「**Degeneration of Period Matrices of Stable Curves**」
の要約

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In the present paper, we study combinatorial anabelian geometry of hyperbolic curves. Before we explain the main question that motivated the theory developed in the present paper, let us recall some basic facts concerning period matrices.

Let $X$ be a stable curve of genus $g$ over an algebraically closed field $k$, $\Gamma_X$ the dual graph of $X$, and $\ell \neq \mathrm{char}(k)$ a prime number. Then one has a natural exact sequence of free $\mathbb{Z}_\ell$-modules

$$0 \longrightarrow M_X^{\mathrm{ver}} \longrightarrow M_X \longrightarrow M_X^{\mathrm{top}} \longrightarrow 0,$$

where $M_X := \pi_1^{\ell\text{-adm}}(X)^{\mathrm{ab}}, M_X^{\mathrm{top}} := \pi_1^{\ell}(\Gamma_X)^{\mathrm{ab}}, M_X^{\mathrm{ver}} := \mathrm{Im}(\bigoplus_{v \in v(\Gamma_X)} \pi_1^{\ell}(X_v \setminus \mathrm{Node}(X))^{\mathrm{ab}} \longrightarrow M_X)$, where $\mathrm{Node}(X)$ denotes the set of nodes of $X$. The stable curve $X$ determines a morphism from $s := \mathrm{Spec}\, k$ to the moduli stack $\overline{\mathcal{M}}_g$, and the pull-back log structure of the natural log structure on $\overline{\mathcal{M}}_g$ determines a log structure on $\mathrm{Spec}\, k$; denote the resulting log scheme by $s^{\log}$ which admits a chart $(\mathrm{Spec}\, k, \bigoplus_{e \in e(\Gamma_X)} \mathbb{N})$. The pro-$\ell$ log étale fundamental group $\pi_1^{\ell}(s^{\log})$ is naturally isomorphic to $\bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$. Therefore, we obtain a natural action of $\bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ on the extension $0 \longrightarrow M_X^{\mathrm{ver}} \longrightarrow M_X \longrightarrow M_X^{\mathrm{top}} \longrightarrow 0$. This extension determines an extension class $[M_X]$, which may be regarded as a homomorphism, which we refer to as the **pro-$\ell$ period matrix morphism** of $X$:

$$f_X : \pi_1^{\ell}(s^{\log}) \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1) \longrightarrow \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)).$$

For each element $a \in \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$, we refer to $f_X(a)$ as the **pro-$\ell$ period matrix** associated to $a$.

If $a = (a_e)_e \in \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e$ is a **positive definite** element (cf. Definition 2.5), then the closed subgroup generated by $a$ can be regard as the image of the the maximal pro-$\ell$ quotient of the inertia group of a $p$-adic local field (cf. the discussion at the beginning of Section 2.2). Thus, by applying Faltings-Chai's theory (or the weight-monodromy conjecture for curves), we know that the pro-$\ell$ period matrix $f_X(a)$ is positive definite, hence also **non-degenerate**. This non-degeneracy property of pro-$\ell$ period matrices is the most non-trivial part in S. Mochizuki's proof of the combinatorial version of the Grothendieck conjecture (=ComGC) for semi-graphs of anabelioids in the case of outer representations of IPSC-type (cf. [15] Corollary 2.8).

More generally, in the theory of combinatorial anabelian geometry, in order to extend results (e.g., the ComGC) in the IPSC-type case to the NN-type case (i.e., the outer Galois action arising from a non-degenerate (= all the coordinates of the element are nonzero), it is natural to attempt to prove the ComGC in the case of outer representations of NN-type case. On the other hand, if one attempts to imitate the proof of the ComGC in the IPSC-type case, one has to consider whether or not the pro-$\ell$ period matrix arising from a **non-degenerate** element of $\pi_1^{\ell}(s^{\log}) \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ is degenerate. It is difficult to determine in general whether or not the pro-$\ell$ period matrix associated to a given **non-degenerate** element is **degenerate**. But at least we can ask which stable curves admit a **non-degenerate** element that gives rise to a **degenerate** pro-$\ell$ period matrix. This question may be formulated as follows:

**Question 0.1.** *Does there exist a criterion to determine whether or not the given stable curve $X$ admits an element $a = (a_e)_e \in \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ such that $a_e \neq 0$ for each $e$ and, moreover, the pro-$\ell$ period matrix $f_X(a)$ is degenerate?*

Our main theorem of the present paper is a criterion as follows (cf. Theorem 2.9):

**Theorem 0.2.** *Let $X$ be a stable curve over an algebraically closed field $k$ and $\Gamma_X$ the dual graph of $X$. Then $X$ is a pro-$\ell$ period matrix degenerate curve (cf. Definition 2.6) if and only if the maximal untangled subgraph $\Gamma_X^{\varnothing}$ (cf. Definition 2.8) of $\Gamma_X$ is not a tree (i.e., $r(\Gamma_X^{\varnothing}) := \mathrm{rank}(\mathrm{H}^1(\Gamma_X^{\varnothing}, \mathbb{Z})) \neq 0$).*

The present paper is organized as follows. In Section 1, we recall some basic facts concerning log structures and log étale fundamental groups of stable curves. In Section 2, we discuss the topic of degeneracy of pro-$\ell$ period matrices of stable curves and prove Theorem 0.2. Finally, we explain the relationship between Theorem 0.2 and the weight-monodromy conjecture.