2\theta')} \omega(-y) = 0.$$

Thus we obtain

$$\frac{(-1)^{k_1+k_2}2\pi i}{\Gamma(1+k_1+s)\Gamma(1+k_2-s)(e^{-2\pi is} - 1)(1-e^{2\pi is})} R_{k_1,k_2}(s, \phi, 1_{D(x)})$$

$$= \int_{\gamma(-2\theta, -2\theta')} \omega(-x, -y) + \int_{\gamma(2\pi - 2\theta, 2\pi - 2\theta')} \omega(-x, -y)$$

$$= (1 + e^{2\pi is}) \int_{\gamma(-2\theta, -2\theta')} \omega(-x, -y).$$

The proposition is proved.

From the proposition, we can evaluate $R_{k_1,k_2}(s, \phi, 1_{D(x)})$ by a residue calculus. The result is the following lemma.

**Lemma 5.27.** For $(k_1, k_2) \in \mathbb{Z}_{\geq 0}^2$ and $x \in \mathbb{L}$, we have

$$R_{k_1,k_2}(s, \phi, 1_{D(x)})$$

$$= (-1)^{k_1+1} e^{\pi i s} (e^{-2\pi is} - 1)(1-e^{2\pi is}) \Gamma(1+k_1+s) \Gamma(1+k_2-s)$$

$$\times \left( \frac{y}{x} \right)^{k_1+k_2+1} \frac{N^{k_1+k_2}}{M^{k_1+k_2+1}} \sum_{0 < k_l \leq N} \phi(kx + ly) B_{k_1+k_2+2} \left( \frac{1}{N} \right).$$

where $k, l$ in the sum runs over all rationals.

**Proof.** From Proposition 5.26, we are to compute

$$\int_{\gamma(-2\theta, -2\theta')} F_{k_1,k_2}(1, u, \phi, (-x, -y)) u^{s-k_2-1} du.$$
Since the poles of $\omega(-x, -y)$ are located at
\[ u \in e^{(1+2\pi i)} \cap e^{(1+2\pi i + 2\pi i)} \]
and
\[ -\pi + 2\pi u y < -2\theta < \pi + 2\pi u x < -2\theta' < \pi + 2\pi u y \]
from the assumption on $\theta, \theta'$, the only pole of $\omega(-x, -y)$ inside $\gamma(-\theta, -\theta')$ is $u = e_{+2\pi i} x$.

For $x, y \in L$, we define $b_{i, j}(\phi, (x, y))$ by the generating function
\[ \sum_{0 < k, l \leq N} \phi(kx + ly) e^{(ku_1 + lu_2)} \frac{u_1^{i-1} u_2^{j-1}}{(1 - e^{N u_1})(1 - e^{N u_2})} = \sum_{i_1, i_2 \in \mathbb{Z}_{\geq 0}} b_{i_1, i_2}(\phi, (x, y)) u_1^{i_1-1} u_2^{i_2-1}. \]

Then we have
\[ F_i((t_1, t_2), \phi, (-x, -y)) = \sum_{i_1 + i_2 = i+2} b_{i_1, i_2}(\phi, (x, y))(xt_1 + \pi t_2)^{i_1-1}(yt_1 + \pi t_2)^{i_2-1}. \]

Thus we have
\[ F_i((1, u), \phi, (-x, -y)) \left( u + \frac{x}{\pi} \right)^{u = -\frac{x}{\pi}} = b_{0, i+2}(\phi, (x, y)) \pi^{i+1} \left( \frac{y\pi - x\pi^2}{\pi} \right)^{i+1}. \]

Hence the residue of $\omega(-x, -y)$ at $u = e_{+2\pi i} x$ is
\[ (-1)^{k_2} e^{\pi i} \sum_{0 < k, l \leq N} \phi(kx + ly) e^{(ku_1 + lu_2)} \frac{u_1^{i-1} u_2^{j-1}}{(1 - e^{N u_1})(1 - e^{N u_2})} \]
with respect to the Taylor expansion in $u_1$, we have
\[ \sum_{i \in \mathbb{Z}_{\geq 0}} b_{0, i}(\phi, (x, y)) u_1^{i-1} = \sum_{0 < k, l \leq N} \phi(kx + ly) e^{u} \frac{e^{iu}}{N(e^{Nu} - 1)} \]
\[ = \sum_{0 < i \leq N} \left( \frac{1}{N} \sum_{0 < k, l \leq N} \phi(kx + ly) \right) e^{iu} \frac{e^{iu}}{e^{Nu} - 1} \]
\[ = \sum_{0 < i \leq N} \left( \frac{1}{N} \sum_{0 < k, l \leq N} \phi(kx + ly) \right) \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{B_i \left( \frac{N}{i} \right)}{i!} (Nu)^{i-1} \]
\[ = \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N^{i-2} e^{xt}}{i!} \sum_{0 < k, l \leq N} \phi(kx + ly) B_i \left( \frac{1}{N} \right), \]
where $B_i(x)$ is the $i$-th Bernoulli polynomial defined by the generating function
\[ \frac{e^{xt}}{e^x - 1} = \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{B_i(x)}{i!} t^{i-1}. \]

Thus we have
\[ b_{0, i}(\phi, (x, y)) = \frac{N^{i-2}}{i!} \sum_{0 < k, l \leq N} \phi(kx + ly) B_i \left( \frac{1}{N} \right). \]
Therefore we have
\[
R_{k_1, k_2}(s, \phi, 1_D(x)) = \frac{(-1)^{k_2}e^{it\pi}(e^{-2\pi is} - 1)(1 - e^{4\pi is})\Gamma(1 + k_1 + s)\Gamma(1 + k_2 - s)}{x^{k_1 + k_2 + 1}} \sum_{0 < k, l \leq N} \frac{\phi(kx + ly)B_{k_1 + k_2 + 2}}{(lN)^{k_1 + k_2 + 1}}.
\]

If we put
\[
\zeta(s, \phi, (x, y)) := \sum_{t \in \mathbb{Q}_{> 0}} \sum_{0 < k \leq N} \frac{\phi(kx + ly)}{\frac{1}{N}},
\]
\(
\zeta(s, \phi, (x, y))
\) is a finite linear combination of Hurwitz zeta functions. Hence \(\zeta(s, \phi, (x, y))\) is meromorphically continued to \(s \in \mathbb{C}\), and we can also express \(R_{k_1, k_2}(s, \phi, 1_D(x))\) as
\[
R_{k_1, k_2}(s, \phi, 1_D(x)) = \frac{(-1)^{k_2}e^{it\pi}(e^{-2\pi is} - 1)(1 - e^{4\pi is})\Gamma(1 + k_1 + s)\Gamma(1 + k_2 - s)}{x^{k_1 + k_2 + 1}} \zeta(-1 - k_1 - k_2, \phi, (x, y)),
\]
\[
\times \frac{(y - x\bar{\gamma})^{k_1 + k_2 + 1}}{(k_1 + k_2 + 2)!}.
\]

In particular, we have the following proposition.

**Proposition 5.28.** For \((k_1, k_2) \in \mathbb{Z}_{> 0}^2\) and \(x \in \hat{\mathbb{L}}\), we have
\[
R_{k_1, k_2}(\phi, 1_D(x)) = (-1)^{k_1+1} \frac{k_2!k_1!}{x^{k_2+1}|\bar{\gamma}|^{k_1+1}} \frac{(y - x\bar{\gamma})^{k_1 + k_2 + 1}}{(k_1 + k_2 + 1)!} \times 2(2\pi i)^2 \zeta(-1 - k_1 - k_2, \phi, (x, y)),
\]
or equivalently,
\[
R_{k_1, k_2}(\phi, 1_D(x)) = (-1)^{k_1+1} \frac{2(2\pi i)^2k_1!k_2!}{x^{k_2+1}|\bar{\gamma}|^{k_1+1}} \times \frac{(y - x\bar{\gamma})^{k_1 + k_2 + 1}}{(k_1 + k_2 + 2)!} \sum_{0 < k, l \leq N} \frac{\phi(kx + ly)B_{k_1 + k_2 + 2}}{(lN)^{k_1 + k_2 + 1}}.
\]

**Example 5.29.** Set \(x = 1, y = \tau\), and \(\phi = 1_\mathbb{L}\) with \(\mathbb{L} = \mathbb{Z}\tau + \mathbb{Z}\), then \(\zeta(s, \phi, (x, y)) = \zeta(s)\) is the Riemann zeta function. Hence we have
\[
R_{k_1, k_2}(1_\mathbb{L}, 1_D(1)) = (-1)^{k_1+1} \frac{k_2!k_1!}{x^{k_2+1}|\bar{\gamma}|^{k_1+1}} \frac{\tau - \bar{\gamma}^{k_1 + k_2 + 1}}{(k_1 + k_2 + 2)!} \times 2(2\pi i)^2.
\]

Combining this result with the Lemma 5.20, we have
\[
\frac{\partial L}{\partial s_i} ((-k_1, -k_2), 1_\mathbb{L}, 1_D(1); \lambda) = \frac{\partial L}{\partial s_i} ((-k_1, -k_2), 1_\mathbb{L}, 1_D(1); e_i)
\]
\[
+ 4\pi i(1 - \lambda)^2 (-1)^{k_1} k_2! k_1! \zeta(-1 - k_1 - k_2) \frac{(\tau - \bar{\gamma})^{k_1 + k_2 + 1}}{(k_1 + k_2 + 2)!}.
\]

For a general fundamental domain \(D\), we can evaluate \(R_{k_1, k_2}(\phi, D)\) by using the following proposition.
Proposition 5.30. Let $D, D'$ be two fundamental domains with $D - D' = (1 - e^{2\pi i}) f$.

Then we have

$$R_{k_1, k_2}(\phi, D - D') = (-1)^{k_1 + k_2} 2(2\pi i)^2 k_1! k_2! \sum_{i=1}^{2} b_i((k_1, k_2), \phi, 1_C).$$

Proof. Since $Z((s_1, s_2), \phi, (1 - e^{2\pi i}) f) = (1 - e^{2\pi i(s_2 - s_1)}) Z((s_1, s_2), \phi, f)$, we have

$$R_{k_1, k_2}(s, \phi, (1 - e^{2\pi i}) f) = (1 - e^{4\pi i}) R_{k_1, k_2}(s, \phi, f).$$

Using Lemma 5.24

$$\frac{(-1)^{k_1 + k_2} 2\pi i}{\Gamma(1 + k_1 + s)\Gamma(1 + k_2 - s)} R_{k_1, k_2}(s, \phi, 1_C) = (e^{-2\pi i s} - 1) \times \int_{\gamma(-\theta)} F_{k_1 + k_2}((1, u), \phi, 1_C) u^{s-k_2-1} du,$$

it follows that

$$\frac{(-1)^{k_1 + k_2} 2\pi i}{\Gamma(1 + k_1 + s)\Gamma(1 + k_2 - s)} R_{k_1, k_2}(s, \phi, (1 - e^{2\pi i}) 1_C) = (e^{-2\pi i s} - 1)(1 - e^{4\pi i}) \int_{\gamma(-\theta)} F_{k_1 + k_2}((1, u), \phi, 1_C) u^{s-k_2-1} du.$$

Thus, by comparing the Taylor coefficient of $s^2$, we have

$$\frac{(-1)^{k_1 + k_2}}{k_1! k_2!} R_{k_1, k_2}(\phi, (1 - e^{2\pi i}) 1_C) = 4\pi i \int_{\gamma(-\theta)} F_{k_1 + k_2}((1, u), \phi, 1_C) u^{k_2-1} du$$

$$= 4\pi i \int_{C_r - C_r} F_{k_1 + k_2}((1, u), \phi, 1_C) u^{k_2-1} du$$

where we denoted by $C_r$ (resp. $C_r$) a counterclockwise circle with sufficiently small radius $\varepsilon$ (resp. sufficiently large radius $r$). Changing the variable by $v = u^{-1}$, we have

$$\int_{-C_r} F_{k_1 + k_2}((1, u), \phi, 1_C) u^{-k_2-1} du = \int_{C_r - 1} F_{k_1 + k_2}((v, 1), \phi, 1_C) v^{-k_2-1} dv.$$

Hence we have

$$\frac{(-1)^{k_1 + k_2}}{k_1! k_2!} R_{k_1, k_2}(\phi, (1 - e^{2\pi i}) 1_C) = 4\pi i \left\{ \int_{C_r} F_{k_1 + k_2}((1, u), \phi, 1_C) u^{-k_2-1} du \right\}$$

$$= + \int_{C_r - 1} F_{k_1 + k_2}((v, 1), \phi, 1_C) v^{-k_2-1} dv$$

$$= 2(2\pi i)^2 \sum_{i=1}^{2} b_i((k_1, k_2), \phi, 1_C).$$

Thus we have

$$R_{k_1, k_2}(\phi, (1 - e^{2\pi i}) f) = (-1)^{k_1 + k_2} 2(2\pi i)^2 k_1! k_2! \sum_{i=1}^{2} b_i((k_1, k_2), \phi, f)$$

for $f \in A(\mathbb{L})$. \qed
5.9. **Rationality of** $L\left((-k, -k), \phi, f; \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. Assume that $\mathbb{Q} \subset F$ with some imaginary quadratic field $F$. Then we have the following important proposition.

**Proposition 5.31.** For $k \in \mathbb{Z}_{\geq 0}$ and $f \in A(L)$, we have

$$L\left((-k, -k), \phi, f; \left(\frac{1}{2}, \frac{1}{2}\right)\right) \in \mathbb{Q}.$$ 

**Proof.** Since

$$Z((s_1, s_2), \phi, f)$$

$$= (2\pi i)k^2 \sum_{i=1}^{2} b_i((k, k), \phi, f)(s_1 + k) + \sum_{i+j \geq 2} c_{ij}((k, k), \phi, f)(s_1 + k)^i(s_2 + k)^j$$

with some $c_{ij}$’s for $k \in \mathbb{Z}_{\geq 0}$, we have

$$L\left((-k, -k) + t \left(\frac{1}{2}, \frac{1}{2}\right), \phi, f\right) = \frac{(2\pi i)k^2}{2} \sum_{i=1}^{2} b_i((k, k), \phi, f)t + O(t^2)$$

$$= \frac{k^2}{2} \sum_{i=1}^{2} b_i((k, k), \phi, f) + O(t).$$

Thus what remains to show is that $\sum_{i=1}^{2} b_i((k, k), \phi, f) \in \mathbb{Q}$ for $f \in A(L)$. From the equalities (5.11) and (5.12), we have $b_i((k_1, k_2), \phi, f) \in F$ and

$$b_1((k_1, k_2), \phi, f) = b_2((k_2, k_1), \phi, f)$$

for $(k_1, k_2) \in \mathbb{Z}^2$. Thus it follows that

$$\sum_{i=1}^{2} b_i((k, k), \phi, f) = b_1((k, k), \phi, f) + b_1((k, k), \phi, f)$$

is rational. This completes the proof of the proposition. \qed

5.10. **The ray class invariant** $\Lambda_i(-k, A)$. We fix an imaginary quadratic field $F$ and a modulus $m$ of $F$. In this section, we construct ray class invariants $\Lambda_i(-k, A)$ for $i \in \{1, 2\}$, $k \in \mathbb{Z}_{\geq 1}$ and $A \in \text{Cl}_m(F)$. For this purpose, we define meromorphic functions $\Lambda_1(s, D, a), \Lambda_2(s, D, a)$ of $s$ as follows.

**Definition 5.32.** For $i \in \{1, 2\}$, a fundamental domain $D$, and a fractional ideal $a$ coprime to $m$, we define $\Lambda_i(s, D, a)$ by

$$\Lambda_i(s, D, a) := \frac{1}{\text{Nm}} N(a)^s \frac{\partial L}{\partial s_i} ((s, s), 1_{a \cap U(m)}, D).$$

**Remark 5.33.** Since $L(s, 1_{a \cap U(m)}, D)$ may have poles only at $s_1 + s_2 \in \mathbb{Z}_{\leq 2}$ from the Dirichlet series expression, $\Lambda_i(s, D, a)$ may have poles only at $2s \in \mathbb{Z}_{\leq 2}$. Moreover, from Proposition 5.17, $L(s, \phi, D)$ admits an expansion of the form

$$L(s, \phi, D) = -\phi(0)\delta_{k, 0} + \sum_{i+j \geq 2} c_{ij}((s_1 + k_1)^i(s_2 + k_2)^j$$

with some $c_{ij}$’s in a neighborhood of $s = -k \in \mathbb{Z}_{\geq 0}$. Hence, $\frac{\partial L}{\partial s_i} ((s, s), \phi, D)$ has no poles at $s \in \mathbb{Z}_{\leq 0}$. Thus $\Lambda_i(s, D, a)$ has no poles at $s \in \mathbb{Z}_{\leq 0}$.
**Definition 5.34.** For $i \in \{1, 2\}$, $k \in \mathbb{Z}_{>0}$ and $A \in \text{Cl}_m(F)$, we define $\Lambda_i(-k, A) \in \mathbb{C}/\mathbb{Q}(1)$ by

$$\Lambda_i(-k, A) := \Lambda_i(-k, D, a) \mod \mathbb{Q}(1)$$

where $D$ is any fundamental domain and $a$ is an element in $A^{-1}$.

**Proposition 5.35.** For $A \in \text{Cl}_m(F)$, $k \in \mathbb{Z}_{>0}$ and $i \in \{1, 2\}$, the above definition of $\Lambda_i(-k, A)$ does not depend on the choice of a fundamental domain $D$ and $a \in A^{-1}$.

**Proof.** Without loss of generality, we can assume that $i = 1$. Put $\phi_a := 1_{a^{-1}U(m)}$ for $a \in A^{-1}$. If $D, D'$ are two fundamental domains, there exists $f \in A(L)$ such that $D - D' = (1 - e^{2\pi i}) f$. Thus we have

$$\frac{\partial L}{\partial s_1}(s, \phi_a, D - D') = \left(1 - e^{2\pi i(s_2 - s_1)}\right) \frac{\partial L}{\partial s_1}(s, \phi_a, f) + 2\pi ie^{2\pi i(s_2 - s_1)} L(s, \phi_a, f).$$

Hence we have

$$\Lambda_1(s, D - D', a) = \frac{2\pi i}{w_m} N(a)^s L((s, s), \phi_a, f).$$

Since

$$\lim_{s \to -k} L((s, s), \phi_a, f) = L \left((-k, -k), \phi_a, f; \left(\frac{1}{2}, \frac{1}{2}\right)\right),$$

we have $\Lambda_1(-k, D - D', a) \in \mathbb{Q}(1)$ for $k \in \mathbb{Z}_{>0}$ from Proposition 5.31.

If $a, a' \in A^{-1}$, then $a^{-1}a'$ is a principle fractional ideal. Hence it is generated by an element $\alpha \in U(m)$. Let $\beta \in \mathbb{C}^*$ be any element with $p(\beta) = \alpha$. Then we have

$$L(s, \phi_{a'}, D) = L(s, \phi_{\alpha a'}, D)$$

$$= \sum_{x \in F} D(x) \phi_{\alpha a}(x)x^{-s_1}\bar{\beta}^{-s_2}$$

$$= \sum_{x \in F} D(\beta x) \phi_{a}(\beta x)(\beta x)^{-s_1}(\bar{\beta}x)^{-s_2}$$

$$= \sum_{x \in L} D(\beta x) \phi_{a}(x)(\beta x)^{-s_1}(\bar{\beta}x)^{-s_2}$$

$$= \beta^{-s_1}\bar{\beta}^{-s_2} L(s, \phi_a, \beta^{-1}D).$$

Thus we have

$$\frac{\partial L}{\partial s_1}(s, \phi_{a'}, D) = -\log\beta \cdot \beta^{-s_1}\bar{\beta}^{-s_2} L(s, \phi_a, \beta^{-1}D) + \beta^{-s_1}\bar{\beta}^{-s_2} \frac{\partial L}{\partial s_1}(s, \phi_a, \beta^{-1}D).$$

$$= -\log\beta \cdot L(s, \phi_{a'}, D) + \beta^{-s_1}\bar{\beta}^{-s_2} \frac{\partial L}{\partial s_1}(s, \phi_a, \beta^{-1}D).$$

Restricting to the diagonal, we have

$$\frac{\partial L}{\partial s_1}((s, s), \phi_{a'}, D) = -\log\beta \cdot L((s, s), \phi_{a'}, D) + N(\alpha)^{-s} \frac{\partial L}{\partial s_1}((s, s), \phi_a, \beta^{-1}D).$$
Since \( L((s, s), \phi_a, D) \) vanishes at \( s = -k \in \mathbb{Z}_{<0} \), we have
\[
N(a')^{-k} \frac{\partial L}{\partial s_1}((-k, -k), \phi_{a'}, D) = N(a')^{-k}N(a)^k \frac{\partial L}{\partial s_1}((-k, -k), \phi_a, \beta^{-1}D)
\]
\[
= N(a)^{-k} \frac{\partial L}{\partial s_1}((-k, -k), \phi_a, \beta^{-1}D).
\]
That is to say
\[
\Lambda_i(-k, D, a') = \Lambda_i(-k, \beta^{-1}D, a).
\]
Since \( \beta^{-1}D \) is again a fundamental domain, the right-hand side equals \( \Lambda_i(-k, D, a) \mod \mathbb{Q}(1) \). Thus the proposition is proved. □

Let \( \Gamma \) be a subgroup of \( \text{Cl}_m(F) \). For \( \mathcal{A}' \in \text{Cl}_m(F)/\Gamma \), we extend the definition by
\[
\Lambda_i(-k, \mathcal{A}') := \sum_{\mathcal{A} \in \varphi^{-1}(\mathcal{A}')} \Lambda_i(-k, \mathcal{A})
\]
where \( \varphi : \text{Cl}_m(F) \to \text{Cl}_m(F)/\Gamma \) is the natural surjection. Let \( H \) be an abelian extension of \( F \) of conductor \( m \), and \( G \) the Galois group \( \text{Gal}(H/F) \). From the class field theory, there is a canonical isomorphism \( \text{rec} : \text{Cl}_m(F)/\Gamma \to G \) with some subgroup \( \Gamma \) of \( \text{Cl}_m(F) \).

**Definition 5.36.** For \( i \in \{1, 2\} \), \( k \in \mathbb{Z}_{>0} \) and \( \eta = \sum_{\sigma \in G} m_\sigma \sigma \in \mathbb{Z}[G] \), we define \( \Lambda_i(-k, \eta) \) by
\[
\Lambda_i(-k, \eta) := \sum_{\sigma \in G} m_\sigma \Lambda_i(-k, \text{rec}^{-1}(\sigma)).
\]

5.11. **Shintani’s decomposition.** We now show that \( \zeta'(-k, \mathcal{A}) \) decomposes as a sum of the partial derivatives \( \Lambda_i(-k, \mathcal{A}) \) for \( i = 1, 2 \).

**Proposition 5.37.** By regarding \( \mathbb{R} \) as a subset of \( \mathbb{C}/\mathbb{Q}(1) \) by \( x \mapsto x + \mathbb{Q}(1) \), we have
\[
\zeta'(-k, \mathcal{A}) = \sum_{i=1}^{2} \Lambda_i(-k, \mathcal{A})
\]
for \( k \in \mathbb{Z}_{>0} \) and \( \mathcal{A} \in \text{Cl}_m(F) \).

**Proof.** Let \( a \in \mathcal{A}^{-1} \), \( \phi_a := 1_{a^{-1}U(m)} \) and \( D \) be a fundamental domain. Since
\[
\zeta(s, \mathcal{A}) = w_m^{-1}N(a)^s L((s, s), \phi_a, D),
\]
we have
\[
\zeta'(s, \mathcal{A}) = w_m^{-1}N(a)^s \left( \log N(a) \cdot L((s, s), \phi_a, D) + \frac{d}{ds} (L((s, s), \phi_a, D)) \right)
\]
If \( k \in \mathbb{Z}_{>0} \) and \( |s| > 0 \) is sufficiently small, \( L((s_1, s_2), \phi_a, D) \) is holomorphic at \( (s_1, s_2) = (-k + s, -k + s) \). Thus we have
\[
L'((-k + s, -k + s), \phi_a, D) = \sum_{i=1}^{2} \frac{\partial L}{\partial s_i}((-k + s, -k + s), \phi_a, D).
\]
Thus we have
\[ \zeta'(-k+s,A) = w_m^{-1}N(a)^{-k} \left\{ \log N(a) \cdot L((-k+s, -k+s), \phi_a, D) + \sum_{i=1}^{2} \frac{\partial L}{\partial s_i}((-k+s, -k+s), \phi_a, D) \right\}. \]

Since \( D \) is a fundamental domain, \( \frac{\partial L}{\partial s}((-k+s, -k+s), \phi_a, D) \) is holomorphic at \( s = 0 \) (see Remark 5.33), and \( L((-k+s, -k+s), \phi_a, D) \) vanishes at \( s = 0 \) from Proposition 5.17. Thus we have

\[ \zeta'(-k,A) = w_m^{-1}N(a)^{-k} \sum_{i=1}^{2} \left[ \frac{\partial L}{\partial s_i}((s,s), \phi_a, D) \right]_{s=-k} = \sum_{i=1}^{2} \Lambda_i(-k,A). \]

Hence the proposition is proved. \( \square \)

We now state one of our main theorems in this article.

**Theorem 5.38.** For \( i \in \{1, 2\} \), \( k \in \mathbb{Z}_{\geq 0} \) and a ray class \( A \in \text{Cl}_m(F) \), we have a well-defined invariant

\[ \Lambda_i(-k,A) \in \mathbb{C}/\mathbb{Q}(1), \]

which refines the leading Taylor coefficient of the partial zeta function at \( s = -k \).

More precisely,

\[ \zeta'(-k,A) = \Lambda_1(-k,A) + \Lambda_2(-k,A). \]

6. Comparison with Zagier-Gangl’s enhanced zeta value

The goal of this section is to show an equality between Zagier-Gangl’s enhanced zeta value \( I_{k+1}(A) \) and the first partial derivative \( \Lambda_1(-k,A) \), i.e.

\[ I_{k+1}(A^{-1}) = \frac{(2\pi i)^k}{k!} \Lambda_1(-k,A), \]

for \( k \in \mathbb{Z}_{\geq 1} \) and an ideal class \( A \) (Theorem 6.16). Let \( 1_D(x) = 1_{C(x)} + 1_{C(x, e_\epsilon x)} \) be a basic fundamental domain. In Section 6.1, we give a Fourier series expansion of \( \frac{\partial L}{\partial s_1}((-2k,0),\phi,1_D(x),(1,0)) \) (Lemma 6.5). In Section 6.2, we give a Fourier series expansion of \( \frac{\partial L}{\partial s_1}((-2k,0),\phi,1_D(x),\left(\frac{1}{2},\frac{1}{2}\right)) \) (Proposition 6.9) by using Lemma 6.5. In Sections 6.4 and 6.3, we investigate the effect of the Maass raising operator and prove certain statements (Corollaries 6.12, 6.14 and 6.15). In Section 6.5, we prove Theorem 6.16 by combining Proposition 6.9 and Corollaries 6.12, 6.14 and 6.15.

6.1. Fourier expansion of \( \frac{\partial L}{\partial s_1}((-k,0),\phi,D,\lambda) \) for \( \lambda = (1,0) \). As we have mentioned in Remark 5.23, \( \frac{\partial L}{\partial s_1}((-k_1, -k_2),\phi,f,\lambda) \) exists for all \((k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \) and \( f \in A(L) \) if \( \lambda \) is the special direction \( (1,0) \). From the definition,

\[ \frac{\partial L}{\partial s_1}((-k,0),\phi,1_D(x),(1,0)) \]

is equal to the derivative at \( s = -k \) of the one variable function \( L((s,0),\phi,1_D(x)) \). Therefore, we first give a Fourier series expansion of \( L((s,0),\phi,1_D(x)) \) in Section
6.1.1. Then, we give a Fourier expansion of \( \frac{\partial L}{\partial x_1}((-k, 0), \phi, 1_{D(x)}), (1, 0) \) in Section 6.1.2. For completeness, we investigate \( L((s, 0), \phi, D) \) for a general fundamental domain \( D \) in Section 6.1.3.

6.1.1. Fourier series expansion of \( L_1(s, \phi, 1_{D(x)}) \). Put \( L_1(s, \phi, f) := L((s, 0), \phi, f) \) for \( f \in A(L) \). Then we have the following proposition.

**Proposition 6.1.** For \( s \in \mathbb{C} \setminus \{0\} \), we have

\[
\left( \frac{N_\phi x e^{\pi/2}}{2\pi} \right)^s \Gamma(s) L_1(s, \phi, 1_{D(x)}) = \eta(1 - s, \phi, x) + \sum_{t \in \mathbb{Z}_{>0}} \sum_{\alpha \in [N_{x,y}]} \left( \phi(\alpha) + e^{-i\pi s}\phi(-\alpha) \right) \frac{e^{\frac{2\pi i x t}{\phi}}} {t^{1-s} (1 - e^{2\pi i x t})},
\]

where \( \eta(s, \phi, x) \) is an analytic continuation of

\[
\sum_{t \in \mathbb{Z}_{>0}} \sum_{\alpha \in [N_{x,y}]} \phi(\alpha) e^{\frac{2\pi i x t}{\phi}}.
\]

**Proof.** Our proof of the proposition is as follows. We first give a contour integral representation of \( L_1(s, \phi, 1_{D(x)}) \). Then, we express the integral as a sum of limits of certain closed contour integrals. Thus, we express \( L((s, 0), \phi, D) \) as a sum of residues, by which we prove the proposition.

Now, let us start with a contour integral representation of \( L_1(s, \phi, 1_{C}) \) for an admissible cone \( C \). Recall the contours \( J(\theta, \psi), J(\theta) \) defined in Section 5.4.2. For \( x \in \mathbb{C}^x \) with \( e_\theta x \in \mathbb{H} \), and \( s \in \mathbb{C} \setminus \mathbb{Z}_{>0} \), we have

\[
x^{-s} = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta)} e^{-p(x)t} t^{s-1} dt.
\]

Put

\[
\mathcal{F}_1(t, \phi, f) := F((t, 0), \phi, f),
\]

\[
F_1(t, \phi, (x, y)) := F((t, 0), \phi, (x, y)),
\]

\[
F_1(t, \phi, x) := F((t, 0), \phi, x).
\]

Thus,

\[
L_1(s, \phi, 1_{C}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta)} \mathcal{F}_1(t, \phi, 1_{C}) t^{s-1} dt
\]

for an admissible cone \( C \) and \( \theta \) with \( e_\theta C \subset \mathbb{H} \).

We now take a basic fundamental domain \( 1_{D(x)} = 1_{C(x)} + 1_{C(x, e_\theta x)} \), which is a sum of characteristic functions of admissible cones

\[
1_{C(x)} + 1_{C(x, y)} + 1_{C(y)} + 1_{C(y, e_\theta x)} + 1_{C(e_\theta x)} + 1_{C(e_\theta x, e_\theta y)} + 1_{C(e_\theta y)} + 1_{C(e_\theta y, e_\theta x)}
\]

with any \( y \) such that \( \bar{\arg}(x) < \bar{\arg}(y) < \bar{\arg}(x) + \pi \). We take \( \theta, \theta' \in \mathbb{R} \) such that

\[
e_\theta C(x, y), e_{\theta'} C(y, e_\theta x) \subset \mathbb{H}
\]
as before. Then one has

\[
L_1(s, \phi, 1_{C(x,y)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta)} F_1(t, \phi, (x,y)) t^{s-1} dt,
\]

\[
L_1(s, \phi, 1_{C(y,e,x)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta')} F_1(t, \phi, (y,-x)) t^{s-1} dt,
\]

\[
L_1(s, \phi, 1_{C(e,x,e,y)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta-\pi)} F_1(t, \phi, (-x,-y)) t^{s-1} dt,
\]

\[
L_1(s, \phi, 1_{C(x,y,e,x)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta')-\pi} F_1(t, \phi, (-y,x)) t^{s-1} dt,
\]

and

\[
L_1(s, \phi, 1_{C(z)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta)} F_1(t, \phi, x) t^{s-1} dt,
\]

\[
L_1(s, \phi, 1_{C(y)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta')} F_1(t, \phi, y) t^{s-1} dt,
\]

\[
L_1(s, \phi, 1_{C(x,z)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta-\pi)} F_1(t, \phi, -x) t^{s-1} dt,
\]

\[
L_1(s, \phi, 1_{C(e,y)}) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta'-\pi)} F_1(t, \phi, -y) t^{s-1} dt.
\]

Here, we denoted \(F_1(t, \phi, -p(y))\) as \(F_1(t, \phi, -y)\) as mentioned before. Thus we obtain a contour integral representation of

\[
L_1(s, \phi, 1_{D(z)}) = L_1(s, \phi, 1_{C(z)}) + L_1(s, \phi, 1_{C(x,y)}) + L_1(s, \phi, 1_{C(y)}) + L_1(s, \phi, 1_{C(e,x)}) + L_1(s, \phi, 1_{C(e,y)}) + L_1(s, \phi, 1_{C(x,e,x)}) + L_1(s, \phi, 1_{C(x,e,y)}) + L_1(s, \phi, 1_{C(y,e,x)}) + L_1(s, \phi, 1_{C(y,e,y)}).
\]

Now, let us modify the integral representation of \(L_1(s, \phi, 1_{D(z)})\). Since

\[
F_1(t, \phi, (x,y)) + F_1(t, \phi, y) + F_1(t, \phi, (y,-x)) = 0,
\]

we see that

\[
L_1(s, \phi, 1_{C(x,y)}) + L_1(s, \phi, 1_{C(y)}) + L_1(s, \phi, 1_{C(x,e,x)})
\]

\[
= \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \left\{ \int_{J(\theta)} F_1(t, \phi, (x,y)) t^{s-1} dt 
\right. 
\]

\[
+ \int_{J(\theta')} (-F_1(t, \phi, (x,y)) - F_1(t, \phi, y)) t^{s-1} dt 
\left. + \int_{J(\theta')} F_1(t, \phi, y) t^{s-1} dt \right\}
\]

\[
= \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta) - J(\theta')} F_1(t, \phi, (x,y)) t^{s-1} dt
\]

\[
= \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{J(\theta + 2\pi, \theta + 2\pi) - J(\theta', \theta)} F_1(t, \phi, (x,y)) t^{s-1} dt
\]

\[
= \frac{1}{\Gamma(s)} \int_{J(\theta', \theta)} F_1(t, \phi, (x,y)) t^{s-1} dt.
\]
Similarly, we have
\[
L_1(s, \phi, 1_{C(e^\phi s, e^\phi s + y)}) + L_1(s, \phi, 1_{C(e^\phi s, e^\phi s + x)}) + L_1(s, \phi, 1_{C(e^\phi s, e^\phi s + x) + y})
= \frac{1}{\Gamma(s)} \int_{J(\theta - \pi, \theta)} F_1(t, \phi, (-x, -y)) t^{s-1} dt.
\]

We have
\[
L_1(s, \phi, 1_{C(x)}) + L_1(s, \phi, 1_{C(x + y)})
= \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \left\{ \int_{J(\theta)} F_1(t, \phi, x) t^{s-1} dt + \int_{J(\theta - \pi)} F_1(t, \phi, -x) t^{s-1} dt \right\}.
\]

Since
\[
\int_{J(\theta - \pi)} F_1(t, \phi, -x) t^{s-1} dt = \int_{J(\theta - \pi)} (-\phi(0) - F_1(t, \phi, x)) t^{s-1} dt
= - \int_{J(\theta - \pi)} F_1(t, \phi, x) t^{s-1} dt
\]
for \( \Re s < 0 \), we have
\[
\int_{J(\theta)} F_1(t, \phi, x) t^{s-1} dt + \int_{J(\theta - \pi)} F_1(t, \phi, -x) t^{s-1} dt
= \int_{J(\theta) - J(\theta - \pi)} F_1(t, \phi, x) t^{s-1} dt
= (e^{2\pi is} - 1) \int_{J(\theta - \pi)} F_1(t, \phi, x) t^{s-1} dt.
\]

Thus we have
\[
L_1(s, \phi, 1_{C(x)}) + L_1(s, \phi, 1_{C(x + y)}) = \frac{1}{\Gamma(s)} \int_{J(\theta - \pi, \theta)} F_1(t, \phi, x) t^{s-1} dt.
\]

Combining these calculations, we have
(6.2)
\[
L_1(s, \phi, 1_{D(x)}) = \frac{1}{\Gamma(s)} \left\{ \int_{J(\theta, \phi)} F_1(t, \phi, (x, y)) t^{s-1} dt
+ \int_{J(\theta - \pi, \phi - \pi)} F_1(t, \phi, (-x, -y)) t^{s-1} dt + \int_{J(\theta - \pi, \phi)} F_1(t, \phi, x) t^{s-1} dt \right\}
\]
for \( \Re s < 0 \).

To calculate the right-hand side of (6.2) in terms of sums of residues, we express each term as a limit of certain closed contour integral as follows. Put \( V := \mathbb{Z}_{\geq 0}^2 \). Then we can choose an infinite increasing sequence \( b_0 < b_1 < \cdots \) of reals such that \( \lim_{n \to \infty} b_n = \infty \) and
\[
\inf \{|v - b_i|; v \in V, i \in \mathbb{Z}_{\geq 0}\}
\]
is not zero. Fix such a sequence \( \{b_i\}_{i \in \mathbb{Z}_{\geq 0}} \), and put \( J \), \( J_n \), \( \gamma_n \), and \( \gamma_n(\theta_1, \theta_2) := \gamma(a, b_n; \theta_1, \theta_2) \) for \( \theta_1, \theta_2 \in \mathbb{R} \) with the same \( a \) as in \( \gamma(\theta_1, \theta_2) \). Since
\[
\lim_{n \to \infty} \int_{I(\varepsilon_1, b_n, \varepsilon_2 b_n)} F_1(t, \phi, (x, y)) t^{s-1} dt = 0
\]
for \( R s < 0 \) and \( \theta_1, \theta_2 \in \mathbb{R} \setminus \{ \arg x + \pi \mathbb{Z} \} \cup \{ \arg y + \pi \mathbb{Z} \} \), we have

\[
(6.3) \quad \int_{J(\theta_1, \theta_2)} F_1(t, \phi, (x, y)) t^{s-1} dt = \lim_{n \to \infty} \int_{J_n(\theta_1, \theta_2)} F_1(t, \phi, (x, y)) t^{s-1} dt
\]

\[
= \lim_{n \to \infty} \int_{\gamma_n(\theta_1, \theta_2)} F_1(t, \phi, (x, y)) t^{s-1} dt.
\]

Similarly, we have

\[
(6.4) \quad \int_{J(\theta_1, \theta_2)} F_1(t, \phi, x) t^{s-1} dt = \lim_{n \to \infty} \int_{\gamma_n(\theta_1, \theta_2)} F_1(t, \phi, x) t^{s-1} dt
\]

for \( R s < 0 \) and \( \theta_1, \theta_2 \in \mathbb{R} \setminus \{ \arg x + \pi \mathbb{Z} \} \).

The contour integrals on the right-hand sides of 6.3 and 6.3 can be calculated by residue calculus. The poles of \( F_1(t, \phi, (\pm x, \pm y)) t^{-1-s} \) are located at

\[
t \in e^{(\pm 1/2)\pi N^0 x} Z_{>0}, e^{(\pm 1/2)\pi N^0 y} N^0 y > 0.
\]

From the conditions

\[
e \phi C(x, y), e \phi C(y, e \pi x) \subset \mathbb{H},
\]

we have

\[
-\frac{\pi}{2} - \arg y < \theta' < -\frac{\pi}{2} - \arg x < \theta < \frac{\pi}{2} - \arg y.
\]

So, all the poles inside the contour \( \gamma_n(\theta', \theta) \) are located at \( e^{-\frac{2\pi}{N_0^0 x}} X_n \), where we put

\[
X_n := \left\{ t \in \mathbb{Z}_{>0} \mid \frac{2\pi t}{N_0^0 x} < b_n \right\}.
\]

Therefore we have

\[
\int_{\gamma_n(\theta', \theta)} F_1(t, \phi, (x, y)) t^{s-1} dt = -2\pi i \sum_{t \in X_n} \text{Res}_{t=e^{-\frac{2\pi}{N_0^0 x} s-1} \frac{2\pi}{N_0^0 x}} F_1(t, \phi, (x, y)) t^{s-1}
\]

\[
= -2\pi i \sum_{t \in X_n} \sum_{\alpha \in \mathbb{N}_0^0: \phi(x) e^{2\pi i \phi x}} \phi(\alpha) e^{\frac{2\pi i}{N_0^0 x}} (e^{-\frac{2\pi}{N_0^0 x}})^{s-1}
\]

\[
= \left( e^{-\frac{2\pi}{N_0^0 x}} \right)^s \sum_{t \in X_n} \sum_{\alpha \in \mathbb{N}_0^0: \phi(x) e^{2\pi i \phi x}} \frac{2\pi i}{1-e^{2\pi i \frac{2\pi}{N_0^0 x}}}.
\]

Thus, by letting \( n \to \infty \), we have

\[
\int_{J(\theta', \theta)} F_1(t, \phi, (x, y)) t^{s-1} dt = \left( e^{-\frac{2\pi}{N_0^0 x}} \right)^s \sum_{t \in \mathbb{Z}_{>0}} \sum_{\alpha \in \mathbb{N}_0^0: \phi(x) e^{2\pi i \phi x}} \frac{2\pi i}{1-e^{2\pi i \frac{2\pi}{N_0^0 x}}}
\]

Similarly we have

\[
\int_{J(\theta' - \pi, \theta - \pi)} F_1(t, \phi, (-x, -y)) t^{s-1} dt = \left( e^{-\frac{2\pi}{N_0^0 x}} \right)^s \sum_{t \in \mathbb{Z}_{>0}} \sum_{\alpha \in \mathbb{N}_0^0: \phi(-x) e^{2\pi i \phi x}} \frac{2\pi i}{1-e^{2\pi i \frac{2\pi}{N_0^0 x}}}
\]

The poles of

\[
F_1(t, \phi, x) t^{s-1} = \frac{\sum_{\alpha \in \mathbb{N}_0^0: \phi(x) e^{-\alpha i \phi x}}}{(1-e^{-N_0^0 x t})} t^{s-1}
\]
are located at \( t \in \mathbb{C}(1/2) \times \mathbb{N}_{\phi x}^* \mathbb{Z}_{>0} \). From the condition \( e^{\theta} \mathcal{C}(x, y) \subset \mathbb{H} \), we have
\[
\theta = \pi - \frac{\pi}{2} - \arg x < \theta,
\]
that is, all the poles inside the contour \( \gamma_n(\theta, \pi, \theta) \) are located at \( e^{-\frac{2\pi}{N_{\phi x}}} X_n \). Therefore we have,
\[
\int_{\gamma_n(\theta, \pi, \theta)} F_1(t, \phi, x) t^{s-1} dt = -2\pi i \sum_{l \in X_n} \text{Res}_{t=e^{-\frac{2\pi}{N_{\phi x}}} l} F_1(t, \phi, x) t^{s-1}
\]
\[
= \left( e^{-\frac{2\pi}{N_{\phi x}}} \right)^s \sum_{l \in X_n} \sum_{\alpha \in [N_{\phi x}, x]} \phi(\alpha) e^{\frac{2\pi i \alpha l}{N_{\phi x}}} l^{1-s}.
\]
Thus we have,
\[
\int_{\gamma(\theta, \pi, \theta)} F_1(t, \phi, x) t^{s-1} dt = \left( e^{-\frac{2\pi}{N_{\phi x}}} \right)^s \sum_{l \in \mathbb{Z}_{>0}} \sum_{\alpha \in [N_{\phi x}, x]} \phi(\alpha) e^{\frac{2\pi i \alpha l}{N_{\phi x}}} l^{1-s}.
\]

From these calculations, we obtain the following Fourier series expression of \( L_1(s, \phi, 1_{D(x)}) \) for \( \Re s < 0 \).
\[
\left( \frac{N_{\phi x} e^{\pi/2}}{2\pi} \right)^s \Gamma(s) L_1(s, \phi, 1_{D(x)})
\]
\[
= \sum_{l \in \mathbb{Z}_{>0}} \frac{\sum_{\alpha \in [N_{\phi x}, x]} \phi(\alpha) e^{\frac{2\pi i \alpha l}{N_{\phi x}}} l}{l^{1-s}} + \sum_{l \in \mathbb{Z}_{>0}} \frac{\sum_{\alpha \in [N_{\phi x}, x]} \phi(\alpha) e^{\frac{2\pi i \alpha l}{N_{\phi x}}} l}{l^{1-s} (1 - e^{2\pi i l})}.
\]
For \( x \in \mathbb{L} \),
\[
\eta(s, \phi, x) = \sum_{l \in \mathbb{Z}_{>0}} \frac{\sum_{\alpha \in [N_{\phi x}, x]} \phi(\alpha) e^{\frac{2\pi i \alpha l}{N_{\phi x}}} l}{l^s}.
\]
converges for \( \Re s > 0 \) and has a meromorphic continuation to \( \mathbb{C} \), since
\[
\eta(s, \phi, x) = N_{\phi}^{s-s} \sum_{0 < m, l \leq N_{\phi}^*} \phi(mx^*) e^{\frac{2\pi i ml}{N_{\phi}^*}} \zeta \left( s, \frac{l}{N_{\phi}^*} \right),
\]
where \( x^* \in \mathbb{Q}_{>0}, N_{\phi}^* \in \mathbb{Z}_{>0} \) are the unique elements satisfying \( \mathbb{Z} x^* = \mathbb{Q} x \cap \mathbb{L}, N_{\phi}^* x^* = N_{\phi} x \),
is just a finite sum of Hurwitz zeta function i.e.
\[
\zeta(s, x) := \sum_{m=0}^{\infty} (m + x)^{-s}.
\]
Hence we obtain Proposition 6.1. \( \square \)

6.1.2. The first derivative \( \frac{d}{ds} \left( -k, \phi, 1_{D(x)} \right) \) for \( k \in \mathbb{Z}_{\geq 0} \). As an immediate consequence of Proposition 6.1, we can obtain Fourier series expansions for the first partial derivative of \( L_1(s, \phi, 1_{D(x)}) \) at non-positive integers as follows.
Lemma 6.5. For \( k \in \mathbb{Z}_{\geq 1} \), we have
\[
\frac{dL_1}{ds}(-k, \phi, 1_{D(x)}) = k! \left( \frac{N_x}{2\pi i} \right)^k \left\{ \eta(1+k, \phi, x) + \sum_{l \in \mathbb{Z}_{>0}} \sum_{0 \leq m < N^*_\phi} \left( \phi(\alpha) + (-1)^k \phi(-\alpha) \right) e^{\frac{2\pi iml}{N^*_\phi}} \right\}
\]
and
\[
\frac{dL_1}{ds}(0, \phi, 1_{D(x)}) = \phi(0) \log \left( \frac{x^* e^{\pi/2}}{2\pi} \right) - \sum_{0 < m < N^*_\phi} \phi(m x^*) \log \left( 1 - e^{\frac{2\pi i m}{N^*_\phi}} \right) + \sum_{l \in \mathbb{Z}_{>0}} \sum_{0 \leq m < N^*_\phi} \left( \phi(\alpha) + \phi(-\alpha) \right) e^{\frac{2\pi iml}{N^*_\phi}} \left( \frac{1}{1 - e^{\frac{2\pi il}{N^*_\phi}}} \right)
\]
with \( x^* \in \mathbb{Q}_{>0}, N^*_\phi \in \mathbb{Z}_{>0} \) being the elements defined by
\[
Z x^* = Q x \cap L, N^*_\phi x^* = N^*_\phi x,
\]
and
\[
1 - e^{\frac{2\pi i}{N^*_\phi}}
\]
in the second sum being considered as lying in the cone \( \mathbb{H} \).

Proof. Since \( \eta(1-s, \phi, x) \) has no poles other than \( s = 0 \), and the Taylor expansion of \( \frac{1}{\Gamma(s)} \) is of the form
\[
\frac{1}{\Gamma(s)} = (-1)^k k!(s + k) + O \left( (s + k)^2 \right),
\]
the case \( k \geq 1 \) immediately follows from the Proposition 6.1.

For \( k = 0 \), we have
\[
\frac{dL_1}{ds}(0, \phi, 1_{D(x)}) = \tilde{\eta}(0) + \sum_{l \in \mathbb{Z}_{>0}} \sum_{0 \leq m < N^*_\phi} \left( \phi(\alpha) + \phi(-\alpha) \right) e^{\frac{2\pi iml}{N^*_\phi}} \left( \frac{1}{1 - e^{\frac{2\pi il}{N^*_\phi}}} \right),
\]
where
\[
\tilde{\eta}(s) = \left( \frac{2\pi}{N^*_\phi x e^{\pi/2}} \right)^s \frac{\eta(1-s, \phi, x)}{\Gamma(s)} = \frac{1}{N^*_\phi} \sum_{0 < m, l \leq N^*_\phi} \phi(m x^*) e^{-\frac{2\pi iml}{N^*_\phi}} \left( \frac{2\pi}{x^* e^{\pi/2}} \right)^s \zeta \left( 1-s, \frac{N^*_\phi}{N^*_\phi} \right).\]

Using the Laurent expansions
\[
\zeta(1-s, a) = -\frac{1}{s} - \frac{\Gamma'(a)}{\Gamma(a)} + O(s) \quad \text{for } a > 0,
\]
\[
\frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3),
\]
we have
\[
\left( \frac{2\pi}{x^*e^{\pi/2}} \right)^s \zeta \left( \frac{1 - s}{\phi} \frac{i}{N^*} \right) \frac{\Gamma \left( \frac{i}{N^*} \right)}{\Gamma \left( \frac{1}{N^*} \right)} = -1 - \left( \gamma + \frac{\Gamma' \left( \frac{i}{N^*} \right)}{\Gamma \left( \frac{i}{N^*} \right)} + \log \left( \frac{2\pi}{x^*e^{\pi/2}} \right) \right) s + O(s^2).
\]

From the equality
\[
\gamma + \frac{\Gamma' (a)}{\Gamma (a)} = \sum_{m=0}^{\infty} \left( \frac{1}{1 + m} - \frac{1}{a + m} \right),
\]
we obtain
\[
\frac{1}{N^*} \sum_{m=1}^{N^*} e^{2\pi i \frac{m}{N^*}} \left( \gamma + \frac{\Gamma' \left( \frac{i}{N^*} \right)}{\Gamma \left( \frac{i}{N^*} \right)} \right) = \begin{cases} \log \left( 1 - e^{2 \pi i / \phi} \right) & \text{for } 0 < m < N^*, \\ 0 & \text{for } m = N^*. \end{cases}
\]
where \(1 - e^{2 \pi i / \phi}\) is regarded as an element of \(\mathbb{C}^\times\) lying in \(\mathbb{H}\). As a consequence, we obtain
\[
\bar{\eta}'(0) = \phi(0) \log \left( \frac{x^*e^{\pi/2}}{2\pi} \right) - \sum_{0 < m < N^*} \phi(mx^*) \log \left( 1 - e^{2 \pi i / \phi} \right),
\]
which completes the proof of the lemma.

\[\square\]

**Example 6.6.** For \(\phi = 1_L\) (so \(N_\phi = 1\)), \(x = x^* = 1\) and \(y = \tau\), we have
\[
\frac{(2\pi i)^{2k}}{2k!} \frac{dL_1}{ds} (-2k, 1; 1, 1_{D(x)}) = \begin{cases} \zeta(1 + 2k) + 2 \sum_{l \in \mathbb{Z} \times \mathbb{Z}, \ell > 0} \frac{e^{2 \pi il \tau}}{l \cdot e^{2 \pi i \tau}} & \text{for } k \geq 1, \\ \frac{e^{2 \pi i \tau}}{2} \log (2\pi) + 2 \sum_{l \in \mathbb{Z} \times \mathbb{Z}, \ell > 0} \frac{e^{2 \pi il \tau}}{l \cdot e^{2 \pi i \tau}} & \text{for } k = 0. \end{cases}
\]

6.1.3. The dependency of \(\frac{dL_1}{ds} (-k, \phi, D)\) on the choice of the fundamental domain \(D\). In Zagier-Gangl's construction of the ideal class invariant \(I_k(A)\), the modular transformation formula for the Eichler integral \(E_{2-2k, \rho}(\tau)\) plays a key role. In this section, we show that the modular transformation formula for \(\bar{E}_{-k, \rho}(\tau)\) is explained by the dependency of \(\frac{dL_1}{ds} (-k, \phi, D)\) on the choice of a fundamental domain \(D\). The reader who are not interested in this fact may skip this section, since the consequence in this section does not have any effect on later sections.

Let \(D, D'\) be two fundamental domains and let \(f \in A(L)\) be such that \(D - D' = (1 - e^{2\pi i}) f\). Since
\[
L_1(s, \phi, (1 - e^{2\pi i}) f) = (1 - e^{-2\pi i s}) L_1(s, \phi, f),
\]
we have
\[
\frac{\partial L_1}{\partial s} (-k, \phi, (1 - e^{2\pi i}) f) = e^{-2\pi i s} \frac{\partial Z}{\partial s} ((s, 0), \phi, f) \bigg|_{s = -k}.
\]
From Proposition 5.16, we have
\[
Z((s, 0), \phi, f) = (-1)^k k! \cdot 2\pi i \cdot b_1((k, 0), \phi, f)(s + k) + O \left( (s + k)^2 \right).
\]
Here, \(b_1((k, 0), \phi, f)\) is the coefficient of \(t^k\) of the Laurent expansion of \(F(t, t u, \phi, f)\). Since \(F((t, t u), \phi, f)\) is holomorphic at \(u = 0\), it is the coefficient of \(t^k\) of the Laurent expansion of \(F_2(t, \phi, f) := F((t, 0), \phi, f)\). Thus we denote simply \(b_1(k, \phi, f)\) for \(b_1((k, 0), \phi, f)\). Then, above observation is stated as the following Proposition.
Proposition 6.7. Let $D, D'$ be fundamental domains. Let $f \in A(\mathbb{L})$ be such that $D - D' = (1 - e^{2\pi i}) f$. Then we have

$$\frac{\partial L_1}{\partial s} (-k, \phi, D - D') = (-1)^k k! \cdot 2\pi i \cdot b_1(k, \phi, f).$$

Using Proposition 6.7, we give an explicit formula for $\frac{\partial L_1}{\partial s} (-k, \phi, D - D')$ in the case where $D = 1_{D(x)}, D' = 1_{D(x')}$ are basic fundamental domains. Without loss of generality, we assume $\arg x < \arg x'$. Since

$$1_{D(x)} - 1_{D(x')} = (1 - e^{2\pi i}) (1_{C(x)} + 1_{C(x,x')}),$$

we have

$$\frac{\partial L_1}{\partial s} (-k, \phi, 1_{D(x)} - 1_{D(x')}) = (-1)^k k! \cdot 2\pi i \cdot b_1(k, \phi, 1_{C(x)} + 1_{C(x,x')}).$$

Since

$$F_1(t, \phi, 1_{C(x)} + 1_{C(x,x')}) = F_1(t, \phi, x) + F_1(t, \phi, (x,x')) = -F_1(t, \phi, (x,-x')),$$

we have

$$F_1(t, \phi, 1_{C(x)} + 1_{C(x,x')}) = \sum_{m,n \in [N]} \phi(mx - nx') \frac{e^{-(mx-nx') t}}{(1 - e^{-N xt})(1 - e^{N xt})} \sum_{i,j \in \mathbb{Z}_{\geq 0}} B_i \left( \frac{n}{N} \right) B_j \left( \frac{n}{N} \right) \frac{(-N xt)^{i-1} (N x')^{j-1}}{i! j!},$$

where $B_i(x)$ is the $i$-th Bernoulli polynomial and $[N] := \{ x \in \mathbb{Q} | 0 < x \leq N \}$. Hence we obtain the formula

$$\frac{\partial L_1}{\partial s} (-k, \phi, 1_{D(x)} - 1_{D(x')}) = -2\pi i \cdot k! \cdot N^k \sum_{m,n \in [N]} \phi(mx - nx') \sum_{i+j=k+2} B_i \left( \frac{n}{N} \right) B_j \left( \frac{n}{N} \right) \frac{x^{i-1} (-x')^{j-1}}{i! j!}.$$

This formula gives a modular transformation for the Eichler integrals, as shown in the following example.

Example 6.8. Let $\phi = 1_{\mathbb{L}}$, and set $p(x) = 1, p(y) = p(x') = \tau, p(y') = -1$. Then we have, for $k \in \mathbb{Z}_{\geq 0},$

$$\frac{\partial L_1}{\partial s} (-2k, \phi, 1_{D(x)} - 1_{D(x')}) = -2\pi i \cdot (2k)! \sum_{i+j=2k+2} B_i B_j \frac{(-\tau)^{j-1}}{i^j},$$

where $B_j$ is the $j$-th Bernoulli number defined by the generating function

$$\frac{1}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} t^{j-1}.$$

Hence we obtain, for $k \geq 1,$

$$E_{-2k}(\tau) - \tau^{2k} E_{-2k} \left( -\frac{1}{\tau} \right) = -(2\pi i)^{2k+1} \sum_{j=0}^{2k+2} \frac{B_{2k+2-j} B_j}{(2k + 2 - j)!} (-\tau)^{j-1},$$

where $E_m(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{(n+m)^m}.$
where we put
\[ E_{-2k}(\tau) := \zeta(1 + 2k) + 2 \sum_{l \in \mathbb{Z}_{>0}} \frac{e^{2\pi il\tau}}{1 + 2k(1 - e^{2\pi il\tau})}. \]

(In fact, this formula is valid for \( k \in \mathbb{Z} \setminus \{0\}. \) For \( k = 0, \) we obtain
\[ E_0(\tau) - E_0 \left( -1 \right) \log \left( \frac{x'}{x} \right) = -(2\pi i) \cdot \sum_{j=0}^{2} \frac{B_{2-j}B_j}{(2-j)!j!} (-\tau)^{j-1} \]
\[ = -(2\pi i) \cdot \left( \frac{1}{4} - \frac{1}{12} \left( \tau + \frac{1}{2}\right) \right), \]
where we put
\[ E_0(\tau) := 2 \sum_{l \in \mathbb{Z}_{>0}} \frac{e^{2\pi il\tau}}{1 - e^{2\pi il\tau}}. \]

Taking \( \exp \) of both sides, one obtains
\[ e^{\frac{2\pi i}{N}} \prod_{m \in \mathbb{Z}_{>0}} \left( 1 - e^{2\pi im(-\frac{1}{2})} \right)^2 = \frac{\tau}{i} e^{\frac{2\pi i}{N}} \prod_{m \in \mathbb{Z}_{>0}} \left( 1 - e^{2\pi im\tau} \right)^2, \]
which is a modular transformation formula for Dedekind eta function.

6.2. Fourier expansion of \( \frac{\partial L}{\partial s_1}((-k, 0), \phi, D, \lambda) \) for \( \lambda = \left( \frac{1}{2}, \frac{1}{2} \right). \)

**Proposition 6.9.** Let \( \tau \in \mathcal{H} \) be an element with \( L = \mathbb{Z}\tau + \mathbb{Z}. \) Then we have
\[ \frac{1}{k!} \left( \frac{2\pi i}{N} \right)^k \frac{\partial L}{\partial s_1} \left( (-k, 0), \phi, 1_{D(1)}; \left( \frac{1}{2}, \frac{1}{2} \right) \right) \]
\[ = \eta(1 + k, \phi, 1 - \zeta(-1-k, \phi, (1, \tau)) \cdot (2\pi i(\tau - \tau))^{k+1} \]
\[ + \sum_{l \in \mathbb{Z}_{>0}} \sum_{\alpha \in [N, 1, \tau]} \left( \phi(\alpha) + (-1)^k \phi(-\alpha) \right) e^{\frac{2\pi i \alpha}{N^k}} \left( \frac{1}{1+k \left( 1 - e^{2\pi i l\tau} \right)} \right), \]
for \( k \in \mathbb{Z}_{>0}, \) and
\[ \frac{\partial L}{\partial s_1} \left( (0, 0), \phi, 1_{D(1)}; \left( \frac{1}{2}, \frac{1}{2} \right) \right) \]
\[ = \phi(0) \log \left( \frac{x^* e^{\pi/2}}{2\pi} \right) \cdot \sum_{0 < m \in N^*_0} \phi(mx^*) \log \left( 1 - e^{\frac{2\pi im}{N^*}} \right) \]
\[ = \zeta(-1, \phi, (1, \tau)) \cdot \frac{2\pi i (\tau - \tau)}{2} + \sum_{l \in \mathbb{Z}_{>0}} \sum_{\alpha \in [N, x^*, y]} \left( \phi(\alpha) + \phi(-\alpha) \right) e^{\frac{2\pi i \alpha}{l \left( 1 - e^{2\pi i l\tau} \right)}}, \]
with \( x^* \in \mathbb{Q}_{>0}, N^*_0 \in \mathbb{Z}_{>0} \) being the elements as in Lemma 6.5.
Proof. From Lemma 5.20 and Proposition 5.28, we have
\[
\frac{\partial L}{\partial s_1} \left( (-k, 0), \phi, 1_{D(1)}; \left( \frac{1}{2}, \frac{1}{2} \right) \right) \\
= \frac{\partial L}{\partial s_1} \left( (-k, 0), \phi, 1_{D(1)}; (1, 0) \right) - \frac{1}{8\pi i} R_{k, \phi}(\phi, 1_{D(1)}) \\
= \frac{\partial L_1}{\partial s} (-k, \phi, 1_{D(1)}) - 2\pi i \zeta \left( -1 - k, \phi, (1, \tau) \right) \left( \tau - \tau \right)^{k+1}
\]
Using Lemma 6.5, we obtain the proposition. □

In particular, for \( \phi = 1_{L} \), we have the following example.

Example 6.10. For \( \phi = 1_{L} \), both \( \zeta(s, \phi, (1, \tau)) \) and \( \eta(s, \phi, 1) \) are Riemann zeta function \( \zeta(s) \). Hence we have
\[
\frac{(2\pi i)^{2k}}{(2k)!} \frac{\partial L}{\partial s_1} \left( (-2k, 0), 1_{L}, 1_{D(1)}; \left( \frac{1}{2}, \frac{1}{2} \right) \right) \\
= \begin{cases} 
\zeta(1 + 2k) + \frac{\zeta(-1 - 2k)}{(2\pi i(\tau - \tau))^2k+1} + 2 \sum_{l \in \mathbb{Z} > 0} \frac{i^{2k} \tau_{-1+k}}{ \tau_{-1+k} e^{2\pi il \tau}}, & \text{for } k \geq 1 \\
- \log(2\pi) + 2\pi i \left( \frac{1}{4} - \frac{1}{21} (\tau - \tau) \right) + 2 \sum_{l \in \mathbb{Z} > 0} \frac{i^{2k} \tau_{-1+k}}{ \tau_{-1+k} e^{2\pi il \tau}}, & \text{for } k = 0.
\end{cases}
\]

The Fourier series for \( -\frac{(2\pi i)^{2k}}{(2k)!} \frac{\partial L}{\partial s_1} \left( (-2k, 0), 1_{L}, 1_{D(1)}; \left( \frac{1}{2}, \frac{1}{2} \right) \right) \) is almost the same as that for the Eichler integral \( \tilde{E}_{-2k, p}(\tau) \) of the Eisenstein series of weight \( 2k + 2 \) (see Section 4.2) except that the holomorphic term
\[
\zeta(-1 - 2k) \frac{(2\pi i \tau)^{2k+1}}{(2k + 1)!}
\]
is replaced by the non-holomorphic term
\[
\frac{1}{2} \zeta \left( -1 - 2k \right) \frac{(2\pi i (\tau - \tau))^2}{(2k + 1)!}.
\]
The effect of this will be discussed later in Section 6.4 and 6.5.

6.3. The effect of Maass raising operator on \( \frac{\partial L}{\partial s_1} \left( (-k_1, -k_2), \phi, D; \lambda \right) \) for \( (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \). For \( s \in \mathbb{C} \), let \( \partial_\tau = \frac{\partial}{\partial \tau} + \frac{s}{\tau} \) be the Maass raising operator. The following lemma shows that \( \Psi_s := (\tau - \tau) \partial_s \) acts as a shifting operator on the Shintani L-values.

Lemma 6.11. For \( s \in \mathbb{C} \), set
\[
\Psi_s := (\tau - \tau) \partial_s,
\]
and \( \delta := (1, -1) = (\mathbf{e}_1 - \mathbf{e}_2) \). Then, for a fundamental domain \( D, k = (k_1, k_2) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \) and a direction \( \lambda \), we have
\[
\Psi_{-k_1} \left( \frac{\partial L}{\partial s_1} (-k, \phi, D; \lambda) \right) = k_1 \frac{\partial L}{\partial s_1} (-k + \delta, \phi, D; \lambda).
\]
Proof. Put \( \psi_s := -s^{-1} \Psi_s \) for \( s \in \mathbb{C} \setminus \{0\} \). Then, for \( x \in \mathbb{L} \) with \( p(x) = m \tau + n \), we have
\[
\psi_s \left( x^{-s} \right) = x^{-s-1} \tau.
\]
Thus, we have, for \( s = (s_1, s_2) \in \mathbb{C}^2 \) with \( \Re(s_1 + s_2) > 2 \),
\[
\psi_{s_1}(L(s, \phi, f)) = \sum_{x \in \mathbb{L}} f(x)\phi(p(x))\psi_{s_1}(x^{-s_1})^{1-s_2}
= \sum_{x \in \mathbb{L}} f(x)\phi(p(x))x^{-s_1-1}^{1-s_2+1}
= L(s + \delta, \phi, f).
\]
Since \( \frac{\partial}{\partial s} s^{-1} = s^{-1} \frac{\partial}{\partial s} - s^{-2} \), we have
\[
\frac{\partial}{\partial s} \psi_{s} = \frac{\partial}{\partial s} (s^{-1} (\tau - \varpi) + 1)
= \psi_{s} \frac{\partial}{\partial s} - s^{-2} (\tau - \varpi).
\]
Thus, we have
\[
\frac{\partial L}{\partial s_1}(s + \delta, \phi, f) = \frac{\partial}{\partial s_1} \psi_{s_1}(L(s, \phi, f))
= \psi_{s_1} \frac{\partial L}{\partial s_1}(s, \phi, f) - s_1^{-2} (\tau - \varpi) L(s, \phi, f).
\]
Let \( D \) be a fundamental domain, \( k = (k_1, k_2) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \) and \( \lambda_1 + \lambda_2 = 1 \). Then, from Proposition 5.17, we have
\[
L(-k + t\lambda, \phi, D) = \frac{O(t^2)}{e^{2\pi i t} - 1}
= O(t).
\]
Hence, it follows that
\[
\frac{\partial L}{\partial s_1}(-k + \delta, \phi, D; \lambda) = \psi_{-k_1} \frac{\partial L}{\partial s_1}(-k, \phi, D; \lambda) - k_1^2 (\tau - \varpi) L(-k, \phi, D; \lambda).
\]
Since \( \Psi_s = -s \psi_s \), the lemma is proved. \( \Box \)

Since \( \Psi_s = (\tau - \tau) \partial_s \), we have
\[
\Psi_{s+j-1} \cdots \Psi_{s+1} \Psi_s = (\tau - \tau)^j \partial_{2j-2+s} \cdots \partial_{s+2} \partial_s
\]
by using the relation \( \partial_s (\tau - \tau) = (\tau - \tau) \partial_{s+1} \) repeatedly. In particular, the operator \( d_k := (\tau - \tau)^k \partial_{-2} \partial_{-4} \cdots \partial_{-2k} \) is equal to \( \Psi_{-k-1} \cdots \Psi_{-2k+1} \Psi_{-2k} \). Hence, we have the following corollary.

**Corollary 6.12.** For a fundamental domain \( D \) and a direction \( \lambda \), we have
\[
d_k \left( \frac{\partial L}{\partial s_1}((-2k, 0), \phi, D; \lambda) \right) = \frac{(2k)!}{k!} \frac{\partial L}{\partial s_1}((-k, -k), \phi, D; \lambda).\]
6.4. The effect of Maass raising operator on the polynomial terms.

**Proposition 6.13.** For $k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, we have

$$d_k(s^s) = k! \sum_{j=0}^{k} \binom{s}{j} \left( \frac{2k-s}{k-j} \right) \tau^{s-j}\tau^j.$$

More generally, if we set $d_{k,a} := \Psi_{a-k-1} \cdots \Psi_{a+1} \Psi_a$ for $a \in \mathbb{C}$, we have

$$d_{k,a}(s^s) = k! \sum_{j=0}^{k} \binom{s}{j} \left( \frac{-a-s}{k-j} \right) \tau^{s-j}\tau^j.$$

**Proof.** We prove the latter statement by induction on $k$. Since the proposition is trivial for $k = 0$, it is sufficient to show the statement for $k+1$ by assuming that for $k$ i.e.

$$d_{k,a}(s^s) = k! \sum_{j=0}^{k} \binom{s}{j} \left( \frac{-a-s}{k-j} \right) \tau^{s-j}\tau^j.$$

Since $d_{k+1,a} = \Psi_{a+k}d_{k,a}$, we have

$$d_{k+1,a}(s^s) = (\tau - \tau)\partial_{a+k}d_{k,a}(s^s)$$

$$= k!(\tau - \tau)\partial_{a+k} \sum_{j=0}^{k} \binom{s}{j} \left( \frac{-a-s}{k-j} \right) \tau^{s-j}\tau^j.$$

Since

$$(\tau - \tau)\partial_{a+k} \left( \tau^{s-j}\tau^j \right) = (s-j)\tau^{s-j-1}\tau^j(\tau - \tau) - (a+k)\tau^{s-j}\tau^j$$

$$= (s-j)\tau^{s-j-1}\tau^j+1 - (a+k+s-j)\tau^{s-j}\tau^j,$$

we have

$$d_{k+1,a}(s^s)$$

$$= k! \sum_{j=0}^{k} \binom{s}{j} \left( \frac{-a-s}{k-j} \right) \left( (s-j)\tau^{s-j-1}\tau^j+1 - (a+k+s-j)\tau^{s-j}\tau^j \right)$$

$$= k! \sum_{j=0}^{k+1} \left( \binom{s}{j-1} \frac{-a-s}{k-j+1} (s-j+1) - \binom{s}{j} \frac{-a-s}{k-j} (a+k+s-j) \right) \tau^{s-j}\tau^j.$$

Since

$$k! \binom{s}{j-1} \frac{-a-s}{k-j+1} (s-j+1) = \frac{(-a-s)!}{(-a-k-1)! (s-j)} \binom{k}{j-1} \binom{-a-k-1}{s-j},$$

$$k! \binom{s}{j} \frac{-a-s}{k-j} (a+k+s-j) = \frac{(-a-s)!}{(-a-k-1)!} \binom{k}{j} \binom{-a-k-1}{s-j},$$

and

$$\binom{k}{j-1} + \binom{k}{j} = \binom{k+1}{j},$$
we obtain
\[
d_{k+1,a}(\tau^s) = \frac{(-a-s)!}{(-a-k-1)!} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{-a-k-1}{s-j} \tau^{s-j}\tau^j
\]
\[
= (k+1)! \sum_{j=0}^{k+1} \binom{s}{j} \binom{-a-s}{k+1-j} \tau^{s-j}\tau^j.
\]

Hence, the proposition is proved. \(\square\)

**Corollary 6.14.** For \(k \in \mathbb{Z}_{\geq 0}\) and \(s \in \mathbb{Z}\) with \(0 \leq s \leq 2k\), \(d_k(\tau^s)\) is a symmetric polynomial in \(\tau, \tau\). More precisely,
\[
d_k(\tau^s) \in k!\mathbb{Z}[\tau + \tau, \tau\tau].
\]

**Proof.** Since \(0 \leq s, 2k-s \leq 2k\), \(\binom{s}{j}(2k-s)_{k-j}\) vanishes except for \(0 \leq j \leq s\), \(0 \leq k-j \leq 2k-s\), i.e. \(\max(0, s-k) \leq j \leq \min(k, s)\). Thus, we have
\[
d_k(\tau^s) = k! \sum_{j=0}^{s} \binom{s}{j} \binom{2k-s}{k-j} \tau^{s-j}\tau^j.
\]

Since \(\binom{s}{j}(2k-s)_{k-j} = \binom{s}{j}(2k-s)_{k-j}\) and \(d_k(\tau^s)\) is a symmetric polynomial in \(\tau, \tau\). \(\square\)

For \(s \geq 2k+1\), \(d_k(\tau^s)\) is no longer a symmetric polynomial in \(\tau, \tau\). Nevertheless, we have the following corollary.

**Corollary 6.15.** For \(k \in \mathbb{Z}_{\geq 0}\), \(d_k(2\tau^{2k+1} - (\tau - \tau)^{2k+1})\) is a symmetric polynomial in \(\tau, \tau\). More precisely, it holds that
\[
d_k(2\tau^{2k+1} - (\tau - \tau)^{2k+1}) \in k!\mathbb{Z}[\tau + \tau, \tau\tau].
\]

**Proof.** Putting \(s = 2k+1\) in the proposition, we have
\[
d_k(\tau^{2k+1}) = k! \sum_{j=0}^{k} (-1)^{k-j} \binom{2k+1}{j} \tau^{2k+1-j}\tau^j.
\]
Using \(\partial_a((\tau - \tau)^s) = (a+s)(\tau - \tau)^{s-1}\) recursively, we have
\[
d_{k,a}(\tau - \tau)^s = k! \binom{-a-s}{k} (\tau - \tau)^s.
\]

Thus, by setting \(a = -2k, s = 2k+1\), we have
\[
d_k((\tau - \tau)^{2k+1}) = (-1)^k k!(\tau - \tau)^{2k+1},
\]
\[
= k! \sum_{j=0}^{k} (-1)^{k-j} \binom{2k+1}{j} \tau^{2k+1-j}\tau^j - \tau^j\tau^{2k+1-j}.
\]

Hence we obtain
\[
d_k(2\tau^{2k+1} - (\tau - \tau)^{2k+1}) = k! \sum_{j=0}^{k} (-1)^{k-j} \binom{2k+1}{j} (\tau^j\tau^{2k+1-j} + \tau^{2k+1-j}\tau^j)
\]
\[
= (2k+1)! \sum_{i+j=2k+1} (-1)^{k-i}(k-i)! \frac{(\tau + \tau)^i (\tau\tau)^j}{i!j!},
\]
which completes the proof. \(\square\)
6.5. The relation with Zagier-Gangl’s enhanced zeta value.

**Theorem 6.16.** Let $F$ be an imaginary quadratic number field, and $A \in \text{Cl}(F)$ an ideal class. Then, for $k \in \mathbb{Z}_{\geq 1}$, we have

$$I_{k+1}(A^{-1}) = \frac{(2\pi i)^k}{k!} \Lambda_1(-k, A),$$

as an element of $\mathbb{C}/\mathbb{Q}(k+1)$.

**Proof.** Let $D$ be a fundamental domain, and $a \in A^{-1}$. Since $a$ is a free $\mathbb{Z}$-module of rank 2, we can take a $\mathbb{Z}$-basis $\omega := (\omega_1, \omega_2) \in (F^\times)^2$ of $a$ i.e.

$$a = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$  

Here, we may assume $\omega_1^{-1} \omega_2 \in \mathcal{H}$ without loss of generality. Now set

$$Q_\omega(X, Y) = N(a)^{-1} (\omega_1 X - \omega_2 Y) \left(\frac{1}{\omega_1 X - \omega_2 Y}\right).$$

Then $Q_\omega(X, Y)$ is a primitive positive definite integral quadratic form in the class that corresponds to $A$. Define $\lambda_\omega \in \mathbb{Z}, \tau_\omega \in \mathcal{H} \cap F$ by $\lambda_\omega = N(\alpha a^{-1})$ and $\tau_\omega = \omega_1^{-1} \omega_2$. Then,

$$Q_\omega(X, Y) = \lambda_\omega (X - \tau_\omega Y) (X - \tau_\omega Y).$$

Note that the the element of $\mathcal{H}$ associated to the quadratic form $Q_\omega(X, Y)$ in Gangl and Zagier’s article [18] is $-\tau_\omega$ (see (4.15)), which is $SL_2(\mathbb{Z})$-equivalent to $\tau_\omega$ associated to $\mathbb{Z}$-basis $\omega'$ of $\alpha^{-1}$ (not of $a$). Since $\omega_1^{-1} a \in A^{-1}$, we have

$$\Lambda_1(-k, A) = \Lambda_1(-k, D, \omega_1^{-1} a) \mod \mathbb{Q}(1)$$

Since we have

$$\tilde{E}_{-2k, \sigma}(\tau) = \frac{(2\pi i)^{2k}}{(2k)!} \frac{\partial L}{\partial s_1} \left(1, -2k, -k_L, 1_{D(1)}; \frac{1}{2}, \frac{1}{2}\right)$$

where $\sigma = \mathbb{Z} + \mathbb{Z} \tau$ and $P$ is a polynomial in $\tau$ of degree at most $2k$, whose coefficients lie in $\mathbb{Q}(2k + 1)$, we obtain

$$\tilde{E}_{-2k, \sigma}(\tau) = \frac{(2\pi i)^{2k}}{(2k)!} \frac{\partial L}{\partial s_1} \left(1, -2k, -k_L, 1_{D(1)}; \frac{1}{2}, \frac{1}{2}\right)$$

From Corollaries 6.14 and 6.15, we see that the last quantity is a symmetric polynomial in $\tau, \tau'$ whose coefficients lie in $\mathbb{Q}(2k + 1)$. Now set $L_\omega := \mathbb{Z} + \mathbb{Z} \tau_\omega$. Since $\tau_\omega + \tau_\omega, \tau_\omega \tau_\omega \sqrt{-1} \in \mathbb{Q}$, we have

$$I_{k+1}(A^{-1}) = \frac{(2\pi i)^k}{k!} \Lambda_1(-k, A)$$

$$= \frac{1}{w_F} \left(\frac{\lambda_\omega}{2\pi i}\right)^k \tilde{E}_{k+1, \sigma}(\tau) \frac{(2\pi i)^{2k}}{(2k)!} \frac{\partial L}{\partial s_1} \left(1, -2k, -k_L, 1_{D(1)}; \frac{1}{2}, \frac{1}{2}\right) \in \mathbb{Q}(k + 1).$$
7. The enhanced conjecture for $\Lambda_1(-k, \mathcal{A})$

Finally, we formulate an enhanced conjecture for the ray class invariant

$$\Lambda_1(-k, \mathcal{A}) = \overline{\Lambda_2(-k, \mathcal{A})},$$

which generalizes Zagier-Gangl's original conjecture for an ideal class to a general ray class. Let

$$\hat{L}_k : B_k(C) \to \mathbb{C}/Q(k)$$

be Zagier-Gangl's enhanced $k$-logarithm. For an abelian extension $H$ of $F$, we regard $H$ as a subfield of $C$ by fixing its complex embedding lying over $\rho$ (the complex embedding of $F$ which we have fixed in the first place), and thus regard $\hat{L}_k$ as a function on $B_k(H)$. Then our conjecture is as follows.

**Conjecture 7.1.** Let $k \in \mathbb{Z}_{\geq 1}$, $F$ an imaginary quadratic field, and $A \in \text{Cl}_m(F)$ a ray class of modulus $m$. Denote by $H_m$ the ray class field of $m$, and by $\text{rec}$ the reciprocity map

$$\text{rec} : \text{Cl}_m(F) \xrightarrow{\cong} \text{Gal}(H_m/F).$$

Then there exists $\xi_{A,k+1} \in B_{k+1}(H_m) \otimes \mathbb{Q}$ such that

$$(2\pi i)^k \Lambda_1(-k, \text{rec}^{-1}(\sigma)A) = \hat{L}_{k+1}(\sigma^{-1}\xi_{A,k+1})$$

for $\sigma \in \text{Gal}(H_m/F)$, as an equality in $C/Q(k)$.

Assuming the conjectural compatibility with the Galois decent of the Bloch group i.e.

$$B_k(H')^{\text{Gal}(H'/H)} = B_k(H),$$

which is believed to hold for a general Galois extension $H'/H$ of a number field, our conjecture is also expressed as

**Conjecture 7.2.** Let $k \in \mathbb{Z}_{\geq 1}$, $F$ an imaginary quadratic field and $H$ a finite abelian extension of $F$. Then, for $\theta \in \mathbb{Z}[\text{Gal}(H/F)]$, there exists $\xi_{\theta,k+1} \in B_{k+1}(H) \otimes \mathbb{Q}$ such that

$$(2\pi i)^k \Lambda_1(-k, \sigma\theta) = \hat{L}_{k+1}(\sigma^{-1}\xi_{\theta,k+1})$$

for $\sigma \in \text{Gal}(H/F)$, as an equality in $C/Q(k)$.

Now, we give several numerical evidence for this conjecture. For Hilbert class field of $F$ (equivalently, for trivial $m$), this conjecture is just a restatement of Zagier-Gangl's enhanced conjecture by Theorem 6.16, so we should check the conjecture for ramified abelian extensions $H$ of $F$ (equivalently, for non-trivial $m$). For this purpose, we briefly describe how to compute both sides of our enhanced conjecture.

### 7.1. Numerical computation of $\Lambda_1(-k, \mathcal{A})$.

Let $F$ be an imaginary quadratic field, $m$ a modulus of $F$, and $\mathcal{A} \in \text{Cl}_m(F)$ a ray class. Then $\Lambda_1(-k, \mathcal{A})$ can be computed in the following way. Let $a$ be a fractional ideal in $\mathcal{A}^{-1}$. Take a $\mathbb{Z}$-basis $(\omega_1, \omega_2)$ of $a$ such that $\Im(\omega_2/\omega_1) > 0$. Put $\tau := \omega_2/\omega_1$ and define $\phi_a : \mathbb{Z} + \mathbb{Z}\tau \to \{0, 1\}$ by

$$\phi_a(x) := 1_{\sigma \in U(m)}(\omega_1 x)$$

where

$$U(m) = \{x \in F^\times \mid x \equiv 1 \mod m\}.$$
Take a positive integer $N$ such that $\phi_a(x + Ny) = \phi_a(x)$ for all $x, y \in \mathbb{Z} + \mathbb{Z}\tau$. Put $w_m := \left|\mathcal{O}_F^\times \cap U(m)\right|$, $c_n := n! \left(\frac{2\pi i}{n}\right)^{-n}$, $T_n := \left(\frac{(\tau - \bar{\tau})}{n!}\right)^n$ and

$$\zeta(s, \phi) := \sum_{l \in \mathbb{Q}_{>0}} \sum_{k \in [N]} \phi(k + l\tau) \frac{1}{l^s}$$

$$\eta(s, \phi) := \sum_{l \in \mathbb{Z}_{>0}} \sum_{m \in [N]} \phi(m)e^{\frac{2\pi i ml}{N}}$$

where $[N] = \{ m \mid m \in \mathbb{Q} \text{ such that } 0 < m \leq N \}$. Then $A_1(-k, A)$ is equal to the following quantity modulo $\mathbb{Q}(1)$.

$$A_1(-k, A, (\omega_1, \omega_2)) := w_m^{-1}N(\omega_1^{-1}a)^{-k} \left\{ (-1)^{k-1}2\pi i(k!)^2 \zeta(-1, \phi_a)(1 + 2k + 1) \right. + c_{2k}\eta(1 + 2k, \phi_a) + k! \sum_{r=0}^{k} \ell_{k+r}T_{k-r} \sum_{l \in \mathbb{Z}_{>0}} l^{-1+(k+r)} \sum_{j \in \mathbb{Q}_{>0}} j^{k-r} \sum_{m \in [N]} (\phi_a(m + j\tau) + \phi_a(-(m + j\tau))) e^{2\pi i(m \tau \text{ mod } 1)} \right\} \left. \bigg|_{\tau = \omega_1^{-1}\omega_2} \right. \in \mathbb{C}.$$

We omit the detail of the derivation of this formula, since it is just a straightforward calculation by the use of Proposition 6.9 and Corollaries 6.12, 6.14, 6.15.

### 7.2. Numerical computation of $\hat{\mathcal{L}}_m(\xi)$

Let $F$ be an imaginary quadratic field, $H$ an abelian extension of $F$ and $m \geq 2$. Then, we compute $\hat{\mathcal{L}}_m(\xi)$ for $\xi \in \mathcal{B}_m(H)$ as follows.

Let $\mu_H$ denote the group of roots of unity in $H$, the set $\Omega \subset \mathbb{C}^\times \times \mathbb{C}$ and a function $\mathcal{F}_m$ on $\Omega$ as in Section 4.1. Let $\text{Li}_k(x)$ be the polylogarithm whose value is taken as in Section 3.1 for $x \in \mathbb{C} \setminus \mathbb{R}_{\geq 1}$ and as $\text{Li}_k(x_+)$ for $x \in \mathbb{R}_{> 1}$. We thus regard $\text{Li}_k(x)$ as a single-valued function on $\mathbb{C} \setminus \{1\}$. Using this-way defined polylogarithms, we can express $\mathcal{F}_m(x, U)$ as mod $\mathbb{Q}(m)$ reduction of the $\mathbb{C}$-valued function

$$\tilde{\mathcal{F}}_m(x, U) := \sum_{k=0}^{m-1} \frac{(-U)^k}{k!} \text{Li}_{m-k}(x),$$

which we use for numerical computation.

Now, fix a homomorphism $L : H^\times / \mu_H \to \mathbb{C}$ such that

$$L(x) \equiv \log x \pmod{\mathbb{Q}(1)}$$

for all $x \in H^\times$ and define

$$\hat{\mathcal{L}}_m(x, L) := \tilde{\mathcal{F}}_m(x, L(x)) + \frac{(-L(x))^{m-1}L(1-x)}{m!} \in \mathbb{C}$$

for $x \in H \setminus \{0, 1\}$. We then define $\hat{\mathcal{L}}_m(\xi, L)$ for $\xi = \sum_i n_i[z_i] \in \mathbb{Z}[H \setminus \{0, 1\}]$, by

$$\hat{\mathcal{L}}_m(\xi, L) := \sum_{i} n_i \hat{\mathcal{L}}_m(z_i, L) \in \mathbb{C}.$$
Then, for \( \xi \in B_m(H) \),
\[
\hat{\Lambda}_m(\xi, L) \quad (\mod \mathbb{Q}(m))
\]
does not depend on the choice of \( L \), nor the choice of representative of \( \xi \) in \( \mathbb{Z}[H \setminus \{0, 1\}] \), and gives the enhanced polylogarithm \( \hat{\Lambda}_m(\xi) \in \mathbb{C}/\mathbb{Q}(m) \).

7.3. **Numerical verification of the enhanced conjecture.** Our enhanced conjecture is equivalent to
\[
(2\pi i)^{-1} \Lambda_1(1-k, \sigma^{-1}) \equiv (2\pi i)^{-k} \hat{\Lambda}_k(\sigma(\xi)) \quad (\mod \mathbb{Q})
\]
for \( k \in \mathbb{Z}_{\geq 2} \). Therefore, we numerically verify the conjecture in this form. In the first three examples, we consider the Bloch groups of degree \( k = 2, 3, 4 \) of the same quadratic extension of \( \mathbb{Q}(\sqrt{-1}) \) respectively, whiles in the fourth example, we consider a cubic extension of \( \mathbb{Q}(\sqrt{-15}) \), the case where the ring of integers is not a PID. In each example, we have verified the equality of the form
\[
(2\pi i)^{-1} \Lambda_1(1-k, \sigma^{-1}) - (2\pi i)^{-k} \hat{\Lambda}_k(\sigma(\xi)) = \text{(some rational)}
\]
to 60-digits precision (meaning that the error is less than \( 10^{-60} \)). Note that the representative of the Bloch groups in the following examples are not necessarily of the simplest forms, since we did not make any effort to obtain such forms.

7.3.1. **Example 1.** Consider the case where \( F = \mathbb{Q}(\sqrt{-1}) \), \( k = 2 \) and
\[
H = F[a]/(a^2 + (\sqrt{-1} - 1)a + 1) = \mathbb{Q}[a]/(a^4 - 2a^3 + 4a^2 - 2a + 1).
\]
The conductor \( m \) of the abelian extension \( H/F \) is \( (4 + 2\sqrt{-1}) \), and the Galois group of the extension is given by \( \text{Gal}(H/F) = \{\text{id}, \sigma\} \simeq \mathbb{Z}/2\mathbb{Z} \) where \( \sigma(a) = a^{-1} \). We embed \( H \) into \( \mathbb{C} \) by
\[
a \mapsto 0.25706586412167716 \ldots + 0.52908551363574612 \ldots i.
\]
Then we have
\[
(2\pi i)^{-1} \Lambda_1(-1, \text{id}, (1, \sqrt{-1}))
\]
\[
= -0.828061172859541643409557 \ldots + 0.1168114059603936677836901 \ldots i,
\]
\[
(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-1}, (3, 3\sqrt{-1}))
\]
\[
= +0.7421727839526208310076223 \ldots - 0.10134361959744827660683 \ldots i.
\]
We put
\[
\begin{align*}
x_0 &= \sqrt{-1} \\
x_1 &= a \\
x_2 &= a + \sqrt{-1}
\end{align*}
\]
and \( g(k_0, k_1, k_2) = [x_0^{k_0}x_1^{k_1}x_2^{k_2}] \). Then \( B_2(H) \otimes \mathbb{Q} \) is generated by \( \xi_1 \) and \( \xi_2 \), where
\[
\xi_1 = 4g(0, -2, -3) + g(-1, 5, 6),
\]
\[
\xi_2 = 24g(0, -1, 0) - g(-1, 5, 6).
\]
Put
\[
\xi = \frac{1}{6} \xi_1 + \frac{1}{5} \xi_2
\]
and take any homomorphism $L : H^x / \mu_H \to \mathbb{C}$ such that
\[
\begin{cases}
\exp(L(x)) = x \\
-\pi < \Im(L(x)) \leq \pi
\end{cases}
\]
for $x \in \{x_1, x_2\}$. Then we have
\[
(2\pi i)^{-2} \hat{\mathcal{L}}_2(\xi, L) = -0.4104366728415097198965112 \ldots + 0.1168114059603936677836901 \ldots i,
\]
\[
(2\pi i)^{-2} \hat{\mathcal{L}}_2(\sigma(\xi), L) = -0.6485911049362680578812664 \ldots - 0.101343619597444827660683 \ldots i.
\]

Therefore, the following equalities hold in high accuracy.
\[
(2\pi i)^{-1} A_1(-1, \text{id}, (1, \sqrt{-1})) - (2\pi i)^{-2} \hat{\mathcal{L}}_2(\xi, L) = -\frac{1}{2^6 3^2 5^2} \times 6013,
\]
\[
(2\pi i)^{-1} A_1(-1, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-2} \hat{\mathcal{L}}_2(\sigma(\xi), L) = \frac{1}{2^6 3^2 5^2} \times 20027.
\]

7.3.2. Example 2. Let $F, H, g$ be as in Example 1 and $k = 3$. Then we have
\[
(2\pi i)^{-1} A_1(-2, \text{id}, (1, \sqrt{-1})) = +22.55139529962232809672931346 \ldots - 0.9653510778209588623198647328 \ldots i,
\]
\[
(2\pi i)^{-1} A_1(-2, \sigma^{-1}, (3, 3\sqrt{-1})) = -25.07639529962232809672931346 \ldots + 0.878122673714396064702349796 \ldots i.
\]

On the other hand, $B_3(H) \otimes \mathbb{Q}$ is generated by $\xi_1$ and $\xi_2$, where
\[
\xi_1 = g(1, 0, 0)
\]
\[
\xi_2 = 2g(0, -1, 0) + 2g(-1, -1, 0) - g(-1, 2, 0) + g(2, -2, 0).
\]

Put
\[
\xi := 1168 \xi_1 + 96 \xi_2
\]

Then we have
\[
(2\pi i)^{-3} \hat{\mathcal{L}}_3(\xi, L) = -4.3661047003767190327068653 \ldots - 0.9653510778209588623198647 \ldots i,
\]
\[
(2\pi i)^{-3} \hat{\mathcal{L}}_3(\sigma(\xi), L) = -3.63389529962232809672931346 \ldots + 0.878122673714396064702349796 \ldots i.
\]

Therefore, the following equalities hold in high accuracy.
\[
(2\pi i)^{-1} A_1(-2, \text{id}, (1, \sqrt{-1})) - (2\pi i)^{-3} \hat{\mathcal{L}}_3(\xi, L) = \frac{1}{2^4 5^2} \times 10767,
\]
\[
(2\pi i)^{-1} A_1(-2, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-3} \hat{\mathcal{L}}_3(\sigma(\xi), L) = -\frac{1}{2^4 5^2} \times 8577.
\]
Then we have

\[(2\pi i)^{-1} \Lambda_1(-3, \text{id}, (1, \sqrt{-1})) = -1649.9567543236663587212631000 \ldots + 17.3784886358384973091620779 \ldots i,
\]
\[(2\pi i)^{-1} \Lambda_1(-3, \sigma^{-1}, (3, 3\sqrt{-1})) = +1955.9608376569969920545964334 \ldots - 16.4972527452257435661551513 \ldots i.\]

On the other hand, \(B_4(H) \otimes \mathbb{Q}\) is generated by \(\xi_1\) and \(\xi_2\), where

\(\xi_1 = g(-1, 0, 0),\)
\(\xi_2 = 6030g(1, -2, -4) - 38592g(-1, -1, -1) + 6300g(2, -2, 0)
- 4288g(-1, 3, 3) + 38592g(2, 1, 1) + 14472g(1, 2, 2)
- 315g(2, -4, 0) - 20160g(0, 1, 0).\)

Put

\[\xi := \frac{1523024}{215}\xi_1 - \frac{24}{43}\xi_2.\]

Then we have

\[(2\pi i)^{-4} \tilde{\mathcal{L}}_4(\xi, L) = 572.6591331924862120797704916 \ldots + 17.3784886358384973091620779 \ldots i,
\]
\[(2\pi i)^{-4} \tilde{\mathcal{L}}_4(\sigma(\xi), L) = 394.9627716847747698323742112 \ldots - 16.4972527452257435661551513 \ldots i.\]

Therefore, the following equalities hold in high accuracy.

\[(2\pi i)^{-1} \Lambda_1(-3, \text{id}, (1, \sqrt{-1})) - (2\pi i)^{-4} \tilde{\mathcal{L}}_4(\xi, L) = -\frac{1}{2^{83}3^25^{13}} \times 27524875151,
\]
\[(2\pi i)^{-1} \Lambda_1(-3, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-4} \tilde{\mathcal{L}}_4(\sigma(\xi), L) = \frac{1}{2^{83}3^25^3} \times 449567443.\]

7.3.4. Example 4. Consider the case where \(F = \mathbb{Q}(\sqrt{-1})\), \(k = 2\) and

\[H = F[a]/(a^3 + a^2 + (-\sqrt{-1} + 1) a + 1)\]
\[= \mathbb{Q}[a]/(a^6 + 2a^5 + 3a^4 + 4a^3 + 4a^2 + 2a + 1).\]

The conductor \(m\) of the abelian extension \(H/F\) is \(3 + 2\sqrt{-1}\), and the Galois group of the extension is given by \(\text{Gal}(H/F) = \{\text{id}, \sigma, \sigma^2\} \simeq \mathbb{Z}/3\mathbb{Z}\), where

\[\sigma(a) = a^3 + \sqrt{-1}.\]

We embed \(H\) into \(\mathbb{C}\) by

\[a \mapsto -1.049136453746963 \ldots - 0.552653068016644 \ldots i.\]

Then we have

\[(2\pi i)^{-1} \Lambda_1(-1, \text{id}, (1, \sqrt{-1})) = -0.4752235692567145469512564 \ldots + 0.0398871847352005271199996 \ldots i,
\]
\[(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-1}, (3, 3\sqrt{-1})) = +1.1870703963535623282072588 \ldots - 0.010841653214714539329223 \ldots i.
\]
\[(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-2}, (7, 7\sqrt{-1})) = -0.9442186219685826530508741 \ldots - 0.0754488906098755935563227 \ldots i.\]
We put

\[
\begin{align*}
    x_0 &= \sqrt{-1} \\
    x_1 &= a^4 + a^3 + 2a^2 + 2a + 1 \\
    x_2 &= a^4 + a^3 + a^2 + a + 1
\end{align*}
\]

and \( g(k_0, k_1, k_2) = [x_0^{k_0} x_1^{k_1} x_2^{k_2}] \). Then \( B_2(H) \otimes \mathbb{Q} \) is generated by \( \xi_1, \xi_2 \) and \( \xi_3 \), where

\[
\begin{align*}
    \xi_1 &= 3g(0,-4,-3) + g(0,-3,-1) + g(-1,4,2), \\
    \xi_2 &= 3g(0,-4,-3) + 3g(0,-1,-2), \\
    \xi_3 &= 7g(0,-3,-1) + 7g(0,-1,-2).
\end{align*}
\]

Put

\[
\xi = -\frac{23}{18} \xi_1 + \frac{23}{9} \xi_2 - \frac{5}{3} \xi_3,
\]

and take any homomorphism \( L : H^e/\mu_H \to \mathbb{C} \) such that

\[
\begin{align*}
    \exp(L(x)) &= x \\
    -\pi &< 3(L(x)) \leq \pi
\end{align*}
\]

for \( x \in \{x_1, x_2\} \). Then we have

\[
\begin{align*}
    (2\pi i)^{-2} \hat{\Gamma}_2(\xi, L) &= 6.5799313452731999829632734 \ldots + 0.0398871847352005271199996 \ldots i, \\
    (2\pi i)^{-2} \hat{\Gamma}_2(\sigma(\xi), L) &= 2.4959646271227330974380280 \ldots - 0.0108416532141714539329223 \ldots i, \\
    (2\pi i)^{-2} \hat{\Gamma}_2(\sigma^2(\xi), L) &= 1.8668123609374002529320318 \ldots - 0.0754488906098755935563227 \ldots i.
\end{align*}
\]

Therefore, the following equalities hold in high accuracy.

\[
\begin{align*}
    (2\pi i)^{-1} \Lambda_1(-1, \text{id}, (1, \sqrt{-1})) - (2\pi i)^{-2} \hat{\Gamma}(\xi, L) &= -\frac{1}{293^{13}} \times 52829, \\
    (2\pi i)^{-1} \Lambda_1(-1, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-2} \hat{\Gamma}(\sigma(\xi), L) &= -\frac{1}{293^{13}} \times 1089, \\
    (2\pi i)^{-1} \Lambda_1(-1, \sigma^{-2}, (7, 7\sqrt{-1})) - (2\pi i)^{-2} \hat{\Gamma}(\sigma^2(\xi), L) &= -\frac{1}{293^{213}} \times 21049.
\end{align*}
\]

7.3.5. Example 5. Consider the case where \( F = \mathbb{Q}(\alpha), \alpha = \frac{1+\sqrt{-15}}{2}, k = 2 \) and

\[
H = F[a]/(a^4 + 4a^3 + (\alpha - 2)a^2 - aa + 1) = \mathbb{Q}[a]/(a^8 + a^7 + a^6 + 5a^5 - 5a^3 + a^2 - a + 1).
\]

The conductor \( \mathfrak{m} \) of the abelian extension \( H/F \) is \( (2\alpha) \), and the Galois group of the extension is given by \( \text{Gal}(H/F) = \{\text{id}, \sigma, \sigma^2, \sigma^3\} \simeq \mathbb{Z}/4\mathbb{Z} \), where

\[
\sigma(a) = -\frac{5}{4}a^7 - 2a^6 - \frac{11}{4}a^5 - 8a^4 - 5a^3 + \frac{9}{4}a^2 + \frac{1}{2}a + \frac{9}{4}.
\]

We embed \( H \) into \( \mathbb{C} \) by

\[
a \mapsto -1.472308583487351 \ldots + 0.228052190401739 \ldots i.
\]
Then we have

\[
(2\pi i)^{-1} \Lambda_1(-1, \text{id}, (1, \alpha)) = -0.5494164885957489658397803 \ldots + 0.4817598174579106040931989 \ldots i,
\]

\[
(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-1}, (3, 1 + \alpha)) = -0.42519244960512455152105 \ldots - 0.078687503428147628771922 \ldots i,
\]

\[
(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-2}, (3, 3\alpha)) = -0.2005835114042510341602196 \ldots - 0.4156085245837687359142919 \ldots i,
\]

\[
(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-3}, (9, 3 + 3\alpha)) = -1.824807503944875444847894 \ldots + 0.0534684668166029952932881 \ldots i.
\]

We put

\[
\begin{align*}
    x_0 &= \frac{5}{2} x^7 + \frac{9}{5} x^6 + \frac{3}{2} x^5 + 5 x^4 + 2 x^3 - \frac{19}{3} x^2 + \frac{1}{3} a - \frac{1}{2} \\
    x_1 &= \frac{1}{2} x^7 + \frac{3}{5} x^5 + x^3 + x^2 - \frac{13}{2} x - \frac{1}{8} \\
    x_2 &= x_0 - a \\
    x_3 &= x_0 - x_1 - 1
\end{align*}
\]

and \(g(k_0, k_1, k_2, k_3) = [x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3}]\). Note that, despite its appearance, \(x_0^3 = -1\). Then \(B_2(H) \otimes \mathbb{Q}\) is generated by \(\xi_1, \xi_2, \xi_3\) and \(\xi_4\), where

\[
\begin{align*}
    \xi_1 &= g(-1, 0, 0, 0), \\
    \xi_2 &= 2g(3, -1, 1, 1) - g(-1, 1, 1, 0) - 2g(2, 1, -1, 1), \\
    \xi_3 &= 6g(3, -1, 1, 1) - 2g(-2, 1, 0, 0) - g(2, 1, -1, -1) + g(0, -1, -2, -1), \\
    \xi_4 &= 2g(3, -1, 1, 1) + 2g(-2, 1, 0, 0) - g(-2, 1, 1, 2).
\end{align*}
\]

Put

\[
\xi := -\frac{23}{5} \xi_1 - \frac{17}{2} \xi_2 + \frac{11}{4} \xi_3 + 6 \xi_4,
\]

and take any homomorphism \(L : H^* / \mu_H \to \mathbb{C}\) such that

\[
\begin{cases}
    \exp(L(x)) = x \\
    -\pi < \Im(L(x)) \leq \pi
\end{cases}
\]

for \(x \in \{x_1, x_2, x_3\}\). Then we have

\[
\begin{align*}
    (2\pi i)^{-2} \hat{L}_2(\xi, L) &= -0.301673433040193410242247 \ldots + 0.4817598174579106040931989 \ldots i \\
    (2\pi i)^{-2} \hat{L}_2(\sigma(\xi), L) &= -4.2600882829388457888485438 \ldots - 0.0798687503428147628771922 \ldots i, \\
    (2\pi i)^{-2} \hat{L}_2(\sigma^2(\xi), L) &= +5.6791039885957489658397803 \ldots - 0.4156085245837687359142919 \ldots i, \\
    (2\pi i)^{-2} \hat{L}_2(\sigma^3(\xi), L) &= -6.9826200503944875444847894 \ldots + 0.0534684668166029952932881 \ldots i.
\end{align*}
\]
Therefore, the following equalities hold in high accuracy.
\[(2\pi i)^{-1} \Lambda_1(-1, \text{id}, (1, \alpha)) - (2\pi i)^{-2} \tilde{L}_2(\xi, L) = -\frac{1}{2^3 3^5} \times 1427,\]
\[(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-1}, (3, 1 + \alpha)) - (2\pi i)^{-2} \tilde{L}_2(\sigma(\xi), L) = -\frac{1}{2^7 5^6} \times 7363,\]
\[(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-2}, (3, 3\alpha)) - (2\pi i)^{-2} \tilde{L}_2(\sigma^2(\xi), L) = -\frac{1}{2^7 5^6} \times 3763,\]
\[(2\pi i)^{-1} \Lambda_1(-1, \sigma^{-3}, (9, 3 + 3\alpha)) - (2\pi i)^{-2} \tilde{L}_2(\sigma^3(\xi), L) = -\frac{1}{2^7 5^6} \times 3301.\]

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REFERENCES