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ASYMPTOTIC ANALYSIS OF GI/G/1-TYPE MARKOV CHAINS

by

TATSUAKI KIMURA

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Preface

The GI/G/1-type Markov chain is a mathematical model for the analysis of various semi-Markovian queueing models. The GI/G/1-type Markov chain is a bivariate Markov chain consisting of two variables: the level and phase. The former represents the additive component such as the number of queuing systems while the latter corresponds to the background component such as system status. Several important classes of structured Markov chains, such as quasi-birth-and-death processes (QBDs), GI/M/1-type and M/G/1-type Markov chains are included in GI/G/1-type Markov chains. Although the stationary distributions of GI/G/1-type Markov chains can be obtained by matrix analytical methods, they cannot be expressed in analytical forms. Thus, it is difficult to compute the stationary distributions of the GI/G/1-type Markov chains in general. Therefore, the asymptotic analysis of the GI/G/1-type Markov chains has recently received considerable attention, for example, the tail asymptotic analysis and the heavy-traffic analysis. Such asymptotic analysis are useful not only for the computation and approximation of the stationary distributions but also for the sensitivity analysis of system parameters of queueing models.

The tail asymptotics of GI/G/1-type Markov chains can be divided into two cases: light-tailed asymptotics and subexponential asymptotics. The former has been studied by many researchers. However, most of the previous studies consider only the case where the tail decay rate is determined by a certain parameter associated with the transition block matrices in the non-boundary levels. In addition, these studies neglect those cases where the decay rate depends on other parameters, such as the convergence radius of the generating function of the transition block matrices in the boundary level. Contrary to the light-tailed asymptotics, less work has been done on the subexponential tail asymptotics. Most recently, Masuyama [42], Kim and Kim [29] presented a weaker sufficient condition for the asymptotic formula than those presented in the literature [7, 40, 63] although their results are limited to the M/G/1-type one. On the other hand, few researchers studied the heavy-traffic asymptotics of the GI/G/1- or M/G/1-type Markov chain. Asmussen [5] presented the heavy-traffic asymptotic formula for the GI/G/1-type Markov chain, in which the stationary distribution of the properly scaled level variable is geometric and independent of the phase variable.

In this thesis, we study the asymptotic analysis of the stationary probability vector of the GI/G/1-type Markov chain. Chapters 2 and 3 study the light-tailed asymptotics and the subexponential tail asymptotics, respectively. We extend the results for the M/G/1-type Markov chain in the previous studies to the GI/G/1-type Markov chain. We also derive new asymptotic formulae for the cases that have not been considered in the literature. In Chapter 4, we derive heavy-traffic limit formulae of the stationary distribution relaxing Asmussen [5]'s sufficient condition. We also derive a heavy-traffic asymptotic formula for the moment of the stationary distribution. To the best of our knowledge, this is the first report on the heavy-traffic asymptotics of the moments.

The main contribution of this thesis is a comprehensive asymptotic analysis of the stationary distribution of the GI/G/1-type Markov chain including new cases that have not been studied in the literature. Although there still remain several research topics in this area, the author hopes this thesis will help in further studies.

Tatsuaki Kimura
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Contents

Chapter 1

Introduction

This chapter provides materials to be required in the following chapters and a brief survey of previous works. Throughout this thesis, we denote matrices and vectors by bold capital letters and bold small letters, respectively, and the empty sum is defined as zero.

1.1 Definition of GI/G/1-type Markov chains

In this section, we describe the GI/G/1-type Markov chain and provide some necessary definitions and assumptions for the subsequent chapters.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. Let $\{(X_n, S_n); n \in \mathbb{Z}_+\}$ denote a discrete-time Markov chain with state space $\mathbb{S} := (\{0\} \times \mathbb{M}_0) \cup (\mathbb{N} \times \mathbb{M})$ such that

$$\begin{aligned} &\mathsf{P}(X_{n+1} = \ell, S_{n+1} = j \mid X_n = k, S_n = i) \\ &\quad = \begin{cases} A_{i,j}(\ell - k), & k \in \mathbb{N}, \quad \ell \ge -k + 1, \quad i \in \mathbb{M}, \quad j \in \mathbb{M}, \\ B_{i,j}(-k), & k \in \mathbb{N}, \quad \ell = -k, \quad i \in \mathbb{M}, \quad j \in \mathbb{M}_0, \\ B_{i,j}(0), & k = 0, \quad \ell = 0, \quad i \in \mathbb{M}_0, \quad j \in \mathbb{M}_0, \\ B_{i,j}(\ell), & k = 0, \quad \ell \in \mathbb{N}, \quad i \in \mathbb{M}_0, \quad j \in \mathbb{M}, \end{cases} \end{aligned} \tag{1.1}$$

where $\mathbb{M}_0 = \{1, 2, \dots, M_0\} \subset \mathbb{N}$ and $\mathbb{M} = \{1, 2, \dots, M\} \subset \mathbb{N}$. We call $\{(X_n, S_n); n \in \mathbb{Z}_+\}$ the *GI/G/1-type Markov chain*. We also call X_n and S_n *level variable* and *phase variable*, respectively.

Let $\boldsymbol{T}$ denote the transition probability matrix of the GI/G/1-type Markov chain $\{(X_n, S_n)\}$ with the transition law (1.1). We then define the block matrices of $\boldsymbol{T}$:

$$\begin{aligned} \boldsymbol{A}(k) &= (A_{i,j}(k))_{(i,j) \in \mathbb{M}^2}, \quad k \in \mathbb{Z}, \\ \boldsymbol{B}(0) &= (B_{i,j}(0))_{(i,j) \in \mathbb{M}_0^2}, \\ \boldsymbol{B}(k) &= (B_{i,j}(k))_{(i,j) \in \mathbb{M}_0 \times \mathbb{M}}, \quad \boldsymbol{B}(-k) = (B_{i,j}(-k))_{(i,j) \in \mathbb{M} \times \mathbb{M}_0}, \quad k \in \mathbb{N}. \end{aligned}$$

Let $\mathbb{L}(0) = \{(0, i); i \in \mathbb{M}_0\} \subset \mathbb{S}$ and $\mathbb{L}(k) = \{(k, i); i \in \mathbb{M}\} \subset \mathbb{S}$ for $k \in \mathbb{N}$. We call $\mathbb{L}(k)$ ($k \in \mathbb{Z}_+$) *level*

k. Using the block matrices $\boldsymbol{A}(k)$'s and $\boldsymbol{B}(k)$'s, we can express $\boldsymbol{T}$ as follows:

$$\boldsymbol{T} = \begin{array}{c} \\ \mathbb{L}(0) \\ \mathbb{L}(1) \\ \mathbb{L}(2) \\ \mathbb{L}(3) \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{L}(0) & \mathbb{L}(1) & \mathbb{L}(2) & \mathbb{L}(3) & \cdots \end{array} \\ \left(\begin{array}{ccccc} \boldsymbol{B}(0) & \boldsymbol{B}(1) & \boldsymbol{B}(2) & \boldsymbol{B}(3) & \cdots \\ \boldsymbol{B}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{B}(-2) & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \boldsymbol{B}(-3) & \boldsymbol{A}(-2) & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) \end{array}, \tag{1.2}$$

where

$$\sum_{\ell=0}^{\infty} \boldsymbol{B}(\ell)\boldsymbol{e} = \boldsymbol{e}, \tag{1.3}$$

$$\boldsymbol{B}(-k)\boldsymbol{e} + \sum_{\ell=-k+1}^{\infty} \boldsymbol{A}(\ell)\boldsymbol{e} = \boldsymbol{e}, \qquad k \in \mathbb{N}. \tag{1.4}$$

Note here that $\boldsymbol{e}$ denotes a column vector of ones with an appropriate order according to the context.

For later use, let $\boldsymbol{\pi} > \boldsymbol{0}$ denote a left eigenvector of $\boldsymbol{A}$ such that $\boldsymbol{\pi A} = \mathrm{sp}(\boldsymbol{A})\boldsymbol{\pi}$ and $\boldsymbol{\pi e} = 1$ (see Theorem 8.4.4 in [26]). Let σ denote

$$\sigma = -\boldsymbol{\pi} \sum_{k\in\mathbb{Z}} k\boldsymbol{A}(k)\boldsymbol{e}. \tag{1.5}$$

If $\boldsymbol{A}$ is stochastic, then $\boldsymbol{\pi}$ is the unique invariant probability vector of $\boldsymbol{A}$ and $-\sigma$ is the conditional mean drift of the level process $\{X_n; n \in \mathbb{Z}_+\}$ with $X_n \geq 1$.

We now make the following assumption.

Assumption 1.1 *(a) $\boldsymbol{T}$ is irreducible and stochastic;*

(b) $\boldsymbol{A} := \sum_{k\in\mathbb{Z}} \boldsymbol{A}(k)$ is irreducible; and

(c) $\boldsymbol{T}$ is positive recurrent.

Under Assumption 1.1, $\boldsymbol{T}$ has the unique and positive stationary probability vector [8, Chapter XI, Proposition 3.1]. We define $\boldsymbol{x} = (x_i(k))_{(k,i)\in\mathbb{S}} > \boldsymbol{0}$ as the stationary probability vector of $\boldsymbol{T}$ and then partition it level-wise, i.e., $\boldsymbol{x} = (\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \dots)$, where $\boldsymbol{x}(\ell) = (x_i(k))_{(k,i)\in\mathbb{L}(\ell)}$ for $\ell \in \mathbb{Z}_+$. Furthermore, for $k = 0, 1, \dots$, let $\overline{\boldsymbol{x}}(k) = \sum_{\ell=k+1}^{\infty} \boldsymbol{x}(\ell)$, which is a positive vector. We call $\overline{\boldsymbol{x}}(k)$'s stationary tail probability vectors of $\boldsymbol{T}$ hereafter.

Remark 1.1 *Note that $\boldsymbol{A}$ is not stochastic in general. In fact, it follows from (1.2) that $\boldsymbol{A}$ is strictly substochastic, i.e., ($\boldsymbol{Ae} \leq \boldsymbol{e}, \neq \boldsymbol{e}$) if and only if*

$$\lim_{k\to\infty} \boldsymbol{B}(-k) \neq \boldsymbol{O},$$

where $\boldsymbol{O}$ denotes a matrix of zeros with an appropriate dimension. Furthermore, if $\boldsymbol{A}$ is irreducible, then $\mathrm{sp}(\boldsymbol{A}) < 1$ if and only if $\boldsymbol{A}$ is strictly substochastic.

1.2 GI/G/1 paradigm

In this section, we describe the GI/G/1 paradigm introduced by Grassmann and Heyman [25]. By considering the censored Markov chain of the original GI/G/1-type Markov chain $\{(X_n, S_n)\}$, we introduce R- and G- matrices and related results, which play a key role throughout this thesis.

For later use, we explain conventions used throughout this thesis. For any matrix $\boldsymbol{X}$, let $[\boldsymbol{X}]_{i,j}$ denote the (i,j)th element of $\boldsymbol{X}$ and $|\boldsymbol{X}|$ denote a matrix such that $[|\boldsymbol{X}|]_{i,j} = |[\boldsymbol{X}]_{i,j}|$. For any square matrix $\boldsymbol{Y}$, let $\mathrm{adj}(\boldsymbol{Y})$ denote the adjugate matrix of $\boldsymbol{Y}$ and $\delta(\boldsymbol{Y})$ denote a maximum-modulus eigenvalue of $\boldsymbol{Y}$ such that its argument $\arg \delta(\boldsymbol{Y})$ is nonnegative and its real part $\mathrm{Re}\, \delta(\boldsymbol{Y})$ is not less than those of the other eigenvalues of maximum modulus. Note here that if $\boldsymbol{Y}$ is nonsingular then $\boldsymbol{Y}^{-1} = \mathrm{adj}(\boldsymbol{Y})/\det(\boldsymbol{Y})$. Note also that if $\boldsymbol{Y}$ is nonnegative then $\delta(\boldsymbol{Y}) = \mathrm{sp}(\boldsymbol{Y})$ (see, e.g., [26, Theorem 8.3.1]) and that if $\boldsymbol{Y}$ is nonnegative and irreducible then $\delta(\boldsymbol{Y})$ is the Perron-Frobenius eigenvalue of $\boldsymbol{Y}$ (see, e.g., [10, Theorem 1.4.4]). Let $\boldsymbol{I}$ denote the identity matrix with an appropriate dimension. For any matrix sequence $\{\boldsymbol{M}(k); k \in \mathbb{Z}_+\}$, let $\overline{\boldsymbol{M}}(k) = \sum_{\ell=k+1}^{\infty} \boldsymbol{M}(\ell)$ for $k \in \mathbb{Z}_+$. Let $\{\boldsymbol{M} * \boldsymbol{N}(k); \in \mathbb{Z}_+\}$ denote the convolution of two matrix sequences $\{\boldsymbol{M}(k); k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{N}(k); k \in \mathbb{Z}_+\}$, i.e.,

$$\boldsymbol{M} * \boldsymbol{N}(k) = \sum_{\ell=0}^{k} \boldsymbol{M}(k-\ell)\boldsymbol{N}(\ell) = \sum_{\ell=0}^{k} \boldsymbol{M}(\ell)\boldsymbol{N}(k-\ell), \qquad k \in \mathbb{Z}_+,$$

where the product of $\boldsymbol{M}(k_1)$ and $\boldsymbol{N}(k_2)$ is well-defined for any $(k_1, k_2) \in \mathbb{Z}_+^2$. In addition, for any square matrix sequence $\{\boldsymbol{H}(k); k \in \mathbb{Z}_+\}$, let $\{\boldsymbol{H}^{*n}(k)\}$ denote the n-fold convolution of $\{\boldsymbol{H}(k)\}$ with itself, i.e., $\boldsymbol{H}^{*1}(k) = \boldsymbol{H}(k)$ ($k \in \mathbb{Z}_+$) and for $n = 2, 3, \dots,$

$$\boldsymbol{H}^{*n}(k) = \boldsymbol{H} * \boldsymbol{H}^{*(n-1)}(k), \quad k \in \mathbb{Z}_+.$$

For convenience, $\boldsymbol{H}^{*0}(0) = \boldsymbol{I}$ and $\boldsymbol{H}^{*0}(k) = \boldsymbol{O}$ for all $k \in \mathbb{N}$. The above conventions for matrices are used for vectors and scalars in an appropriate manner. Finally, the superscript "$\top$" represents the transpose operator for vectors and matrices.

For any fixed $k \in \mathbb{Z}_+$, we partition $\boldsymbol{T}$ at level k, as follows:

$$\boldsymbol{T} = \begin{array}{c} \\ \cup_{\ell=0}^{k}\mathbb{L}(\ell) \\ \cup_{\ell=k+1}^{\infty}\mathbb{L}(\ell) \end{array} \begin{array}{c} \begin{array}{cc} \cup_{\ell=0}^{k}\mathbb{L}(\ell) & \cup_{\ell=k+1}^{\infty}\mathbb{L}(\ell) \end{array} \\ \left(\begin{array}{cc} \boldsymbol{T}^{\leqslant k} & \boldsymbol{U}^{[k]} \\ \boldsymbol{D}^{[k]} & \boldsymbol{T}^{>k} \end{array} \right) \end{array},$$

where $\boldsymbol{T}^{\leqslant k}$ (resp. $\boldsymbol{T}^{>k}$) is the transient transition probability matrix of an absorbing Markov chain restricted to the levels $0, 1, \dots, k$ (resp. $k+1, k+2, \dots$). We then define $\boldsymbol{T}^{[k]}$ ($k \in \mathbb{Z}_+$) as

$$\boldsymbol{T}^{[k]} = \boldsymbol{T}^{\leqslant k} + \boldsymbol{U}^{[k]}(\boldsymbol{I} - \boldsymbol{T}^{>k})^{-1}\boldsymbol{D}^{[k]}. \tag{1.6}$$

Note here that $\boldsymbol{T}^{[k]}$ is the transition probability matrix of the censored Markov chain obtained by observing $\{(X_n, S_n)\}$ only when it is in levels 0 through k. Note also that $\boldsymbol{T}^{[k]}$ is an irreducible stochastic matrix because the original transition probability matrix $\boldsymbol{T}$ is irreducible and recurrent (see Assumption 1.1). Therefore, $\boldsymbol{T}^{[k]}$ has an unique invariant measure up to scalar multiples and

$$(\boldsymbol{x}(0), \boldsymbol{x}(1), \dots, \boldsymbol{x}(k))\boldsymbol{T}^{[k]} = (\boldsymbol{x}(0), \boldsymbol{x}(1), \dots, \boldsymbol{x}(k)). \tag{1.7}$$

Let $\boldsymbol{T}^{[k]}_{\nu,\eta}$ ($\nu,\eta \in \{0,1,\dots,k\}$) denote a submatrix of $\boldsymbol{T}^{[k]}$, whose (i,j)th element represents the probability that the censored Markov chain moves from state $(\nu,i) \in \mathbb{S}$ to $(\eta,j) \in \mathbb{S}$ in one step. The structure (1.2) of $\boldsymbol{T}$ implies that $\boldsymbol{T}^{[k]}_{k-\ell,k}$ and $\boldsymbol{T}^{[k]}_{k,k-\ell}$ are independent of k if $\ell \in \{0,1,\dots,k-1\}$ and $k \in \mathbb{N}$. Thus, for simplicity, we define $\boldsymbol{\Phi}(\ell)$ ($\ell \in \mathbb{Z}$) as

$$
\begin{aligned}
\boldsymbol{\Phi}(\ell) &= \boldsymbol{T}^{[k]}_{k-\ell,k}, \qquad \ell \in \{0,1,\dots,k-1\},\ k \in \mathbb{N},\\
\boldsymbol{\Phi}(-\ell) &= \boldsymbol{T}^{[k]}_{k,k-\ell}, \qquad \ell \in \{0,1,\dots,k-1\},\ k \in \mathbb{N}.
\end{aligned} \tag{1.8}
$$

Note here that for any fixed $\nu \in \mathbb{N}$, $[\boldsymbol{\Phi}(0)]_{i,j}$ represents the probability of hitting state (ν,j) for the first time before entering the levels $0,1,\dots,\nu-1$, given that it starts with state (ν,i), i.e.,

$$
[\boldsymbol{\Phi}(0)]_{i,j} = \mathsf{P}(S_{T_{\downarrow\nu}} = j \mid X_0 = \nu, S_0 = i),
$$

where $T_{\downarrow\ell} = \inf\{n \in \mathbb{N}; X_n = \ell < X_m\ (m = 1,2,\dots,n-1)\}$. Thus $\sum_{n=0}^{\infty}(\boldsymbol{\Phi}(0))^n = (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}$ exists because $\boldsymbol{T}^{[k]}$ is irreducible. The following result characterizes the matrices $\{\boldsymbol{\Phi}(k)\}$.

Proposition 1.1 (Theorem 1 in [25]) *$\{\boldsymbol{\Phi}(k); k \in \mathbb{Z}\}$ is the minimal nonnegative solution of the following equations.*

$$
\begin{aligned}
\boldsymbol{\Phi}(k) &= \boldsymbol{A}(k) + \sum_{m=1}^{\infty} \boldsymbol{\Phi}(k+m)(\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}\boldsymbol{\Phi}(-m), \qquad k \in \mathbb{Z}_+,\\
\boldsymbol{\Phi}(-k) &= \boldsymbol{A}(-k) + \sum_{m=1}^{\infty} \boldsymbol{\Phi}(m)(\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}\boldsymbol{\Phi}(-k-m), \qquad k \in \mathbb{Z}_+.
\end{aligned}
$$

Remark 1.2 *The proof of Theorem 1 in [25] is based on induction and probabilistic interpretation, which are valid without the recurrence of $\boldsymbol{T}$.*

Let $\boldsymbol{G}$ and $\boldsymbol{G}(k)$ ($k \in \mathbb{N}$) denote

$$
\boldsymbol{G} = \sum_{k=1}^{\infty} \boldsymbol{G}(k), \qquad \boldsymbol{G}(k) = (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}\boldsymbol{\Phi}(-k), \qquad k \in \mathbb{N}, \tag{1.9}
$$

respectively. Note that for any fixed $\nu \in \mathbb{N}$, $[\boldsymbol{G}(k)]_{i,j}$ represents the probability of hitting state (ν,j) when the Markov chain $\{(X_n,S_n)\}$ enters the levels $0,1,\dots,\nu+k-1$ for the first time, given that it starts with state $(\nu+k,i)$, i.e.,

$$
[\boldsymbol{G}(k)]_{i,j} = \mathsf{P}(X_{T_{<k+\nu}} = \nu, S_{T_{<k+\nu}} = j \mid X_0 = k+\nu, S_0 = i), \qquad k \in \mathbb{N},
$$

where $T_{<\ell} = \inf\{n \in \mathbb{N}; X_n < \ell \le X_m\ (m = 1,2,\dots,n-1)\}$. For convenience, let $\boldsymbol{G}(0) = \boldsymbol{O}$. Furthermore, let $\boldsymbol{L}(k)$ ($k \in \mathbb{Z}_+$) denote

$$
\boldsymbol{L}(k) = \sum_{n=0}^{\infty} \boldsymbol{G}^{*n}(k), \qquad k \in \mathbb{Z}_+, \tag{1.10}
$$

where $\boldsymbol{L}(0) = \boldsymbol{I}$. Note that $[\boldsymbol{L}(k)]_{i,j}$ represents the probability that, for any fixed $n \ge k+1$, the first passage time to $\sqcup_{m=0}^{n-k}\mathbb{L}(m)$ ends with state $(n-k,j)$ starting with state (n,i), i.e.,

$$
[\boldsymbol{L}(k)]_{i,j} = \mathsf{P}(S_{T_{\downarrow\nu}} = j \mid X_0 = k+\nu, S_0 = i).
$$

It follows from (1.10) that

$$\widehat{\boldsymbol{L}}(z) := \sum_{k=0}^{\infty} z^{-k}\boldsymbol{L}(k) = (\boldsymbol{I} - \widehat{\boldsymbol{G}}(z))^{-1}, \qquad |z| \ge 1, \tag{1.11}$$

where $\widehat{\boldsymbol{G}}(z) = \sum_{k=0}^{\infty} z^{-k}\boldsymbol{G}(k)$.

We now define $\boldsymbol{R}_0(k)$ and $\boldsymbol{R}(k)$ ($k \in \mathbb{Z}_+$) denote $M_0 \times M$ and $M \times M$ matrices, respectively, such that

$$\begin{aligned} \boldsymbol{R}(k) &= \boldsymbol{\Phi}(k)(\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}, & (1.12)\\ \boldsymbol{R}_0(k) &= \boldsymbol{T}_{0,k}^{[k]}(\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}, & (1.13) \end{aligned}$$

For convenience, let $\boldsymbol{R}(0) = \boldsymbol{O}$ and $\boldsymbol{R}_0(0) = \boldsymbol{O}$. In addition, $\boldsymbol{R}^{*0}(0) = \boldsymbol{I}$ and $\boldsymbol{R}^{*0}(k) = \boldsymbol{O}$ for all $k \in \mathbb{N}$. For any fixed $n \in \mathbb{N}$, $[\boldsymbol{R}(k)]_{i,j}$ represents the expected number of visits to state $(n+k, j)$ starting with state (n, i) and until entering to $\sqcup_{m=0}^{n+k-1}\mathbb{L}(m)$. We also define $\boldsymbol{R}_0(k)$ ($k \in \mathbb{N}$) as an $M_0 \times M$ matrix such that $[\boldsymbol{R}_0(k)]_{i,j}$ represents the expected number of visits to state (k, j) starting with state $(0, i)$ and until entering to $\sqcup_{m=0}^{k-1}\mathbb{L}(m)$. Formally, for $k \in \mathbb{N}$,

$$\begin{aligned} [\boldsymbol{R}_0(k)]_{i,j} &= \mathsf{E}\left[\sum_{n=1}^{T_{<k}} \mathbb{1}(X_n = k, S_n = j) \,\middle|\, X_0 = 0, S_0 = i \right], \\ [\boldsymbol{R}(k)]_{i,j} &= \mathsf{E}\left[\sum_{n=1}^{T_{<k+\nu}} \mathbb{1}(X_n = k+\nu, S_n = j) \,\middle|\, X_0 = \nu \in \mathbb{N}, S_0 = i \right], \end{aligned}$$

where $\mathbb{1}(\chi)$ denotes the indicator function of an event χ. It follows from the definitions of $\boldsymbol{R}_0(k)$, $\boldsymbol{R}(k)$, $\boldsymbol{L}(k)$ and $\boldsymbol{\Phi}(0)$ that

$$\begin{aligned} \boldsymbol{R}_0(k) &= \left[\boldsymbol{B}(k) + \sum_{m=1}^{\infty} \boldsymbol{B}(k+m)\boldsymbol{L}(m) \right] (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}, \quad k \in \mathbb{N}, & (1.14)\\ \boldsymbol{R}(k) &= \left[\boldsymbol{A}(k) + \sum_{m=1}^{\infty} \boldsymbol{A}(k+m)\boldsymbol{L}(m) \right] (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}, \quad k \in \mathbb{N}, & (1.15) \end{aligned}$$

which hold without the recurrence of $\boldsymbol{T}$. We now define $\widehat{\boldsymbol{R}}_0(z)$, $\widehat{\boldsymbol{R}}(z)$ and $\widehat{\boldsymbol{B}}(z)$ as

$$\widehat{\boldsymbol{R}}_0(z) = \sum_{k=1}^{\infty} z^k \boldsymbol{R}_0(k), \quad \widehat{\boldsymbol{R}}(z) = \sum_{k=1}^{\infty} z^k \boldsymbol{R}(k), \quad \widehat{\boldsymbol{B}}(z) = \sum_{k=1}^{\infty} z^k \boldsymbol{B}(k),$$

respectively. Propositions 1.2 and 1.3 below show the connection between the generating functions $\widehat{\boldsymbol{B}}(z)$ and $\widehat{\boldsymbol{R}}_0(z)$, and that between $\widehat{\boldsymbol{A}}(z)$, $\widehat{\boldsymbol{R}}(z)$ and $\widehat{\boldsymbol{G}}(z)$, which play an important role in this thesis.

Proposition 1.2 (Theorem 1 and Lemma 3 in [41]) *Let r_{R_0}, r_R, r_G, r_{A_+}, r_{A_-} and r_B denote the convergence radii of $\widehat{\boldsymbol{R}}_0(z)$, $\widehat{\boldsymbol{R}}(z)$, $\widehat{\boldsymbol{G}}(1/z) = \sum_{k=1}^{\infty} z^k\boldsymbol{G}(k)$, $\sum_{k=1}^{\infty} z^k\boldsymbol{A}(k)$, $\sum_{k=1}^{\infty} z^k\boldsymbol{A}(-k)$ and $\widehat{\boldsymbol{B}}(z)$, respectively. It then holds that $r_{R_0} = r_B \ge 1$, $r_R = r_{A_+} \ge 1$ and $r_G = r_{A_-} \ge 1$.*

Proposition 1.3 (Zhao et al. [68], Theorem 14) *If Assumption 1.1 (a) and (b) hold, then*

$$\boldsymbol{I} - \widehat{\boldsymbol{A}}(z) = (\boldsymbol{I} - \widehat{\boldsymbol{R}}(z))(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widehat{\boldsymbol{G}}(z)), \qquad |z| \in I_A, \tag{1.16}$$

where $I_A = \{z \ge 0; \sum_{k\in\mathbb{Z}} z^k \boldsymbol{A}(k)$ *is finite*$\} \supseteq (r_{A_-}, r_{A_+})$*. Equation (1.16) is called the RG-factorization of* $\widehat{\boldsymbol{A}}(z)$*.*

Remark 1.3 *Propositions 1.2 and 1.3 do not necessarily require Assumption 1.1 (c).*

The following proposition characterizes $\boldsymbol{R}$ and $\boldsymbol{G}$ when $\boldsymbol{A}$ is strictly substochastic.

Proposition 1.4 *Let* $\boldsymbol{R} = \sum_{k=1}^{\infty} \boldsymbol{R}(k)$*. If* $\boldsymbol{A}$ *is irreducible and strictly substochastic, then (i)* $\mathrm{sp}(\boldsymbol{G}) < 1$*; (ii)* $\mathrm{sp}(\boldsymbol{R}) < 1$*; and (iii)* $\mathrm{sp}(\sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell)) < 1$*, where* $\mathrm{sp}(\,\cdot\,)$ *denotes the spectral radius of a matrix in parentheses.*

Proof. Equation (1.16) yields

$$\det(\boldsymbol{I} - \boldsymbol{A}) = \det(\boldsymbol{I} - \boldsymbol{R}) \det(\boldsymbol{I} - \boldsymbol{\Phi}(0)) \det(\boldsymbol{I} - \boldsymbol{G}).$$

It thus follows from $\mathrm{sp}(\boldsymbol{A}) < 1$ that

$$\det(\boldsymbol{I} - \boldsymbol{G}) \neq 0, \quad \det(\boldsymbol{I} - \boldsymbol{R}) \neq 0. \tag{1.17}$$

Note here that by definition,

$$\sum_{k=1}^{N} \sum_{j\in\mathbb{M}} [\boldsymbol{G}(k)]_{i,j} = \mathsf{P}(T_{<N} < \infty \mid X_0 = N, S_0 = i), \quad \text{for all } N \in \mathbb{N},$$

which shows that $\boldsymbol{G}\boldsymbol{e} \le \boldsymbol{e}$ and thus $\mathrm{sp}(\boldsymbol{G}) \le 1$ (see Theorem 8.1.22 in [26]). Furthermore, $\mathrm{sp}(\boldsymbol{R}) \le 1$ due to the duality of the R- and G-matrices (see [67]). Therefore, it follows from Theorem 8.3.1 in [26] and (1.17) that (i) $\mathrm{sp}(\boldsymbol{G}) < 1$ and (ii) $\mathrm{sp}(\boldsymbol{R}) < 1$.

Finally, we prove (iii). From (1.8), we have

$$\boldsymbol{\Phi}(-k) \ge \boldsymbol{O}, \quad \boldsymbol{0} \le \sum_{\ell=0}^{k-1} \boldsymbol{\Phi}(-\ell)\boldsymbol{e} \le \boldsymbol{e}, \quad \text{for all } k \in \mathbb{N},$$

which implies that $\mathrm{sp}(\sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell)) \le 1$ (see Theorem 8.1.22 in [26]). Thus it suffices to prove that $\sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell)$ does not have the eigenvalue *one*. Indeed, (1.9) yields

$$(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \boldsymbol{G}) = \boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell).$$

Therefore we have $\det(\boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell)) \neq 0$ because $\boldsymbol{I} - \boldsymbol{\Phi}(0)$ is nonsingular and $\mathrm{sp}(\boldsymbol{G}) < 1$. □

1.3 Matrix-product form of stationary distribution

This section discusses the stationary distribution $\{\boldsymbol{x}(k)\}$ under Assumption 1.1. It follows from (1.7) that

$$\boldsymbol{x}(k) = \left[\boldsymbol{x}(0)\boldsymbol{T}_{0,k}^{[k]} + \sum_{\ell=1}^{k-1}\boldsymbol{x}(\ell)\boldsymbol{\Phi}(k-\ell)\right](\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}, \qquad k \in \mathbb{N}, \tag{1.18}$$

In terms of $\boldsymbol{R}(k)$ and $\boldsymbol{R}_0(k)$, we can rewrite (1.18) as

$$\boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{R}_0(k) + \sum_{\ell=1}^{k}\boldsymbol{x}(\ell)\boldsymbol{R}(k-\ell), \qquad k \in \mathbb{N}, \tag{1.19}$$

where we use $\boldsymbol{R}(0) = \boldsymbol{O}$. It then follows from (1.19) that

$$\boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{R}_0 * \boldsymbol{F}(k), \qquad k \in \mathbb{N}, \tag{1.20}$$

where $\boldsymbol{F}(k)$ $(k \in \mathbb{Z}_+)$ is given by

$$\boldsymbol{F}(k) = \sum_{n=0}^{\infty}\boldsymbol{R}^{*n}(k). \tag{1.21}$$

Thus, $\overline{\boldsymbol{x}}(k)$ is given by

$$\overline{\boldsymbol{x}}(k) = \boldsymbol{x}(0)\overline{\boldsymbol{R}_0 * \boldsymbol{F}}(k), \qquad k \in \mathbb{Z}_+. \tag{1.22}$$

It follows from Theorem 23 of [68] that if T is positive recurrent then $\delta(\widehat{\boldsymbol{R}}(1)) < 1$ and thus (see [26, Theorem 8.1.18])

$$|\delta(\widehat{\boldsymbol{R}}(z))| \le \delta(|\widehat{\boldsymbol{R}}(z)|) \le \delta(\widehat{\boldsymbol{R}}(|z|)) \le \delta(\widehat{\boldsymbol{R}}(1)) < 1, \quad |z| \le 1. \tag{1.23}$$

This implies that $\boldsymbol{I} - \widehat{\boldsymbol{R}}(z)$ is nonsingular for $|z| \le 1$. Therefore, from (1.21), we obtain

$$\widehat{\boldsymbol{F}}(z) := \sum_{k=0}^{\infty} z^k \boldsymbol{F}(k) = (\boldsymbol{I} - \boldsymbol{R}(z))^{-1}, \qquad |z| \le 1. \tag{1.24}$$

Furthermore, let $\widehat{\boldsymbol{x}}(z) = \sum_{k=1}^{\infty} z^k \boldsymbol{x}(k)$. Combining (1.24) and (1.20) yields

$$\widehat{\boldsymbol{x}}(z) = \boldsymbol{x}(0)\widehat{\boldsymbol{R}}_0(z)(\boldsymbol{I} - \widehat{\boldsymbol{R}}(z))^{-1}, \qquad |z| \le 1. \tag{1.25}$$

Letting $z = 1$ in (1.25), we have

$$\overline{\boldsymbol{x}}(0) = \boldsymbol{x}(0)\boldsymbol{R}_0(\boldsymbol{I} - \boldsymbol{R})^{-1}, \tag{1.26}$$

where $\boldsymbol{R} = \widehat{\boldsymbol{R}}(1)$ and $\boldsymbol{R}_0 = \widehat{\boldsymbol{R}}_0(1)$.

We now define $\boldsymbol{\kappa} = (\kappa_i)_{i \in \mathbb{M}_0} > \boldsymbol{0}$ as the (unique) stationary probability vector of the irreducible stochastic matrix $\boldsymbol{T}^{[0]}$. We then have

$$\boldsymbol{x}(0) = \frac{1}{\sum_{i \in \mathbb{M}_0} \kappa_i \mathsf{E}[\tau_0 \mid X_0 = 0, S_0 = i]} \cdot \boldsymbol{\kappa}, \tag{1.27}$$

where $\tau_0 = \inf\{n \in \mathbb{N}; X_n = 0\}$.

1.4 Period of MAdP associated with GI/G/1-type Markov chain

In this section, we consider a MAdP $\{(\breve{X}_n, \breve{S}_n); n \in \mathbb{Z}_+\}$ with state space $\mathbb{Z} \times \mathbb{M}$ and kernel $\{\boldsymbol{A}(k); k \in \mathbb{Z}\}$. Let τ denote the period of the MAdP $\{(\breve{X}_n, \breve{S}_n)\}$ (see Definition B.1 in Appendix B). The period of MAdP has strong relationship with the asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ in many cases. Thus, we provide several important results related to the period of MAdP as preliminaries in this section. More detailed explanation and the proofs of the results are summarized in Appendix B.

For simplicity, we write $(k_1, j_1) \to (k_2, j_2)$ when there exists a path from state (k_1, j_1) to state (k_2, j_2) with some positive probability. In this section, we set the following condition:

Condition 1.1

(a) $\boldsymbol{A}$ *is irreducible (Assumption 1.1 (b)); and*

(b) for each $i \in \mathbb{M}$ *there exists some* $k_i \in \mathbb{Z}\backslash\{0\}$ *such that* $(0, i) \to (k_i, i)$, *or equivalently,* $\sum_{n=1}^{\infty} A_{i,i}^{*n}(k_i) > 0$, *where* $\{\boldsymbol{A}^{*n}(k) := (A_{i,j}^{*n}(k))_{i,j\in\mathbb{M}}; k \in \mathbb{Z}\}$ *denotes the* n*-fold convolution of* $\{\boldsymbol{A}(k); k \in \mathbb{Z}\}$, *i.e.,* $\boldsymbol{A}^{*1}(k) = \boldsymbol{A}(k)$ *and* $\boldsymbol{A}^{*n}(k) = \sum_{\ell\in\mathbb{Z}} \boldsymbol{A}^{*(n-1)}(k - \ell)\boldsymbol{A}(\ell)$ *for* $n \ge 2$.

Under Condition 1.1, the period of MAdP τ is well-defined (see Definition B.1), and τ is the largest positive integer such that

$$[\boldsymbol{A}(k)]_{i,j} > 0 \text{ only if } k \equiv p(j) - p(i) \pmod{\tau}, \tag{1.28}$$

where p is some function from $\mathbb{M}$ to $\{0, 1, \dots, \tau - 1\}$ (see Lemma B.2).

Example 1.1 *We suppose*

$$\begin{aligned}
\boldsymbol{A}(0) &= \boldsymbol{O}, \quad \boldsymbol{A}(1) = \begin{pmatrix} 0 & 0 & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}, \\
\boldsymbol{A}(-2) &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \boldsymbol{A}(-1) = \begin{pmatrix} 0 & 0 & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}.
\end{aligned}$$

Let $p(0) = p(1) = 1$ *and* $p(2) = 0$. *It then follows that*

$$[\boldsymbol{A}(k)]_{i,j} > 0 \textit{ only if } k \equiv p(j) - p(i) \; (\textit{mod } 2),$$

and thus the period of MAdP with kernel $\{\boldsymbol{A}(k)\}$ *is equal to two.*

Let $\boldsymbol{\mu}(z)$ and $\boldsymbol{v}(z)$ denote left- and right-eigenvectors of $\widehat{\boldsymbol{A}}(z)$ corresponding to an eigenvalue $\delta(\widehat{\boldsymbol{A}}(z))$, which are normalized such that

$$\boldsymbol{\mu}(z)\boldsymbol{\Delta}_M(z/|z|)\boldsymbol{e} = 1, \qquad \boldsymbol{\mu}(z)\boldsymbol{v}(z) = 1, \tag{1.29}$$

where $\boldsymbol{\Delta}_M(z)$ denotes an $M \times M$ diagonal matrix whose ith ($i \in \mathbb{M}$) diagonal element is equal to $z^{-p(i)}$.

Remark 1.4 *For $z \in I_A \setminus \{0\}$, $\widehat{\boldsymbol{A}}(z)$ is a nonnegative irreducible matrix and thus $\delta(\widehat{\boldsymbol{A}}(z))$ is the Perron-Frobenius eigenvalue with the left and right eigenvectors $\boldsymbol{\mu}(z) > \boldsymbol{0}$ and $\boldsymbol{v}(z) > \boldsymbol{0}$. The positivity of $\boldsymbol{\mu}(z)$ and $\boldsymbol{v}(z)$ follows from the Perron-Frobenius theorem (see, e.g., [10, Theorem 1.4.4]) and the normalizing condition (1.29). Furthermore, if $\delta(\boldsymbol{A}) = \delta(\widehat{\boldsymbol{A}}(1)) = 1$, i.e., $\boldsymbol{A}$ is stochastic, then $\boldsymbol{\mu}(1) = \boldsymbol{\pi}$ and $\boldsymbol{v}(1) = \boldsymbol{e}$.*

Propositions 1.5 and 1.6 below are the fundamental results on the connection between the period τ of MAdP $\{(\breve{X}_n, \breve{S}_n); n \in \mathbb{Z}_+\}$ and the maximum-modulus eigenvalues of the generating function of the kernel $\{\boldsymbol{A}(k); k \in \mathbb{Z}\}$. These results are used to prove Lemma 1.4, presented in the next section.

Proposition 1.5 *Let $\omega_x = \exp(2\pi\sqrt{-1}/x)$ for $x \ge 1$. If Condition 1.1 is satisfied, then the following hold for all $y \in I_A$ and $\nu = 0, 1, \dots, \tau - 1$.*

(i) $\delta(\widehat{\boldsymbol{A}}(y\omega_\tau^\nu)) = \delta(\widehat{\boldsymbol{A}}(y))$*, both of which are simple eigenvalues.*

(ii) $\boldsymbol{\mu}(y\omega_\tau^\nu) = \boldsymbol{\mu}(y)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}$ *and* $\boldsymbol{v}(y\omega_\tau^\nu) = \boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(y)$.

Proposition 1.6 *Suppose that Condition 1.1 is satisfied and $\delta(\widehat{\boldsymbol{A}}(y)) = 1$ for some $y \in I_A$. Let ω denote an arbitrary complex number such that $|\omega| = 1$. Under these conditions, $\delta(\widehat{\boldsymbol{A}}(y\omega)) = 1$ if and only if $\omega^\tau = 1$. Therefore,*

$$\tau = \max\{n \in \mathbb{N}; \delta(\widehat{\boldsymbol{A}}(y\omega_n)) = 1\}.$$

Furthermore, if $\delta(\widehat{\boldsymbol{A}}(y\omega)) = 1$, the eigenvalue is simple.

Finally, we present a lemma, which provides a sufficient condition under which Condition 1.1 (b) holds. It also implies that the period τ is well-defined under Assumption 1.1 and $\boldsymbol{A}$ is stochastic.

Lemma 1.1 *Suppose that Assumption 1.1 holds. If $\boldsymbol{A}$ is stochastic, then Condition 1.1 (b) holds.*

Proof. We prove this lemma by contradiction. To this end, we assume that the negation of Condition 1.1 (b), i.e., there exists some $i_0 \in \mathbb{M}$ such that

$$(0, i_0) \to (k, i_0) \quad \text{only if } k = 0. \tag{1.30}$$

Since $\boldsymbol{A}$ is an irreducible stochastic matrix,

$$\sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} \boldsymbol{A}^{*n}(k) = \sum_{n=1}^{\infty} \boldsymbol{A}^n > \boldsymbol{O},$$

which diverges to infinity. Therefore, for each pair of phases $i, j \in \mathbb{M}$, there exists some $k_{i,j} \in \mathbb{Z}$ such that $(0, i) \to (k_{i,j}, j)$. Using this fact, we have the following path.

$$(0, i_0) \to (k_{i_0,j}, j) \to (k_{i_0,j} + k_{j,i_0}, i_0) \quad \text{for all } j \in \mathbb{M}. \tag{1.31}$$

Combining (1.30) and (1.31) leads to

$$k_{i_0,j} = -k_{j,i_0} \quad \text{for all } j \in \mathbb{M}. \tag{1.32}$$

Furthermore, it follows from (1.30) and (1.32) that, for each $j \in \mathbb{M}$, $k_{i_0,j}$ and thus k_{j,i_0} must be uniquely determined. Indeed, suppose that there exist some $j_0 \in \mathbb{M}$ and integer $k'_{i_0,j_0} \neq k_{i_0,j_0}$ such that $(0, i_0) \to (k'_{i_0,j_0}, j_0)$. We then have

$$(0, i_0) \quad \to \quad (k'_{i_0,j_0}, j_0) \to (k'_{i_0,j_0} + k_{j_0,i_0}, i_0).$$

Note here that (1.32) yields $k'_{i_0,j_0} + k_{j_0,i_0} = k'_{i_0,j_0} - k_{i_0,j_0} \neq 0$, which is the contradiction to (1.30).

The above argument shows that, for each $j \in \mathbb{M}$,

$$(0, i_0) \quad \to \quad (k, j) \quad \text{only if } k = k_{i_0,j},$$

which implies there exists some $K \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} [\boldsymbol{A}^{*n}(k)\boldsymbol{e}]_{i_0} = 0 \quad \text{for all } |k| \geq K, \tag{1.33}$$

where $[\,\cdot\,]_i$ denotes the ith element of the vector in square brackets.

We now note that $\boldsymbol{B}(k)\boldsymbol{e} + \sum_{\ell=k+1}^{\infty} \boldsymbol{A}(\ell)\boldsymbol{e} = \boldsymbol{e}$ for all $k \leq -1$. Thus, since $\boldsymbol{A}$ is stochastic,

$$\boldsymbol{B}(k)\boldsymbol{e} = \boldsymbol{e} - \sum_{\ell=k+1}^{\infty} \boldsymbol{A}(\ell)\boldsymbol{e} = \sum_{\ell=-\infty}^{k} \boldsymbol{A}(\ell)\boldsymbol{e} \quad \text{for all } k \leq -1. \tag{1.34}$$

From (1.33) and (1.34), we have

$$[\boldsymbol{B}(k)\boldsymbol{e}]_{i_0} = 0 \quad \text{for all } k \leq -K. \tag{1.35}$$

Equations (1.33) and (1.35) imply that level zero is not reachable from states $\{(k, i_0); k \geq K\}$. Note here that the Markov chain $\{(X_n, J_n); n \in \mathbb{Z}_+\}$ with transition probability matrix $\boldsymbol{T}$ follows the same law as the MAdP $\{(\breve{X}_n, \breve{S}_n); n \in \mathbb{Z}_+\}$ while the former Markov chain is away from level zero. Thus, (1.33) and (1.35) are inconsistent with the irreducibility of $\boldsymbol{T}$. Consequently, there exists no integer $i_0 \in \mathbb{M}$ such that (1.30) holds (i.e., Condition 1.1 (b) is true). $\square$

1.5 Roots of fundamental equation of MAdP

In this section, we study the characteristics of the solutions to the following equation:

$$\varphi(z) := \delta(\widehat{\boldsymbol{A}}(z)) - 1 = 0, \qquad r_{A_-} < |z| < r_{A_+}, \tag{1.36}$$

which is called *the fundamental equation of MAdP* hereafter. Roots of the fundamental equation of MAdP are strongly related to not only the recurrence of $\boldsymbol{T}$ but also the structures of R- and G-matrices, which also have impacts on the tail asymptotics of $\{\boldsymbol{x}(k)\}$. In this section, we discuss roots of the fundamental equation of MAdP and provide some related results. The proofs of the results in this section are omitted in this version of the thesis.

Since $\delta(\widehat{\boldsymbol{A}}(z))$ is the Perron-Frobenius eigenvalue of $\widehat{\boldsymbol{A}}(z)$, $\delta(\widehat{\boldsymbol{A}}(z))$ and thus $\varphi(z)$ are differentiable and convex for $z \in (r_{A_-}, r_{A_+})$ (see Andrew et al. [3, Theorem 2.1] and the theorem of Kingman [35]). Therefore, we obtain the following result.

Lemma 1.2 *If Assumption 1.1 (a) and (b) are satisfied, then the following hold:*

(i) The fundamental equation (1.36) has at most two roots $\theta_+ \in [1, r_{A_+})$ and $\theta_- \in (r_{A_-}, 1]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\delta(\widehat{\boldsymbol{A}}(z))\bigg|_{z=\theta_+} \ge 0, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\delta(\widehat{\boldsymbol{A}}(z))\bigg|_{z=\theta_-} \le 0. \tag{1.37}$$

(ii) Suppose that the roots θ_+ and θ_- exist. If $\theta_+ \neq \theta_-$, then they are simple roots.

(iii) The root θ_+ exists if and only if $\lim_{y\uparrow r_{A_+}} \delta(\widehat{\boldsymbol{A}}(y)) > 1$.

(iv) The root θ_- exists if and only if $\lim_{y\downarrow r_{A_-}} \delta(\widehat{\boldsymbol{A}}(y)) > 1$.

Remark 1.5 *A typical sufficient condition for the two roots is that $\widehat{\boldsymbol{A}}(z)$ is meromorphic (see Gail et al. [22, Lemma 2]).*

We now present a result on the connection between the recurrence of $\boldsymbol{T}$ and the location of the roots θ_+ and θ_-. It should be noted that if the root θ_+ (resp. θ_-) is simple then either of the following is true: the root θ_- (resp. θ_+) does not exist; or the root θ_- (resp. θ_+) exists and the two roots are different.

Lemma 1.3 *Under Assumption 1.1 (a) and (b), the following are true:*

(i) If $\varphi(1) < 1$, then $\boldsymbol{T}$ is positive recurrent.

(ii) The root $\theta_+ = 1$ is simple if and only if $\boldsymbol{T}$ is transient.

(iii) If $\theta_- = 1$, then $\boldsymbol{T}$ is recurrent.

(iv) If the root $\theta_- = 1$ is simple and $\sum_{k=1}^{\infty} k\boldsymbol{B}(k)\boldsymbol{e}$ is finite, then $\boldsymbol{T}$ is positive recurrent.

(v) If $\theta_- = \theta_+ = 1$, then $\boldsymbol{T}$ is null recurrent.

$$\frac{\mathrm{d}}{\mathrm{d}z}\delta(\widehat{\boldsymbol{A}}(z))\bigg|_{z=1} = \boldsymbol{\pi}\widehat{\boldsymbol{A}}'(1)\boldsymbol{e} = -\sigma, \tag{1.38}$$

Remark 1.6 *The condition '$\varphi(1) < 1$' in Lemma 1.3 (i) includes the following four subcases: (A) both θ_+ and θ_- exist satisfying $\theta_- < 1 < \theta_+$; (B) $\theta_+ > 1$ exists and θ_- does not exist; (C) $\theta_- < 1$ exists and θ_+ does not exist; and (D) neither θ_- nor θ_+ exists. $\boldsymbol{A}$ is strictly substochastic in all the subcases.*

Remark 1.7 *Lemma 1.3 implies that if $\boldsymbol{T}$ is positive recurrent, i.e., Assumption 1.1 (c) holds, then θ_+ does not exist or $\theta_+ > 1$.*

Let $\lambda_i^{(A)}(z)$ $(i = 1, 2, \dots, M)$ denote the eigenvalues of $\widehat{\boldsymbol{A}}(z)$ such that $\delta(\widehat{\boldsymbol{A}}(z)) = \lambda_1^{(A)}(z)$ and

$$|\delta(\widehat{\boldsymbol{A}}(z))| = |\lambda_1^{(A)}(z)| \ge |\lambda_2^{(A)}(z)| \ge |\lambda_3^{(A)}(z)| \ge \cdots \ge |\lambda_M^{(A)}(z)|. \tag{1.39}$$

We then have

$$\det(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z)) = \{1 - \delta(\widehat{\boldsymbol{A}}(z))\} \prod_{i=2}^{M} (1 - \lambda_i^{(A)}(z)). \tag{1.40}$$

Thus, the roots θ_+ and θ_- (if any) of the fundamental equation (1.36) are those of the following equation:

$$\det(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z)) = 0, \qquad r_{A_-} < |z| < r_{A_+}. \tag{1.41}$$

The following lemma is used to prove several asymptotic formulae for $\{\overline{\boldsymbol{x}}(k)\}$ presented in Sections 2.3 and 3.2. To shorten the statements of this lemma and the subsequent results, we denote by ε_0 a special symbol representing a sufficiently small positive number, which may take different values in different places hereafter. In addition, let $\mathbb{C}$ denote the set of complex numbers.

Lemma 1.4 *Suppose that Assumption 1.1 (a) and (b) and Condition 1.1 are satisfied. If there exists the root* $\theta_+ \in [1, r_{A_+})$ *(resp.* $\theta_- \in (r_{A_-}, 1]$*) of the fundamental equation (1.36), then*

(i) the equation (1.41) has exactly τ *roots* $\{\theta_+\omega_\tau^\nu; \nu = 0, 1, \dots, \tau - 1\}$ *(resp.* $\{\theta_-\omega_\tau^\nu; \nu = 0, 1, \dots, \tau - 1\}$*) in the domain* $\{z \in \mathbb{C}; 1 \le |z| < \theta_+ + \varepsilon_0\}$ *(resp.* $\{z \in \mathbb{C}; \theta_- - \varepsilon_0 < |z| \le 1\}$*); and*

(ii) the τ *roots are all simple.*

Next we show the connection between the root θ_+ and R-matrix, and that between the root θ_- and G-matrix, which are summarized in Lemmas 1.5 and 1.6 below.

Lemma 1.5 *Suppose that Assumption 1.1 (a) and (b) are satisfied. If the root* $\theta_- \in (r_{A_-}, 1]$ *of the fundamental equation (1.36) exists, then the following hold:*

(i) the matrix $\boldsymbol{G} := \widehat{\boldsymbol{G}}(1)$ *has an exactly one irreducible class, denoted by* $\mathbb{M}_\bullet^{\rm G} \subseteq \mathbb{M}$*. Thus,* $\boldsymbol{G}$ *is irreducible or, after an appropriate permutation of rows and columns,* $\boldsymbol{G}$ *takes the following form:*

$$\boldsymbol{G} = \begin{matrix} & \begin{matrix} \mathbb{M}_\bullet^{\rm G} & \mathbb{M}_{\rm T}^{\rm G} \end{matrix} \\ \begin{matrix} \mathbb{M}_\bullet^{\rm G} \\ \mathbb{M}_{\rm T}^{\rm G} \end{matrix} & \begin{pmatrix} \boldsymbol{G}_\bullet & \boldsymbol{O} \\ \boldsymbol{G}_\circ & \boldsymbol{G}_{\rm T} \end{pmatrix} \end{matrix}, \qquad \mathbb{M}_{\rm T}^{\rm G} := \mathbb{M} \setminus \mathbb{M}_\bullet^{\rm G}, \tag{1.42}$$

where $\boldsymbol{G}_\bullet$ *is irreducible (and possibly equal to* $\boldsymbol{G}$*),* $\boldsymbol{G}_{\rm T}$ *is strictly lower triangular (if it is not null, i.e.,* $\boldsymbol{G}_\bullet \neq \boldsymbol{G}$*) and* $\boldsymbol{G}_\circ$ *does not have, in general, a special structure.*

Furthermore, let $\boldsymbol{\psi}(z)$ *denote*

$$\boldsymbol{\psi}(z) \quad = \quad \boldsymbol{\mu}(z)(\boldsymbol{I} - \widehat{\boldsymbol{R}}(z))(\boldsymbol{I} - \boldsymbol{\Phi}(0)). \tag{1.43}$$

If θ_- *is simple, then the statements (ii)–(iv) below are true.*

(ii) $\mathrm{sp}(\boldsymbol{R}) = \delta(\boldsymbol{R}) < \theta_-^{-1}$.

(iii) The vector $\boldsymbol{\psi}(\theta_-)$ *is a left-eigenvector of* $\widehat{\boldsymbol{G}}(\theta_-)$ *corresponding to the simple eigenvalue* $\delta(\widehat{\boldsymbol{G}}(\theta_-)) = 1$.

(iv) The vector $\boldsymbol{\psi}(\theta_-)$ satisfies

$$
\begin{aligned}
[\boldsymbol{\psi}(\theta_-)]_i &> 0, && i \in \mathbb{M}^{\mathrm{G}}_{\bullet}, \\
[\boldsymbol{\psi}(\theta_-)]_i &= 0, && i \in \mathbb{M}^{\mathrm{G}}_{\mathrm{T}} = \mathbb{M} \setminus \mathbb{M}^{\mathrm{G}}_{\bullet}.
\end{aligned}
$$

In addition, if Condition 1.1 is satisfied, then the statements (v) and (vi) below hold.

(v) For $\nu = 0, 1, \dots, \tau - 1$,

$$
\boldsymbol{\psi}(\theta_-) = \boldsymbol{\mu}(\theta_-)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}(\boldsymbol{I} - \widehat{\boldsymbol{R}}(\theta_-\omega_\tau^\nu))(\boldsymbol{I} - \boldsymbol{\Phi}(0))\boldsymbol{\Delta}_M(\omega_\tau^\nu). \tag{1.44}
$$

(vi) $\omega^\tau = 1$ if and only if $\delta(\widehat{\boldsymbol{G}}(\theta_-\omega)) = 1$, which is a simple eigenvalue.

Lemma 1.6 *Suppose that Assumption 1.1 (a) and (b) are satisfied. If the root $\theta_+ \in [1, r_{A_+})$ of the fundamental equation (1.36) exists, then the following hold:*

(i) the matrix $\boldsymbol{R}$ has an exactly one irreducible class $\mathbb{M}^{\mathrm{R}}_{\bullet} \subseteq \mathbb{M}$. Thus, $\boldsymbol{R}$ is irreducible or, by permutation of rows and columns, $\boldsymbol{R}$ takes the following form:

$$
\boldsymbol{R} = \begin{array}{c} \\ \mathbb{M}^{\mathrm{R}}_{\bullet} \\ \mathbb{M}^{\mathrm{R}}_{\mathrm{T}} \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{M}^{\mathrm{R}}_{\bullet} & \mathbb{M}^{\mathrm{R}}_{\mathrm{T}} \end{array} \\ \begin{pmatrix} \boldsymbol{R}_{\bullet} & \boldsymbol{R}_{\circ} \\ \boldsymbol{O} & \boldsymbol{R}_{\mathrm{T}} \end{pmatrix} \end{array}, \qquad \mathbb{M}^{\mathrm{R}}_{\mathrm{T}} := \mathbb{M} \backslash \mathbb{M}^{\mathrm{R}}_{\bullet},
$$

where $\boldsymbol{R}_{\bullet}$ is irreducible (and possibly equal to $\boldsymbol{R}$), $\boldsymbol{R}_{\mathrm{T}}$ is strictly upper triangular (if it is not null, i.e., $\boldsymbol{R}_{\bullet} \neq \boldsymbol{R}$) and $\boldsymbol{R}_{\circ}$ does not have, in general, a special structure.

Furthermore, let $\boldsymbol{r}(z)$ denote

$$
\boldsymbol{r}(z) \quad = \quad (\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widehat{\boldsymbol{G}}(z))\boldsymbol{v}(z). \tag{1.45}
$$

If the root θ_+ is simple, then the statements (ii)–(iv) below are true.

(ii) $\mathrm{sp}(\boldsymbol{G}) = \delta(\boldsymbol{G}) < \theta_+$.

(iii) The vector $\boldsymbol{r}(\theta_+)$ is a right-eigenvector of $\widehat{\boldsymbol{R}}(\theta_+)$ corresponding to the simple eigenvalue $\delta(\widehat{\boldsymbol{R}}(\theta_+)) = 1$.

(iv) The vector $\boldsymbol{r}(\theta_+)$ satisfies

$$
\begin{aligned}
[\boldsymbol{r}(\theta_+)]_i &> 0, && i \in \mathbb{M}^{\mathrm{R}}_{\bullet}, \\
[\boldsymbol{r}(\theta_+)]_i &= 0, && i \in \mathbb{M}^{\mathrm{R}}_{\mathrm{T}} = \mathbb{M} \setminus \mathbb{M}^{\mathrm{R}}_{\bullet}.
\end{aligned}
$$

In addition, if Condition 1.1 is satisfied, then the statements (v) and (vi) below hold.

(v) For $\nu = 0, 1, \dots, \tau - 1$,

$$
\boldsymbol{r}(\theta_+) = \boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widehat{\boldsymbol{G}}(\theta_+\omega_\tau^\nu))\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta_+). \tag{1.46}
$$

(vi) $\omega^{\tau} = 1$ *if and only if* $\delta(\widehat{\boldsymbol{R}}(\theta_+\omega)) = 1$, *which is a simple eigenvalue.*

Proof. The proof of this lemma is omitted in this version of the thesis. □

Using Lemmas 1.5 and 1.6, we can prove Lemmas 1.7 and 1.8 below.

Lemma 1.7 *Suppose that Assumption 1.1 (a) and (b) hold. If there exist the roots* $\theta_+ \in [1, r_{A_+})$ *and* $\theta_- \in (r_{A_-}, 1]$ *of the fundamental equation (1.36), then* $\mathbb{M}_\bullet^{\mathrm{R}} \cap \mathbb{M}_\bullet^{\mathrm{G}} \neq \emptyset$.

Lemma 1.8 *Suppose that Assumption 1.1 (a) and (b) hold. Under these conditions, the following are true:*

(i) *If the root* $\theta_+ \in [1, r_{A_+})$ *exists, then* θ_+ *is equal to a root of the equation* $\delta(\widehat{\boldsymbol{R}}(z))) - 1 = 0$.

(ii) *If the root* $\theta_- \in (r_{A_-}, 1]$ *exists, then* θ_- *is equal to a root of the equation* $\delta(\widehat{\boldsymbol{G}}(z))) - 1 = 0$.

Remark 1.8 *The statement of Lemma 1.8 is different from the statement (iii)'s of Lemmas 1.5 and 1.6 because the former includes the case in which the root* θ_+ *or* θ_- *is not simple.*

1.6 Sufficient conditions for positive recurrence

In this section, we provide two sets of sufficient conditions for the positive recurrence of $\boldsymbol{T}$. As mentioned in Remark 1.1, $\boldsymbol{A}$ is not always stochastic. Therefore, we consider both cases where $\boldsymbol{A}$ is stochastic and $\boldsymbol{A}$ is strictly substochastic in this thesis. The following propositions show the sufficient conditions for the positive recurrence of $\boldsymbol{T}$ corresponding to each case.

Proposition 1.7 (Proposition 3.1 in Chapter XI of [8]) *Under the assumption that* $\boldsymbol{T}$ *and* $\boldsymbol{A}$ *are irreducible and stochastic,* $\boldsymbol{T}$ *is positive recurrent if and only if* $\sigma > 0$ *and* $\sum_{k=1}^{\infty} k\boldsymbol{B}(k)\boldsymbol{e} < \infty$.

Proposition 1.8 *Suppose* $\boldsymbol{T}$ *is irreducible and stochastic. If* $\boldsymbol{A}$ *is irreducible and strictly substochastic, then* $\boldsymbol{T}$ *is positive recurrent.*

Proof. Proposition 1.4 implies that $\lim_{k\to\infty} \boldsymbol{R}^k = \boldsymbol{O}$ and $(\boldsymbol{I} - \boldsymbol{G})^{-1}$ exists. Furthermore, from (1.14), we have

$$\begin{aligned}
\boldsymbol{R}_0 &:= \sum_{k=1}^{\infty} \boldsymbol{R}_0(k) \\
&= \left[\sum_{k=1}^{\infty} \boldsymbol{B}(k) + \sum_{m=1}^{\infty} \left(\sum_{k=1}^{\infty} \boldsymbol{B}(k+m)\right) \boldsymbol{L}(m)\right] (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1} \\
&\le \sum_{k=1}^{\infty} \boldsymbol{B}(k) \left[\boldsymbol{I} + \sum_{m=1}^{\infty} \boldsymbol{L}(m)\right] (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1} \\
&= \sum_{k=1}^{\infty} \boldsymbol{B}(k)(\boldsymbol{I} - \boldsymbol{G})^{-1}(\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1} < \infty,
\end{aligned}$$

where the last equality follows from (1.11). As a result, it follows from Theorem 3.4 in [66] that $\boldsymbol{T}$ is positive recurrent. □

1.7 Asymptotic analysis of GI/G/1-type Markov chains

In this section, we give a brief survey of previous studies on the asymptotic analysis of GI/G/1-type Markov chains. Various semi-Markovian queueing models are, by the embedded Markov chain method, reduced to GI/G/1-type Markov chains including quasi-birth-and-death processes (QBDs), GI/M/-type and M/G/1-type Markov chains [25, 37, 50]. It is not, in general, easy to compute the stationary distributions of GI/G/1-type Markov chains due to lack of the skip-free property [37, 50]. Therefore, the asymptotic analysis of the GI/G/1-type Markov chains has recently received considerable attention.

There are many studies on the tail asymptotics of the stationary vectors of the GI/G/1-type Markov chains including M/G/1-type ones. The tail asymptotics can be divided into two cases: the light-tailed asymptotics and subexponential-tail asymptotics. For the light-tailed asymptotics, Abate et al. [1] first presented a necessary condition so that the stationary vector of the M/G/1-type Markov chain has a geometric decay rate by making use of the Tauberian theorem. Falkenberg [19], Gail et al. [23] and Møller [49] showed similar sufficient conditions. By using the Markov renewal theory, Takine [63] considered a more general case, in which the periodicity appears in the geometric decay of $\{\overline{\boldsymbol{x}}(k)\}$. Kimura et al. [30] showed that this period of the asymptotic formula is equal to a divisor of the period of an MAdP with kernel $\{\boldsymbol{A}(k); k \in \mathbb{Z}\}$. As for the GI/G/1-type Markov chain, Li and Zhao [41] and Tai [61] considered the case in which the MAdP with kernel $\{\boldsymbol{A}(k)\}$ is aperiodic. In addition, Li et al. [39], Miyazawa [46], Miyazawa and Zhao [48], Miyazawa [47], Ozawa [51] considered the case where the phase space is infinite. However, most of the studies above only focused on the typical case where the transition block matrices in the non-boundary levels have a dominant impact on the decay rate of the stationary tail probability vectors and their decay is aperiodic. Thus, there have been a few studies on other cases in which the decay rate is determined by the convergence radius of the generating functions $\widehat{\boldsymbol{B}}(z)$ or $\sum_{k=1}^{\infty} z^k \boldsymbol{A}(k)$.

Contrary to the light-tailed asymptotics, much less work has been done on the subexponential tail asymptotics of the stationary distribution of the GI/G/1- or M/G/1-type Markov chain. Asmussen and Møller [7] studied the subexponential tail asymptotics of the M/G/1-type Markov chain and derived an asymptotic formula assuming the subexponentiality of level increments $\{\boldsymbol{A}(k); k = 1, 2, \dots\}$ and $\{\boldsymbol{B}(k); k = 1, 2, \dots\}$ and several additional conditions. Li and Zhao [40] provided a subexponential asymptotic formula for the GI/G/1-type Markov chain, however, some of their results are incorrect because they include "the inverse of a singular matrix". Takine [63] studied the subexponential tail asymptotics of the M/G/1-type Markov chain without the condition assumed in [7]. However, Masuyama [42] pointed out that Takine's proof needs an additional condition that the G-matrix is aperiodic. Furthermore, Masuyama [42] presented a weaker sufficient condition for the asymptotic formula than those presented in the literature [7, 40, 63]. Recently, Kim and Kim [29] improved Masuyama [42]'s sufficient condition in the case where the G-matrix is periodic.

In addition to the tail asymptotics, few researchers studied the heavy-traffic asymptotics of the GI/G/1- or M/G/1-type Markov chain. Asmussen [5] considered the heavy-traffic asymptotics of the GI/G/1-type Markov chain and proved that the diffusion-scaled level process converges weakly to a reflected Brownian motion as the mean drift in level $-\sigma$ goes to zero. As a corollary, Asmussen [5] also presented the asymptotic formula, in which the stationary distribution of the properly scaled level variable is geometric and independent of the phase variable. Falin [18] proved the heavy-traffic limit for the M/G/1-type Markov chain under weaker conditions for the matrices $\{\boldsymbol{A}(k); k = 1, 2, \dots\}$ and $\{\boldsymbol{B}(k); k = 1, 2, \dots\}$ than in [5]. However, they assumed several additional constraints, such as $\boldsymbol{A}(k) = \boldsymbol{B}(k)$ for all $k \geq 1$. As far as we know, there are no studies on the heavy-traffic limits of the moments of the stationary distribution of the GI/G/1-type Markov chain.

1.8 Overview of thesis

In this thesis, we study the asymptotic analysis of the stationary probability vector of the GI/G/1-type Markov chain.

In Chapter 2, we study the light-tailed asymptotics. We consider three cases: (i) the tail decay rate is determined by the root θ_+ of the fundamental equation of MAdP (see Section 1.5) associated with the transition block matrices $\{\boldsymbol{A}(k)\}$; (ii) by the convergence radius r_B of the generating function of the transition block matrices $\{\boldsymbol{B}(k); k = 1, 2, \dots\}$; and (iii) by the convergence radius r_{A_+} of $\sum_{k=1}^{\infty} z^k \boldsymbol{A}(k)$. Most of the previous studies [19, 23, 49, 63, 30] focused on the case (i) although they limited to the M/G/1-type one. Thus, we extend the existing asymptotic formula for the M/G/1-type Markov chain to the GI/G/1-type one. Contrary, there are a few studies for the case (ii). In this case, we present general asymptotic formulae that include, as special cases, the existing results in the literature [40, 30] by using completely different approach to them. In case (iii), we derive new asymptotic formulae. To the best of our knowledge, such formulae have not been reported in the literature.

Chapter 3 considers the subexponential asymptotics of the tail stationary distributions of the GI/G/1-type Markov chain in two cases: (i) $\boldsymbol{A}$ is stochastic; and (ii) $\boldsymbol{A}$ is strictly substochastic. For case (i), Masuyama [42], Kim and Kim [29] recently derived the subexponential asymptotic formula for the M/G/1-type Markov chain under weaker conditions than those of earlier studies [7, 40, 63]. Thus, we extend these results to the GI/G/1-type Markov chain. As for case (ii), the subexponential asymptotics has not been studied as far as we know. We also study the locally subexponential asymptotics of the stationary distributions in both cases (i) and (ii).

In Chapter 4, we consider the heavy-traffic limits of the stationary distribution and their moments of the GI/G/1-type Markov chain. Asmussen [5] first studied the heavy traffic asymptotics for the GI/G/1-type Markov chain and showed that the stationary distribution of the properly scaled level variable is geometric and independent of the phase variable in the heavy-traffic limit. We prove such heavy-traffic asymptotic formula of the stationary distribution of the GI/G/1-type Markov chain under a weaker condition than Asmussen's, by a characteristic function approach. Using a similar approach, we also present a heavy-traffic asymptotic formula for the moments of the stationary distribution, which is not reported in the literature.

Chapter 5 concludes this thesis and provides several suggestions for future research.

The results discussed in Chapter 2 is mainly based on [34], Chapter 3 on [32] and Chapter 4 on [31] including new topics that will be submitted soon. Furthermore, important results for the tail asymptotics presented in Appendix A is based on [30].

Chapter 2

Light-Tailed Asymptotics

2.1 Introduction

In this chapter, we study the light-tailed asymptotics of the stationary distribution of the GI/G/1-type Markov chain. To proceed, we first make the following assumption for r_{A_+} and r_B throughout this chapter.

Assumption 2.1 *(a)* $r_{A_+} > 1$, *and (b)* $r_B > 1$.

Under Assumptions 1.1 and 2.1, the sequences $\{\boldsymbol{x}(k)\}$ and $\{\overline{\boldsymbol{x}}(k)\}$ are light-tailed (see Li and Zhao [41, Theorem 2]). Furthermore, it follows that $\{\boldsymbol{x}(k)\}$ is light-tailed if and only if the convergence radius of the generating function $\widehat{\boldsymbol{x}}(z)$ is greater than one, or equivalently,

$$r := \left[\limsup_{k\to\infty}\{\boldsymbol{x}(k)\boldsymbol{e}\}^{1/k}\right]^{-1} > 1. \tag{2.1}$$

Note also that (2.1) is equivalent to

$$r = \left[\limsup_{k\to\infty}\{\overline{\boldsymbol{x}}(k)\boldsymbol{e}\}^{1/k}\right]^{-1} > 1, \tag{2.2}$$

because

$$\widehat{\overline{\boldsymbol{x}}}(z) := \sum_{k=0}^{\infty} z^k \overline{\boldsymbol{x}}(k) = \frac{\widehat{\boldsymbol{x}}(1) - \widehat{\boldsymbol{x}}(z)}{1-z}, \quad z \in \mathbb{C}, |z| < r. \tag{2.3}$$

In what follows, we overview the previous studies on the light-tailed asymptotics for $\{\boldsymbol{x}(k)\}$ and/or $\{\overline{\boldsymbol{x}}(k)\}$. Most of the previous studies assume that $\{\boldsymbol{A}(k)\}$ has a dominant impact on the decay rate $1/r$ of $\{\boldsymbol{x}(k)\}$ and thus $\{\overline{\boldsymbol{x}}(k)\}$. More specifically, those studies assume that r is equal to the real and minimum-modulus root θ_+ of the equation $\det(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z)) = 0$ with $r_{A_-} < |z| < r_{A_+}$, which is equal to a root of the fundamental equation of the MAdP with kernel $\{\boldsymbol{A}(k)\}$ (for details, see section 1.5).

Several researchers considered the M/G/1-type Markov chain, which is a special case of the GI/G/1-type Markov chain. Using the Tauberian theorem, Abate et al. [1] presented a necessary condition for the existence of a positive vector $\boldsymbol{d}$ such that

$$\lim_{k\to\infty} \theta^k \boldsymbol{x}(k) = \boldsymbol{d}. \tag{2.4}$$

Conversely, Falkenberg [19], Gail et al. [23] and Møller [49] showed (almost the same) sufficient conditions under which

$$\lim_{k\to\infty} \theta^k \overline{\boldsymbol{x}}(k) = (\theta - 1)^{-1}\boldsymbol{d}. \tag{2.5}$$

Note here that (2.4) implies (2.5), whereas the converse is not true. Takine [63] considered a more general case and proved that there exists some positive integer h such that

$$\lim_{n\to\infty} \theta^{nh+\ell} \overline{\boldsymbol{x}}(nh+\ell) = \boldsymbol{d}_\ell, \qquad \text{for } \ell = 0, 1, \dots, h-1, \tag{2.6}$$

where $\boldsymbol{d}_\ell$'s ($\ell = 0, 1, \dots, h-1$) are some positive vectors such that at least two of them are different if $h \geq 2$. Equation (2.6) shows that the periodicity appears in the geometric decay of $\{\overline{\boldsymbol{x}}(k)\}$ in general. The largest number h satisfying (2.6) is called the period of the geometric decay of $\{\overline{\boldsymbol{x}}(k)\}$. Making use of the Markov renewal theory, Takine [63] also derived two expressions of $\boldsymbol{d}_\ell$ ($\ell = 0, 1, \dots, h-1$): one is for a special case of $h = 1$; and the other a general case (i.e., $h \geq 1$) (see Theorems 2 and 3 therein). However, the expression of $\boldsymbol{d}_l$ in the general case is somewhat complicated and thus it is difficult to confirm that the general formula (2.6) with $h = 1$ is equivalent to the special one (2.5) for $h = 1$. Kimura et al. [30] presented another expression of $\boldsymbol{d}_\ell$ ($\ell = 0, 1, \dots, h-1$) in the general case by locating maximum-order minimum-modulus poles (called *dominant poles* hereafter) of the generating function of $\{\overline{\boldsymbol{x}}(k)\}$. This alternative expression is readily reduced to the one for the special case of $h = 1$. Kimura et al. [30] also showed that h in (2.6) is equal to a divisor of the period τ of an MAdP with kernel $\{\boldsymbol{A}(k); k \in \mathbb{Z}\}$.

As for the GI/G/1-type Markov chain, Li and Zhao [41] and Tai [61] considered the case in which the MAdP with kernel $\{\boldsymbol{A}(k)\}$ is aperiodic, i.e., $\tau = 1$. Li and Zhao [41] presented an asymptotic formula like (2.4). Tai [61] provided sufficient conditions under which $\{\boldsymbol{x}(k)\}$ is asymptotically geometric and is light-tailed but not exactly geometric. He also discussed the decay rate $1/r$ of $\{\boldsymbol{x}(k)\}$.

It should be noted that the decay rate $1/r$ can be determined by either the convergence radius r_{A_+} of $\sum_{k=1}^{\infty} z^k \boldsymbol{A}(k)$ or that r_B of $\widehat{\boldsymbol{B}}(z)$; more specifically, $r = r_{A_+}$ or $r = r_B$. However, there have been a few studies on such cases. Kimura et al. [30] and Li and Zhao [41] discussed the case in which $r = r_B$ and $z = r_B$ is a pole of $\widehat{\boldsymbol{B}}(z)$. To the best of our knowledge, there have been no studies on the case of $r = r_{A_+}$.

In this chapter, we study the light-tailed asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ in three cases: (i) $r = \theta$; (ii) $r = r_B$; and (iii) $r = r_{A_+}$. We first consider the case (i). Applying the techniques in Kimura et al. [30], we derive a geometric asymptotic formula for $\{\overline{\boldsymbol{x}}(k)\}$ and then show that the period h of the geometric decay is equal to a divisor of the period τ of the MAdP with kernel $\{\boldsymbol{A}(k)\}$. These results are the generalizations of the corresponding ones in Kimura et al. [30]. We next divide the case (ii) into two subcases: (ii.a) $r = r_B < \theta$; and (ii.b) $r = r_B = \theta$ and then present the following formulae:

$$\text{Case (ii.a):} \qquad \overline{\boldsymbol{x}}(k) = r_B^{-k} k^{\phi} \boldsymbol{c}_0(k) + o(r_B^{-k} k^{\phi})\boldsymbol{e}^\top, \qquad \phi \in \mathbb{R} := (-\infty, \infty), \tag{2.7}$$

$$\text{Case (ii.b):} \qquad \overline{\boldsymbol{x}}(k) = \begin{cases} r_B^{-k} k^{\phi+1} \boldsymbol{c}_1(k) + o(r_B^{-k} k^{\phi+1})\boldsymbol{e}^\top, & \phi > -1, \\ r_B^{-k} \boldsymbol{c}_2(k) + o(r_B^{-k})\boldsymbol{e}^\top, & \phi < -1, \\ r_B^{-k} (\boldsymbol{c}_1(k) + \boldsymbol{c}_2(k)) + o(r_B^{-k})\boldsymbol{e}^\top, & \phi = -1, \end{cases} \tag{2.8}$$

where $\boldsymbol{c}_i$ ($i = 0, 1, 2$) is some vector-valued function on $\mathbb{Z}_+$ such that $\limsup_{k\to\infty} \boldsymbol{c}_i(k) \geq \boldsymbol{0}, \neq \boldsymbol{0}$; and $f(x) = o(g(x))$ represents $\lim_{x\to\infty} |f(x)/g(x)| = 0$. Note here that ϕ can take all values in $\mathbb{R}$. Actually, Kimura et al. [30] and Li and Zhao [41] obtained similar formulae to (2.7) and (2.8), though ϕ is restricted to positive integers. Therefore, our results are more general than the corresponding ones in Kimura et al. [30] and Li and Zhao [41]. Finally for the case (iii), we prove that

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{c} \geq \boldsymbol{0}, \neq \boldsymbol{0},$$

assuming that there exists some subexponential random variable Y such that

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{C}_A \geq \boldsymbol{O}, \neq \boldsymbol{O}.$$

As far as we know, any asymptotic formula for the case (iii) has not been reported in the literature. We assume that the phase space is finite to study how the tail of the level increment impacts on that of the stationary distribution. The infiniteness of the phase space makes the situation much more complicated (see [39, 46, 48, 47, 51]), which is beyond the scope of this thesis.

The reminder of this chapter is organized as follows. In Section 2.2, we first study the asymptotics of $\{\boldsymbol{F}(k)\}$ and $\{\boldsymbol{L}(k)\}$. These results are required for deriving the asymptotic formulae of $\{\overline{\boldsymbol{x}}(k)\}$, which are presented in Section 2.3. The proofs of the results in this chapter are omitted in this version of the thesis due to copyright reasons.

2.2 Asymptotic analysis of $\{F(k)\}$ and $\{L(k)\}$

In this section, we study the asymptotics of $\{\boldsymbol{F}(k)\}$ and $\{\boldsymbol{L}(k)\}$, which depend on the existence of the root $\theta_+ \in [1, r_{A_+})$ and $\theta_- \in (r_{A_-}, 1]$ of the fundamental equation (1.36).

2.2.1 Case where θ_+ and θ_- exist

We first consider the cases where the roots θ_+ and θ_- exist. We begin with the following additional lemmas.

For simplicity, in what follows, we write $f(x) = O(g(x))$ to represent $\limsup_{x\to\infty} |f(x)|/|g(x)| < \infty$.

Lemma 2.1 *Suppose that Assumption 1.1 together with Condition 1.1 is satisfied. If there exists a root* $\theta \in \{\theta_-, \theta_+\}$ *(*$\theta_+ \in [1, r_{A_+})$*,* $\theta_- \in (r_{A_-}, 1]$*) of the fundamental equation (1.36), then, for* $\nu = 0, 1, \dots, \tau - 1$*,*

$$\mathrm{adj}(\boldsymbol{I} - \widehat{\boldsymbol{A}}(\theta\omega_\tau^\nu)) = \prod_{m=2}^{M} (1 - \lambda_m^{(A)}(\theta\omega_\tau^\nu))\boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu) \neq \boldsymbol{O}. \tag{2.9}$$

Lemma 2.2 *Suppose that Assumption 1.1 together with Condition 1.1 is satisfied. If there exists a root* $\theta \in \{\theta_-, \theta_+\}$ *(*$\theta_+ \in [1, r_{A_+})$*,* $\theta_- \in (r_{A_-}, 1]$*) of the fundamental equation (1.36), then, for* $\nu = 0, 1, \dots, \tau - 1$*,*

$$\lim_{z\to\theta\omega_\tau^\nu} \left(1 - \frac{z}{\theta\omega_\tau^\nu}\right) \left(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z)\right)^{-1} = \frac{\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta)\boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}}{\theta(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}})|_{y=\theta}}. \tag{2.10}$$

Using the above lemmas, we now provide asymptotic formulae for $\{\boldsymbol{F}(k)\}$ and $\{\boldsymbol{L}(k)\}$.

Lemma 2.3 *Suppose that Assumption 1.1 together with Condition 1.1 is satisfied. If there exists the root $\theta_+ \in [1, r_{A_+})$ of the fundamental equation (1.36), then*

$$\boldsymbol{F}(k) = \theta_+^{-k}\left[\widetilde{\boldsymbol{F}}_{k-\lfloor k/\tau \rfloor \tau} + O((1+\varepsilon_0)^{-k})\boldsymbol{e}\boldsymbol{e}^\top\right], \tag{2.11}$$

where $\widetilde{\boldsymbol{F}}_\ell$ ($\ell = 0,1,\dots,\tau-1$) is given by

$$[\widetilde{\boldsymbol{F}}_\ell]_{i,j} = \begin{cases} \dfrac{\tau[\boldsymbol{r}(\theta_+)]_i[\boldsymbol{\mu}(\theta_+)]_j}{\theta_+ (\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))\mid_{y=\theta_+}} > 0, & \text{if } i \in \mathbb{M}_\bullet^{\mathrm{R}},\ p(i) \equiv p(j) - \ell \pmod{\tau}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Lemma 2.4 *Suppose that Assumption 1.1 together with Condition 1.1 is satisfied. If there exists the root $\theta_- \in (r_{A_-}, 1]$ of the fundamental equation (1.36), then*

$$\boldsymbol{L}(k) = \theta_-^{k}\left[\widetilde{\boldsymbol{L}}_{k-\lfloor k/\tau \rfloor \tau} + O((1+\varepsilon_0)^{-k})\boldsymbol{e}\boldsymbol{e}^\top\right], \tag{2.13}$$

where $\widetilde{\boldsymbol{L}}_\ell$ ($\ell = 0,1,\dots,\tau-1$) is given by

$$[\widetilde{\boldsymbol{L}}_\ell]_{i,j} = \begin{cases} -\dfrac{\tau[\boldsymbol{v}(\theta_-)]_i[\boldsymbol{\psi}(\theta_-)]_j}{\theta_- (\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))\mid_{y=\theta_-}} > 0, & \text{if } j \in \mathbb{M}_\bullet^{\mathrm{G}},\ p(j) \equiv p(i) - \ell \pmod{\tau}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

2.2.2 Case where θ_+ or θ_- does not exist

Next we consider the case where θ_+ (resp. θ_-) does not exist, i.e., $\lim_{y\to r_{A_+}} \delta(\widehat{\boldsymbol{A}}(y)) < 1$ (resp. $\lim_{y\to r_{A_-}} \delta(\widehat{\boldsymbol{A}}(r_{A_-})) < 1$). To state the asymptotic formulae for $\{\boldsymbol{F}(k)\}$ and $\{\boldsymbol{L}(k)\}$, we use the subexponential class (of probability distributions) and its related classes in this section. The definitions and related results are summarized in Chapter C.

We now make a condition on the right tail of $\{\boldsymbol{A}(k); k \in \mathbb{Z}\}$, i.e., the tail of $\{\boldsymbol{A}(k); k \in \mathbb{N}\}$.

Condition 2.1 *There exists some random variable Y in $\mathbb{Z}_+$ with finite positive mean such that*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{C}_A \ge \boldsymbol{O}, \ne \boldsymbol{O}. \tag{2.15}$$

Remark 2.1 *Condition 2.1 implies that $\widehat{\boldsymbol{A}}(r_{A_+}) = \sum_{k=-\infty}^{\infty}(r_{A_+})^k \boldsymbol{A}(k)$ is finite.*

Under Condition 2.1, we obtain asymptotic formulae for $\{\overline{\boldsymbol{F}}(k)\}$ and $\{\boldsymbol{F}(k)\}$.

Lemma 2.5 *Suppose that Assumption 1.1 and Condition 2.1 are satisfied. If $Y \in \mathcal{S}^*$ and $\delta(\widehat{\boldsymbol{A}}(r_{A_+})) < 1$, then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{C}_F, \tag{2.16}$$

$$\lim_{k\to\infty} \frac{\boldsymbol{F}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \left(1 - \frac{1}{r_{A_+}}\right)\boldsymbol{C}_F, \tag{2.17}$$

where $\boldsymbol{C}_F$ is given by

$$\boldsymbol{C}_F = (\boldsymbol{I} - \widehat{\boldsymbol{R}}(r_{A_+}))^{-1}\boldsymbol{C}_A(\boldsymbol{I} - \widehat{\boldsymbol{A}}(r_{A_+}))^{-1}, \tag{2.18}$$

which has no zero-columns and -rows.

To show similar formulae for $\{\overline{\boldsymbol{L}}(k)\}$ and $\{\boldsymbol{L}(k)\}$, we make a condition on the left tail of $\{\boldsymbol{A}(k); k \in \mathbb{Z}\}$, i.e., the tail of $\{\boldsymbol{A}(-k); k \in \mathbb{N}\}$.

Condition 2.2 *There exists some random variable Y in $\mathbb{Z}_+$ with finite positive mean such that*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(-k)}{r_{A_-}^k \mathsf{P}(Y > k)} = \boldsymbol{C}_{A_-} \geq \boldsymbol{O}, \neq \boldsymbol{O}. \tag{2.19}$$

Remark 2.2 *Condition 2.2 implies that $\widehat{\boldsymbol{A}}(r_{A_-}) = \sum_{k=-\infty}^{\infty}(r_{A_-})^k \boldsymbol{A}(k)$ is finite.*

Lemma 2.6 *Suppose that Assumption 1.1 together with Condition 2.2 is satisfied. If $Y \in \mathcal{S}^*$ and $\delta(\widehat{\boldsymbol{A}}(r_{A_-})) < 1$, then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{L}}(k)}{r_{A_-}^k \mathsf{P}(Y > k)} = \boldsymbol{C}_L, \tag{2.20}$$

$$\lim_{k\to\infty} \frac{\boldsymbol{L}(k)}{r_{A_-}^k \mathsf{P}(Y > k)} = (1 - r_{A_-})\boldsymbol{C}_L, \tag{2.21}$$

where $\boldsymbol{C}_L$ is given by

$$\boldsymbol{C}_L = (\boldsymbol{I} - \widehat{\boldsymbol{A}}(r_{A_-}))^{-1}\boldsymbol{C}_{A_-}(\boldsymbol{I} - \widehat{\boldsymbol{G}}(r_{A_-}))^{-1}, \tag{2.22}$$

which has no zero-columns and -rows.

2.3 Asymptotic analysis of $\{\overline{x}(k)\}$

In this subsection, we present several asymptotic formulae for $\{\overline{\boldsymbol{x}}(k)\}$. As shown in (2.3), $\widehat{\overline{\boldsymbol{x}}}(z) = \sum_{k=0}^{\infty} z^k \overline{\boldsymbol{x}}(k)$ is expressed in terms of $\widehat{\boldsymbol{x}}(z)$. Substituting (1.25) into (2.3), we have

$$\widehat{\overline{\boldsymbol{x}}}(z) = \frac{\widehat{\boldsymbol{x}}(1)}{1-z} - \frac{\boldsymbol{x}(0)\widehat{\boldsymbol{R}}_0(z)}{1-z}(\boldsymbol{I} - \widehat{\boldsymbol{R}}(z))^{-1}, \qquad |z| < r, \tag{2.23}$$

where $\widehat{\overline{\boldsymbol{x}}}(z)$ is holomorphic for all $|z| < r$ due to (2.2). Combining (2.23) with Proposition 1.3, we obtain

$$\widehat{\overline{\boldsymbol{x}}}(z) = \frac{\widehat{\boldsymbol{x}}(1)}{1-z} - \frac{\boldsymbol{x}(0)\widehat{\boldsymbol{R}}_0(z)(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widehat{\boldsymbol{G}}(z))}{1-z} \frac{\mathrm{adj}(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z))}{\det(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z))}. \tag{2.24}$$

The matrix-valued function

$$\widehat{\boldsymbol{R}}_0(z)(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widehat{\boldsymbol{G}}(z))\mathrm{adj}(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z))$$

in the right hand side of (2.24) is holomorphic for $r_{A_-} < |z| < \min(r_{A_+}, r_B)$ (which follows from Proposition 1.2 and Assumption 2.1). It should be noted that $z = \theta_+$ (if any) is one of the roots of $\det(\boldsymbol{I} - \widehat{\boldsymbol{A}}(z)) = 0$ and thus can be a dominant singularity of $\widehat{\overline{\boldsymbol{x}}}(z)$. As a result, (2.24) implies that if $\overline{\boldsymbol{x}}(k)$ has a certain geometric decay rate θ such that $1 < \theta < \min(r_B, r_{A_+})$, then the root θ_+ of the fundamental equation (1.36) exists and θ is equal to θ_+. Equation (2.24) also implies that there are three possibilities: $r = \theta_+$; $r = r_B$; or $r = r_{A_+}$. In what follows, we discuss the three cases individually.

2.3.1 Case where $r = \theta_+$

We first consider the case where $r = \theta_+$, which includes subcases $r = \theta_+ < r_B$; and $r = \theta_+ = r_B$. However, the latter case is discussed in Section 2.3.2, which covers the case where $r = r_B$.

Theorem 2.1 ($r = \theta_+ < r_B$) *Suppose that Assumptions 1.1 and 2.1 together with Condition 1.1 are satisfied; and that the root $\theta_+ \in (1, r_{A_+})$ of the fundamental equation (1.36) exists. If $\theta_+ < r_B$, then*

$$\lim_{n\to\infty} \theta_+^{n\tau+\ell}\overline{\boldsymbol{x}}(n\tau+\ell) = \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^\ell} c(\omega_\tau^\nu)\boldsymbol{\mu}(\theta_+)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1} > \boldsymbol{0}, \quad \ell = 0,1,\dots,\tau-1, \tag{2.25}$$

where

$$c(\omega_\tau^\nu) = \frac{\boldsymbol{x}(0)\widehat{\boldsymbol{R}}_0(\theta_+\omega_\tau^\nu)(\boldsymbol{I}-\boldsymbol{\Phi}(0))(\boldsymbol{I}-\widehat{\boldsymbol{G}}(\theta_+\omega_\tau^\nu))\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta_+)}{\theta_+(\theta_+\omega_\tau^\nu-1)\cdot(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))|_{y=\theta_+}}. \tag{2.26}$$

Theorem 2.1 is a generalization of Theorem 5 of Li and Zhao [41], where $\tau = 1$ is assumed. In addition, Theorem 2.1 includes Theorem 3.1 of Kimura et al. [30] as a special case where the Markov chain $\{(X_n, S_n)\}$ is assumed to be of M/G/1 type. To verify this, we suppose that the Markov chain $\{(X_n, S_n)\}$ is of M/G/1 type. We then have $\widehat{\boldsymbol{G}}(z) = \boldsymbol{G}/z$ and $\boldsymbol{L}(k) = \boldsymbol{G}^k$ for $k \in \mathbb{N}$, where $\boldsymbol{G}$ is the G-matrix of the M/G/1-type Markov chain. We also define $\boldsymbol{\Phi}_0(k) = \sum_{m=k}^\infty \boldsymbol{B}(m)\boldsymbol{G}^{m-k}$ for $k \in \mathbb{N}$. It follows from (1.14) that

$$\widehat{\boldsymbol{R}}_0(z)(\boldsymbol{I}-\boldsymbol{\Phi}(0))(\boldsymbol{I}-\boldsymbol{G}/z) = \widehat{\boldsymbol{B}}(z) - \boldsymbol{\Phi}_0(1)\boldsymbol{G}, \tag{2.27}$$

which is equivalent to the second equation of Proposition 2.2 of Kimura et al. [30]. Applying (2.27) to (2.26), we obtain

$$\begin{aligned} c(\omega_\tau^\nu) &= \frac{\boldsymbol{x}(0)(\widehat{\boldsymbol{B}}(\theta_+\omega_\tau^\nu) - \boldsymbol{\Phi}_0(1)\boldsymbol{G})\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta_+)}{\theta_+(\theta_+\omega_\tau^\nu-1)\cdot(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))|_{y=\theta_+}} \\ &= \frac{\boldsymbol{x}(0)[\widehat{\boldsymbol{B}}(\theta_+\omega_\tau^\nu) - \boldsymbol{\Phi}_0(1)(\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}\boldsymbol{A}(-1)]\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta_+)}{\theta_+(\theta_+\omega_\tau^\nu-1)\cdot(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))|_{y=\theta_+}} \\ &= \frac{[\boldsymbol{x}(0)\widehat{\boldsymbol{B}}(\theta_+\omega_\tau^\nu) - \boldsymbol{x}(1)\boldsymbol{A}(-1)]\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta_+)}{\theta_+(\theta_+\omega_\tau^\nu-1)\cdot(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))|_{y=\theta_+}}, \end{aligned} \tag{2.28}$$

where we use $\boldsymbol{G} = (\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}\boldsymbol{A}(-1)$ and $\boldsymbol{x}(1) = \boldsymbol{x}(0)\boldsymbol{\Phi}_0(1)(\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}$ in the last two equalities. Substituting (2.28) into the right hand side of (2.25) yields

$$\sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^l} \frac{[\boldsymbol{x}(0)\widehat{\boldsymbol{B}}(\theta_+\omega_\tau^\nu) - \boldsymbol{x}(1)\boldsymbol{A}(-1)]\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta_+)}{\theta_+(\theta_+\omega_\tau^\nu-1)\cdot(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))|_{y=\theta_+}} \cdot \boldsymbol{\mu}(\theta_+)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1},$$

which is equivalent to $\boldsymbol{c}_l$ in Theorem 3.1 of Kimura et al. [30].

2.3.2 Case where $r = r_B$

We move on to the second case, which is divided into two subcases. One is where the root θ_+ of the fundamental equation (1.36) may not exist; and the other is where the root θ_+ exists and equals to r_B. In both subcases, the asymptotics of $\{\overline{\boldsymbol{B}}(k)\}$ has a dominant impact on that of $\{\overline{\boldsymbol{x}}(k)\}$. To proceed further, we specify the tail asymptotics of $\{\overline{\boldsymbol{B}}(k)\}$.

Condition 2.3 *There exists some $\phi \in \mathbb{R}$ such that*

$$[\overline{\boldsymbol{B}}(k)]_{i,j} = r_B^{-k} k^{\phi} [\breve{\boldsymbol{B}}(k)]_{i,j} + o(r_B^{-k} k^{\phi}), \qquad (i,j) \in \mathbb{M}_0 \times \mathbb{M}, \tag{2.29}$$

where $\{\breve{\boldsymbol{B}}(k); k \in \mathbb{Z}_+\}$ is a bounded sequence of nonnegative matrices such that

$$\limsup_{k\to\infty} \breve{\boldsymbol{B}}(k) \ \ \ge \ \ \boldsymbol{O}, \neq \boldsymbol{O}. \tag{2.30}$$

Remark 2.3 *Kimura et al. [30] studied a special case where $\widehat{\boldsymbol{B}}(z)$ is meromorphic for $|z| < r_B + \varepsilon_0$, i.e., $\phi \in \mathbb{N}$ and, for some $m, N \in \mathbb{N}$,*

$$\breve{\boldsymbol{B}}(k) = \frac{1}{(m-1)!} \sum_{n=0}^{N-1} \frac{1}{\zeta_n^k} \frac{1}{r_B\zeta_n - 1} \lim_{z\to r_B\zeta_n} \left(1 - \frac{z}{r_B\zeta_n}\right)^m \widehat{\boldsymbol{B}}(z),$$

where ζ_n's $(n = 0, 1, \dots, N-1)$ are complex numbers such that $|\zeta_n| = 1$ and $0 = \arg(\zeta_0) < \arg(\zeta_1) < \cdots < \arg(\zeta_{N-1}) < 2\pi$ (see Assumption 3.2 therein). In other words, $\{r_B\zeta_n\}$ are the poles of order m of $\widehat{\boldsymbol{B}}(z)$ on the circle $\{z; |z| = r_B\}$. Li and Zhao [40] considered a similar case.

We now define $\boldsymbol{\beta}(k; y)$ $(k \in \mathbb{Z}_+, y > 1)$ as

$$\begin{aligned}\boldsymbol{\beta}(k; y) \ &= \ \boldsymbol{x}(0) \sum_{\ell=0}^{k} \left[\breve{\boldsymbol{B}}(k-\ell) + \sum_{m=1}^{\infty} y^{-m} \breve{\boldsymbol{B}}(k-\ell+m) \boldsymbol{L}(m) \right] \\ &\qquad \times (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1} y^{\ell} \boldsymbol{F}(\ell).\end{aligned} \tag{2.31}$$

We then present the theorem on the first subcase, i.e., where the existence of θ_+ is not necessarily assumed.

Theorem 2.2 **($r = r_B$)** *Suppose that Assumptions 1.1 and 2.1 together with Condition 2.3 are satisfied; and that $r_B < r_{A_+}$ and $\delta(\widehat{\boldsymbol{A}}(r_B)) < 1$. Under these conditions, the following asymptotic formula holds:*

$$\overline{\boldsymbol{x}}(k) \ = \ r_B^{-k} k^{\phi} \boldsymbol{\beta}(k; r_B) + o(r_B^{-k} k^{\phi}) \boldsymbol{e}^{\top}, \tag{2.32}$$

where, $\{\boldsymbol{\beta}(k; r_B); k \in \mathbb{Z}_+\}$ is a bounded sequence of nonnegative vectors. In addition, suppose that either of the conditions (a), (b) and (c) below holds.

(a) Condition 1.1 is satisfied and the roots $\theta_+ \in (1, r_{A_+})$ and $\theta_- \in (r_{A_-}, 1]$ of the fundamental equation (1.36) exist;

(b) Condition 2.1 is satisfied, $Y \in \mathcal{S}^*$ and $\delta(\widehat{\boldsymbol{A}}(r_{A_+})) < 1$; or

(c) Condition 2.2 is satisfied, $Y \in \mathcal{S}^*$ and $\delta(\widehat{\boldsymbol{A}}(r_{A_-})) < 1$.

It then holds that

(i) $\limsup_{k\to\infty} \boldsymbol{\beta}(k; r_B) > \boldsymbol{0}$*; and*

(ii) if $\liminf_{k\to\infty} \breve{\boldsymbol{B}}(k) \ge \boldsymbol{O}$ *and* $\liminf_{k\to\infty} \breve{\boldsymbol{B}}(k) \neq \boldsymbol{O}$*, then* $\liminf_{k\to\infty} \boldsymbol{\beta}(k; r_B) > \boldsymbol{0}$.

Corollary 2.1 *Suppose that Assumptions 1.1 and 2.1 together with Condition 2.3 are satisfied; and that* $r_B < r_{A_+}$ *and* $\delta(\widehat{\boldsymbol{A}}(r_B)) < 1$*. Furthermore, if*

$$\breve{\boldsymbol{B}}(\infty) := \lim_{k\to\infty} \breve{\boldsymbol{B}}(k) \ge \boldsymbol{O}, \ne \boldsymbol{O}, \tag{2.33}$$

then

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{r_B^{-k} k^{\phi}} = \boldsymbol{x}(0)\breve{\boldsymbol{B}}(\infty)(\boldsymbol{I} - \widehat{\boldsymbol{A}}(r_B))^{-1}. \tag{2.34}$$

Remark 2.4 *We provide an example of a queueing model corresponding to the case where* $r = r_B$ *as follows. We consider a MAP/GI/1 queueing model with exceptional service, which represents the service for customers arriving to the queue when the server is vacant, i.e., the system is empty. For instance, the exceptional service time can be considered as the original service time including warming-up time, which is required for restarting the stopped server. We assume that the original service time and exceptional service time follow i.i.d. distributions. The queue length distribution at departure of this queueing model results in the stationary distribution of a certain M/G/1-type Markov chain. Furthermore, since the exceptional service time is longer than the original one, the impact of the jump size from the boundary level of the corresponding M/G/1-type Markov chain is larger than those from the non-boundary levels. In other words,* $\{\boldsymbol{B}(k); k = 0, 1, \dots\}$ *of the transition matrix of the Markov chain have dominant impact on the stationary distribution. As a result,* $r_B < \min(\theta_+, r_{A_+})$*, and thus the decay rate* r *of the stationary distribution can be equal to* r_B*.*

The following theorem is concerned with the second subcase, i.e., where $\theta_+ = r_B$.

Theorem 2.3 ($r = r_B = \theta_+$**)** *Suppose that Assumptions 1.1 and 2.1 together with Conditions 1.1 and 2.3 are satisfied. Suppose that the root* $\theta_+ \in (1, r_{A_+})$ *of the fundamental equation (1.36) exists and* $\theta_+ = r_B$*. Let* $\boldsymbol{\xi}(k; r_B) = k^{-1}\boldsymbol{\beta}(k; r_B) \ge \boldsymbol{0}$ *for* $k \in \mathbb{N}$*. Under these conditions, the following hold:*

(i) the asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ *is given by*

$$\overline{\boldsymbol{x}}(k) = r_B^{-k}\boldsymbol{x}(0)\boldsymbol{R}_0\widetilde{\overline{\boldsymbol{F}}}_{k-\lfloor k/\tau\rfloor\tau} + o(r_B^{-k})\boldsymbol{e}^{\top}, \qquad \phi < -1, \tag{2.35}$$

$$\overline{\boldsymbol{x}}(k) = r_B^{-k}\left[\boldsymbol{x}(0)\boldsymbol{R}_0\widetilde{\overline{\boldsymbol{F}}}_{k-\lfloor k/\tau\rfloor\tau} + \boldsymbol{\xi}(k; r_B)\right] + o(r_B^{-k})\boldsymbol{e}^{\top}, \qquad \phi = -1, \tag{2.36}$$

$$\overline{\boldsymbol{x}}(k) = r_B^{-k}k^{\phi+1}\boldsymbol{\xi}(k; r_B) + o(r_B^{-k}k^{\phi+1})\boldsymbol{e}^{\top}, \qquad \phi > -1;\ \textit{and} \tag{2.37}$$

(ii) for all $k \in \mathbb{N}$*,*

$$\boldsymbol{x}(0)\boldsymbol{R}_0\widetilde{\overline{\boldsymbol{F}}}_{k-\lfloor k/\tau\rfloor\tau} > \boldsymbol{0},$$

where $\widetilde{\overline{\boldsymbol{F}}}_{\ell}$ $(\ell = 0, \dots, \tau - 1)$ *is a nonnegative and nonzero matrix such that*

$$\widetilde{\overline{\boldsymbol{F}}}_{\ell} = \sum_{\nu=0}^{\tau-1} \frac{(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widehat{\boldsymbol{G}}(r_B\omega_\tau^\nu))}{(r_B\omega_\tau^\nu - 1)(\omega_\tau^\nu)^{\ell}} \cdot \frac{\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(r_B)\boldsymbol{\mu}(r_B)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}}{r_B(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y))\,|_{y=r_B}}. \tag{2.38}$$

In addition, suppose that

$$\liminf_{k\to\infty} \breve{\boldsymbol{B}}(k) \ge \boldsymbol{O}, \ne \boldsymbol{O}, \tag{2.39}$$

and either of the conditions (a) and (b) below holds.

(a) the root $\theta_- \in (r_{A_-}, 1]$ of the fundamental equation (1.36) exists; or

(b) Condition 2.2 is satisfied, $Y \in \mathcal{S}^*$ and $\delta(\widehat{\boldsymbol{A}}(r_{A_-})) < 1$.

It then holds that

(iii) $\limsup_{k\to\infty} \boldsymbol{\xi}(k; r_B) \ge \liminf_{k\to\infty} \boldsymbol{\xi}(k; r_B) > \mathbf{0}$.

Corollary 2.2 *Suppose that Assumptions 1.1 and 2.1 together with Conditions 1.1 and 2.3 are satisfied. Suppose that the root $\theta_+ \in (1, r_{A_+})$ of the fundamental equation (1.36) exists and $\theta_+ = r_B$. Furthermore, if (2.33) holds, then, for $\ell = 0, 1, \dots, \tau - 1$,*

$$\lim_{n\to\infty} \frac{\overline{\boldsymbol{x}}(n\tau + \ell)}{r_B^{-n\tau-\ell}} = \boldsymbol{x}(0)\boldsymbol{R}_0\widetilde{\overline{\boldsymbol{F}}}_\ell, \qquad \phi < -1, \tag{2.40}$$

$$\lim_{n\to\infty} \frac{\overline{\boldsymbol{x}}(n\tau + \ell)}{r_B^{-n\tau-\ell}} = \boldsymbol{x}(0)\boldsymbol{R}_0\widetilde{\overline{\boldsymbol{F}}}_\ell + \boldsymbol{\xi}(\infty; r_B), \qquad \phi = -1, \tag{2.41}$$

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{r_B^{-k} k^{\phi+1}} = \boldsymbol{\xi}(\infty; r_B), \qquad \phi > -1, \tag{2.42}$$

where

$$\boldsymbol{\xi}(\infty, r_B) = \frac{\boldsymbol{x}(0)\breve{\boldsymbol{B}}(\infty)\boldsymbol{v}(r_B)\boldsymbol{\mu}(r_B)}{r_B(\mathrm{d}/\mathrm{d}y)\delta(\widehat{\boldsymbol{A}}(y)) \mid_{y=r_B}} > \mathbf{0}. \tag{2.43}$$

2.3.3 Case where $r = r_{A_+}$

Finally, we consider two subcases: $r = r_{A_+} < r_B$ and $r = r_{A_+} = r_B$. It is assumed in both subcases that $\delta(\widehat{\boldsymbol{A}}(r_{A_+})) < 1$ and thus the root θ_+ does not exist (see Lemma 1.2).

The following theorem provides the asymptotic formula for the first subcase.

Theorem 2.4 ($r = r_{A_+} < r_B$) *Suppose that Assumptions 1.1 and 2.1 together with Condition 2.1 are satisfied. If $Y \in \mathcal{S}^*$, $\delta(\widehat{\boldsymbol{A}}(r_{A_+})) < 1$ and $r_{A_+} < r_B$, then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{x}(0)\widehat{\boldsymbol{R}}_0(r_{A_+})\boldsymbol{C}_F \ge \mathbf{0}, \ne \mathbf{0}, \tag{2.44}$$

where $\boldsymbol{C}_F$ is given in (2.18).

The asymptotic formula for the second subcase is proved under the following condition.

Condition 2.4 *There exists some random variable Y in $\mathbb{Z}_+$ with finite positive mean such that*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{C}_A \ge \boldsymbol{O}, \quad \lim_{k\to\infty} \frac{\overline{\boldsymbol{B}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{C}_B \ge \boldsymbol{O},$$

where $\boldsymbol{C}_A \ne \boldsymbol{O}$ or $\boldsymbol{C}_B \ne \boldsymbol{O}$.

Theorem 2.5 ($r = r_{A_+} = r_B$) *Suppose that Assumptions 1.1 and 2.1 together with Condition 2.4 are satisfied. If* $Y \in \mathcal{S}^*$, $\delta(\widehat{\boldsymbol{A}}(r_{A_+})) < 1$ *and* $r_{A_+} = r_B$, *then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{r_{A_+}^{-k}\mathsf{P}(Y > k)} = \boldsymbol{x}(0)\left[\boldsymbol{C}_B(\boldsymbol{I} - \widehat{\boldsymbol{A}}(r_{A_+}))^{-1} + \widehat{\boldsymbol{R}}_0(r_{A_+})\boldsymbol{C}_F\right] \ge \boldsymbol{0}, \ne \boldsymbol{0}. \tag{2.45}$$

Chapter 3

Heavy-Tailed Asymptotics

3.1 Introduction

This chapter studies the subexponential asymptotics of the stationary distribution of the GI/G/1-type Markov chain. We briefly review the literature related to the subexponential asymptotics. For this purpose, let Y denote a random variable in $\mathbb{Z}_+$, and for a while, assume that

$$\lim_{k\to\infty} \frac{\sum_{\ell=k+1}^{\infty} \boldsymbol{A}(\ell)}{\mathsf{P}(Y > k)} = \boldsymbol{C}_1 \ge \boldsymbol{O}, \quad \lim_{k\to\infty} \frac{\sum_{\ell=k+1}^{\infty} \boldsymbol{B}(\ell)}{\mathsf{P}(Y > k)} = \boldsymbol{C}_2 \ge \boldsymbol{O},$$

with $\boldsymbol{C}_1 \neq \boldsymbol{O}$ or $\boldsymbol{C}_2 \neq \boldsymbol{O}$. Asmussen and Møller [7] consider two cases: (a) Y is regularly varying; and (b) Y belongs to both the subexponential class $\mathcal{S}$ (see Definition C.2) and the maximum domain of attraction of the Gumbel distribution (see, e.g., [17, Section 3.3]). For the two cases, they show that under some additional conditions,

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\mathsf{P}(Y_{\rm e} > k)} = \boldsymbol{c}_1 > \boldsymbol{0}, \quad Y_{\rm e} \in \mathcal{S}, \tag{3.1}$$

where $Y_{\rm e}$ denotes the discrete equilibrium random variable of Y, distributed with $\mathsf{P}(Y_{\rm e} = k) = \mathsf{P}(Y > k)/\mathsf{E}[Y]$ ($k \in \mathbb{Z}_+$). Note here that $Y \in \mathcal{S}$ does not necessarily imply $Y_{\rm e} \in \mathcal{S}$ and vice versa (see [60, Remark 3.5]).

Li and Zhao [40] show the subexponential tail asymptotics (3.1) under the condition that $\boldsymbol{C}_2 = \boldsymbol{O}$ and Y belongs to a subclass $\mathcal{S}^*$ of $\mathcal{S}$ (see Definition C.3). Note here that $Y \in \mathcal{S}^*$ implies $Y \in \mathcal{S}$ and $Y_{\rm e} \in \mathcal{S}$ (see Proposition C.2). Although Li and Zhao [40] derive some other asymptotic formulae for $\{\overline{\boldsymbol{x}}(k)\}$, those formulae are incorrect due to "the inverse of a singular matrix" (detailed explanation can be found in [42]).

Takine [63] proves that the subexponential tail asymptotics (3.1) holds for an M/G/1-type Markov chain, assuming that $Y_{\rm e} \in \mathcal{S}$ but not necessarily $Y \in \mathcal{S}$. Thus Takine's result shows that $Y \in \mathcal{S}$ is not a necessary condition for the subexponential decay of $\{\overline{\boldsymbol{x}}(k)\}$. However, Masuyama [42] points out that Takine's proof needs an additional condition that the G-matrix is aperiodic. Furthermore, Masuyama [42] presents a weaker sufficient condition for (3.1) than those presented in the literature [7, 40, 63], though his result is limited to the M/G/1-type Markov chain. Recently, Kim and Kim [29] improve Masuyama [42]'s sufficient condition in the case where the G-matrix is periodic.

In this chapter, we study the subexponential decay of the tail probabilities $\{\overline{\boldsymbol{x}}(k)\}$ in two cases: (i) $\boldsymbol{A}$ is stochastic (i.e., $\boldsymbol{A}\boldsymbol{e} = \boldsymbol{e}$); and (ii) $\boldsymbol{A}$ is strictly substochastic (i.e., $\boldsymbol{A}\boldsymbol{e} \le \boldsymbol{e}, \neq \boldsymbol{e}$). For the case (i), we generalize Masuyama [42]'s and Kim and Kim [29]'s results to the GI/G/1-type Markov chain. The obtained sufficient condition for the subexponential tail asymptotics (3.1) is weaker than those presented in Asmussen and Møller [7] and Li and Zhao [40]. As for the case (ii), we present a subexponential asymptotic formula such that

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\mathsf{P}(Y>k)} = \boldsymbol{c}_2 > \boldsymbol{0}, \quad Y \in \mathcal{S}.$$

It should be noted that the embedded queue length process of a BMAP/GI/1 queue with disasters falls into the case (ii) (see, e.g., [58]). As far as we know, the subexponential asymptotics in the case (ii) has not been studied in the literature. Therefore, this is the first report on the subexponential asymptotics in the case (ii).

We also study the locally subexponential asymptotics of the stationary probabilities $\{\boldsymbol{x}(k)\}$. In the case (i) (i.e., $\boldsymbol{A}$ is stochastic), we prove the following formula under some technical conditions:

$$\lim_{k\to\infty} \frac{\boldsymbol{x}(k)}{\mathsf{P}(Y_{\rm e}=k)} = \boldsymbol{c}_3 > \boldsymbol{0}, \quad Y \in \mathcal{S}^*.$$

Furthermore, in the case (ii) (i.e., $\boldsymbol{A}$ is strictly substochastic), we assume that Y is *locally subexponential with span one* (i.e., $Y \in \mathcal{S}_{\rm loc}(1)$; see Definition C.5). We then show that

$$\lim_{k\to\infty} \frac{\boldsymbol{x}(k)}{\mathsf{P}(Y=k)} = \boldsymbol{c}_4 > \boldsymbol{0}, \quad Y \in \mathcal{S}_{\rm loc}(1),$$

with some technical conditions. For the reader's convenience, Appendix C.3 presents simple examples of the case where the stationary distribution is locally subexponential.

The rest of this chapter is divided into two sections. In Sections 3.2 and 3.3, we study the subexponential tail asymptotics and locally subexponential asymptotics, respectively, of the stationary distribution.

3.2 Ordinal subexponential asymptotics

This section studies the subexponential decay of the tail probabilities $\{\overline{\boldsymbol{x}}(k)\}$, under the following assumption.

Assumption 3.1 *Either of (I) and (II) is satisfied:*

(I) Assumption 1.1 holds, $\boldsymbol{A}$ is stochastic, and $\sum_{k\in\mathbb{Z}} |k|\boldsymbol{A}(k) < \infty$; or

(II) Assumption 1.1 holds and $\boldsymbol{A}$ is strictly substochastic.

Assumption 3.1 (I) and (II) are considered in subsections 3.2.1 and 3.2.2, respectively.

3.2.1 Case of stochastic $\boldsymbol{A}$

Lemma 3.1 *Under Assumption 3.1 (I),*

$$\sigma = \boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{R})(\boldsymbol{I}-\boldsymbol{\Phi}(0))\sum_{k=1}^{\infty} k\boldsymbol{G}(k)\boldsymbol{e} \in (0,\infty), \tag{3.2}$$

where σ is defined in (1.5).

Proof. We have $0 < \sigma < \infty$ due to (1.5), Proposition 1.7 and the third condition of Assumption 3.1 (I). Furthermore, since $\sigma = -\boldsymbol{\pi}(\mathrm{d}/\mathrm{d}z)\widehat{\boldsymbol{A}}(z)|_{z=1}\boldsymbol{e}$ and $(\mathrm{d}/\mathrm{d}z)\widehat{\boldsymbol{G}}(z)|_{z=1} = -\sum_{k=1}^{\infty} k\boldsymbol{G}(k)$, we obtain (3.2) by differentiating (1.16) with respect to z, pre-multiplying by $\boldsymbol{\pi}$, post-multiplying by $\boldsymbol{e}$ and letting $z = 1$. □

Since $\boldsymbol{A}$ is stochastic, the root of the fundamental equation θ_- is equal to 1 (see (1.36) and Lemma 1.3). In addition, Lemma 1.1 implies that if Assumption 3.1 (I) holds, the condition of Lemma 2.4 is satisfied. Thus, substituting $\theta_- = 1$ into (2.13) and using (1.43), $(\mathrm{d}/\mathrm{d}z)\delta(\widehat{\boldsymbol{A}}(z))|_{z=1} = \boldsymbol{\pi}\widehat{\boldsymbol{A}}'(1)\boldsymbol{e} = -\sigma$, $\boldsymbol{v}(1) = \boldsymbol{e}$, and $\boldsymbol{\mu}(1) = \boldsymbol{\pi}$ yield

$$\lim_{n\to\infty} [\boldsymbol{L}(n\tau + l)]_{i,j} = \begin{cases} \dfrac{\tau[\boldsymbol{\psi}]_j}{\sigma}, & \text{if } j \in \mathbb{M}_{\bullet}^{\mathrm{G}}, p(j) \equiv p(i) - l \pmod{\tau}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

where

$$\boldsymbol{\psi} = \boldsymbol{\psi}(1) = \boldsymbol{\pi}(\boldsymbol{I} - \boldsymbol{R})(\boldsymbol{I} - \boldsymbol{\Phi}(0)). \tag{3.4}$$

For $\ell = 0, 1, \dots, \tau - 1$, let $\mathbb{M}^{(\ell)} = \{j \in \mathbb{M}_{\bullet}^{\mathrm{G}}; p(j) = \ell\}$ and $|\mathbb{M}^{(\ell)}|$ denote the cardinality of $\mathbb{M}^{(\ell)}$. Furthermore, let $\boldsymbol{\psi}^{(\ell)}$ denote a subvector of $\boldsymbol{\psi}$ corresponding to $\mathbb{M}^{(\ell)}$, and $\boldsymbol{e}^{(\ell)}$ denote an $|\mathbb{M}^{(\ell)}| \times 1$ vector of ones. Using these notations, (3.3) can be rewritten as

$$\lim_{n\to\infty} \boldsymbol{L}(n\tau + \ell) = \tau \boldsymbol{E}\boldsymbol{H}_{\ell}, \tag{3.5}$$

where

$$\boldsymbol{E} = \begin{matrix} \mathbb{M}^{(0)} \\ \mathbb{M}^{(1)} \\ \vdots \\ \mathbb{M}^{(\tau-2)} \\ \mathbb{M}^{(\tau-1)} \end{matrix} \begin{pmatrix} \boldsymbol{e}^{(0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{e}^{(1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{e}^{(\tau-2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{e}^{(\tau-1)} \end{pmatrix}, \tag{3.6}$$

and

$$\sigma \boldsymbol{H}_{\ell} = \begin{array}{c} \begin{matrix} \mathbb{M}^{(0)} & \mathbb{M}^{(1)} & \cdots & \mathbb{M}^{(\tau-\ell-1)} & \mathbb{M}^{(\tau-\ell)} & \mathbb{M}^{(\tau-\ell+1)} & \cdots & \mathbb{M}^{(\tau-1)} \end{matrix} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\psi}^{(\tau-\ell)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \boldsymbol{\psi}^{(\tau-l+1)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\psi}^{(\tau-1)} \\ \boldsymbol{\psi}^{(0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\psi}^{(1)} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\psi}^{(\tau-\ell-1)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \end{array}.$$

Remark 3.1 *Suppose the Markov chain $\{(X_n, S_n)\}$ is of M/G/1-type. It then follows that $\boldsymbol{L}(n) = \boldsymbol{G}^n$ for $n = 1, 2, \dots$. Furthermore, it is easy to see that $\boldsymbol{\psi}/\sigma$ is a stationary probability vector of $\boldsymbol{G}$ and therefore $[\boldsymbol{\psi}]_j = 0$ for all $j \in \mathbb{M}_{\mathrm{T}}$ (see Lemma 1.5). We now define $\boldsymbol{\psi}_{\bullet}^{(\ell)}$ ($\ell = 0, 1, \dots, \tau - 1$) as a subvector of $\boldsymbol{\psi}$ corresponding to $\mathbb{M}_{\bullet}^{(\ell)} := \{j \in \mathbb{M}_{\bullet} \cap \mathbb{M}^{(\ell)}\}$. As a result, (3.5) yields*

$$\lim_{n\to\infty} \frac{1}{\tau}\boldsymbol{G}^{n\tau} = \begin{array}{c} \\ \mathbb{M}_{\bullet}^{(0)} \\ \mathbb{M}_{\bullet}^{(1)} \\ \vdots \\ \mathbb{M}_{\bullet}^{(\tau-1)} \\ \mathbb{M}_{\mathrm{T}}^{(0)} \\ \mathbb{M}_{\mathrm{T}}^{(1)} \\ \vdots \\ \mathbb{M}_{\mathrm{T}}^{(\tau-1)} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{M}_{\bullet}^{(0)} & \mathbb{M}_{\bullet}^{(1)} & \cdots & \mathbb{M}_{\bullet}^{(\tau-1)} & \mathbb{M}_{\mathrm{T}} \end{array} \\ \begin{pmatrix} \boldsymbol{e}\boldsymbol{\psi}_{\bullet}^{(0)}/\sigma & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{e}\boldsymbol{\psi}_{\bullet}^{(1)}/\sigma & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{e}\boldsymbol{\psi}_{\bullet}^{(\tau-1)}/\sigma & \boldsymbol{O} \\ \boldsymbol{e}\boldsymbol{\psi}_{\bullet}^{(0)}/\sigma & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{e}\boldsymbol{\psi}_{\bullet}^{(1)}/\sigma & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{e}\boldsymbol{\psi}_{\bullet}^{(\tau-1)}/\sigma & \boldsymbol{O} \end{pmatrix} \end{array}, \tag{3.7}$$

where $\mathbb{M}_{\mathrm{T}}^{(\ell)} = \mathbb{M}^{(\ell)} \setminus \mathbb{M}_{\bullet}^{(\ell)}$ ($\ell = 0, 1, \dots, \tau - 1$). Note here that $\boldsymbol{\psi}_{\bullet}^{(\ell)}\boldsymbol{e} = \sigma/\tau$ for all $\ell = 0, 1, \dots, \tau - 1$ because $(1/\tau)\boldsymbol{G}^{n\tau}\boldsymbol{e} = \boldsymbol{e}/\tau$ for all $n = 1, 2, \dots$. As a result, the limit (3.7) is consistent with the equation (14) in [42], where $\sum_{\nu=1}^{\tau} \boldsymbol{f}_{\nu} = \boldsymbol{e}$ and each element of $\boldsymbol{f}_{\nu}$ ($\nu = 1, 2, \dots, \tau$) is equal to one or zero.

Lemma 3.2 *If Assumption 3.1 (I) holds, then*

$$\lim_{n\to\infty} \sum_{\ell=0}^{\tau-1} \boldsymbol{L}(n\tau + \ell) = \frac{\tau}{\sigma}\boldsymbol{e}\boldsymbol{\psi}. \tag{3.8}$$

Proof. We obtain (3.8) by combining (3.5) and

$$\sigma \sum_{\ell=0}^{\tau-1} \boldsymbol{H}_{\ell} = \boldsymbol{e}\boldsymbol{\psi}. \tag{3.9}$$

□

We now make the following assumption.

Assumption 3.2 *There exists some random variable Y in $\mathbb{Z}_+$ with positive finite mean such that*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k)\boldsymbol{e}}{\mathsf{P}(Y > k)} = \frac{\boldsymbol{c}_A}{\mathsf{E}[Y]}, \quad \lim_{k\to\infty} \frac{\overline{\boldsymbol{B}}(k)\boldsymbol{e}}{\mathsf{P}(Y > k)} = \frac{\boldsymbol{c}_B}{\mathsf{E}[Y]}, \tag{3.10}$$

where $\boldsymbol{c}_A$ and $\boldsymbol{c}_B$ are nonnegative $M \times 1$ and $M_0 \times 1$ vectors, respectively, satisfying $\boldsymbol{c}_A \neq \boldsymbol{0}$ or $\boldsymbol{c}_B \neq \boldsymbol{0}$.

Lemma 3.3 *Suppose Assumptions 3.1 (I) and 3.2 hold. If Y_{e} is long-tailed (i.e., $Y_{\mathrm{e}} \in \mathcal{L}$; see Definition C.1), then*

$$\lim_{k\to\infty} \sum_{m=1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}} > k)} = \frac{\boldsymbol{c}_A\boldsymbol{\pi}(\boldsymbol{I} - \boldsymbol{R})(\boldsymbol{I} - \boldsymbol{\Phi}(0))}{\sigma}, \tag{3.11}$$

$$\lim_{k\to\infty} \sum_{m=1}^{\infty} \frac{\overline{\boldsymbol{B}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}} > k)} = \frac{\boldsymbol{c}_B\boldsymbol{\pi}(\boldsymbol{I} - \boldsymbol{R})(\boldsymbol{I} - \boldsymbol{\Phi}(0))}{\sigma}. \tag{3.12}$$

Proof. Equations (3.8) and (3.10) show that for any $\varepsilon > 0$ there exists some $m_* := m_*(\varepsilon) \in \mathbb{N}$ such that for all $m \ge m_*$ and $\ell = 0, 1, \dots, \tau - 1$,

$$\boldsymbol{e}(\tau\boldsymbol{\psi} - \varepsilon\boldsymbol{e}^\top) \le \sum_{\ell=0}^{\tau-1} \boldsymbol{L}(\lfloor m/\tau \rfloor \tau + l) \le \boldsymbol{e}(\tau\boldsymbol{\psi} + \varepsilon\boldsymbol{e}^\top), \tag{3.13}$$

$$\frac{1}{\mathsf{E}[Y]}(\boldsymbol{c}_A - \varepsilon\boldsymbol{e}) \le \frac{\overline{\boldsymbol{A}}(\lfloor m/\tau \rfloor \tau + \ell)\boldsymbol{e}}{\mathsf{P}(Y > m)} \le \frac{1}{\mathsf{E}[Y]}(\boldsymbol{c}_A + \varepsilon\boldsymbol{e}). \tag{3.14}$$

Furthermore, since $Y_{\rm e} \in \mathcal{L}$ and $\boldsymbol{L}(m) \le \boldsymbol{e}\boldsymbol{e}^\top$ for all $m = 1, 2, \dots$, we have

$$\begin{aligned}
\limsup_{k\to\infty} \sum_{m=1}^{m_*-1} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e} > k)} &\le \sum_{m=1}^{m_*-1} \limsup_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{e}\boldsymbol{e}^\top}{\mathsf{P}(Y > k+m)} \limsup_{k\to\infty} \frac{\mathsf{P}(Y > k+m)}{\mathsf{P}(Y_{\rm e} > k+m)} \\
&\qquad \times \limsup_{k\to\infty} \frac{\mathsf{P}(Y_{\rm e} > k+m)}{\mathsf{P}(Y_{\rm e} > k)} \\
&= \boldsymbol{O},
\end{aligned} \tag{3.15}$$

where the last equality follows from (3.10) and the fact that $Y_{\rm e} \in \mathcal{L}$ has a heavier tail than Y (see Corollary 3.3 in [60]).

On the other hand,

$$\begin{aligned}
\sum_{m=m_*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e} > k)} &\le \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \sum_{\ell=0}^{\tau-1} \frac{\overline{\boldsymbol{A}}(k+m'\tau+\ell)\boldsymbol{L}(m'\tau+\ell)}{\mathsf{P}(Y_{\rm e} > k)} \\
&\le \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m'\tau)}{\mathsf{P}(Y_{\rm e} > k)} \sum_{\ell=0}^{\tau-1} \boldsymbol{L}(m'\tau+\ell) \\
&\le \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m'\tau)\boldsymbol{e}}{\mathsf{P}(Y_{\rm e} > k)} \left(\frac{\tau}{\sigma}\boldsymbol{\psi} + \varepsilon\boldsymbol{e}^\top \right),
\end{aligned} \tag{3.16}$$

where the second inequality holds because $\{\overline{\boldsymbol{A}}(k); k \in \mathbb{Z}_+\}$ is nonincreasing, and where the last inequality is due to (3.13). Note here that (3.10) implies for all sufficiently large k,

$$\sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m'\tau)\boldsymbol{e}}{\mathsf{P}(Y_{\rm e} > k)} \le (\boldsymbol{c}_A + \varepsilon\boldsymbol{e}) \cdot \frac{1}{\mathsf{E}[Y]} \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\mathsf{P}(Y > k+m'\tau)}{\mathsf{P}(Y_{\rm e} > k)},$$

from which and Proposition C.1 it follows that

$$\limsup_{k\to\infty} \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m'\tau)\boldsymbol{e}}{\mathsf{P}(Y_{\rm e} > k)} \le \frac{\boldsymbol{c}_A + \varepsilon\boldsymbol{e}}{\tau}. \tag{3.17}$$

Combining (3.16) and (3.17) and letting $\varepsilon \downarrow 0$ yield

$$\limsup_{k\to\infty} \sum_{m=m_*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e} > k)} \le \frac{\boldsymbol{c}_A\boldsymbol{\psi}}{\sigma}. \tag{3.18}$$

As a result, from (3.15) and (3.18), we have

$$\limsup_{k\to\infty} \sum_{m=1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e} > k)} \le \frac{\boldsymbol{c}_A\boldsymbol{\psi}}{\sigma}. \tag{3.19}$$

Next we consider the lower limit. It follows from (3.13) and (3.14) that

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}}>k)} &\ge \sum_{m=m_*}^{\infty}\frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}}>k)}\\ &\ge \sum_{m'=\lfloor m_*/\tau\rfloor+1}^{\infty}\sum_{\ell=0}^{\tau-1}\frac{\overline{\boldsymbol{A}}(k+m'\tau+\ell)\boldsymbol{L}(m'\tau+\ell)}{\mathsf{P}(Y_{\mathrm{e}}>k)}\\ &\ge \sum_{m'=\lfloor m_*/\tau\rfloor+1}^{\infty}\frac{\overline{\boldsymbol{A}}(k+m'\tau+\tau)}{\mathsf{P}(Y_{\mathrm{e}}>k)}\sum_{\ell=0}^{\tau-1}\boldsymbol{L}(m'\tau+\ell)\\ &\ge \sum_{m'=\lfloor m_*/\tau\rfloor+2}^{\infty}\frac{\overline{\boldsymbol{A}}(k+m'\tau)\boldsymbol{e}}{\mathsf{P}(Y_{\mathrm{e}}>k)}\left(\frac{\tau}{\sigma}\boldsymbol{\psi}-\varepsilon\boldsymbol{e}^{\top}\right), \end{aligned}\tag{3.20}$$

where the third inequality requires the fact that $\{\overline{\boldsymbol{A}}(k)\}$ is nonincreasing. Furthermore, the following can be shown in a very similar way to (3.17):

$$\liminf_{k\to\infty}\sum_{m'=\lfloor m_*/\tau\rfloor+2}^{\infty}\frac{\overline{\boldsymbol{A}}(k+m'\tau)\boldsymbol{e}}{\mathsf{P}(Y_{\mathrm{e}}>k)}\ge\frac{\boldsymbol{c}_A-\varepsilon\boldsymbol{e}}{\tau}.$$

Combining this with (3.20) and letting $\varepsilon\downarrow 0$ yield

$$\liminf_{k\to\infty}\sum_{m=1}^{\infty}\frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}}>k)}\ge\frac{\boldsymbol{c}_A\boldsymbol{\psi}}{\sigma}.\tag{3.21}$$

Finally, (3.11) follows from (3.19), (3.21) and (3.4). Equation (3.12) can be proved in the same way, and thus the proof is omitted.

□

Lemma 3.4 *Suppose Assumptions 3.1 (I) and 3.2 hold. If* $Y_{\mathrm{e}}\in\mathcal{L}$*, then*

$$\lim_{k\to\infty}\frac{\overline{\boldsymbol{R}}(k)}{\mathsf{P}(Y_{\mathrm{e}}>k)} = \frac{\boldsymbol{c}_A\boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{R})}{\sigma},\tag{3.22}$$

$$\lim_{k\to\infty}\frac{\overline{\boldsymbol{R}_0}(k)}{\mathsf{P}(Y_{\mathrm{e}}>k)} = \frac{\boldsymbol{c}_B\boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{R})}{\sigma}.\tag{3.23}$$

Proof. It follows from (1.15) that

$$\overline{\boldsymbol{R}}(k)=\left[\overline{\boldsymbol{A}}(k)+\sum_{m=1}^{\infty}\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)\right](\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}.\tag{3.24}$$

Note that Corollary 3.3 in [60] and (3.10) yield

$$\limsup_{k\to\infty}\frac{\overline{\boldsymbol{A}}(k)}{\mathsf{P}(Y_{\mathrm{e}}>k)}\le\limsup_{k\to\infty}\frac{\overline{\boldsymbol{A}}(k)\boldsymbol{e}\boldsymbol{e}^{\mathrm{t}}}{\mathsf{P}(Y>k)}\limsup_{k\to\infty}\frac{\mathsf{P}(Y>k)}{\mathsf{P}(Y_{\mathrm{e}}>k)}=\boldsymbol{O}.$$

Thus from (3.24), we have

$$\lim_{k\to\infty}\frac{\overline{\boldsymbol{R}}(k)}{\mathsf{P}(Y_{\mathrm{e}}>k)}=\lim_{k\to\infty}\sum_{m=1}^{\infty}\frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}}>k)}(\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}.\tag{3.25}$$

Substituting (3.11) into (3.25), we obtain (3.22). Similarly, we can prove (3.23). □

The following theorem presents a subexponential asymptotic formula for $\{\overline{\boldsymbol{x}}(k)\}$.

Theorem 3.1 *Suppose Assumptions 3.1 (I) and 3.2 hold. If $Y_{\rm e} \in \mathcal{S}$, then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\mathsf{P}(Y_{\rm e} > k)} = \frac{\boldsymbol{x}(0)\boldsymbol{c}_B + \overline{\boldsymbol{x}}(0)\boldsymbol{c}_A}{\sigma} \cdot \boldsymbol{\pi} \tag{3.26}$$

Proof. It follows from (1.21) that

$$\sum_{k=0}^{\infty} \boldsymbol{F}(k) = (\boldsymbol{I} - \boldsymbol{R})^{-1}. \tag{3.27}$$

Thus using (3.27) and Lemma 6 in [28], we have

$$\begin{aligned}\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}}(k)}{\mathsf{P}(Y_{\rm e} > k)} &= \lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{\overline{\boldsymbol{R}^{*n}}(k)}{\mathsf{P}(Y_{\rm e} > k)} \\ &= (\boldsymbol{I} - \boldsymbol{R})^{-1} \lim_{k\to\infty} \frac{\overline{\boldsymbol{R}}(k)}{\mathsf{P}(Y_{\rm e} > k)} (\boldsymbol{I} - \boldsymbol{R})^{-1}.\end{aligned}$$

Substituting (3.22) into the above equation yields

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}}(k)}{\mathsf{P}(Y_{\rm e} > k)} = \frac{(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{c}_A\boldsymbol{\pi}}{\sigma}. \tag{3.28}$$

Finally, applying Proposition C.3 to (1.22) and using (3.23) and (3.28) lead to

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\mathsf{P}(Y_{\rm e} > k)} = \frac{\boldsymbol{x}(0)}{\sigma} \left[\boldsymbol{c}_B\boldsymbol{\pi} + \boldsymbol{R}_0(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{c}_A\boldsymbol{\pi}\right],$$

from which and (1.26) we have (3.26). □

Remark 3.2 *Theorem 3.1 is a generalization of Theorem 1 in [29] to the GI/G/1-type Markov chain. In fact, the latter extends the corollary of Theorem 3.1 in [42] (Corollary 3.1 therein) to the case where the G-matrix is periodic.*

3.2.2 Case of strictly substochastic $\boldsymbol{A}$

In this subsection, we make the following assumption in addition to Assumption 3.1 (II):

Assumption 3.3 *There exists some random variable Y in $\mathbb{Z}_+$ such that*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k)}{\mathsf{P}(Y > k)} = \boldsymbol{C}_A, \quad \lim_{k\to\infty} \frac{\overline{\boldsymbol{B}}(k)}{\mathsf{P}(Y > k)} = \boldsymbol{C}_B, \tag{3.29}$$

where $\boldsymbol{C}_A$ and $\boldsymbol{C}_B$ are nonnegative $M \times M$ and $M_0 \times M$ matrices, respectively, satisfying $\boldsymbol{C}_A \neq \boldsymbol{O}$ or $\boldsymbol{C}_B \neq \boldsymbol{O}$.

Lemma 3.5 *Suppose Assumptions 3.1 (II) and 3.3 hold. If $Y \in \mathcal{L}$, then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{R}}(k)}{\mathsf{P}(Y > k)} = \boldsymbol{C}_A \left(\boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell)\right)^{-1}, \tag{3.30}$$

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{R}_0}(k)}{\mathsf{P}(Y > k)} = \boldsymbol{C}_B \left(\boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell)\right)^{-1}. \tag{3.31}$$

Proof. From (1.15) and (3.29), we have

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{R}}(k)}{\mathsf{P}(Y>k)} = \left[\boldsymbol{C}_A + \lim_{k\to\infty}\sum_{m=1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)}{\mathsf{P}(Y>k)}\boldsymbol{L}(m)\right](\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}. \tag{3.32}$$

Note here that under Assumption 3.1 (II), $\mathrm{sp}(\boldsymbol{G}) < 1$ (see Proposition 1.4) and thus (1.11) yields

$$\sum_{m=1}^{\infty} \boldsymbol{L}(m) = (\boldsymbol{I}-\boldsymbol{G})^{-1}\boldsymbol{G} < \infty, \tag{3.33}$$

from which and (3.29) it follows that for $k = 0, 1, \dots,$

$$\sum_{m=1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)}{\mathsf{P}(Y>k)}\boldsymbol{L}(m) \le \sup_{k\in\mathbb{Z}_+} \frac{\overline{\boldsymbol{A}}(k)}{\mathsf{P}(Y>k)} \sum_{m=1}^{\infty}\boldsymbol{L}(m) < \infty.$$

Therefore applying the dominated convergence theorem to (3.32) and using (3.29) and $Y \in \mathcal{L}$, we obtain

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\overline{\boldsymbol{R}}(k)}{\mathsf{P}(Y>k)} \\
&= \left[\boldsymbol{C}_A + \sum_{m=1}^{\infty} \lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k+m)}{\mathsf{P}(Y>k+m)} \frac{\mathsf{P}(Y>k+m)}{\mathsf{P}(Y>k)}\boldsymbol{L}(m)\right](\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1} \\
&= \boldsymbol{C}_A\left[\boldsymbol{I} + (\boldsymbol{I}-\boldsymbol{G})^{-1}\boldsymbol{G}\right](\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1} \\
&= \boldsymbol{C}_A(\boldsymbol{I}-\boldsymbol{G})^{-1}(\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}.
\end{aligned} \tag{3.34}$$

From (1.9), we have

$$\begin{aligned}
(\boldsymbol{I}-\boldsymbol{G})^{-1} &= \left[\boldsymbol{I} - (\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}\sum_{\ell=1}^{\infty}\boldsymbol{\Phi}(-\ell)\right]^{-1} \\
&= \left(\boldsymbol{I} - \sum_{\ell=0}^{\infty}\boldsymbol{\Phi}(-\ell)\right)^{-1}(\boldsymbol{I}-\boldsymbol{\Phi}(0)).
\end{aligned} \tag{3.35}$$

Finally, substituting (3.35) into (3.34) yields (3.30). Equation (3.31) can be proved in the same way. □

Theorem 3.2 *Suppose Assumptions 3.1 (II) and 3.3 hold. If $Y \in \mathcal{S}$, then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\mathsf{P}(Y>k)} = [\boldsymbol{x}(0)\boldsymbol{C}_B + \overline{\boldsymbol{x}}(0)\boldsymbol{C}_A](\boldsymbol{I}-\boldsymbol{A})^{-1} > \boldsymbol{0}. \tag{3.36}$$

Proof. Applying Proposition C.3 to (1.22) and using (3.27) and (3.31), we have

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\mathsf{P}(Y>k)} = \boldsymbol{x}(0)\boldsymbol{C}_B\left(\boldsymbol{I} - \sum_{\ell=0}^{\infty}\boldsymbol{\Phi}(-\ell)\right)^{-1}(\boldsymbol{I}-\boldsymbol{R})^{-1} + \boldsymbol{x}(0)\boldsymbol{R}_0 \lim_{k\to\infty}\frac{\overline{\boldsymbol{F}}(k)}{\mathsf{P}(Y>k)}, \tag{3.37}$$

where $\boldsymbol{F}(k)$ is given in (1.21). Furthermore, it follows from Lemma 6 in [28] and (3.30) that

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}}(k)}{\mathsf{P}(Y>k)} = (\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{C}_A\left(\boldsymbol{I} - \sum_{\ell=0}^{\infty}\boldsymbol{\Phi}(-\ell)\right)^{-1}(\boldsymbol{I}-\boldsymbol{R})^{-1}.$$

Substituting the above equation into (3.37) and using (1.26), we have

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\mathsf{P}(Y>k)} = [\boldsymbol{x}(0)\boldsymbol{C}_B + \overline{\boldsymbol{x}}(0)\boldsymbol{C}_A] \left(\boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell) \right)^{-1} (\boldsymbol{I}-\boldsymbol{R})^{-1}. \tag{3.38}$$

Note here that (3.35) yields

$$\left(\boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell) \right)^{-1} (\boldsymbol{I}-\boldsymbol{R})^{-1} = (\boldsymbol{I}-\boldsymbol{G})^{-1} (\boldsymbol{I}-\boldsymbol{\Phi}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R})^{-1} = (\boldsymbol{I}-\boldsymbol{A})^{-1}, \tag{3.39}$$

where the second equality follows from Proposition 1.3. As a result, we obtain (3.36) by combining (3.38) with (3.39).

It is easy to show that the right hand side of (3.36) is positive. Indeed, $(\boldsymbol{I}-\boldsymbol{A})^{-1} > \boldsymbol{O}$ due to the irreducibility of $\boldsymbol{A}$. In addition, $\boldsymbol{x}(0)\boldsymbol{C}_B + \overline{\boldsymbol{x}}(0)\boldsymbol{C}_A \ge \boldsymbol{0}, \neq \boldsymbol{0}$ because $\boldsymbol{x}(0) > \boldsymbol{0}$ and $\overline{\boldsymbol{x}}(0) > \boldsymbol{0}$; and $\boldsymbol{C}_A \neq \boldsymbol{O}$ or $\boldsymbol{C}_B \neq \boldsymbol{O}$. Therefore, $(\boldsymbol{x}(0)\boldsymbol{C}_B + \overline{\boldsymbol{x}}(0)\boldsymbol{C}_A)(\boldsymbol{I}-\boldsymbol{A})^{-1} > \boldsymbol{0}$. □

3.3 Locally subexponential asymptotics

This section considers the locally subexponential asymptotics of the stationary distribution.

3.3.1 Case of stochastic A

In this subsection, we proceed under Assumption 3.1 (I) and the following assumption:

Assumption 3.4 *There exists some random variable Y in $\mathbb{Z}_+$ with positive finite mean such that*

$$\lim_{k\to\infty} \frac{\boldsymbol{A}(k)\boldsymbol{E}}{\mathsf{P}(Y=k)} = \frac{\boldsymbol{C}_A^{\mathrm{E}}}{\mathsf{E}[Y]}, \quad \lim_{k\to\infty} \frac{\boldsymbol{B}(k)\boldsymbol{E}}{\mathsf{P}(Y=k)} = \frac{\boldsymbol{C}_B^{\mathrm{E}}}{\mathsf{E}[Y]}, \tag{3.40}$$

where $\boldsymbol{E}$ is given in (3.6), and where $\boldsymbol{C}_A^{\mathrm{E}}$ and $\boldsymbol{C}_B^{\mathrm{E}}$ are nonnegative $M\times\tau$ and $M_0\times\tau$ matrices, respectively, satisfying $\boldsymbol{C}_A^{\mathrm{E}} \neq \boldsymbol{O}$ or $\boldsymbol{C}_B^{\mathrm{E}} \neq \boldsymbol{O}$.

Lemma 3.6 *Suppose Assumptions 3.1 (I) and 3.4 hold. Furthermore, suppose either of the following is satisfied: Y is locally long-tailed with span one (i.e., $Y \in \mathcal{L}_{\mathrm{loc}}(1)$; see Definition C.4); or $Y \in \mathcal{L}$ and $\{\mathsf{P}(Y=k)\}$ is eventually nonincreasing. We then have*

$$\lim_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}}=k)} = \boldsymbol{C}_A^{\mathrm{E}}\boldsymbol{e}\frac{\boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{R})(\boldsymbol{I}-\boldsymbol{\Phi}(0))}{\sigma}, \tag{3.41}$$

$$\lim_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{B}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\mathrm{e}}=k)} = \boldsymbol{C}_B^{\mathrm{E}}\boldsymbol{e}\frac{\boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{R})(\boldsymbol{I}-\boldsymbol{\Phi}(0))}{\sigma}. \tag{3.42}$$

Proof. We give the proof of (3.41) only. Equation (3.42) can be proved in the same way. It follows from (3.5), $\boldsymbol{E}\boldsymbol{e} = \boldsymbol{e}$ and (3.4) that for $\varepsilon > 0$ there exists some $m_* := m_*(\varepsilon) \in \mathbb{N}$ such that for all $m \ge m_*$ and $\ell = 0,1,\dots,\tau-1$,

$$\boldsymbol{E}(\tau\boldsymbol{H}_\ell - \varepsilon\boldsymbol{e}\boldsymbol{e}^\top) \le \boldsymbol{L}(m) \le \boldsymbol{E}(\tau\boldsymbol{H}_\ell + \varepsilon\boldsymbol{e}\boldsymbol{e}^\top), \quad m \equiv \ell \pmod{\tau}, \tag{3.43}$$

$$\frac{1}{\mathsf{E}[Y]}(\boldsymbol{C}_A^{\mathrm{E}} - \varepsilon\boldsymbol{e}\boldsymbol{e}^\top) \le \frac{\boldsymbol{A}(m)\boldsymbol{E}}{\mathsf{P}(Y=m)} \le \frac{1}{\mathsf{E}[Y]}(\boldsymbol{C}_A^{\mathrm{E}} + \varepsilon\boldsymbol{e}\boldsymbol{e}^\top). \tag{3.44}$$

Thus from (3.40), $\boldsymbol{L}(m) \le \boldsymbol{E}\boldsymbol{e}\boldsymbol{e}^{\top}$ and $Y \in \mathcal{L}$ (see Remark C.2), we have

$$\lim_{k\to\infty} \sum_{m=1}^{m_*-1} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e}=k)} \le \mathsf{E}[Y] \sum_{m=1}^{m_*-1} \lim_{k\to\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{E}\boldsymbol{e}\boldsymbol{e}^{\top}}{\mathsf{P}(Y=k+m)} \frac{\mathsf{P}(Y=k+m)}{\mathsf{P}(Y>k)} = \boldsymbol{O}.$$

Using this and (3.43), we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e}=k)} &= \limsup_{k\to\infty} \sum_{m=m_*}^{\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e}=k)} \\
&= \limsup_{k\to\infty} \sum_{\ell=0}^{\tau-1} \sum_{\substack{m\ge m_* \\ m\equiv \ell \,(\mathrm{mod}\,\tau)}} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e}=k)} \\
&\le \sum_{\ell=0}^{\tau-1} \left[\limsup_{k\to\infty} \sum_{\substack{m\ge m_* \\ m\equiv \ell \,(\mathrm{mod}\,\tau)}} \frac{\boldsymbol{A}(k+m)\boldsymbol{E}}{\mathsf{P}(Y_{\rm e}=k)} \right] (\tau \boldsymbol{H}_{\ell} + \varepsilon \boldsymbol{e}\boldsymbol{e}^{\top}). \qquad (3.45)
\end{aligned}$$

Furthermore, it follows from (3.44) and Proposition C.4 that

$$\begin{aligned}
\limsup_{k\to\infty} \sum_{\substack{m\ge m_* \\ m\equiv \ell \,(\mathrm{mod}\,\tau)}} \frac{\boldsymbol{A}(k+m)\boldsymbol{E}}{\mathsf{P}(Y_{\rm e}=k)} &\le \frac{\boldsymbol{C}_A^{\rm E} + \varepsilon \boldsymbol{e}\boldsymbol{e}^{\top}}{\mathsf{E}[Y]} \limsup_{k\to\infty} \sum_{\substack{m\ge m_* \\ m\equiv \ell \,(\mathrm{mod}\,\tau)}} \frac{\mathsf{P}(Y=k+m)}{\mathsf{P}(Y_{\rm e}=k)} \\
&= \frac{\boldsymbol{C}_A^{\rm E} + \varepsilon \boldsymbol{e}\boldsymbol{e}^{\top}}{\tau}. \qquad (3.46)
\end{aligned}$$

Substituting (3.46) into (3.45) and letting $\varepsilon \downarrow 0$, we obtain

$$\limsup_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e}=k)} \le \boldsymbol{C}_A^{\rm E} \sum_{\ell=0}^{\tau-1} \boldsymbol{H}_{\ell} = \frac{1}{\sigma} \boldsymbol{C}_A^{\rm E} \boldsymbol{e}\boldsymbol{\psi},$$

where we use (3.9) in the last equality. Similarly, we can show that

$$\liminf_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e}=k)} \ge \frac{1}{\sigma} \boldsymbol{C}_A^{\rm E} \boldsymbol{e}\boldsymbol{\psi}.$$

As a result,

$$\lim_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e}=k)} = \frac{1}{\sigma} \boldsymbol{C}_A^{\rm E} \boldsymbol{e}\boldsymbol{\psi},$$

from which and (3.4) we have (3.41). □

Remark 3.3 *Lemma 3.6 is proved by using Proposition C.4, which requires either that $Y \in \mathcal{L}_{\rm loc}(1)$ or that $Y \in \mathcal{L}$ and $\{\mathsf{P}(Y=k)\}$ is eventually nonincreasing.*

Lemma 3.7 *Under the same assumptions as in Lemma 3.6,*

$$\lim_{k\to\infty} \frac{\boldsymbol{R}(k)}{\mathsf{P}(Y_{\rm e}=k)} = \boldsymbol{C}_A^{\rm E} \boldsymbol{e} \frac{\boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{R})}{\sigma}, \qquad (3.47)$$

$$\lim_{k\to\infty} \frac{\boldsymbol{R}_0(k)}{\mathsf{P}(Y_{\rm e}=k)} = \boldsymbol{C}_B^{\rm E} \boldsymbol{e} \frac{\boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{R})}{\sigma}. \qquad (3.48)$$

Proof. It follows from $\boldsymbol{E}\boldsymbol{e} = \boldsymbol{e}$, (3.40) and $Y \in \mathcal{L}$ that

$$\lim_{k\to\infty} \frac{\boldsymbol{A}(k)}{\mathsf{P}(Y_{\rm e} = k)} \le \mathsf{E}[Y] \lim_{k\to\infty} \frac{\boldsymbol{A}(k)\boldsymbol{E}\boldsymbol{e}\boldsymbol{e}^\top}{\mathsf{P}(Y > k)} \frac{\mathsf{P}(Y = k)}{\mathsf{P}(Y > k)} = \boldsymbol{O}.$$

Thus from (1.15), we have

$$\lim_{k\to\infty} \frac{\boldsymbol{R}(k)}{\mathsf{P}(Y_{\rm e} = k)} = \lim_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)\boldsymbol{L}(m)}{\mathsf{P}(Y_{\rm e} = k)} (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}. \tag{3.49}$$

Substituting (3.41) into (3.49) yields (3.47). Similarly, we can readily show (3.48). □

We now obtain a locally subexponential asymptotic formula for $\{\boldsymbol{x}(k)\}$.

Theorem 3.3 *Suppose Assumptions 3.1 (I) and 3.4 hold. Furthermore, suppose (i) $Y_{\rm e}$ is locally subexponential with span one (i.e., $Y_{\rm e} \in \mathcal{S}_{\rm loc}(1)$; see Definition C.5); and (ii) $Y \in \mathcal{L}_{\rm loc}(1)$ or $\{\mathsf{P}(Y = k)\}$ is eventually nonincreasing. We then have*

$$\lim_{k\to\infty} \frac{\boldsymbol{x}(k)}{\mathsf{P}(Y_{\rm e} = k)} = \frac{\boldsymbol{x}(0)\boldsymbol{C}_B^{\rm E}\boldsymbol{e} + \overline{\boldsymbol{x}}(0)\boldsymbol{C}_A^{\rm E}\boldsymbol{e}}{\sigma} \cdot \boldsymbol{\pi}. \tag{3.50}$$

Remark 3.4 *According to Definition C.5 and Proposition C.5, $Y_{\rm e} \in \mathcal{S}_{\rm loc}(1)$ is equivalent to $Y \in \mathcal{S}^*$. Thus since $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L}$, the assumptions of Theorem 3.3 are sufficient for those of Lemma 3.6.*

Proof of Theorem 3.3. Proposition C.9 yields

$$\begin{aligned} \lim_{k\to\infty} \frac{\boldsymbol{F}(k)}{\mathsf{P}(Y_{\rm e} = k)} &= \lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{\boldsymbol{R}^{*n}(k)}{\mathsf{P}(Y_{\rm e} = k)} \\ &= (\boldsymbol{I} - \boldsymbol{R})^{-1} \lim_{k\to\infty} \frac{\boldsymbol{R}(k)}{\mathsf{P}(Y_{\rm e} = k)} (\boldsymbol{I} - \boldsymbol{R})^{-1}, \end{aligned}$$

from which and (3.47) it follows that

$$\lim_{k\to\infty} \frac{\boldsymbol{F}(k)}{\mathsf{P}(Y_{\rm e} = k)} = \frac{(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{C}_A^{\rm E}\boldsymbol{e}\boldsymbol{\pi}}{\sigma}. \tag{3.51}$$

Furthermore applying Proposition C.8 to (1.20) and using (3.48) and (3.51), we obtain

$$\lim_{k\to\infty} \frac{\boldsymbol{x}(k)}{\mathsf{P}(Y_{\rm e} = k)} = \frac{\boldsymbol{x}(0)}{\sigma} \left[\boldsymbol{C}_B^{\rm E}\boldsymbol{e}\boldsymbol{\pi} + \boldsymbol{R}_0(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{C}_A^{\rm E}\boldsymbol{e}\boldsymbol{\pi}\right].$$

Substituting (1.26) into the above equation yields (3.50). □

We present another asymptotic formula.

Assumption 3.5 *There exists some random variable Y in $\mathbb{Z}_+$ with positive finite mean such that*

$$\lim_{k\to\infty} \frac{\boldsymbol{A}(k)\boldsymbol{e}}{\mathsf{P}(Y = k)} = \frac{\boldsymbol{c}_A}{\mathsf{E}[Y]}, \quad \lim_{k\to\infty} \frac{\boldsymbol{B}(k)\boldsymbol{e}}{\mathsf{P}(Y = k)} = \frac{\boldsymbol{c}_B}{\mathsf{E}[Y]},$$

where $\boldsymbol{c}_A$ and $\boldsymbol{c}_B$ are nonnegative $M \times 1$ and $M_0 \times 1$ vectors, respectively, satisfying $\boldsymbol{c}_A \neq \boldsymbol{0}$ or $\boldsymbol{c}_B \neq \boldsymbol{0}$.

Theorem 3.4 *Suppose Assumptions 3.1 (I) and 3.5 hold. Furthermore, suppose (i)* $Y_{\rm e} \in \mathcal{S}_{\rm loc}(1)$*; (ii)* $Y \in \mathcal{L}_{\rm loc}(1)$ *or* $\{\mathsf{P}(Y = k)\}$ *is eventually nonincreasing; and (iii)* $\{\boldsymbol{A}(k); k \in \mathbb{Z}_+\}$ *and* $\{\boldsymbol{B}(k); k \in \mathbb{N}\}$ *are eventually nonincreasing. We then have*

$$\lim_{k\to\infty} \frac{\boldsymbol{x}(k)}{\mathsf{P}(Y_{\rm e} = k)} = \frac{\boldsymbol{x}(0)\boldsymbol{c}_B + \overline{\boldsymbol{x}}(0)\boldsymbol{c}_A}{\sigma} \cdot \boldsymbol{\pi}.$$

Proof. This theorem can be proved in a very similar way to Theorem 3.1. For doing this, we require an additional condition that $\{\boldsymbol{A}(k); k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{B}(k); k \in \mathbb{N}\}$ are eventually nonincreasing, i.e., there exists some $k_* \in \mathbb{N}$ such that $\boldsymbol{A}(k) \ge \boldsymbol{A}(k+1)$ and $\boldsymbol{B}(k) \ge \boldsymbol{B}(k+1)$ for all $k \ge k_*$. The details are omitted. □

Remark 3.5 *Since* $\boldsymbol{E}\boldsymbol{e} = \boldsymbol{e}$*, Assumption 3.5 is sufficient for Assumption 3.4. Thus Theorem 3.4 is not a corollary of Theorem 3.3.*

Remark 3.6 *We give an example of a queueing model that has locally subexponential-tail asymptotics in the case of stochastic* $\boldsymbol{A}$ *in Appendix C.3.1.*

3.3.2 Case of strictly substochastic $\boldsymbol{A}$

In addition to Assumption 3.1 (II), we assume the following:

Assumption 3.6 *There exists some random variable* Y *in* $\mathbb{Z}_+$ *such that*

$$\lim_{k\to\infty} \frac{\boldsymbol{A}(k)}{\mathsf{P}(Y = k)} = \boldsymbol{C}_A, \quad \lim_{k\to\infty} \frac{\boldsymbol{B}(k)}{\mathsf{P}(Y = k)} = \boldsymbol{C}_B, \tag{3.52}$$

where $\boldsymbol{C}_A$ *and* $\boldsymbol{C}_B$ *are nonnegative* $M \times M$ *and* $M_0 \times M$ *matrices, respectively, satisfying* $\boldsymbol{C}_A \ne \boldsymbol{O}$ *or* $\boldsymbol{C}_B \ne \boldsymbol{O}$.

Lemma 3.8 *Suppose Assumptions 3.1 (II) and 3.6 hold. If* $Y \in \mathcal{L}_{\rm loc}(1)$*; and* $r_{A_-} > 1$ *or* $\{\mathsf{P}(Y = k)\}$ *is eventually nonincreasing, then*

$$\lim_{k\to\infty} \frac{\boldsymbol{R}(k)}{\mathsf{P}(Y = k)} = \boldsymbol{C}_A \left(\boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell) \right)^{-1}, \tag{3.53}$$

$$\lim_{k\to\infty} \frac{\boldsymbol{R}_0(k)}{\mathsf{P}(Y = k)} = \boldsymbol{C}_B \left(\boldsymbol{I} - \sum_{\ell=0}^{\infty} \boldsymbol{\Phi}(-\ell) \right)^{-1}. \tag{3.54}$$

Proof. From (1.15) and (3.52), we have

$$\lim_{k\to\infty} \frac{\boldsymbol{R}(k)}{\mathsf{P}(Y = k)} = \left[\boldsymbol{C}_A + \lim_{k\to\infty} \sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)}{\mathsf{P}(Y = k)} \boldsymbol{L}(m) \right] (\boldsymbol{I} - \boldsymbol{\Phi}(0))^{-1}. \tag{3.55}$$

To apply the dominated convergence theorem to (3.55), we show that for all sufficiently large k,

$$\sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)}{\mathsf{P}(Y = k)} \boldsymbol{L}(m) < \infty.$$

Suppose $\{\mathsf{P}(Y = k)\}$ is eventually nonincreasing. We then have for all sufficiently large k,

$$\sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)}{\mathsf{P}(Y=k)} \boldsymbol{L}(m) \le \sup_{m' \in \mathbb{N}} \frac{\boldsymbol{A}(k+m')}{\mathsf{P}(Y=k+m')} \sum_{m=1}^{\infty} \boldsymbol{L}(m) < \infty,$$

where the last inequality is due to (3.33) and (3.52). On the other hand, suppose $r_{A_-} > 1$. It then follows from Proposition 1.2 that $\{\boldsymbol{G}(k)\}$ is light-tailed, i.e.,

$$\sum_{k=1}^{\infty} r^k \boldsymbol{G}(k) < \infty \quad \text{for all } 1 < r < r_{A_-}. \tag{3.56}$$

Note here that $\widehat{\boldsymbol{G}}(1/z) = \sum_{k=1}^{\infty} z^k \boldsymbol{G}(k)$ and $\mathrm{sp}(\widehat{\boldsymbol{G}}(1)) < 1$ (see Proposition 1.4). Thus according to Theorem 8.1.18 in [26],

$$\mathrm{sp}(\widehat{\boldsymbol{G}}(1/z)) = 1 \quad \text{only if } 1 < z \le r_{A_-}. \tag{3.57}$$

The equations (1.11), (3.56) and (3.57) imply that there exists some $r > 1$ such that

$$\sum_{m=1}^{\infty} r^m \boldsymbol{L}(m) < \infty.$$

Furthermore, it follows from Assumption 3.6 and $Y \in \mathcal{L}_{\mathrm{loc}}(1)$ that for any $\varepsilon > 0$ there exists some $k_0 \in \mathbb{Z}_+$ such that for all $k \ge k_0$,

$$\frac{\boldsymbol{A}(k+m)}{\mathsf{P}(Y=k)} \le (\boldsymbol{C}_A + \varepsilon \boldsymbol{e}\boldsymbol{e}^\top) \frac{\mathsf{P}(Y=k+m)}{\mathsf{P}(Y=k)} \le (1+\varepsilon)^m (\boldsymbol{C}_A + \varepsilon \boldsymbol{e}\boldsymbol{e}^\top), \quad m \in \mathbb{Z}_+.$$

Therefore, for $0 < \varepsilon \le r - 1$ and $k \ge k_0$,

$$\sum_{m=1}^{\infty} \frac{\boldsymbol{A}(k+m)}{\mathsf{P}(Y=k)} \boldsymbol{L}(m) \le (\boldsymbol{C}_A + \varepsilon \boldsymbol{e}\boldsymbol{e}^\top) \sum_{m=1}^{\infty} (1+\varepsilon)^m \boldsymbol{L}(m) < \infty.$$

As a result, applying the dominated convergence theorem to (3.55) and following the proof of Lemma 3.5, we can prove (3.53). Equation (3.54) can be proved in the same way. □

Using Lemma 3.8, we can readily prove the following theorem. The proof is very similar to that of Theorem 3.2 and thus is omitted.

Theorem 3.5 *Suppose Assumptions 3.1 (II) and 3.6 hold. If $Y \in \mathcal{S}_{\mathrm{loc}}(1)$; and $r_{A_-} > 1$ or $\{\mathsf{P}(Y = k)\}$ is eventually nonincreasing, then*

$$\lim_{k\to\infty} \frac{\boldsymbol{x}(k)}{\mathsf{P}(Y=k)} \quad = \quad [\boldsymbol{x}(0)\boldsymbol{C}_B + \overline{\boldsymbol{x}}(0)\boldsymbol{C}_A](\boldsymbol{I} - \boldsymbol{A})^{-1} > \boldsymbol{0}.$$

Remark 3.7 *In Appendix C.3.2, we provide an example of a discrete-time single-server queue with disasters. This queueing model has locally subexponential-tail asymptotics corresponding to the case of strictly stochastic $\boldsymbol{A}$.*

Chapter 4

Heavy-Traffic Asymptotics

4.1 Introduction

In this chapter, we study the heavy-traffic limits of the stationary distribution and moments of the GI/G/1-type Markov chain. To begin with, we review the previous studies. Asmussen [5] assumed that

$$\sum_{k=1}^{\infty} k^2 \boldsymbol{B}(k) < \infty, \quad \sum_{k\in\mathbb{Z}} |k^3| \boldsymbol{A}(k) < \infty, \tag{4.1}$$

and then proved that the diffusion-scaled level process converges weakly to a reflected Brownian motion as the mean drift in level $-\sigma$ goes to zero. It should be noted that if $\sigma = 0$ and Assumption 1.1 (a) and (b) hold then $\boldsymbol{T}$ is null-recurrent [8, Chapter XI, Proposition 3.1]. Asmussen [5] also presented the asymptotic formula for the stationary distribution of the following form:

$$\mathsf{P}(\sigma X > x, S = i) \to e^{-x/\gamma}\pi_i, \quad x \ge 0,\ i \in \mathbb{M}, \quad \text{as } \sigma \downarrow 0, \tag{4.2}$$

where $\gamma > 0$ is a certain parameter and (X, S) denotes a random vector distributed according to the stationary distribution of $\{(X_n, S_n)\}$, i.e., $\mathsf{P}(X = k, S = i) = x_i(k)$ for $(k, i) \in \mathbb{S}$. Falin [18] proved the heavy-traffic limit (4.2) for the M/G/1-type Markov chain under the following conditions:

$$\sum_{k=1}^{\infty} k \boldsymbol{B}(k) < \infty, \quad \sum_{k\in\mathbb{Z}} k^2 \boldsymbol{A}(k) < \infty, \tag{4.3}$$

$$\boldsymbol{x}(0) \to \boldsymbol{0}, \quad \text{as } \sigma \downarrow 0, \tag{4.4}$$

$$\boldsymbol{A} = \boldsymbol{B}(0) + \sum_{k=1}^{\infty} \boldsymbol{B}(k). \tag{4.5}$$

In this chapter, we prove the heavy-traffic limit (4.2) for the GI/G/1-type Markov chain by the characteristic function approach. The proof does not require Falin [18]'s additional conditions (4.4) and (4.5). Furthermore, our assumption is weaker than Asmussen's. More specifically, Asmussen's heavy-traffic limit theorem requires that (4.1) holds. On the other hand, we assume the following:

$$\sum_{k=1}^{\infty} k \boldsymbol{B}(k) < \infty, \quad \sum_{k\in\mathbb{Z}} k^2 \boldsymbol{A}(k) < \infty.$$

We also present a heavy-traffic asymptotic formula for the moments of the stationary distribution. We assume that

$$\sum_{k=1}^{\infty} k^m \boldsymbol{B}(k) < \infty, \quad \sum_{k=1}^{\infty} k^m \boldsymbol{A}(k) < \infty,$$

under which we prove that there exists some $\eta > 0$ such that, for $i \in \mathbb{M}$ and $m \in \mathbb{N}$,

$$\mathsf{E}[(\sigma X)^m \mathbb{1}(S = i)] \to m!\gamma^m \pi_i, \quad \text{as } \sigma \downarrow 0.$$

As far as we know, there are no studies on the heavy-traffic limits of the moments of the stationary distribution of the GI/G/1-type Markov chain.

The rest of this chapter is organized as follows. In Section 4.2, we summarize preliminary results. Section 4.3 provides the heavy-traffic asymptotic formula of the stationary distribution. Section 4.4 presents a heavy-traffic asymptotic formula for the moments of the stationary distribution.

4.2 Preliminaries

In this section, we provide preliminary results for the heavy-traffic asymptotics. We first introduce the characteristic function of $\{\boldsymbol{x}(k); k \in \mathbb{N}\}$. We then parameterize the Markov chain to consider the heavy traffic limit and discuss the boundedness and continuity of the related vectors and matrices to the Markov chain.

4.2.1 Characteristic functions and related results

In this chapter, we use a characteristic function approach in stead of a generating function one considered in Chapters 2 and 3. To this end, we first provide necessary notations and results as preliminaries in this subsection. We write i as the imaginary unit, i.e., $\mathrm{i} = \sqrt{-1}$ hereafter.

Let $\widetilde{\boldsymbol{A}}(\xi) = \sum_{k\in\mathbb{Z}} e^{\mathrm{i}\xi k}\boldsymbol{A}(k)$, $\widetilde{\boldsymbol{G}}(\xi) = \sum_{k=1}^{\infty} e^{-\mathrm{i}\xi k}\boldsymbol{G}(k)$, $\widetilde{\boldsymbol{R}}(\xi) = \sum_{k=1}^{\infty} e^{\mathrm{i}\xi k}\boldsymbol{R}(k)$ and $\widetilde{\boldsymbol{R}}_0(\xi) = \sum_{k=1}^{\infty} e^{\mathrm{i}\xi k}\boldsymbol{R}_0(k)$ for $\xi \in \mathbb{R} := (-\infty, \infty)$, respectively. It then follows from Proposition 1.3 that

$$\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi) = (\boldsymbol{I} - \widetilde{\boldsymbol{R}}(\xi))(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widetilde{\boldsymbol{G}}(\xi)). \tag{4.6}$$

The above equation is the RG-factorization expressed by characteristic functions.

In addition, we assume the following condition throughout this chapter.

Assumption 4.1 *$\boldsymbol{A}$ is stochastic.*

Remark 4.1 *Assumptions 1.1 and 4.1 imply that $\sigma > 0$ and $\sum_{k=1}^{\infty} k\boldsymbol{B}(k)\boldsymbol{e} < \infty$ (see Proposition 1.7).*

Let $\widetilde{\boldsymbol{x}}(\xi) = \sum_{k=1}^{\infty} e^{\mathrm{i}\xi k}\boldsymbol{x}(k)$. It then follows from (1.22) that

$$\widetilde{\boldsymbol{x}}(\xi)(\boldsymbol{I} - \widetilde{\boldsymbol{R}}(\xi)) = \boldsymbol{x}(0)\widetilde{\boldsymbol{R}}_0(\xi). \tag{4.7}$$

From (4.6) and (4.7), we have

$$\widetilde{\boldsymbol{x}}(\xi)(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi)) = \boldsymbol{x}(0)\widetilde{\boldsymbol{R}}_0(\xi)(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widetilde{\boldsymbol{G}}(\xi)), \tag{4.8}$$

which yields, for $\xi \in \mathbb{R}$ such that $\phi(\xi) \neq 0$,

$$\widetilde{\boldsymbol{x}}(\xi) = \boldsymbol{x}(0)\widetilde{\boldsymbol{R}}_0(\xi)(\boldsymbol{I} - \boldsymbol{\Phi}(0))(\boldsymbol{I} - \widetilde{\boldsymbol{G}}(\xi))\frac{\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))}{\phi(\xi)}, \tag{4.9}$$

where

$$\phi(\xi) = \det(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi)), \qquad \xi \in \mathbb{R}. \tag{4.10}$$

Let $\widetilde{\delta}(\xi) := \delta(\widetilde{\boldsymbol{A}}(\xi))$ ($\xi \in \mathbb{R}$) denote a maximum-modulus eigenvalue of $\widetilde{\boldsymbol{A}}(\xi)$, whose imaginary part is nonnegative and whose real part is not less than those of the other eigenvalues of maximum modulus. Let $\widetilde{\boldsymbol{\mu}}(\xi) = (\widetilde{\mu}_i(\xi))_{i\in\mathbb{M}}$ and $\widetilde{\boldsymbol{v}}(\xi) = (\widetilde{v}_i(\xi))_{i\in\mathbb{M}}$ denote the left- and right-eigenvectors of $\widetilde{\boldsymbol{A}}(\xi)$ corresponding to eigenvalue $\widetilde{\delta}(\xi)$, which are normalized such that $\mathrm{Re}(\widetilde{\mu}_1(\xi)) \geq 0$, $\sum_{i\in\mathbb{M}} |\widetilde{\mu}_i(\xi)| = 1$ and $\widetilde{\boldsymbol{\mu}}(\xi)\widetilde{\boldsymbol{v}}(\xi) = 1$. Let $\widetilde{\Gamma}(\xi)$ denote

$$\widetilde{\Gamma}(\xi) = \prod_{i=2}^{M}(1 - \widetilde{\lambda}_i^{(A)}(\xi)),$$

where $\widetilde{\lambda}_i^{(A)}(\xi)$ ($m = 2, 3, \dots, M$) denote the eigenvalues of $\widetilde{\boldsymbol{A}}(\xi)$ such that $|\widetilde{\delta}(\xi)| \geq |\widetilde{\lambda}_2^{(A)}(\xi)| \geq |\widetilde{\lambda}_3^{(A)}(\xi)| \geq \cdots \geq |\widetilde{\lambda}_M^{(A)}(\xi)|$. We then have

$$\phi(\xi) = (1 - \widetilde{\delta}(\xi))\widetilde{\Gamma}(\xi). \tag{4.11}$$

Since $\widetilde{\boldsymbol{A}}(0) = \boldsymbol{A}$ is an irreducible stochastic matrix (see Assumptions 1.1 (b) and 4.1), the eigenvalue $\widetilde{\delta}(0)$ is simple (see, e.g., [10, Theorem 1.4.4]) and

$$\widetilde{\boldsymbol{\mu}}(0) \;=\; \boldsymbol{\pi}, \qquad \widetilde{\boldsymbol{v}}(0) = \boldsymbol{e}. \tag{4.12}$$

The fact $\sigma > 0$ (see Remark 4.1) implies that $\mathrm{i}\xi \sum_{k\in\mathbb{Z}} k e^{\mathrm{i}\xi k}\boldsymbol{A}(k)$ is finite and thus $\widetilde{\boldsymbol{A}}(\xi)$ is differentiable for all $\xi \in \mathbb{R}$. Therefore, it follows from Theorem 2.1 of [3] that $\widetilde{\delta}(\xi)$, $\widetilde{\boldsymbol{\mu}}(\xi)$ and $\widetilde{\boldsymbol{v}}(\xi)$ are differentiable for all $\xi \in \mathbb{R}$. Furthermore, we obtain

$$\widetilde{\delta}'(0) = -\mathrm{i}\sigma, \tag{4.13}$$

by differentiating both sides of $\widetilde{\delta}(\xi) = \widetilde{\boldsymbol{\mu}}(\xi)\widetilde{\boldsymbol{A}}(\xi)\widetilde{\boldsymbol{v}}(\xi)$ and letting $\xi = 0$.

Remark 4.2 *If $(\mathrm{d}^m/\mathrm{d}\xi^m)\widetilde{\boldsymbol{A}}(\xi)\ |_{\xi=0}$ exists, i.e., $\widetilde{\boldsymbol{A}}(\xi)$ is m-times differentiable at $\xi = 0$, then $\widetilde{\delta}(\xi)$, $\widetilde{\boldsymbol{\mu}}(\xi)$ and $\widetilde{\boldsymbol{v}}(\xi)$ are m-times differentiable at $\xi = 0$, which indicates that $\widetilde{\Gamma}(\xi)$ is also m-times differentiable at $\xi = 0$. These facts can be easily confirmed because $\widetilde{\boldsymbol{A}}(\xi)$ is m-times differentiable for all $\xi \in \mathbb{R}$; and $\widetilde{\delta}(\xi)$ is a simple eigenvalue of $\widetilde{\boldsymbol{A}}(\xi)$. Discussions on the differentiability and continuity of eigen-vectors and values of matrix-valued functions are summarized in Appendix D.*

In what follows, we assume the following.

Assumption 4.2 $\sum_{k\in\mathbb{Z}} k^2 \boldsymbol{A}(k) < \infty$.

Under Assumption 4.2, $(\mathrm{d}^2/\mathrm{d}\xi^2)\widetilde{\boldsymbol{A}}(\xi)\mid_{\xi=0}$ exists and thus $\delta(\xi)$, $\widetilde{\boldsymbol{\mu}}(\xi)$, $\widetilde{\boldsymbol{v}}(\xi)$ and $\widetilde{\Gamma}(\xi)$ are all twice differentiable at $\xi = 0$ (see Remark 4.2). Therefore, by differentiating (4.11) with respect to ξ and using $\widetilde{\delta}(0) = 1$ and (4.13), we obtain

$$\phi'(0) = \mathrm{i}\sigma\widetilde{\Gamma}(0) = \mathrm{i}\sigma\prod_{i=2}^{M}(1-\widetilde{\lambda}_i^{(A)}(0)), \tag{4.14}$$

$$\phi''(0) = -\widetilde{\delta}''(0)\widetilde{\Gamma}(0) - 2\mathrm{i}\sigma\widetilde{\Gamma}'(0). \tag{4.15}$$

The following proposition gives the expression of $\widetilde{\delta}''(0)$.

Proposition 4.1 *If Assumptions 1.1 (a) and (b), 4.1 and 4.2 hold, then*

$$\widetilde{\delta}''(0) = 2\sigma^2 - \boldsymbol{\pi}\sum_{k\in\mathbb{Z}}k^2\boldsymbol{A}(k)\boldsymbol{e} - 2\boldsymbol{\pi}\sum_{k\in\mathbb{Z}}k\boldsymbol{A}(k)(\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{e}\boldsymbol{\pi})^{-1}\boldsymbol{\beta}_A. \tag{4.16}$$

Proof. From the definition of $\widetilde{\boldsymbol{\mu}}(\xi)$ and $\widetilde{\delta}(\xi)$, we have

$$\widetilde{\boldsymbol{\mu}}(\xi)\tilde{\boldsymbol{A}}(\xi) = \widetilde{\delta}(\xi)\widetilde{\boldsymbol{\mu}}(\xi). \tag{4.17}$$

Differentiating both sides of (4.17) twice with respect to ξ, post-multiplying them by $\boldsymbol{e}$ and letting $\xi = 0$, we obtain

$$\widetilde{\delta}''(0) = -\boldsymbol{\pi}\sum_{k\in\mathbb{Z}}k^2\boldsymbol{A}(k)\boldsymbol{e} + \mathrm{i}2\widetilde{\boldsymbol{\mu}}'(0)\boldsymbol{\beta}_A, \tag{4.18}$$

where we use $\widetilde{\delta}(0) = 1$, $\widetilde{\boldsymbol{\mu}}(0) = \boldsymbol{\pi}$, $\widetilde{\boldsymbol{\mu}}'(\xi)\boldsymbol{e} = 0$ for all $\xi \in \mathbb{R}$, and (4.13). Furthermore, it follows from (4.17) that

$$\widetilde{\boldsymbol{\mu}}(\xi)[\widetilde{\delta}(\xi)\boldsymbol{I} - \tilde{\boldsymbol{A}}(\xi) + \boldsymbol{e}\widetilde{\boldsymbol{\mu}}(\xi)] = \widetilde{\boldsymbol{\mu}}(\xi).$$

By differentiating the above equation and letting $\xi = 0$, we have

$$\widetilde{\boldsymbol{\mu}}'(0)(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi}) = \boldsymbol{\pi}(\tilde{\boldsymbol{A}}'(0) - \widetilde{\delta}'(0)\boldsymbol{I}),$$

which leads to

$$\widetilde{\boldsymbol{\mu}}'(0) = \mathrm{i}\sigma\boldsymbol{\pi} + \mathrm{i}\boldsymbol{\pi}\sum_{k\in\mathbb{Z}}k\boldsymbol{A}(k)(\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{e}\boldsymbol{\pi})^{-1}. \tag{4.19}$$

Finally, (4.16) is obtained by substituting (4.19) into (4.18). □

The following lemma will be used to prove Proposition 4.2 and Lemma 4.17 in Section 4.3.

Lemma 4.1 *If Assumptions 1.1 (a) and (b), 4.1 and 4.2 hold, then*

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I}-\widetilde{\boldsymbol{A}}(\xi))\right|_{\xi=0} &= \mathrm{i}\widetilde{\Gamma}(0)(\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{e}\boldsymbol{\pi})^{-1}[\boldsymbol{\beta}_A\boldsymbol{\pi}+\sigma\boldsymbol{I}] + \widetilde{\Gamma}'(0)\boldsymbol{e}\boldsymbol{\pi} \\ &\quad +\mathrm{i}\widetilde{\Gamma}(0)\boldsymbol{e}\boldsymbol{\pi}\left[\sigma\boldsymbol{I} + \sum_{k\in\mathbb{Z}}k\boldsymbol{A}(k)(\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{e}\boldsymbol{\pi})^{-1}\right].\end{aligned} \tag{4.20}$$

Proof. From the definition, it is clear that

$$(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi)) = (1 - \widetilde{\delta}(\xi))\widetilde{\Gamma}(\xi)\boldsymbol{I}. \tag{4.21}$$

By differentiating the above equation with respect to ξ and letting $\xi = 0$, we obtain

$$(\boldsymbol{I} - \boldsymbol{A}) \left.\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))\right|_{\xi=0} - \mathrm{i}\widetilde{\Gamma}(0)\boldsymbol{\beta}_A\boldsymbol{\pi} = \mathrm{i}\sigma\widetilde{\Gamma}(0)\boldsymbol{I}, \tag{4.22}$$

where we use (4.13) and the following fact (see Lemma 2.1):

$$\lim_{\xi\to 0} \mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi)) = \boldsymbol{e}\boldsymbol{\pi}\widetilde{\Gamma}(0). \tag{4.23}$$

On the other hand, pre-multiplying (4.21) by $\boldsymbol{e}\widetilde{\boldsymbol{\mu}}(\xi)$ leads to

$$\boldsymbol{e}\widetilde{\boldsymbol{\mu}}(\xi)\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi)) = \widetilde{\Gamma}(\xi)\boldsymbol{e}\widetilde{\boldsymbol{\mu}}(\xi).$$

Differentiating the above equation with respect to ξ and letting $\xi = 0$ yields

$$\boldsymbol{e}\boldsymbol{\pi} \left.\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))\right|_{\xi=0} = \widetilde{\Gamma}'(0)\boldsymbol{e}\boldsymbol{\pi} + \widetilde{\Gamma}(0)\boldsymbol{e}\widetilde{\boldsymbol{\mu}}'(0), \tag{4.24}$$

where we use (4.23) and $\widetilde{\boldsymbol{\mu}}'(0)\boldsymbol{e} = 0$. It thus follows from (4.22) and (4.24) that

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi}) &\left.\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))\right|_{\xi=0} \\ &= \mathrm{i}\widetilde{\Gamma}(0)\left(\boldsymbol{\beta}_A\boldsymbol{\pi} + \sigma\boldsymbol{I}\right) + \widetilde{\Gamma}'(0)\boldsymbol{e}\boldsymbol{\pi} + \widetilde{\Gamma}(0)\boldsymbol{e}\widetilde{\boldsymbol{\mu}}'(0).\end{aligned}$$

Finally, combining (4.19) with the above equation leads to (4.20). □

Using (4.9), (4.14) and Lemma 4.1, we have Proposition 4.2 below, which is needed to prove Lemma 4.17 in Section 4.3.

Proposition 4.2 *If Assumptions 1.1 and 4.1 hold, then*

$$\widetilde{\boldsymbol{x}}(0)\boldsymbol{e} = \frac{1}{\sigma}\boldsymbol{x}(0)\boldsymbol{R}_0(\boldsymbol{I} - \boldsymbol{\Phi}(0))\left[\boldsymbol{\beta}_G + (\boldsymbol{I} - \boldsymbol{G})(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1}\boldsymbol{\beta}_A\right], \tag{4.25}$$

where $\boldsymbol{R}_0 = \sum_{k=1}^{\infty} \boldsymbol{R}_0(k)$ *and* $\boldsymbol{\beta}_G = \sum_{k=1}^{\infty} k\boldsymbol{G}(k)\boldsymbol{e}$*.*

Remark 4.3 *Under Assumptions 1.1 and 4.1, it holds that* $\sigma \in (0,\infty)$ *and* $\sum_{k=1}^{\infty} k\boldsymbol{B}(k)\boldsymbol{e} < \infty$ *(see Remark 4.1), which implies that* $\boldsymbol{R}_0 < \infty$ *(see [68, Lemma 25]). In addition, Assumption 4.2 implies that* $\boldsymbol{\beta}_G < \infty$ *(see Proposition 3.1 in Chapter XI of [8]).*

Proof. Post-multiplying both sides of (4.9) by $\boldsymbol{e}$, we obtain

$$\widetilde{\boldsymbol{x}}(\xi)\boldsymbol{e} \;=\; \boldsymbol{x}(0)\widetilde{\boldsymbol{R}}_0(\xi)(\boldsymbol{I} - \boldsymbol{\Phi}(0))\frac{\boldsymbol{I} - \widetilde{\boldsymbol{G}}(\xi)}{\xi}\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))\frac{\xi}{\phi(\xi)}\boldsymbol{e}. \tag{4.26}$$

From the definition of the adjugate matrix, each element of $\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))$ is infinitely differentiable at $\xi = 0$. It thus follows from Taylor's theorem that there exists $s := s(\xi)$ such that $\lim_{\xi\to 0} s(\xi) = 0$ and

$$\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi)) = \mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(0)) + \xi\frac{\mathrm{d}}{\mathrm{d}\xi}\left.\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))\right|_{\xi=0} + \frac{\xi^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left.\mathrm{adj}(\boldsymbol{I} - \widetilde{\boldsymbol{A}}(\xi))\right|_{\xi=s}. \tag{4.27}$$

On the other hand, using l'Hôpital's rule, we have

$$\lim_{\xi\to 0}\frac{(\boldsymbol{I}-\widetilde{\boldsymbol{G}}(\xi))\boldsymbol{e}}{\xi} \;=\; \mathrm{i}\boldsymbol{\beta}_G. \tag{4.28}$$

Therefore, combining (4.28) with (4.20), (4.23), and (4.27) yields

$$\begin{aligned}
&\lim_{\xi\to 0}\frac{(\boldsymbol{I}-\widetilde{\boldsymbol{G}}(\xi))}{\xi}\mathrm{adj}(\boldsymbol{I}-\widetilde{\boldsymbol{A}}(\xi)) \\
&= \widetilde{\Gamma}(0)\lim_{\xi\to 0}\frac{(\boldsymbol{I}-\widetilde{\boldsymbol{G}}(\xi))\boldsymbol{e}}{\xi}\boldsymbol{\pi} + (\boldsymbol{I}-\boldsymbol{G})\frac{\mathrm{d}}{\mathrm{d}\xi}\,\mathrm{adj}(\boldsymbol{I}-\widetilde{\boldsymbol{A}}(\xi))\Big|_{\xi=0} \\
&= \mathrm{i}\widetilde{\Gamma}(0)\left(\boldsymbol{\beta}_G\boldsymbol{\pi} + (\boldsymbol{I}-\boldsymbol{G})(\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{e}\boldsymbol{\pi})^{-1}\left[\boldsymbol{\beta}_A\boldsymbol{\pi}+\sigma\boldsymbol{I}\right]\right).
\end{aligned} \tag{4.29}$$

Furthermore, using l'Hôpital's rule to (4.14), we obtain

$$\lim_{\xi\to 0}\frac{\xi}{\phi(\xi)} \;=\; \frac{1}{\phi'(0)} = \frac{1}{\mathrm{i}\sigma\widetilde{\Gamma}(0)}. \tag{4.30}$$

As a result, applying (4.29) and (4.30) to (4.26) yields (4.25). □

Remark 4.4 *It follows from (4.6), (4.29) and (4.30) that*

$$(\boldsymbol{I}-\boldsymbol{R})^{-1} = \frac{1}{\sigma}(\boldsymbol{I}-\boldsymbol{\Phi}(0))\left(\boldsymbol{\beta}_G\boldsymbol{\pi} + (\boldsymbol{I}-\boldsymbol{G})(\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{e}\boldsymbol{\pi})^{-1}\left[\boldsymbol{\beta}_A\boldsymbol{\pi}+\sigma\boldsymbol{I}\right]\right) \ge \boldsymbol{I},$$

which implies that

$$(\boldsymbol{I}-\boldsymbol{\Phi}(0))\left[\boldsymbol{\beta}_G + (\boldsymbol{I}-\boldsymbol{G})(\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{e}\boldsymbol{\pi})^{-1}\boldsymbol{\beta}_A\right] \ge \sigma\boldsymbol{e}. \tag{4.31}$$

Note that $\boldsymbol{G}$ is stochastic (see [66, Theorem 3.4]). In addition, $\boldsymbol{G}$ has exactly one irreducible class due to the irreducibility of $\boldsymbol{A}$ (see Proposition 2.5.1 in [32]). Let $\boldsymbol{g} = (g_i)_{i\in\mathbb{M}}$ denote the unique stationary probability vector of $\boldsymbol{G}$. We then have the following result.

Proposition 4.3 *Under Assumptions 1.1 and 4.1, it holds that*

$$\sum_{i\in\mathbb{M}} g_i\mathsf{E}[\tau_0 \mid X_0 = 1, S_0 = i] = \frac{\boldsymbol{g}\boldsymbol{\beta}_G}{\sigma}. \tag{4.32}$$

Lemma 4.2 below relates the finiteness of $\{\boldsymbol{A}(k); k\in\mathbb{N}\}$, $\{\boldsymbol{A}(-k); k\in\mathbb{N}\}$ and $\{\boldsymbol{B}(k); k\in\mathbb{N}\}$ to that of $\{\boldsymbol{R}(k); k\in\mathbb{N}\}$, $\{\boldsymbol{G}(k); k\in\mathbb{N}\}$ and $\{\boldsymbol{R}_0(k); k\in\mathbb{N}\}$, respectively. This result is directly connected with the sufficient condition for the heavy traffic limit of the m-th moment of the stationary distribution, which will be shown in Lemma 4.18 and Theorem 4.2.

Lemma 4.2 *If Assumptions 1.1 (a) and (b) and 4.1 hold, then the following are true for any $m\in\mathbb{N}$:*

(i) If $\sum_{k=1}^{\infty} k^{m+1}\boldsymbol{A}(k) < \infty$, then $\sum_{k=1}^{\infty} k^{m}\boldsymbol{R}(k) < \infty$.

(ii) If $\sum_{k=1}^{\infty} k^{m+1}\boldsymbol{A}(-k) < \infty$, then $\sum_{k=1}^{\infty} k^{m}\boldsymbol{G}(k) < \infty$.

(iii) If $\sum_{k=1}^{\infty} k^{m+1}\boldsymbol{B}(k) < \infty$, then $\sum_{k=1}^{\infty} k^{m}\boldsymbol{R}_0(k) < \infty$.

Proof. Owing to the duality between the G-matrices and the R-matrices [67], we can prove the statements (i) and (ii) in the same way. Furthermore, the statement (iii) is also proved in the same way according to the definitions of $\boldsymbol{R}(k)$ and $\boldsymbol{R}_0(k)$ (see (1.12) and (1.13)). Thus, the proof of statement (ii) and (iii) are omitted.

Let $\boldsymbol{T}_+$ denote a submatrix of $\boldsymbol{T}$, which is obtained by deleting the first block row and column:

$$\boldsymbol{T}_+ = \begin{pmatrix} \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \boldsymbol{A}(3) & \cdots \\ \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{A}(-2) & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \boldsymbol{A}(-3) & \boldsymbol{A}(-2) & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $\boldsymbol{T}$ is irreducible and recurrent, $\boldsymbol{U} := \sum_{n=0}^{\infty}(\boldsymbol{T}_+)^n$ is finite.

As with $\boldsymbol{T}_+$, we partition $\boldsymbol{U}$ as $(\boldsymbol{U}(\ell,m))_{\ell,m\in\mathbb{N}}$, where $\boldsymbol{U}(\ell,m)$ is an $M \times M$ matrix whose (i,j)th element represents

$$\mathsf{E}\left[\sum_{n=0}^{\tau_0-1} \mathbb{1}(X_n = m,\ S_n = j) \mid X_0 = \ell,\ S_0 = i\right].$$

We then have

$$\boldsymbol{R}(k) = \sum_{\ell=k}^{\infty} \boldsymbol{A}(\ell)\boldsymbol{U}(\ell-k,1).$$

Note here that there exists some $b \in (0,\infty)$ such that $\boldsymbol{U}(\ell,1)\boldsymbol{e} \le b\boldsymbol{e}$ for all $\ell \in \mathbb{N}$ (see the proof of Lemma 25 in [68]). Thus, since $\sum_{k=1}^{\infty} k^{m+1}\boldsymbol{A}(k) < \infty$ we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} k^m \boldsymbol{R}(k)\boldsymbol{e} &\le b\sum_{k=1}^{\infty} k^m \sum_{\ell=k}^{\infty} \boldsymbol{A}(\ell)\boldsymbol{e} \\ &= b\sum_{\ell=1}^{\infty}\left[\sum_{k=1}^{\ell} k^m\right]\boldsymbol{A}(\ell)\boldsymbol{e} \\ &= \frac{b}{m+1}\sum_{\ell=1}^{\infty}\left[\sum_{j=0}^{m}(-1)^j\binom{m+1}{j}B_j\ell^{m+1-j}\right]\boldsymbol{A}(\ell)\boldsymbol{e} < \infty, \end{aligned}$$

where we use the Faulhaber's formula (see e.g. [16]) in the last equality and $B_j < \infty$ ($j \in \mathbb{N}$) are Bernoulli numbers such that

$$B_0 = 1, \qquad B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k, \qquad n \in \mathbb{N}.$$

□

4.2.2 α-parametrization

To consider the heavy traffic limit, we parameterize $\{(X_n, S_n)\}$ with a parameter $\alpha \ge 0$, which is denoted by $\{({}_{(\alpha)}X_n, {}_{(\alpha)}S_n)\}$. All the vectors, matrices and functions associated with $\{({}_{(\alpha)}X_n, {}_{(\alpha)}S_n)\}$ are also denoted with a subscript "(α)", e.g., ${}_{(\alpha)}\boldsymbol{x}(k)$, ${}_{(\alpha)}\boldsymbol{T}$, ${}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi)$ etc. In this subsection, we prove the boundedness and right-continuity at $\alpha = 0$ of the vectors, matrices related to the Markov chain $\{{}_{(\alpha)}X_n, {}_{(\alpha)}S_n\}$. These results are needed to show our main results in the following sections.

In what follows, we make the following assumption.

Assumption 4.3

(a) Assumptions 1.1 (a) and (b), 4.1 and 4.2 hold for all $\alpha \ge 0$*;*

(b) Assumption 1.1 (c) holds for all $\alpha > 0$*;*

(c) as $\alpha \downarrow 0$*,* ${}_{(\alpha)}\sigma$ *converges to zero from above and* ${}_{(0)}\sigma = 0$*; and*

(d) ${}_{(\alpha)}\boldsymbol{A} = \boldsymbol{A}$ *for all* $\alpha \ge 0$*.*

(e) $\sup_{\alpha\ge0}\sum_{k\in\mathbb{Z}}^{\infty} k^2\,{}_{(\alpha)}\boldsymbol{A}(k) < \infty$*; and*

(f) the sequences of the matrices $\{{}_{(\alpha)}\boldsymbol{A}(k); k \in \mathbb{Z}\}$ *and* $\{{}_{(\alpha)}\boldsymbol{B}(k); k \in \mathbb{Z}\}$ *are uniformly right-continuous at* $\alpha = 0$*, i.e., for any* $\varepsilon > 0$*, there exists some* $\alpha_0 > 0$ *such that, for all* $k \in \mathbb{Z}$ *and* $0 \le \alpha < \alpha_0$*,*

$$|{}_{(\alpha)}\boldsymbol{A}(k) - {}_{(0)}\boldsymbol{A}(k)| < \varepsilon, \quad |{}_{(\alpha)}\boldsymbol{B}(k) - {}_{(0)}\boldsymbol{B}(k)| < \varepsilon.$$

Remark 4.5 *Assumption 4.3 implies that* ${}_{(\alpha)}\boldsymbol{T}$ *is recurrent for all* $\alpha \ge 0$ *[8, Chapter XI, Proposition 3.1].*

Under Assumption 4.3 (f), we can show the right-continuities at $\alpha = 0$ of several matrices and vectors related to the Markov chain ${}_{(\alpha)}\boldsymbol{T}$ (e.g., ${}_{(\alpha)}\boldsymbol{\Phi}(k)$, ${}_{(\alpha)}\boldsymbol{R}(k)$, ${}_{(\alpha)}\boldsymbol{G}(k)$, ${}_{(\alpha)}\boldsymbol{\kappa}$) as shown in Lemmas 4.4–4.9 below. To begin with, we provide the basic lemma below.

Lemma 4.3 *Let* ${}_{(\alpha)}f(x)$ *denote a right-continuous function for* $x \in \mathbb{R}$ *such that* $\lim_{\alpha\downarrow0}{}_{(\alpha)}f(x) = {}_{(0)}f(x)$ *for all* $x \in \mathbb{R}$*. If there exists some* ${}_{(\alpha)}c \in \mathbb{R}$ *such that* $\lim_{\alpha\downarrow0}{}_{(\alpha)}c = {}_{(0)}c$*, then*

$$\lim_{\alpha\downarrow0}{}_{(\alpha)}f({}_{(\alpha)}c) = {}_{(0)}f({}_{(0)}c).$$

Proof. Let $\alpha_0 > 0$ denote a sufficiently small number. We then have for any $\varepsilon > 0$, ${}_{(\alpha)}f(x) \le {}_{(0)}f(x) + \varepsilon$ and ${}_{(\alpha)}c \le {}_{(0)}c + \varepsilon$ for $0 \le \alpha \le \alpha_0$. Furthermore, since f is right-continues at ${}_{(0)}c$, there exists some $\delta > 0$ such that for all $\alpha \ge 0$,

$$|{}_{(\alpha)}f({}_{(0)}c) - {}_{(\alpha)}f({}_{(0)}c + \delta)| \le \varepsilon.$$

Thus, we obtain for $0 \le \alpha \le \alpha_0$,

$$\begin{aligned} &|{}_{(\alpha)}f({}_{(\alpha)}c) - {}_{(0)}f({}_{(0)}c)| \\ &\le |{}_{(\alpha)}f({}_{(\alpha)}c) - {}_{(\alpha)}f({}_{(0)}c)| + |{}_{(\alpha)}f({}_{(0)}c) - {}_{(0)}f({}_{(0)}c)| \le 2\varepsilon, \end{aligned}$$

which completes the proof. □

Lemma 4.4 *If Assumption 4.3 holds, then* $\lim_{\alpha\downarrow0}{}_{(\alpha)}\boldsymbol{\Phi}(k) = {}_{(0)}\boldsymbol{\Phi}(k)$ *for all* $k \in \mathbb{Z}$*.*

Proof. From the definition of ${}_{(\alpha)}\boldsymbol{\Phi}(k)$ ($k \in \mathbb{Z}$), it suffices to prove that $\lim_{\alpha\downarrow 0} {}_{(\alpha)}\boldsymbol{T}^{[k]} = {}_{(0)}\boldsymbol{T}^{[k]}$ (see (1.6)). According to Assumption 4.3 (f), for any $\varepsilon > 0$, there exists some α_0 such that, for $0 \le \alpha \le \alpha_0$,

$$ {}_{(\alpha)}\boldsymbol{T}^{\le k}\boldsymbol{e} \le {}_{(0)}\boldsymbol{T}^{\le k}\boldsymbol{e} + \varepsilon\boldsymbol{e}, \tag{4.33}$$

$$ {}_{(\alpha)}\boldsymbol{T}^{>k}\boldsymbol{e} \le {}_{(0)}\boldsymbol{T}^{>k}\boldsymbol{e} + \varepsilon\boldsymbol{e}, \tag{4.34}$$

$$ {}_{(\alpha)}\boldsymbol{U}^{[k]}\boldsymbol{e} \le {}_{(0)}\boldsymbol{U}^{[k]}\boldsymbol{e} + \varepsilon\boldsymbol{e}, \qquad {}_{(\alpha)}\boldsymbol{D}^{[k]}\boldsymbol{e} \le {}_{(0)}\boldsymbol{D}^{[k]}\boldsymbol{e} + \varepsilon\boldsymbol{e}. \tag{4.35}$$

Since ${}_{(\alpha)}\boldsymbol{T}$ is irreducible and recurrent for any $\alpha \ge 0$, there exists some $\gamma_0 \in (0,1)$ such that

$$ {}_{(0)}\boldsymbol{U}^{[k]}\boldsymbol{e} \le \gamma_0\boldsymbol{e}, \qquad {}_{(0)}\boldsymbol{D}^{[k]}\boldsymbol{e} \le \gamma_0\boldsymbol{e}. $$

It thus follows from (4.35) and the above inequalities that there exists some $\gamma_1 \in (\gamma_0, 1)$ such that, for $0 \le \alpha \le \alpha_0$,

$$ {}_{(\alpha)}\boldsymbol{U}^{[k]}\boldsymbol{e} \le \gamma_1\boldsymbol{e}, \qquad {}_{(\alpha)}\boldsymbol{D}^{[k]}\boldsymbol{e} \le \gamma_1\boldsymbol{e}. \tag{4.36}$$

Note that $\sum_{m=0}^{\infty}({}_{(\alpha)}\boldsymbol{T}^{>k})^m = (\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{T}^{>k})^{-1} < \infty$ for all $\alpha \ge 0$. Therefore, there exists some $m_* \in \mathbb{N}$ such that

$$ ({}_{(0)}\boldsymbol{T}^{>k})^{m_*}\boldsymbol{e} \le \gamma_0\boldsymbol{e}. \tag{4.37}$$

Inequalities (4.34) and (4.37) imply that there exists some $\gamma_2 \in (\gamma_0, 1)$ such that, for $0 \le \alpha \le \alpha_0$,

$$ ({}_{(\alpha)}\boldsymbol{T}^{>k})^{m_*}\boldsymbol{e} \le \gamma_2\boldsymbol{e}, $$

from which and ${}_{(\alpha)}\boldsymbol{T}^{>k}\boldsymbol{e} \le \boldsymbol{e}$, we have

$$ \sum_{m=0}^{\infty}({}_{(\alpha)}\boldsymbol{T}^{>k})^m\boldsymbol{e} = \sum_{i=1}^{m_*-1}\sum_{\ell=1}^{\infty}({}_{(\alpha)}\boldsymbol{T}^{>k})^{\ell m_*+i}\boldsymbol{e} \le \sum_{i=1}^{m_*-1}\sum_{\ell=1}^{\infty}\gamma_2\boldsymbol{e} = \frac{m_*}{1-\gamma_2}\boldsymbol{e}. \tag{4.38}$$

It follows from (4.33), (4.36) and (4.38) that

$$ \begin{aligned} {}_{(\alpha)}\boldsymbol{T}^{[k]}\boldsymbol{e} &= {}_{(\alpha)}\boldsymbol{T}^{\le k}\boldsymbol{e} + {}_{(\alpha)}\boldsymbol{U}^{[k]}(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{T}^{>k})^{-1}{}_{(\alpha)}\boldsymbol{D}^{[k]}\boldsymbol{e} \\ &\le {}_{(0)}\boldsymbol{T}^{\le}\boldsymbol{e} + \varepsilon\boldsymbol{e} + \frac{m_*\gamma_1^2}{1-\gamma_2}\boldsymbol{e}, \end{aligned} $$

which implies that ${}_{(\alpha)}\boldsymbol{T}^{[k]}$ is bounded for $\alpha \in [0, \alpha_0]$ and right-continuous at $\alpha = 0$. □

Lemma 4.5 *If Assumption 4.3 holds, then* $\lim_{\alpha\downarrow 0}(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))^{-1} = (\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))^{-1}$.

Proof. From Lemma 4.4, for any $\varepsilon > 0$, there exists some $\alpha_0 > 0$ such that

$$ {}_{(\alpha)}\boldsymbol{\Phi}(0) \le {}_{(0)}\boldsymbol{\Phi}(0) + \varepsilon\boldsymbol{e}\boldsymbol{e}^{\top}, \quad \text{for } 0 \le \alpha \le \alpha_0. \tag{4.39}$$

Since $\{({}_{(\alpha)}X_n, {}_{(\alpha)}S_n)\}$ is irreducible and recurrent for all $\alpha \ge 0$, $\sum_{m=0}^{\infty}[{}_{(\alpha)}\boldsymbol{\Phi}(0)]^m = (\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))^{-1} < \infty$ and ${}_{(\alpha)}\boldsymbol{\Phi}(0)\boldsymbol{e} \le \boldsymbol{e}$ for all $\alpha \ge 0$. Thus, there exists some $\gamma_0 \in (0,1)$ and $m_* \in \mathbb{N}$ such that

$$ [{}_{(0)}\boldsymbol{\Phi}(0)]^{m_*}\boldsymbol{e} \le \gamma_0\boldsymbol{e}. \tag{4.40}$$

Inequalities (4.39) and (4.40) show that, for any $\gamma \in (\gamma_0, 1)$, there exists some $\gamma_* > 0$ such that

$$[{}_{(\alpha)}\boldsymbol{\Phi}(0)]^{m_*}\boldsymbol{e} \le \gamma_*\boldsymbol{e}, \qquad \text{for } 0 \le \alpha \le \alpha_0.$$

Using the above inequality and ${}_{(\alpha)}\boldsymbol{\Phi}(0)\boldsymbol{e} \le \boldsymbol{e}$ and the similar argument to (4.38), we obtain for $0 \le \alpha \le \alpha_0$,

$$\sum_{m=0}^{\infty}[{}_{(\alpha)}\boldsymbol{\Phi}(0)]^m\boldsymbol{e} \quad \le \quad \frac{m_*}{1-\gamma_*}\boldsymbol{e}, \tag{4.41}$$

which implies that $(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))^{-1}$ is bounded for $\alpha \in [0, \alpha_0]$ and right-continuous at $\alpha = 0$. □

Lemma 4.6 *If Assumption 4.3 holds, then, for all $k \in \mathbb{N}$, the following are true: (i)* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{R}(k) = {}_{(0)}\boldsymbol{R}(k)$; *(ii)* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{R}_0(k) = {}_{(0)}\boldsymbol{R}_0(k)$; *and (iii)* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{G}(k) = {}_{(0)}\boldsymbol{G}(k)$.

Proof. From Lemma 4.4, ${}_{(\alpha)}\boldsymbol{\Phi}(k)$ ($k \in \mathbb{Z}$) and ${}_{(\alpha)}\boldsymbol{T}_{0,k}^{[k]}$ ($k \in \mathbb{N}$) are right-continuous at $\alpha = 0$. According to Lemma 4.5, $(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))^{-1}$ is also right-continuous at $\alpha = 0$. Therefore, the statements (i)–(iii) are obvious from the equations (1.12), (1.13) and (1.9), respectively. □

Lemma 4.7 *If Assumption 4.3 holds, then for all $\xi \in \mathbb{R}$ the following are true: (i)* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi) = {}_{(0)}\widetilde{\boldsymbol{A}}(\xi)$; *(ii)* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\widetilde{\boldsymbol{R}}(\xi) = {}_{(0)}\widetilde{\boldsymbol{R}}(\xi)$; *(iii)* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\widetilde{\boldsymbol{R}}_0(\xi) = {}_{(0)}\widetilde{\boldsymbol{R}}_0(\xi)$; *and (iv)* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\widetilde{\boldsymbol{G}}(\xi) = {}_{(0)}\widetilde{\boldsymbol{G}}(\xi)$.

Proof. Note that for any $(i,j) \in \mathbb{M}^2$ and $\alpha \ge 0$,

$$|[{}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi)]_{i,j}| \le \sum_{k=1}^{\infty}[{}_{(\alpha)}\boldsymbol{A}(k)]_{i,j} \le [\boldsymbol{A}]_{i,j}.$$

Therefore, using the dominated convergence theorem, we obtain

$$\lim_{\alpha\downarrow 0}{}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi) = \sum_{k\in\mathbb{Z}} e^{\mathrm{i}\xi k}\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{A}(k) = {}_{(0)}\widetilde{\boldsymbol{A}}(\xi).$$

Next, we consider the statements (ii)–(iv). By definition, the spectral radius of ${}_{(\alpha)}\boldsymbol{\Phi}(k)$, ${}_{(\alpha)}\boldsymbol{\Phi}(-k)$, and ${}_{(\alpha)}\boldsymbol{T}_{0,k}^{[k]}$ are less than or equal to 1, which means that $\sum_{k=1}^{\infty}{}_{(\alpha)}\boldsymbol{\Phi}(k)\boldsymbol{e} \le \boldsymbol{e}$, $\sum_{k=1}^{\infty}{}_{(\alpha)}\boldsymbol{\Phi}(-k)\boldsymbol{e} \le \boldsymbol{e}$, and $\sum_{k=1}^{\infty}{}_{(\alpha)}\boldsymbol{T}_{0,k}^{[k]}\boldsymbol{e} \le \boldsymbol{e}$. Combining these facts with (1.12), (1.13), and (1.9), we can prove the statements (ii)–(v) in the similar way to the statement (i). □

Lemma 4.8 *If Assumption 4.3 holds, then* $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{\kappa} = {}_{(0)}\boldsymbol{\kappa}$.

Proof. Since ${}_{(\alpha)}\boldsymbol{\kappa}$ is the unique stationary vector of ${}_{(\alpha)}\boldsymbol{T}^{[0]}$ for all $\alpha \ge 0$, ${}_{(\alpha)}\boldsymbol{\kappa} = {}_{(\alpha)}\boldsymbol{\kappa}{}_{(\alpha)}\boldsymbol{T}^{[0]}$ and ${}_{(\alpha)}\boldsymbol{\kappa}\boldsymbol{e} = 1$. Therefore, we obtain

$$\begin{aligned} {}_{(\alpha)}\boldsymbol{\kappa} - {}_{(0)}\boldsymbol{\kappa} &= {}_{(\alpha)}\boldsymbol{\kappa}{}_{(\alpha)}\boldsymbol{T}^{[0]} - {}_{(\alpha)}\boldsymbol{\kappa}{}_{(0)}\boldsymbol{T}^{[0]} + {}_{(\alpha)}\boldsymbol{\kappa}{}_{(0)}\boldsymbol{T}^{[0]} - {}_{(0)}\boldsymbol{\kappa}{}_{(0)}\boldsymbol{T}^{[0]} \\ &= {}_{(\alpha)}\boldsymbol{\kappa}\left({}_{(\alpha)}\boldsymbol{T}^{[0]} - {}_{(0)}\boldsymbol{T}^{[0]}\right) + \left({}_{(\alpha)}\boldsymbol{\kappa} - {}_{(0)}\boldsymbol{\kappa}\right){}_{(0)}\boldsymbol{T}^{[0]}. \end{aligned} \tag{4.42}$$

Note here that $\boldsymbol{K}_* \triangleq (\boldsymbol{I} - {}_{(0)}\boldsymbol{T}^{[0]} + \boldsymbol{e}{}_{(0)}\boldsymbol{\kappa})^{-1}$ exists. Thus, (4.42) leads to

$${}_{(\alpha)}\boldsymbol{\kappa} - {}_{(0)}\boldsymbol{\kappa} = {}_{(0)}\boldsymbol{\kappa}\left({}_{(\alpha)}\boldsymbol{T}^{[0]} - {}_{(0)}\boldsymbol{T}^{[0]}\right)\boldsymbol{K}_*. \tag{4.43}$$

Note also that $\lim_{\alpha\downarrow 0} {}_{(\alpha)}\boldsymbol{T}^{[0]} = {}_{(0)}\boldsymbol{T}^{[0]}$ (see Lemma 4.4), which implies that there exists some $\alpha_0 > 0$ such that ${}_{(\alpha)}\boldsymbol{T}^{[0]} \le {}_{(0)}\boldsymbol{T}^{[0]} + \varepsilon \boldsymbol{e}\boldsymbol{e}^{\top}$ for all $\varepsilon > 0$ and $\alpha \in [0, \alpha_0]$. It thus follows from (4.43) that, for $\alpha \in [0, \alpha_0]$ and $i \in \mathbb{M}_0$,

$$\left| \left[{}_{(\alpha)}\boldsymbol{\kappa} - {}_{(0)}\boldsymbol{\kappa} \right]_i \right| \le \sum_{j=1}^{M_0} \left| \left[{}_{(0)}\boldsymbol{\kappa} \left({}_{(\alpha)}\boldsymbol{T}^{[0]} - {}_{(0)}\boldsymbol{T}^{[0]} \right) \right]_j \right| |[\boldsymbol{K}_*]_{ij}| \le \varepsilon \sum_{j=1}^{M_0} |[\boldsymbol{K}_*]_{ij}|,$$

which completes the proof. □

Lemma 4.9 *If Assumption 4.3 holds and* $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^2 {}_{(\alpha)}\boldsymbol{A}(-k) < \infty$, *then* $\lim_{\alpha\downarrow 0} {}_{(\alpha)}\boldsymbol{\beta}_G = {}_{(0)}\boldsymbol{\beta}_G$.

Proof. We can easily confirm that if $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^2 {}_{(\alpha)}\boldsymbol{A}(-k) < \infty$, then $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k_{(\alpha)}\boldsymbol{G}(k) < \infty$ in the similar way to Proposition 4.2. Thus, using the dominated convergence theorem, we have

$$\lim_{\alpha\downarrow 0} {}_{(\alpha)}\boldsymbol{\beta}_G = \sum_{k}^{\infty} \lim_{\alpha\downarrow 0} k_{(\alpha)}\boldsymbol{G}(k)\boldsymbol{e} = {}_{(0)}\boldsymbol{\beta}_G.$$

□

Lemma 4.10 *If Assumption 4.3 (a)–(c) hold, then the following are true for any* $m \in \mathbb{N}$*:*

(i) If $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^{m+1} {}_{(\alpha)}\boldsymbol{A}(k) < \infty$, *then* $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^{m} {}_{(\alpha)}\boldsymbol{R}(k) < \infty$;

(ii) If $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^{m+1} {}_{(\alpha)}\boldsymbol{A}(-k) < \infty$, *then* $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^{m} {}_{(\alpha)}\boldsymbol{G}(k) < \infty$; *and*

(iii) If $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^{m+1} {}_{(\alpha)}\boldsymbol{B}(k) < \infty$, *then* $\sup_{\alpha\ge 0} \sum_{k=1}^{\infty} k^{m} {}_{(\alpha)}\boldsymbol{R}_0(k) < \infty$.

Proof. This lemma can be proved in the same way to the proof of Lemma 4.2 by using Lemma 4.5; thus, we omit the proof. □

Remark 4.6 *Lemma 4.2 presents the existence condition of moments for each* α *whereas Lemma 4.10 assures the boundedness of them for* $\alpha \ge 0$.

4.3 Heavy-traffic limit of stationary distribution

In this section, we consider the heavy-traffic limit for the stationary distribution. The following theorem shows the heavy-traffic asymptotic formula.

Theorem 4.1 *If Assumption 4.3 holds, then*

$$\lim_{\alpha\downarrow 0} {}_{(\alpha)}\widetilde{\boldsymbol{x}}({}_{(\alpha)}\sigma\xi) = \frac{1}{1 - \mathrm{i}\xi\gamma}\boldsymbol{\pi}, \quad \xi \in \mathbb{R}, \tag{4.44}$$

where

$$\gamma = \frac{1}{2}\boldsymbol{\pi}\sum_{k\in\mathbb{Z}} k^2 {}_{(0)}\boldsymbol{A}(k)\boldsymbol{e} + \boldsymbol{\pi}\sum_{k\in\mathbb{Z}} k_{(0)}\boldsymbol{A}(k)(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1} {}_{(0)}\boldsymbol{\beta}_A. \tag{4.45}$$

Remark 4.7 *Theorem 4.1 shows that*

$$\lim_{\alpha\downarrow 0} \mathsf{P}({}_{(\alpha)}\sigma_{(\alpha)}X > x, {}_{(\alpha)}S = i) = e^{-x/\gamma}\pi_i, \quad x \ge 0,\ i \in \mathbb{M}.$$

In what follows, we present a complete proof of Theorem 4.1, following the several lemmas necessary for the proof.

Lemma 4.11 $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{x}(0) = \boldsymbol{0}$.

Proof. Recall here that, for all $\alpha \ge 0$, ${}_{(\alpha)}\boldsymbol{T}$ is recurrent (see Remark 4.5) and thus ${}_{(\alpha)}\boldsymbol{G}$ is stochastic [68]. Combining this fact and the definition of ${}_{(\alpha)}\boldsymbol{\beta}_G$, we have ${}_{(\alpha)}\boldsymbol{g}_{(\alpha)}\boldsymbol{\beta}_G \ge {}_{(\alpha)}\boldsymbol{g}_{(\alpha)}\boldsymbol{G}\boldsymbol{e} = 1$ for all $\alpha \ge 0$. It follows from this inequality and Proposition 4.3 that

$$\lim_{\alpha\downarrow 0}\sum_{i\in\mathbb{M}} {}_{(\alpha)}g_i \mathsf{E}[{}_{(\alpha)}\tau_0 \mid {}_{(\alpha)}X_0 = 1, {}_{(\alpha)}S_0 = i_1] \ge \lim_{\alpha\downarrow 0}\frac{1}{{}_{(\alpha)}\sigma} = \infty, \tag{4.46}$$

which implies that there exists at least one $i_1 \in \mathbb{M}$ such that

$$\lim_{\alpha\downarrow 0}\mathsf{E}[{}_{(\alpha)}\tau_0 \mid {}_{(\alpha)}X_0 = 1, {}_{(\alpha)}S_0 = i_1] = \infty.$$

It should be noted that the recurrence of ${}_{(\alpha)}\boldsymbol{T}$ implies that there exists some $i_0 \in \mathbb{M}_0$ such that the Markov chain $\{({}_{(\alpha)}X_n, {}_{(\alpha)}S_n)\}$ reaches state $(1, i_1)$ from state $(0, i_0)$ avoiding level zero with probability 1. Thus, since $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{\kappa} = {}_{(0)}\boldsymbol{\kappa} > \boldsymbol{0}$ (see Lemma 4.8), we obtain

$$\lim_{\alpha\downarrow 0}\sum_{i\in\mathbb{M}_0} {}_{(\alpha)}\kappa_i \mathsf{E}[{}_{(\alpha)}\tau_0 \mid {}_{(\alpha)}X_0 = 0, {}_{(\alpha)}S_0 = i] = \infty.$$

Finally, applying this to (1.27) yields $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\boldsymbol{x}(0) = \boldsymbol{0}$. □

Lemma 4.12 ${}_{(0+)}\widetilde{\boldsymbol{x}}(0) := \lim_{\alpha\downarrow 0}{}_{(\alpha)}\widetilde{\boldsymbol{x}}(0) = \boldsymbol{\pi}$.

Proof. From (4.8), we have

$${}_{(\alpha)}\widetilde{\boldsymbol{x}}(0)(\boldsymbol{I} - \boldsymbol{A}) = {}_{(\alpha)}\boldsymbol{x}(0)_{(\alpha)}\boldsymbol{R}_0(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{G}). \tag{4.47}$$

Note here that for all $\alpha \ge 0$, ${}_{(\alpha)}\boldsymbol{\Phi}(0)$ is substochastic and ${}_{(\alpha)}\boldsymbol{G} = {}_{(\alpha)}\widetilde{\boldsymbol{G}}(0)$ is stochastic because ${}_{(\alpha)}\boldsymbol{T}$ is irreducible and recurrent. Note also that for all $\alpha \ge 0$, ${}_{(\alpha)}\boldsymbol{R}_0 = {}_{(\alpha)}\widetilde{\boldsymbol{R}}_0(0) < \infty$ due to $\sum_{k=1}^{\infty} k_{(\alpha)}\boldsymbol{B}(k)\boldsymbol{e} < \infty$ (see Remark 4.3). It thus follows from (4.47), Lemmas 4.7 and 4.11 that

$${}_{(0+)}\widetilde{\boldsymbol{x}}(0)(\boldsymbol{I} - \boldsymbol{A}) = \boldsymbol{0},$$

which implies that ${}_{(0+)}\widetilde{\boldsymbol{x}}(0) = c\boldsymbol{\pi}$ for some $c \ge 0$. Furthermore, since ${}_{(\alpha)}\widetilde{\boldsymbol{x}}(0)\boldsymbol{e} + {}_{(\alpha)}\boldsymbol{x}(0)\boldsymbol{e} = 1$ $(\forall\alpha > 0)$, Lemma 4.11 yields ${}_{(0+)}\widetilde{\boldsymbol{x}}\boldsymbol{e} = c = 1$. □

Lemma 4.13

$$\lim_{\alpha\downarrow 0}\frac{({}_{(\alpha)}\sigma)^2}{{}_{(\alpha)}\phi({}_{(\alpha)}\sigma\xi)} = \frac{1}{\mathrm{i}\xi + (\mathrm{i}\xi)^2\dfrac{{}_{(0)}\widetilde{\delta}''(0)}{2}}\frac{1}{{}_{(0)}\widetilde{\Gamma}(0)}. \tag{4.48}$$

Proof. Note that $|_{(\alpha)}\phi''(\xi)| < \infty$ for all $\xi \in \mathbb{R}$ and all $\alpha \ge 0$ due to Assumption 4.3 (d) (see (4.15)). Note also that $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\phi(\xi) = {}_{(0)}\phi(\xi)$ for all $\xi \in \mathbb{R}$ because ${}_{(\alpha)}\phi(\xi)$ is a polynomial function of ξ whose coefficients are polynomials of elements of ${}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi)$. Furthermore, ${}_{(\alpha)}\phi(0) = \det(\boldsymbol{I} - \boldsymbol{A}) = 0$. It thus follows from Taylor's theorem that there exists some ${}_{(\alpha)}c := {}_{(\alpha)}c(\xi) \in (0, {}_{(\alpha)}\sigma\xi)$ such that $\lim_{\alpha\downarrow 0}{}_{(\alpha)}c = {}_{(0)}c$ and

$$
{}_{(\alpha)}\phi({}_{(\alpha)}\sigma\xi) \;=\; {}_{(\alpha)}\phi'(0)\cdot({}_{(\alpha)}\sigma\xi) + \frac{{}_{(\alpha)}\phi''({}_{(\alpha)}c)}{2}({}_{(\alpha)}\sigma\xi)^2. \tag{4.49}
$$

Since $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\phi''(\xi) = {}_{(0)}\phi''(\xi)$ for all ξ and $\lim_{\xi\downarrow 0}{}_{(\alpha)}\phi''(\xi) = {}_{(\alpha)}\phi''(0)$ for all $\alpha \ge 0$, Lemma 4.3 yields $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\phi''({}_{(\alpha)}c) = {}_{(0)}\phi''({}_{(0)}c)$. Therefore (4.49) leads to

$$
\lim_{\alpha\downarrow 0}\frac{{}_{(\alpha)}\phi({}_{(\alpha)}\sigma\xi)}{({}_{(\alpha)}\sigma)^2} = \xi\lim_{\alpha\downarrow 0}\frac{{}_{(\alpha)}\phi'(0)}{{}_{(\alpha)}\sigma} + \xi^2\frac{{}_{(\alpha)}\phi''(0)}{2}. \tag{4.50}
$$

From (4.14) and (4.15), we have

$$
\begin{aligned}
\xi\lim_{\alpha\downarrow 0}\frac{{}_{(\alpha)}\phi'(0)}{{}_{(\alpha)}\sigma} &= \mathrm{i}\xi\,{}_{(0)}\widetilde{\Gamma}(0), &(4.51)\\
\xi^2\frac{{}_{(0)}\phi''^{(0)}(0)}{2} &= (\mathrm{i}\xi)^2\frac{{}_{(0)}\widetilde{\delta}''(0)}{2}\,{}_{(0)}\widetilde{\Gamma}(0). &(4.52)
\end{aligned}
$$

Substituting (4.51) and (4.52) into (4.50) yields (4.48). □

Lemma 4.14

$$
\lim_{\alpha\downarrow 0}\frac{\boldsymbol{e} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi)\boldsymbol{e}}{{}_{(\alpha)}\sigma} = \mathrm{i}\xi\,{}_{(0)}\boldsymbol{\beta}_G. \tag{4.53}
$$

Proof. Let $\boldsymbol{\Delta}_{{}_{(\alpha)}\beta_G}$ denote a diagonal matrix whose jth diagonal element is equal to the jth element of ${}_{(\alpha)}\boldsymbol{\beta}_G$. We then have

$$
\begin{aligned}
\frac{\boldsymbol{e} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi)\boldsymbol{e}}{{}_{(\alpha)}\sigma} &= \boldsymbol{\Delta}_{{}_{(\alpha)}\boldsymbol{\beta}_G}\cdot\frac{1 - e^{-\mathrm{i}_{(\alpha)}\sigma\xi}}{{}_{(\alpha)}\sigma}\cdot\boldsymbol{\Delta}^{-1}_{{}_{(\alpha)}\boldsymbol{\beta}_G}\frac{\boldsymbol{e} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi)\boldsymbol{e}}{1 - e^{-\mathrm{i}_{(\alpha)}\sigma\xi}}\\
&= \boldsymbol{\Delta}_{{}_{(\alpha)}\boldsymbol{\beta}_G}\cdot\frac{1 - e^{-\mathrm{i}_{(\alpha)}\sigma\xi}}{{}_{(\alpha)}\sigma}\cdot\sum_{k=0}^{\infty}e^{(-\mathrm{i}_{(\alpha)}\sigma\xi)k}\sum_{m=k+1}^{\infty}\boldsymbol{\Delta}^{-1}_{{}_{(\alpha)}\boldsymbol{\beta}_G}\,{}_{(\alpha)}\boldsymbol{G}(m)\boldsymbol{e}. \qquad (4.54)
\end{aligned}
$$

Using Lemma 4.9 and the dominated convergence theorem, we obtain

$$
\lim_{\alpha\downarrow 0}\sum_{k=0}^{\infty}e^{(-\mathrm{i}_{(\alpha)}\sigma\xi)k}\sum_{m=k+1}^{\infty}\boldsymbol{\Delta}^{-1}_{{}_{(\alpha)}\boldsymbol{\beta}_G}\,{}_{(\alpha)}\boldsymbol{G}(m)\boldsymbol{e} = \boldsymbol{e}. \tag{4.55}
$$

Note here that

$$
\lim_{\alpha\downarrow 0}\frac{1 - e^{-\mathrm{i}_{(\alpha)}\sigma\xi}}{{}_{(\alpha)}\sigma} = \mathrm{i}\xi. \tag{4.56}
$$

Finally, applying (4.55) and (4.56) to (4.54), we have (4.53). □

Lemma 4.15

$$\lim_{\alpha\downarrow 0} \mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}({}_{(\alpha)}\sigma\xi)) = \boldsymbol{e}\boldsymbol{\pi}{}_{(0)}\widetilde{\Gamma}(0). \tag{4.57}$$

Proof. From the definition of adjugate matrix, $\mathrm{adj}(\boldsymbol{I}-{}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi))_{ij}$ represents the (i,j)-minor of $\boldsymbol{I}-{}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi)$, which is a polynomial of elements of $\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi)$. Therefore Lemma 4.7 (i) implies that

$$\lim_{\alpha\downarrow 0} \mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi)) = \mathrm{adj}(\boldsymbol{I} - {}_{(0)}\widetilde{\boldsymbol{A}}(\xi)), \qquad \xi \in \mathbb{R}.$$

Thus, using Lemma 4.3 yields

$$\lim_{\alpha\downarrow 0} \mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}({}_{(\alpha)}\sigma\xi)) = \mathrm{adj}(\boldsymbol{I} - {}_{(0)}\widetilde{\boldsymbol{A}}(0)). \tag{4.58}$$

Note here that (4.23) leads to

$$\lim_{\xi\to 0} \mathrm{adj}(\boldsymbol{I} - {}_{(0)}\tilde{\boldsymbol{A}}(\xi)) = \boldsymbol{e}\boldsymbol{\pi}{}_{(0)}\widetilde{\Gamma}(0). \tag{4.59}$$

From (4.58), we have (4.57). □

Lemma 4.16

$$\lim_{\alpha\downarrow 0} \frac{1}{{}_{(\alpha)}\sigma}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi))\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}({}_{(\alpha)}\sigma\xi)) \quad = \quad \mathrm{i}\xi{}_{(0)}\widetilde{\Gamma}(0){}_{(0)}\boldsymbol{d}_0\boldsymbol{\pi}, \tag{4.60}$$

where

$$_{(0)}\boldsymbol{d}_0 = {}_{(0)}\boldsymbol{\beta}_G + (\boldsymbol{I} - {}_{(0)}\boldsymbol{G})(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1}{}_{(0)}\boldsymbol{\beta}_A < \infty. \tag{4.61}$$

Proof. Note first that $|\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi))| < \infty$ for all $\xi \in \mathbb{R}$ and all $\alpha \ge 0$. It follows from (4.57) and Taylor's theorem that there exists some ${}_{(\alpha)}s := {}_{(\alpha)}s(\xi)$ such that ${}_{(\alpha)}s(\xi) \in (0, {}_{(\alpha)}\sigma\xi)$ and $\lim_{\alpha\downarrow 0}{}_{(\alpha)}s = {}_{(0)}s$ and

$$\begin{aligned}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}({}_{(\alpha)}\sigma\xi)) \quad &= \quad \mathrm{adj}(\boldsymbol{I} - {}_{(0)}\widetilde{\boldsymbol{A}}(0)) + ({}_{(\alpha)}\sigma\xi)\,\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi))\bigg|_{\xi=0} \\ &\quad +({}_{(\alpha)}\sigma\xi)^2\,\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi))\bigg|_{\xi={}_{(\alpha)}s}.\end{aligned} \tag{4.62}$$

From (4.53) and (4.57), we have

$$\lim_{\alpha\downarrow 0} \frac{1}{{}_{(\alpha)}\sigma}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi))\mathrm{adj}(\boldsymbol{I} - {}_{(0)}\widetilde{\boldsymbol{A}}(0)) \quad = \quad \mathrm{i}\xi{}_{(0)}\widetilde{\Gamma}(0){}_{(0)}\boldsymbol{\beta}_G\boldsymbol{\pi}. \tag{4.63}$$

Furthermore, from (4.21), we obtain

$$\begin{aligned}\lim_{\alpha\downarrow 0} \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi))\bigg|_{\xi=0} &= \mathrm{i}_{(0)}\widetilde{\Gamma}(0)(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1}{}_{(0)}\boldsymbol{\beta}_A\boldsymbol{\pi} \\ +\boldsymbol{e}\boldsymbol{\pi}\left({}_{(0)}\widetilde{\Gamma}'(0)\boldsymbol{I} + +\mathrm{i}_{(0)}\widetilde{\Gamma}(0)\sum_{k\in\mathbb{Z}} k{}_{(0)}\boldsymbol{A}(k)(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1}\right),\end{aligned}$$

from which and (4.53) it follows that

$$\lim_{\alpha\downarrow 0}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi)) \ \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}({}_{(\alpha)}\sigma\xi))\Big|_{\xi=0}$$
$$= \ \mathrm{i}_{(0)}\widetilde{\Gamma}(0)(\boldsymbol{I} - {}_{(0)}\boldsymbol{G})(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1}{}_{(0)}\boldsymbol{\beta}_A\boldsymbol{\pi}. \tag{4.64}$$

According to the definition of the adjugate matrix, it is clear that for any $\alpha \ge 0$,

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi))\Big|_{\xi=0} < \infty.$$

Thus, (4.62) leads to

$$\lim_{\alpha\downarrow 0}\frac{1}{{}_{(\alpha)}\sigma}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi))\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}({}_{(\alpha)}\sigma\xi))$$
$$= \ \lim_{\alpha\downarrow 0}\frac{(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi))}{{}_{(\alpha)}\sigma}\mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(0))$$
$$+\xi\lim_{\alpha\downarrow 0}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi))\frac{\mathrm{d}}{\mathrm{d}\xi} \ \mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}(\xi))\Big|_{\xi=0}.$$

As a result, substituting (4.63) and (4.64) into the above equation yields (4.60) and (4.61). □

Lemma 4.17

$$\lim_{\alpha\downarrow 0}\frac{{}_{(\alpha)}\boldsymbol{x}(0)}{{}_{(\alpha)}\sigma} = \nu_* \cdot {}_{(0)}\boldsymbol{\kappa},$$

where ν_* is a finite positive number such that

$$\nu_* = \left[{}_{(0)}\boldsymbol{\kappa}_{(0)}\boldsymbol{R}_0(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0\right]^{-1}. \tag{4.65}$$

Proof. It follows from (1.27) and Lemma 4.11 that ${}_{(\alpha)}\boldsymbol{x}(0) = {}_{(\alpha)}\psi_{(\alpha)}\boldsymbol{\kappa}$ for some ${}_{(\alpha)}\psi > 0$ such that $\lim_{\alpha\downarrow 0}{}_{(\alpha)}\psi = 0$. Thus, from Proposition 4.2 and ${}_{(\alpha)}\widetilde{\boldsymbol{x}}(0)\boldsymbol{e} = 1 - {}_{(\alpha)}\psi$, we have

$$\lim_{\alpha\downarrow 0}\frac{{}_{(\alpha)}\psi}{{}_{(\alpha)}\sigma} \ = \ \lim_{\alpha\downarrow 0}(1 - {}_{(\alpha)}\psi)\left[{}_{(\alpha)}\boldsymbol{\kappa}_{(\alpha)}\boldsymbol{R}_0(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))\right.$$
$$\left.\times\left({}_{(\alpha)}\boldsymbol{\beta}_G + (\boldsymbol{I} - {}_{(0)}\boldsymbol{G})(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1}{}_{(\alpha)}\boldsymbol{\beta}_A\right)\right]^{-1} = \nu_*,$$

where we use Lemmas 4.7, 4.8, and 4.9 in the second equality. □

Next, we prove $\nu_* \in (0,\infty)$. Since ${}_{(\alpha)}\boldsymbol{T}$ is irreducible, ${}_{(\alpha)}\boldsymbol{R}_0$ has no zero column. It thus follows from ${}_{(\alpha)}\boldsymbol{\kappa} > \boldsymbol{0}$ ($\forall\alpha \ge 0$) that

$${}_{(\alpha)}\boldsymbol{\kappa}_{(\alpha)}\boldsymbol{R}_0 > \boldsymbol{0}, \qquad \text{for all } \alpha \ge 0.$$

Furthermore, (4.31) implies that $(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))_{(\alpha)}\boldsymbol{d}_0 \ge \boldsymbol{0}$ for all $\alpha \ge 0$. Thus, it is sufficient to show that $(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0 \ne \boldsymbol{0}$. Note here that ${}_{(\alpha)}\boldsymbol{\Phi}(0)$ does not have an eigenvalue 1. In addition, if ${}_{(\alpha)}\boldsymbol{d}_0 = \boldsymbol{0}$ (see (4.61)),

$${}_{(\alpha)}\boldsymbol{g}_{(\alpha)}\boldsymbol{d}_0 = {}_{(\alpha)}\boldsymbol{g}_{(\alpha)}\boldsymbol{\beta}_G = 0,$$

which contradicts with ${}_{(\alpha)}\boldsymbol{g}_{(\alpha)}\boldsymbol{\beta}_G \ge {}_{(\alpha)}\boldsymbol{g}_{(\alpha)}\boldsymbol{G}\boldsymbol{e} = 1$. Therefore, we obtain, for all $\alpha \ge 0$, ${}_{(\alpha)}\boldsymbol{d}_0 \ne \boldsymbol{0}$ and thus

$$
{}_{(\alpha)}\boldsymbol{\kappa}_{(\alpha)}\boldsymbol{R}_0(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0))_{(\alpha)}\boldsymbol{d}_0 > 0 \quad \text{for all } \alpha > 0.
$$

Finally, the positivity of ν (i.e., $\nu > 0$) follows from the fact that for all $\alpha \ge 0$, ${}_{(\alpha)}\boldsymbol{R}_0 < \infty$ (see Remark 4.3) and ${}_{(\alpha)}\boldsymbol{\beta}_G < \infty$ (see Lemma 4.2 (ii)). □

We now provide the proof of Theorem 4.1.

Proof of Theorem 4.1. Lemma 4.12 shows that (4.44) holds for $\xi = 0$. Thus, we consider the case of $\xi \ne 0$. From (4.9), we have

$$
\begin{aligned}
\lim_{\alpha\downarrow 0} {}_{(\alpha)}\widetilde{\boldsymbol{x}}({}_{(\alpha)}\sigma\xi) &= \lim_{\alpha\downarrow 0} \frac{{}_{(\alpha)}\boldsymbol{x}(0)}{{}_{(\alpha)}\sigma} {}_{(\alpha)}\widetilde{\boldsymbol{R}}_0({}_{(\alpha)}\sigma\xi)(\boldsymbol{I} - {}_{(\alpha)}\boldsymbol{\Phi}(0)) \\
&\quad \times \frac{\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{G}}({}_{(\alpha)}\sigma\xi)}{{}_{(\alpha)}\sigma} \mathrm{adj}(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{A}}({}_{(\alpha)}\sigma\xi)) \frac{({}_{(\alpha)}\sigma)^2}{{}_{(\alpha)}\phi({}_{(\alpha)}\sigma\xi)}.
\end{aligned} \tag{4.66}
$$

Applying Lemmas 4.7, 4.13, 4.16, and 4.17 to (4.66) yields

$$
\lim_{\alpha\downarrow 0} {}_{(\alpha)}\widetilde{\boldsymbol{x}}({}_{(\alpha)}\sigma\xi) = \frac{1}{1 - \mathrm{i}\xi \dfrac{-{}_{(0)}\widetilde{\delta}''(0)}{2}} \boldsymbol{\pi},
$$

from which and (4.16) we have (4.44). □

4.4 Heavy-traffic limit of moments

In this section, we provide the heavy-traffic asymptotic formula for the moments of the stationary distribution. For simplicity, for any function f (including vectors and matrix functions), let $f^{(n)}$ $(n \in \mathbb{N})$ denote an n times differential of f hereafter. Before giving the main theorem, we provide the following lemma.

Lemma 4.18 *Suppose that Assumption 4.3 holds. If* $\sup_{\alpha\ge 0}\sum_{k=1}^{\infty} k^{m+1}{}_{(\alpha)}\boldsymbol{A}(k) < \infty$ *for some* $m \in \mathbb{Z}_+$, *then,*

$$
\lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma \frac{\mathrm{d}^m}{\mathrm{d}\xi^m} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1} = \frac{m!}{(1 - \mathrm{i}\xi\gamma)^{m+1}} (\mathrm{i}\gamma)^m (\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0\boldsymbol{\pi}, \tag{4.67}
$$

where γ *and* ${}_{(0)}\boldsymbol{d}_0$ *are given in (4.45) and (4.61), respectively.*

Proof. We prove this lemma by induction. By applying Lemmas 4.14–4.16 to (4.6), we can easily confirm that (4.67) holds when $m = 0$, i.e.,

$$
\lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi))^{-1} = \frac{1}{1 - \mathrm{i}\xi\gamma}(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0\boldsymbol{\pi}. \tag{4.68}
$$

Next, we assume that (4.67) holds for $m \le n$ $(n \in \mathbb{N})$ under the condition that

$$
\sup_{\alpha\ge 0}\sum_{k=1}^{\infty} k^{n+1}{}_{(\alpha)}\boldsymbol{A}(k) < \infty.
$$

By differentiating $(n+1)$ times the equation

$$\left[\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right] \cdot \left[\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right]^{-1} = \boldsymbol{I},$$

with respect to ξ, we obtain

$$\begin{aligned} &\left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right) \frac{\mathrm{d}^{n+1}}{\mathrm{d}\xi^{n+1}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1} \\ &= \sum_{\ell=0}^{n} \binom{n+1}{\ell} ({}_{(\alpha)}\sigma)^{n+1-\ell} {}_{(\alpha)}\widetilde{\boldsymbol{R}}^{(n+1-\ell)}({}_{(\alpha)}\sigma\xi) \frac{\mathrm{d}^{\ell}}{\mathrm{d}\xi^{\ell}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1}. \end{aligned} \tag{4.69}$$

Note here that, for any $\ell \geq 2$ and $m \leq n$,

$$\lim_{\alpha\downarrow 0} ({}_{(\alpha)}\sigma)^{\ell} \frac{\mathrm{d}^{m}}{\mathrm{d}\xi^{m}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1} = \boldsymbol{O}.$$

Furthermore, if $\sup_{\alpha\geq 0}\sum_{k=1}^{\infty} k^{n+2}{}_{(\alpha)}\boldsymbol{A}(k) < \infty$, then $\sup_{\alpha\geq 0}\sum_{k=1}^{\infty} k^{n+1}{}_{(\alpha)}\boldsymbol{R}(k) < \infty$ due to Lemma 4.10. Thus, applying the dominated convergence theorem to (4.69) leads to

$$\begin{aligned} &\lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma \frac{\mathrm{d}^{n+1}}{\mathrm{d}\xi^{n+1}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right) \\ &= \lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1} \\ &\times \lim_{\alpha\downarrow 0} \left[\sum_{\ell=0}^{n} \binom{n+1}{l} \left({}_{(\alpha)}\sigma\right)^{n+1-l} {}_{(\alpha)}\widetilde{\boldsymbol{R}}^{(n+1-l)}({}_{(\alpha)}\sigma\xi) \frac{\mathrm{d}^{\ell}}{\mathrm{d}\xi^{\ell}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1}\right] \\ &= \lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1} \\ &\times (n+1)\cdot \lim_{\alpha\downarrow 0} {}_{(\alpha)}\widetilde{\boldsymbol{R}}'({}_{(\alpha)}\sigma\xi) \cdot \lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma \frac{\mathrm{d}^{n}}{\mathrm{d}\xi^{n}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1}. \end{aligned}$$

Substituting the induction assumption (4.67) and (4.68) into the above equation yields

$$\begin{aligned} &\lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma \frac{\mathrm{d}^{n+1}}{\mathrm{d}\xi^{n+1}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right) \\ &= \frac{(n+1)!}{(1-\mathrm{i}\xi\gamma)^{n+2}} (\mathrm{i}\gamma)^{n} (\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))\boldsymbol{d}_0\boldsymbol{\pi} \left[\sum_{k=0}^{\infty} \mathrm{i}k_{(0)}\boldsymbol{R}(k)\right] (\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0\boldsymbol{\pi}. \end{aligned} \tag{4.70}$$

In addition, by differentiating twice (4.6) with respect to ξ and letting $\xi = 0$, we obtain

$$\begin{aligned} \sum_{k\in\mathbb{Z}} k^2{}_{(0)}\boldsymbol{A}(k) &= \sum_{k\in\mathbb{N}} k^2{}_{(0)}\boldsymbol{R}(k)(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))(\boldsymbol{I} - {}_{(0)}\boldsymbol{G}) \\ &\quad +2\sum_{k\in\mathbb{N}} k_{(0)}\boldsymbol{R}(k)(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0)) \sum_{k\in\mathbb{N}} k_{(0)}\boldsymbol{G}(k) \\ &\quad +(\boldsymbol{I} - {}_{(0)}\boldsymbol{R})(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0)) \sum_{k\in\mathbb{N}} k^2{}_{(0)}\boldsymbol{G}(k). \end{aligned} \tag{4.71}$$

Note that ${}_{(0)}\boldsymbol{T}$ is null-recurrent because ${}_{(0)}\sigma = 0$. Thus, we have $\boldsymbol{\pi}(\boldsymbol{I} - {}_{(0)}\boldsymbol{R}) = \boldsymbol{O}$ (see [67, Theorem 15]). It then follows from (4.71) that

$$\boldsymbol{\pi}\sum_{k=1}^{\infty} k_{(0)}\boldsymbol{R}(k)(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{\beta}_G = \frac{1}{2}\boldsymbol{\pi}\sum_{k\in\mathbb{Z}} k^2{}_{(0)}\boldsymbol{A}(k)\boldsymbol{e}. \tag{4.72}$$

Furthermore, by differentiating (4.6) with respect to ξ and letting $\xi = 0$ and $\alpha = 0$, and pre-multiplying $\boldsymbol{\pi}$, we have

$$\boldsymbol{\pi}\sum_{k\in\mathbb{Z}} k_{(0)}\boldsymbol{A}(k) = \boldsymbol{\pi}\sum_{k\in\mathbb{N}} k_{(0)}\boldsymbol{R}(k)(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))(\boldsymbol{I} - {}_{(0)}\boldsymbol{G}), \tag{4.73}$$

where we use $\boldsymbol{\pi}(\boldsymbol{I} - {}_{(0)}\boldsymbol{R}) = \boldsymbol{0}$. As a result, combining (4.72) and (4.73) with (4.61) leads to

$$\begin{aligned}
&\boldsymbol{\pi}\sum_{k=0}^{\infty} k_{(0)}\boldsymbol{R}(k)(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0 \\
&= \frac{1}{2}\boldsymbol{\pi}\sum_{k\in\mathbb{Z}} k^2{}_{(0)}\boldsymbol{A}(k)\boldsymbol{e} + \boldsymbol{\pi}\sum_{k\in\mathbb{Z}} k_{(0)}\boldsymbol{A}(k)(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi})^{-1}{}_{(0)}\boldsymbol{\beta}_A = \gamma.
\end{aligned} \tag{4.74}$$

Therefore, applying (4.74) to (4.70) leads to

$$\lim_{\alpha\downarrow 0} {}_{(\alpha)}\sigma \frac{\mathrm{d}^{n+1}}{\mathrm{d}\xi^{n+1}} \left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right) = \frac{(n+1)!}{(1-\mathrm{i}\xi\gamma)^{n+2}} (\mathrm{i}\gamma)^{n+1} (\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0\boldsymbol{\pi},$$

which shows that (4.67) holds when $m = n+1$ under the condition that $\sup_{\alpha\ge 0}\sum_{k=1}^{\infty} k^{n+2}{}_{(\alpha)}\boldsymbol{A}(k) < \infty$. □

Theorem 4.2 *Suppose that Assumption 4.3 holds. If*

$$\sup\sum_{k=1}^{\infty} k^{m+1}{}_{(\alpha)}\boldsymbol{A}(k) < \infty, \qquad \sup\sum_{k=1}^{\infty} k^{m+1}{}_{(\alpha)}\boldsymbol{B}(k) < \infty,$$

for any $m \in \mathbb{Z}_+$, then,

$$\lim_{\alpha\downarrow 0} \frac{\mathrm{d}^m}{\mathrm{d}\xi^m} {}_{(\alpha)}\widetilde{\boldsymbol{x}}({}_{(\alpha)}\sigma\xi)\Big|_{\xi=0} = m!\,(\mathrm{i}\gamma)^m \cdot \boldsymbol{\pi}, \tag{4.75}$$

where γ is given in (4.45).

Proof. By differentiating m times (4.7) with respect to ξ, we obtain

$$\begin{aligned}
\frac{\mathrm{d}^m}{\mathrm{d}\xi^m} {}_{(\alpha)}\widetilde{\boldsymbol{x}}^{(m)}({}_{(\alpha)}\sigma\xi) &= {}_{(\alpha)}\boldsymbol{x}(0)\sum_{\ell=0}^{m}\binom{m}{\ell}\left({}_{(\alpha)}\sigma\right)^{\ell}{}_{(\alpha)}\widetilde{\boldsymbol{R}}_0^{(\ell)}({}_{(\alpha)}\sigma\xi) \\
&\quad \times \frac{\mathrm{d}^{m-\ell}}{\mathrm{d}\xi^{m-\ell}}\left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1}.
\end{aligned} \tag{4.76}$$

According to Lemma 4.10, $\sup\sum_{k=1}^{\infty} k^{m+1}{}_{(\alpha)}\boldsymbol{B}(k) < \infty$ implies that $\sup\sum_{k=1}^{\infty} k^{\ell}{}_{(\alpha)}\boldsymbol{R}_0(k) < \infty$ for all $\ell \le m$. Thus, applying the dominated convergence theorem, Lemmas 4.17 and 4.18 to (4.76), we obtain

$$\begin{aligned}
&\lim_{\alpha\downarrow 0}\frac{\mathrm{d}^m}{\mathrm{d}\xi^m}{}_{(\alpha)}\widetilde{\boldsymbol{x}}^{(m)}({}_{(\alpha)}\sigma\xi) \\
&= \lim_{\alpha\downarrow 0}\frac{{}_{(\alpha)}\boldsymbol{x}(0)}{{}_{(\alpha)}\sigma}\sum_{\ell=0}^{m}\binom{m}{\ell}\left({}_{(\alpha)}\sigma\right)^{\ell}{}_{(\alpha)}\widetilde{\boldsymbol{R}}_0^{(\ell)}({}_{(\alpha)}\sigma\xi)\left({}_{(\alpha)}\sigma\right)\frac{\mathrm{d}^{m-\ell}}{\mathrm{d}\xi^{m-\ell}}\left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1} \\
&= \lim_{\alpha\downarrow 0}\frac{{}_{(\alpha)}\boldsymbol{x}(0)}{{}_{(\alpha)}\sigma}{}_{(\alpha)}\widetilde{\boldsymbol{R}}_0({}_{(\alpha)}\sigma\xi)\cdot\lim_{\alpha\downarrow 0}\left[\left({}_{(\alpha)}\sigma\right)\frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\left(\boldsymbol{I} - {}_{(\alpha)}\widetilde{\boldsymbol{R}}({}_{(\alpha)}\sigma\xi)\right)^{-1}\right] \\
&= \frac{m!(\mathrm{i}\gamma)^m\nu_*}{(1-\mathrm{i}\xi\gamma)^{m+1}}{}_{(0)}\boldsymbol{\kappa}_{(0)}\boldsymbol{R}_0(\boldsymbol{I} - {}_{(0)}\boldsymbol{\Phi}(0))_{(0)}\boldsymbol{d}_0\boldsymbol{\pi} = \frac{m!(\mathrm{i}\gamma)^m}{(1-\mathrm{i}\xi\gamma)^{m+1}}\boldsymbol{\pi},
\end{aligned}$$

where we use (4.65) in the last equality. Consequently, letting $\xi = 0$ in the above equation yields (4.75). □

Chapter 5

Conclusion

5.1 Summary of results

This thesis studied the asymptotic analysis of the stationary distribution of the GI/G/1-type Markov chain. The summary of the results is listed as follows.

(a) We studied the light-tailed asymptotics and showed asymptotic formulae for the cases where the decay rate is determined by (i) a root θ_+ of the fundamental equation of MAdP; (ii) the convergence radius of $\widehat{\boldsymbol{B}}(z)$; and (iii) that of $\widehat{\boldsymbol{A}}(z)$. We extended the previous results for the M/G/1-type Markov chains to the GI/G/1-type for the cases (i) and (ii), and derived completely new asymptotics formulae in the case (iii).

(b) We studied the subexponential tail asymptotics for the cases where $\boldsymbol{A}$ is stochastic and is strictly substochastic. In the former case, we extended the result for the M/G/1-type Markov chain to the GI/G/1-type one. Furthermore, in the latter case, we derived new asymptotic formulae, which is not considered in the literature.

(c) We conducted the heavy-traffic asymptotics of the GI/G/1-type Markov chain. We derived the heavy-traffic formula under weaker conditions than those of the previous studies. We also showed the heavy-traffic limit of the moments of the stationary distributions, which is not reported in previous studies.

5.2 Future work

This thesis studies the asymptotic behaviors of the stationary distribution of *a special type of* block-structured Markov chains from the several standpoints: tail asymptotics and heavy traffic. However, there remain many research problems in this area. The author describes some future work below.

(a) Although we derived various asymptotic formulae in this thesis, their errors or rates of the convergence were not considered. To apply the results in this thesis as approximate formulae, Theoretical understanding to their accuracy is necessary.

(b) As well as heavy-traffic limit, the light-traffic limit for the GI/G/1-type Markov chain has not been studied and is limited to a GI/G/1 queueing model. Thus, we leave this as future work.

(c) Although the GI/G/1-type Markov chain is a general model that appears in the analysis of various queueing models, a *level-dependent* GI/G/1-type Markov chain is a more general one. It is known that retrial queueing models or queues with impatient customers can be directly connected to level-dependent GI/G/1-type Markov chains, in which increments of levels are not homogeneous and depend on the current levels. The analysis of such models is a challenging task because it is difficult to apply the technique used in this thesis due to the high flexibility of the models.

Appendix A

Tail Asymptotics of Nonnegative Sequences

Let $\{x_k; k = 0, 1, \dots\}$ denote a sequence of nonnegative numbers including infinite positive-numbers. Let σ denote

$$\sigma = \sup \left\{ |z|; \sum_{k=0}^{\infty} x_k z^k < \infty, z \in \mathbb{C} \right\},$$

which is called the convergence radius of the power series. Let $f(z)$ denote the generating function of $\{x_k; k = 0, 1, \dots\}$. We then have

$$f(z) = \sum_{k=0}^{\infty} x_k z^k, \qquad |z| < \sigma. \tag{A.1}$$

Furthermore, by definition, $f(z)$ is holomorphic inside the convergence radius.

In what follows, we make the following assumption.

Assumption A.1 *$f(z)$ is meromorphic in the domain $\{z \in \mathbb{C}; |z| \le \sigma\}$, and the point $z = \sigma$ is an $\breve{m}$th pole of $f(z)$, where $\breve{m}$ is some finite positive integer.*

Lemma A.1 *Under Assumption A.1, any pole of $f(z)$ on $C(0, \sigma)$ is of order less than or equal to $\breve{m}$.*

Proof. We define $g(z)$ as

$$g(z) = f(z) \left(1 - \frac{z}{\sigma}\right)^{\breve{m}}.$$

From (A.1), we have for any $\varepsilon > 0$,

$$g(\sigma - \varepsilon) = f(\sigma - \varepsilon) \left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}} = \sum_{k=0}^{\infty} x_k (\sigma - \varepsilon)^k \left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}}. \tag{A.2}$$

It thus follows from (A.1) and (A.2) that for any $\omega_* \in \mathbb{C}$ such that $|\omega_*| = 1$ and $\omega_* \ne 1$,

$$\begin{aligned}
\liminf_{\substack{z=(\sigma-\varepsilon)\omega_* \\ \varepsilon\downarrow 0}} \left| f(z) \left(1 - \frac{z}{\sigma\omega_*}\right)^{\breve{m}} \right| &= \liminf_{\varepsilon\downarrow 0} \left| \sum_{k=0}^{\infty} x_k (\sigma - \varepsilon)^k (\omega_*)^k \left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}} \right| \\
&\le \limsup_{\varepsilon\downarrow 0} \sum_{k=0}^{\infty} x_k (\sigma - \varepsilon)^k \left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}} \\
&= \limsup_{\varepsilon\downarrow 0} g(\sigma - \varepsilon) = g(\sigma) < \infty, \qquad \text{(A.3)}
\end{aligned}$$

where the last inequality holds because $g(z)$ is holomorphic in some neighborhood of $z = \sigma$. Let $\breve{m}_*$ denote

$$\breve{m}_* = \inf \left\{ m \in \mathbb{N} \cup \{0\}; \lim_{z \to \sigma\omega_*} \left| f(z) \left(1 - \frac{z}{\sigma\omega_*}\right)^m \right| < \infty \right\},$$

where $f(z)(1 - z/(\sigma\omega_*))^m$ is meromorphic in the domain $\{z \in \mathbb{C}; |z| \le \sigma\}$ for $m = 0, 1, \dots$. Thus, if $\breve{m}_* > \breve{m}$, we have

$$\liminf_{\substack{z=(\sigma-\varepsilon)\omega_* \\ \varepsilon \downarrow 0}} \left| f(z) \left(1 - \frac{z}{\sigma\omega_*}\right)^{\breve{m}} \right| \ge \liminf_{z \to \sigma\omega_*} \left| f(z) \left(1 - \frac{z}{\sigma\omega_*}\right)^{\breve{m}} \right| = \infty,$$

which contradicts (A.3). As a result, $\breve{m}_* \le \breve{m}$, which implies that this lemma is true. □

According to Lemma A.1, we introduce the following definition.

Definition A.1 *Under Assumption A.1, a dominant pole of $f(z)$ is a pole that is located on its convergence radius $C(0, \sigma)$ and is of the same order as that of pole $z = \sigma$. Thus the order of any dominant pole of $f(z)$ is equal to $\breve{m}$.*

We make the following assumption, in addition to Assumption A.1.

Assumption A.2 *There exist exactly P ($P \ge 1$) dominant poles, σ_j's ($j = 0, 1, \dots, P-1$), of $f(z)$, where $\sigma_0 = \sigma$ and $0 = \arg\sigma_0 < \arg\sigma_1 < \cdots < \arg\sigma_{P-1} < 2\pi$.*

Remark A.1 *Since $f(z)$ is the generating function of the nonnegative sequence $\{x_k\}$, the set $\{\sigma_j; j = 0, 1, \dots, P-1\}$ consists of one or two real numbers and $\lfloor (P-1)/2 \rfloor$ pairs of conjugate complex numbers. Therefore $\sigma_j \sigma_{P-j} = \sigma^2$ for $j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor$.*

Theorem A.1 *Let $a_{1,k} = 1/(\sigma + \varepsilon_0)^k$ ($k = 0, 1, \dots$), where $\varepsilon_0 > 0$ is a sufficiently small number; and for $m = 2, 3, \dots$, $a_{m,k} = k^{m-2}/\sigma^k$ ($k = 0, 1, \dots$). If Assumptions A.1 and A.2 hold, then the following are true.*

(a) The sequence $\{x_k; k = 0, 1, \dots\}$ satisfies

$$\begin{aligned} x_k &= \binom{k + \breve{m} - 1}{\breve{m} - 1} \frac{1}{\sigma^k} \xi_k + O(a_{\breve{m},k}) \\ &= \frac{k^{\breve{m}-1}}{(\breve{m}-1)!} \frac{1}{\sigma^k} \xi_k + O(a_{\breve{m},k}), \end{aligned} \tag{A.4}$$

where

$$\xi_k = \sum_{j=0}^{P-1} \left(\frac{\sigma}{\sigma_j}\right)^k \lim_{z \to \sigma_j} \left(1 - \frac{z}{\sigma_j}\right)^{\breve{m}} f(z). \tag{A.5}$$

(b) $\limsup_{k\to\infty} \xi_k > 0$.

(c) $\xi_k \ge 0$ *for all* $k = 0, 1, \dots$.

(d) *In addition, if $\{x_k\}$ is nonincreasing and $(\arg \sigma_j)/\pi$ is a rational number for any $j = 0, 1, \ldots, P-1$, then $\xi_k > 0$ for all $k = 0, 1, \ldots$.*

Proof. *Statement (a).* It follows from Assumption A.1 that there exists some $R > \sigma$ such that $f(z)$ is holomorphic in the domain $\{z \in \mathbb{C}; \sigma < |z| \le R\}$. We can choose P positive numbers r_j's ($j = 0, 1, \ldots, P-1$) such that all the $C(\sigma_j, r_j)$'s are strictly inside $C(0, R)$ and any two of them have no intersection. Let $\mathbb{D}$ denote

$$\mathbb{D} = \{z; |z| < R\} \Bigg\backslash \bigcup_{j=0}^{P-1} \{z; |z - \sigma_j| \le r_j\}.$$

Clearly $f(z)$ is holomorphic in domain $\mathbb{D} \cup C(0,R) \cup C(\sigma_0, r_0) \cup \cdots \cup C(\sigma_{P-1}, r_{P-1})$. Thus by the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \sum_{j=0}^{P-1} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \qquad z \in \mathbb{D}, \tag{A.6}$$

where the integrals are taken counter-clockwise.

We now consider the first term in (A.6). For any $z \in \mathbb{D}$ and $\zeta \in C(0, R)$, we have $|z/\zeta| < 1$ and therefore

$$\frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \frac{z^n}{\zeta^n} \mathrm{d}\zeta, \qquad z \in \mathbb{D}. \tag{A.7}$$

Since $f(\zeta)$ is holomorphic for $\zeta \in C(0, R)$, there exists some $f_{\max} > 0$ such that

$$|f(\zeta)| \le f_{\max}, \qquad \zeta \in C(0, R). \tag{A.8}$$

Thus for any fixed $z \in \mathbb{D}$,

$$\left| \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \frac{z^n}{\zeta^n} \right| \le \frac{f_{\max}}{R} \sum_{n=0}^{\infty} \left| \frac{z}{R} \right|^n = \frac{f_{\max}}{R} \frac{1}{1 - \left| \dfrac{z}{R} \right|} < \infty, \qquad \zeta \in C(0, R),$$

which shows that the order of summation and integration on the right hand side of (A.7) is interchangeable. As a result, it follows from (A.7) that

$$\frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta \right) z^n = \sum_{n=0}^{\infty} c_n z^n, \tag{A.9}$$

where

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta, \qquad n = 0, 1, \ldots. \tag{A.10}$$

Next we consider the second term in (A.6). Since $|\zeta - \sigma_j|/|z - \sigma_j| < 1$ for any $z \in \mathbb{D}$ and $\zeta \in C(\sigma_j, r_j)$,

$$\frac{1}{\zeta - z} = \frac{1}{(\sigma_j - z) - (\sigma_j - \zeta)} = \frac{1}{\sigma_j - z} \cdot \frac{1}{1 - \dfrac{\sigma_j - \zeta}{\sigma_j - z}} = \frac{1}{\sigma_j - z} \sum_{n=0}^{\infty} \left(\frac{\sigma_j - \zeta}{\sigma_j - z} \right)^n.$$

Thus we have

$$\frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} \sum_{n=1}^{\infty} \frac{f(\zeta)(\sigma_j - \zeta)^{n-1}}{(\sigma_j - z)^n} \mathrm{d}\zeta, \qquad z \in \mathbb{D}.$$

In a way very similar to the right hand side of (A.7), we can confirm that the order of summation and integration in the above equation is interchangeable, and then obtain

$$\frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} f(\zeta)(\zeta - \sigma_j)^{n-1} \mathrm{d}\zeta \right) \frac{1}{(\sigma_j - z)^n}, \qquad z \in \mathbb{D}. \tag{A.11}$$

Since $z = \sigma_j$ is an $\breve{m}$th order pole,

$$\frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} f(\zeta)(\zeta - \sigma_j)^{n-1} \mathrm{d}\zeta = 0, \qquad \text{for all } n = \breve{m} + 1, \breve{m} + 2, \dots,$$

from which and (A.11) we have

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \\ &\quad = \sum_{n=1}^{\breve{m}} (-1)^{n-1} \left(\frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} f(\zeta)(\zeta - \sigma_j)^{n-1} \mathrm{d}\zeta \right) \frac{1}{(\sigma_j - z)^n} \\ &\quad = -\sum_{n=1}^{\breve{m}} \sigma_j^{\breve{m}} \left[(-1)^{n+\breve{m}} \left(\frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)(1 - \zeta/\sigma_j)^{\breve{m}}}{(\zeta - \sigma_j)^{\breve{m}-n+1}} \mathrm{d}\zeta \right) \right] \frac{1}{(\sigma_j - z)^n} \\ &\quad = -\sum_{n=1}^{\breve{m}} \frac{\sigma_j^{\breve{m}} c_{j,n}}{(\sigma_j - z)^n}, \end{aligned} \tag{A.12}$$

where

$$c_{j,n} = (-1)^{n+\breve{m}} \cdot \frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)(1 - \zeta/\sigma_j)^{\breve{m}}}{(\zeta - \sigma_j)^{\breve{m}-n+1}} \mathrm{d}\zeta. \tag{A.13}$$

Substituting (A.9) and (A.12) into (A.6), we have

$$f(z) = \sum_{n=0}^{\infty} c_n z^n + \sum_{j=0}^{P-1} \sum_{n=1}^{\breve{m}} \frac{\sigma_j^{\breve{m}} c_{j,n}}{(\sigma_j - z)^n}, \qquad z \in \mathbb{D},$$

and therefore

$$\sum_{n=0}^{\infty} x_n z^n = \sum_{n=0}^{\infty} c_n z^n + \sum_{j=0}^{P-1} \sum_{n=1}^{\breve{m}} \frac{\sigma_j^{\breve{m}} c_{j,n}}{(\sigma_j - z)^n}, \qquad z \in \mathbb{D} \cap \{z \in \mathbb{C}; |z| < \sigma\}. \tag{A.14}$$

Differentiating both sides of (A.14) k times with respect to z, dividing them by $k!$ and letting $z = 0$ yield

$$x_k = c_k + \sum_{j=0}^{P-1} \sum_{n=1}^{\breve{m}} \sigma_j^{\breve{m}-n} c_{j,n} \binom{k+n-1}{n-1} \frac{1}{\sigma_j^k}. \tag{A.15}$$

It follows from (A.8) and (A.10) that

$$|c_k| \le \frac{1}{2\pi} \oint_{C(0,R)} \left| \frac{f(\zeta)}{\zeta^{k+1}} \right| \mathrm{d}\zeta \le \frac{1}{2\pi} \oint_{C(0,R)} \frac{f_{\max}}{R^{k+1}} \mathrm{d}\zeta = \frac{f_{\max}}{R^k},$$

which leads to

$$\lim_{k\to\infty} \left| \frac{c_k}{\dfrac{1}{\sigma_j^k}} \right| = \lim_{k\to\infty} |c_k| \sigma^k \le \lim_{k\to\infty} f_{\max} \left(\frac{\sigma}{R} \right)^k = 0, \qquad \text{for all } j = 0, 1, \dots, P-1, \tag{A.16}$$

where we use $|\sigma_j| = \sigma$ ($j = 0, 1, \dots, P-1$) and $0 < \sigma/R < 1$. From (A.15) and (A.16), we have

$$\begin{aligned} x_k &= \binom{k + \breve{m} - 1}{\breve{m} - 1} \frac{1}{\sigma^k} \sum_{j=0}^{P-1} \left(\frac{\sigma}{\sigma_j} \right)^k c_{j,\breve{m}} + O(a_{\breve{m},k}) \\ &= \frac{k^{\breve{m}-1}}{(\breve{m}-1)!} \frac{1}{\sigma^k} \sum_{j=0}^{P-1} \left(\frac{\sigma}{\sigma_j} \right)^k c_{j,\breve{m}} + O(a_{\breve{m},k}). \end{aligned} \tag{A.17}$$

Note here that (A.13) yields

$$c_{j,\breve{m}} = \frac{1}{2\pi \mathrm{i}} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)(1 - \zeta/\sigma_j)^{\breve{m}}}{\zeta - \sigma_j} \mathrm{d}\zeta = \lim_{\zeta \to \sigma_j} \left(1 - \frac{\zeta}{\sigma_j} \right)^{\breve{m}} f(\zeta), \tag{A.18}$$

where we use the Cauchy integral formula in the last equality. As a result, statement (a) is true.

Statement (b). From (A.4) and the definition of $\{a_{\breve{m},k}\}$, we have

$$x_k = \frac{k^{\breve{m}-1}}{(\breve{m}-1)!} \frac{1}{\sigma^k} \xi_k + o\left(\frac{k^{\breve{m}-1}}{\sigma^k} \right). \tag{A.19}$$

We now suppose $\limsup_{k\to\infty} \xi_k \le 0$. Equation (A.19) yields

$$\limsup_{k\to\infty} \frac{x_k}{k^{\breve{m}-1} \sigma^{-k}} = 0,$$

which implies that for any $\varepsilon > 0$ there exists some positive integer $K_\varepsilon \ge \breve{m} - 1$ such that $x_k < \varepsilon (k^{\breve{m}-1}/\sigma^k)$ for all $k = K_\varepsilon, K_\varepsilon + 1, \dots$. Thus we have

$$f(y) \le \sum_{k=0}^{K_\varepsilon - 1} y^k x_k + \varepsilon \sum_{k=K_\varepsilon}^{\infty} k^{\breve{m}-1} \left(\frac{y}{\sigma} \right)^k, \qquad 0 \le y < \sigma. \tag{A.20}$$

Note that for $\ell = 1, 2, \dots,$

$$\sum_{k=\ell}^{\infty} k(k-1) \cdots (k - \ell + 1) \left(\frac{y}{\sigma} \right)^k = (-1)^{\ell+1} \ell! \frac{\sigma y^\ell}{(y - \sigma)^{\ell+1}}. \tag{A.21}$$

Note also that there exists an $(\breve{m}-1)$-tuple $(b_1, b_2, \dots, b_{\breve{m}-1})$ of real numbers such that

$$k^{\breve{m}-1} = \sum_{\ell=1}^{\breve{m}-1} b_\ell \cdot k(k-1) \cdots (k - \ell + 1). \tag{A.22}$$

It follows from (A.20), (A.21) and (A.22) that for any $\varepsilon > 0$,

$$0 \le \limsup_{y \uparrow \sigma} \left(1 - \frac{y}{\sigma}\right)^{\breve{m}} f(y) \le \varepsilon b_{\breve{m}-1} (\breve{m} - 1)!.$$

Letting $\varepsilon \to 0$ in the above inequality, we have $\lim_{y \uparrow \sigma} \{1 - (y/\sigma)\}^{\breve{m}} f(y) = 0$, which is inconsistent with Assumption A.1.

Statement (c). It follows from (A.18), Assumption A.2 and Remark A.1 that $c_{0,\breve{m}}$ is a real number and $(c_{j,\breve{m}}, c_{P-j,\breve{m}})$ $(j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor)$ is a pair of complex conjugates, and thus ξ_k is a real number such that

$$\xi_k = y_0 + \sum_{j=1}^{\lfloor (P-1)/2 \rfloor} y_j \cos(2\pi k \alpha_j), \qquad k = 0, 1, \dots, \tag{A.23}$$

where $y_j \in \mathbb{R}$ $(j = 0, 1, \dots, \lfloor (P-1)/2 \rfloor)$ and $0 \le \alpha_j < 1$ $(j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor)$.

In what follows, we assume $\xi_{k_0} < 0$ for some nonnegative integer k_0 and then prove the following.

Claim: There exists some $b > 0$ such that $\xi_k < -b$ for infinitely many k's.

If this is true, (A.19) implies that $x_k < 0$ for a sufficiently large k, which contradicts the fact that $x_k \ge 0$ for all $k = 0, 1, \dots$. As a result, for all $k = 0, 1, \dots$, ξ_k must be nonnegative, i.e., the statement (c) is true.

We split $\mathcal{A} \triangleq \{\alpha_j; j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor\}$ into rational numbers and irrational numbers. We then define $\mathcal{A}_0$ as the set of the rational numbers of $\mathcal{A}$. Next we choose an irrational number α_{j_1} from $\mathcal{A} \backslash \mathcal{A}_0$ (if any) and let $\mathcal{A}_1 = \{\alpha_j \in \mathcal{A} \backslash \mathcal{A}_0; \alpha_j/\alpha_{j_1} \text{ is rational}\}$. Furthermore, we choose an irrational number α_{j_2} from $\mathcal{A} \backslash (\mathcal{A}_0 \cup \mathcal{A}_1)$ (if any) and let $\mathcal{A}_2 = \{\alpha_j \in \mathcal{A} \backslash (\mathcal{A}_0 \cup \mathcal{A}_1); \alpha_j/\alpha_{j_2} \text{ is rational}\}$. Repeating this procedure, we can obtain $\tilde{P}$ sets, $\mathcal{A}_j$'s $(j = 1, 2, \dots, \tilde{P})$, where $\tilde{P}$ may be equal to zero, i.e., all members of $\mathcal{A}$ may be rational. Let $\tilde{\alpha}_j$ $(j = 0, 1, \dots, \tilde{P})$ denote some number such that all members of $\mathcal{A}_j$ are multiples of $\tilde{\alpha}_j$. From Definition A.2, $\tilde{\alpha}_j$'s $(j = 0, 1, \dots, \tilde{P})$ are linearly independent over the rationals. Note here that for $n = 1, 2, \dots$,

$$\cos(nt) = T_n(\cos t), \qquad t \in \mathbb{R},$$

where $T_n(t)$'s $(n = 1, 2, \dots)$ denote the Chebyshev polynomials of the first kind. It thus follows from (A.23) that there exist some polynomial functions $\psi^{(\mathcal{A}_j)}$'s $(j = 0, 1, \dots, \tilde{P})$ on $\mathbb{R}$ such that

$$\xi_k = y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k \tilde{\alpha}_0) + \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi k \tilde{\alpha}_j), \qquad k = 0, 1, \dots, \tag{A.24}$$

where $\psi^{(\mathcal{A}_j)} \circ \cos(\cdot)$ denotes a composite function $\psi^{(\mathcal{A}_j)}(\cos(\cdot))$ of functions $\psi^{(\mathcal{A}_j)}(\cdot)$ and $\cos(\cdot)$. Since $\tilde{\alpha}_0$ is rational, there exists some $g \in \mathbb{N}$ such that

$$\psi^{(\mathcal{A}_0)} \circ (2\pi(ng + k)\tilde{\alpha}_0) = \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k \tilde{\alpha}_0), \qquad \text{for all } k, n = 0, 1, \dots. \tag{A.25}$$

Therefore in the case of $\tilde{P} = 0$, it follows from (A.24) and (A.25) that

$$\xi_{ng+k_0} = y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k_0 \tilde{\alpha}_0) = \xi_{k_0} < 0, \qquad \text{for all } n = 0, 1, \dots,$$

which implies the above claim.

We next consider the case of $\tilde{P} \geq 1$. Since $g\tilde{\alpha}_1$, $g\tilde{\alpha}_2$, $\ldots$, $g\tilde{\alpha}_{\tilde{P}}$ are linearly independent over the rationals, it follows from Proposition A.1 that for any $\varepsilon > 0$ and any $\boldsymbol{t} \triangleq (t_1, t_2, \ldots, t_{\tilde{P}}) \in \mathbb{R}^{\tilde{P}}$, there exist integers $n_* := n_*(\varepsilon, \boldsymbol{t})$ and $l_j := l_j(\varepsilon, \boldsymbol{t})$ ($j = 1, 2, \ldots, \tilde{P}$) such that

$$|(n_* g + k_0)\tilde{\alpha}_j - l_j - t_j| < \frac{\varepsilon}{2\pi}, \qquad j = 1, 2, \ldots, \tilde{P}.$$

Thus since $\psi^{(\mathcal{A}_j)} \circ \cos(2\pi x)$ is a continuous function of x, there exists some $\delta := \delta(\varepsilon) > 0$ such that $\lim_{\varepsilon \downarrow 0} \delta = 0$ and

$$|\psi^{(\mathcal{A}_j)} \circ \cos(2\pi(n_* g + k_0)\tilde{\alpha}_j) - \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j)| < \delta, \qquad j = 1, 2, \ldots, \tilde{P}. \tag{A.26}$$

It follows from (A.24), (A.25) and (A.26) that

$$\begin{aligned}
&\left| \xi_{n_* g + k_0} - \left(y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k_0 \tilde{\alpha}_0) + \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j) \right) \right| \\
&\qquad \leq \sum_{j=1}^{\tilde{P}} \left| \psi^{(\mathcal{A}_j)} \circ \cos(2\pi(n_* g + k_0)\tilde{\alpha}_j) - \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j) \right| < \tilde{P}\delta.
\end{aligned} \tag{A.27}$$

We define $V_+(k)$ and $V_-(k)$ ($k = 0, 1, \ldots$) as

$$\begin{aligned}
V_+(k) &= y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k \tilde{\alpha}_0) + \max_{\boldsymbol{t} \in \mathbb{R}^{\tilde{P}}} \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j), \\
V_-(k) &= y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k \tilde{\alpha}_0) + \min_{\boldsymbol{t} \in \mathbb{R}^{\tilde{P}}} \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j),
\end{aligned}$$

respectively. It follows from the above definition and (A.24) that $V_-(k_0) \leq \xi_{k_0} \leq V_+(k_0)$. Furthermore, (A.27) implies that $\{\xi_{ng+k_0}; n = 0, 1, \ldots\}$ is dense in the interval $[V_-(k_0), V_+(k_0)]$. Thus there exist infinitely many n's such that $\xi_{ng+k_0} < V_-(k_0)/2 < 0$. This completes the proof of the statement (c).

Statement (d). We prove this by reduction to absurdity, assuming $\xi_{\hat{k}} \leq 0$ for some nonnegative integer $\hat{k}$. Since $(\arg \sigma_j)/\pi$ is a rational number for any $j = 0, 1, \ldots, P-1$, there exist a positive integer g and nonnegative integers $\ell_0, \ell_1, \ldots, \ell_{P-1}$ such $\sigma_j = \sigma \exp(\mathrm{i}2\pi \ell_j / g)$ ($j = 0, 1, \ldots, P-1$). Clearly, $\xi_{ng+\hat{k}} \leq 0$ for all $n = 0, 1, \ldots$. It thus follows from (A.19) that for any $\varepsilon > 0$ there exists some nonnegative integer $\hat{n}$ such that $\hat{n}g + \hat{k} \geq \breve{m} - 1$ and

$$x_{ng+\hat{k}} \leq \varepsilon \frac{(ng + \hat{k})^{\breve{m}-1}}{\sigma^{ng+\hat{k}}}, \qquad \text{for all } n = \hat{n}, \hat{n}+1, \ldots.$$

Since $\{x_k\}$ is nonincreasing,

$$x_{ng+\hat{k}+\ell} \leq \varepsilon \frac{(ng + \hat{k})^{\breve{m}-1}}{\sigma^{ng+\hat{k}}}, \qquad n = \hat{n}, \hat{n}+1, \ldots, \ \ \ell = 0, 1, \ldots, g-1,$$

which yields for $0 \le y < \sigma$,

$$\begin{aligned} 0 \le f(y) &\le \sum_{k=0}^{\hat{n}g+\hat{k}-1} y^k x_k + \varepsilon \sum_{n=\hat{n}}^{\infty} \frac{(ng+\hat{k})^{\breve{m}-1}}{\sigma^{ng+\hat{k}}} y^{ng+\hat{k}} \sum_{\ell=0}^{g-1} y^\ell \\ &\le C_1 + \varepsilon \frac{1-\sigma^g}{1-\sigma} \sum_{n=\hat{n}}^{\infty} (ng+\hat{k})^{\breve{m}-1} \left(\frac{y}{\sigma}\right)^{ng+\hat{k}} \\ &\le C_1 + \varepsilon C_2 \sum_{k=\breve{m}-1}^{\infty} k^{\breve{m}-1} \left(\frac{y}{\sigma}\right)^k \le C_1 + \varepsilon C_2 \sum_{k=\breve{m}-1}^{\infty} k^{\breve{m}-1} \left(\frac{y}{\sigma}\right)^{k-\breve{m}+1}, \end{aligned} \tag{A.28}$$

where $C_1 = \sum_{k=0}^{\hat{n}g+\hat{k}-1} \sigma^k x_k < \infty$ and $C_2 = (1-\sigma^g)/(1-\sigma)$. Note here that the second last inequality in (A.28) follows from $\hat{n}g + \hat{k} \ge \breve{m} - 1$ and the last one follows from $0 \le y/\sigma < 1$.

Let $\phi(y) = \sum_{k=0}^{\infty} (y/\sigma)^k = -\sigma(y-\sigma)^{-1}$ for $0 \le y < \sigma$. We then have for $0 \le y < \sigma$,

$$\frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} \phi(y) = \sum_{k=0}^{\infty} \frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} \left(\frac{y}{\sigma}\right)^k = \frac{1}{\sigma^{\breve{m}-1}} \sum_{k=\breve{m}-1}^{\infty} k(k-1)\cdots(k-\breve{m}+2) \left(\frac{y}{\sigma}\right)^{k-\breve{m}+1}.$$

Thus, for $1 \le \ell \le \breve{m} - 1$,

$$\sum_{k=\breve{m}-1}^{\infty} k(k-1)\cdots(k-\ell+1) \left(\frac{y}{\sigma}\right)^{k-\breve{m}+1} \le \sigma^{\breve{m}-1} \frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} \phi(y).$$

Using this inequality and (A.22), we can bound $f(y)$ in (A.28) as follows.

$$0 \le f(y) \le C_1 + \varepsilon C \frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} \phi(y),$$

where $C = C_2 \sigma^{\breve{m}-1} \sum_{\ell=1}^{\breve{m}-1} b_\ell$. Furthermore,

$$\frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} \phi(y) = -\sigma \frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} (y-\sigma)^{-1} = \sigma(-1)^{\breve{m}}(\breve{m}-1)!(y-\sigma)^{-\breve{m}}.$$

As a result,

$$0 \le \limsup_{y \uparrow \sigma} \left(1 - \frac{y}{\sigma}\right)^{\breve{m}} f(y) \le \varepsilon C (\breve{m}-1)! \sigma^{-\breve{m}+1}.$$

Letting $\varepsilon \to 0$ in the above inequality, we have $\lim_{y\uparrow\sigma} (1 - y/\sigma)^{\breve{m}} f(y) = 0$, which contradicts Assumption A.1. □

Remark A.2 *A result similar to the statement (a) is given in Theorem 5.2.1 in [65]. Furthermore, when $P = 1$, (A.5) is reduced to eq. (2) at p. 238 in [9].*

Remark A.3 *Suppose the candidates for the dominant poles are σ_j's ($j = 0, 1, \ldots, P-1$) and at least one of them is indeed a dominant pole (according to Assumption A.1, $z = \sigma$ is a dominant pole). For σ_j not a dominant pole, we have*

$$\lim_{z \to \sigma_j} \left(1 - \frac{z}{\sigma_j}\right)^{\breve{m}} f(z) = 0.$$

Thus the statements (a)–(d) of Theorem A.1 still hold, though the right hand side of (A.5) may include some null terms.

A.1 Kronecker's approximation theorem

The following is Kronecker's approximation theorem. For details, see, e.g., Theorem 7.10 in [4].

Proposition A.1 *Let γ_i's ($i = 1, 2, \ldots, n$) are arbitrary real numbers. Let β_i's ($i = 1, 2, \cdots, n$) arbitrary real numbers such that $\beta_1, \beta_2, \ldots, \beta_n$ and 1 are linearly independent over the rationals (see Definition A.2 below). For any $\varepsilon > 0$, there exist an $(n + 1)$-tuple $(k, l_1, l_2, \ldots, l_n)$ of integers such that*

$$|k\beta_i - l_i - \gamma_i| < \varepsilon, \qquad \textit{for all } i = 1, 2, \ldots, n, \tag{A.29}$$

and thus for any $\varepsilon > 0$ and any $\tilde{\gamma}_i \in [0, 1]$,

$$|k\beta_i - \lfloor k\beta_i \rfloor - \tilde{\gamma}_i| < \varepsilon, \qquad \textit{for all } i = 1, 2, \ldots, n,$$

which implies that $k\beta_i - \lfloor k\beta_i \rfloor$ ($k \in \mathbb{Z}$) is dense in the interval $[0, 1]$.

Definition A.2 *Arbitrary real numbers β_i's ($i = 1, 2, \ldots, n$) are said to be linearly independent over the rationals (equivalently integers) if there exists no set of nonzero rational numbers q_i's ($i = 1, 2, \ldots, n$) such that*

$$\beta_1 q_1 + \beta_2 q_2 + \cdots + \beta_n q_n = 0. \tag{A.30}$$

Therefore if β_i's are linearly independent over the rationals, (A.30) implies that $q_1 = q_2 = \cdots = q_n = 0$.

Appendix B

Period of Markov Additive Processes

This appendix summarizes fundamental results of the period of MAdPs. In fact, most of the results described here are already implied in [2, 59], though that is done in not an accessible way. Furthermore, an MAdP related to the Markov chain of M/G/1 type (and slightly more general one) are discussed in [21].

We consider an MAdP $\{(\Gamma_n, J_n); n = 0, 1, \dots\}$, where the level variable Γ_n takes a value in $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and the phase variable J_n takes a value in $\mathbb{J} \triangleq \{1, 2, \dots, J\}$. Let $\mathbf{\Gamma}(k)$ $(k \in \mathbb{Z})$ denote a $J \times J$ matrix whose (i,j)th $(i,j \in \mathbb{J})$ element represents

$$\Pr[\Gamma_{n+1} = k_0 + k, J_{n+1} = j \mid \Gamma_n = k_0, J_n = i],$$

for any fixed $k_0 \in \mathbb{Z}$. For simplicity, we denote the MAdP $\{(\Gamma_n, J_n); n = 0, 1, \dots\}$ with kernel $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$ by MAdP $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$. For any two states (k_1, j_1) and (k_2, j_2) in $\mathbb{Z} \times \mathbb{J}$, we write $(k_1, j_1) \to (k_2, j_2)$ when there exists a path from (k_1, j_1) to (k_2, j_2) with some positive probability.

Assumption B.1

(a) $\mathbf{\Gamma} \triangleq \sum_{k \in \mathbb{Z}} \mathbf{\Gamma}(k)$ *is irreducible.*

(b) For each $j \in \mathbb{J}$*, there exists a nonzero integer* k_j *such that* $(0, j) \to (k_j, j)$.

Let $\mathbb{K}_j$ $(j \in \mathbb{J})$ denote

$$\mathbb{K}_j = \{k \in \mathbb{Z} \backslash \{0\};\ (0,j) \to (k,j)\},$$

which is well-defined under Assumption B.1.

Lemma B.1 *Let* $d_j = \gcd\{k \in \mathbb{K}_j\}$ *for* $j \in \mathbb{J}$*. If Assumption B.1 holds, then* d_j*'s* $(j \in \mathbb{J})$ *are all identical.*

Proof. Under Assumption B.1, for any $i, j \in \mathbb{J}$ $(i \neq j)$ there exist integers $k_{i,j}$ and $k_{j,i}$ $(k_{i,j} + k_{j,i} \neq 0)$ such that $(0, i) \to (k_{i,j}, j)$ and $(0, j) \to (k_{j,i}, i)$. Let $\mathbb{K}_{j \to i \to j}$ denote

$$\mathbb{K}_{j \to i \to j} = \{k_{j,i} + k_{i,j}\} \cup \{k_{j,i} + k + k_{i,j};\ k \in \mathbb{K}_i\}. \tag{B.1}$$

Clearly $\mathbb{K}_{j \to i \to j} \subseteq \mathbb{K}_j$ and therefore

$$\gcd\{k \in \mathbb{K}_{j \to i \to j}\} \geq \gcd\{k \in \mathbb{K}_j\} = d_j. \tag{B.2}$$

In what follows, we prove $\gcd\{k \in \mathbb{K}_{j\to i\to j}\} \le d_i$, from which and (B.2) it follows that $d_j \le d_i$. Interchanging i and j in the proof of $d_j \le d_i$, we can readily show that $d_i \le d_j$. Therefore we have $d_i = d_j$.

Since $(0,i) \to (k_{i,j}, j) \to (k_{i,j} + k_{j,i}, i)$, we have $k_{i,j} + k_{j,i} \in \mathbb{K}_i$ and therefore $k_{i,j} + k_{j,i} = a_0 d_i$ for some integer $a_0 \ne 0$. Note here that $\mathbb{K}_i$ has at least two elements because

$$\{k_{i,j} + k_{j,i}\} \cup \{k_{i,j} + k + k_{j,i};\ k \in \mathbb{K}_j\} \subseteq \mathbb{K}_i.$$

Thus there exists a couple of nonzero integers (a_1, a_2) such that $\{a_1 d_i, a_2 d_i\} \subseteq \mathbb{K}_i$ and $\gcd\{a_1, a_2\} = 1$, due to $d_i = \gcd\{k \in \mathbb{K}_i\}$. It follows from (B.1) and $k_{i,j} + k_{j,i} = a_0 d_i$ that

$$\begin{aligned}
\mathbb{K}_{j\to i\to j} &\supseteq \{k_{j,i} + k_{i,j}\} \cup \{k_{j,i} + a_1 d_i + k_{i,j},\ k_{j,i} + a_2 d_i + k_{i,j}\} \\
&= \{a_0 d_i\} \cup \{a_0 d_i + a_1 d_i,\ a_0 d_i + a_2 d_i\} \\
&= \{a_0 d_i, (a_0 + a_1) d_i, (a_0 + a_2) d_i\},
\end{aligned}$$

which leads to $\gcd\{k \in \mathbb{K}_{j\to i\to j}\} \le \gcd\{a_0 d_i, (a_0 + a_1) d_i, (a_0 + a_2) d_i\} = d_i$. □

Definition B.1 *According to Lemma B.1, we write* d *to represent* d_j*'s and refer to the constant* d *as the period of MAdP* $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$.

We choose a state $i_0 \in \mathbb{J}$ and then define $\mathbb{J}_0^{(i_0)}$ as

$$\mathbb{J}_0^{(i_0)} = \{j \in \mathbb{J};\ (0, i_0) \to (k, j),\ k \equiv 0 \pmod{d}\}.$$

We also define $\mathbb{J}_m^{(i_0)}$ ($m = 1, 2, \dots, d-1$) as

$$\mathbb{J}_m^{(i_0)} = \{j \in \mathbb{J};\ (0, i_0) \to (k, j),\ k \equiv m \pmod{d}\}.$$

Since $\mathbf{\Gamma}$ is irreducible, each $j \in \mathbb{J}$ must belong to at least one of $\{\mathbb{J}_m^{(i_0)}; m = 0, 1, \dots, d-1\}$. Furthermore, for any $i \in \mathbb{J}_m^{(i_0)}$,

$$(0, i) \to (k, i_0) \text{ only if } k \equiv -m \pmod{d},$$

which implies that $\mathbb{J}_{m_1}^{(i_0)} \cap \mathbb{J}_{m_2}^{(i_0)} = \emptyset$ for $m_1 \not\equiv m_2$ (if not, it would hold that $(0, i_0) \to (k, i_0)$ for some $k \equiv m_1 - m_2 \not\equiv 0 \pmod{d}$). Thus $\mathbb{J}_0^{(i_0)} + \mathbb{J}_1^{(i_0)} + \cdots + \mathbb{J}_{d-1}^{(i_0)} = \mathbb{J}$ and there exists an injective function q_0 from $\mathbb{J}$ to $\{0, 1, \dots, d-1\}$ such that $j \in \mathbb{J}_{q_0(j)}^{(i_0)}$. It follows from the definition of $\mathbb{J}_m^{(i_0)}$'s that

$$[\mathbf{\Gamma}(k)]_{i,j} > 0 \text{ only if } k \equiv q_0(j) - q_0(i) \pmod{d}.$$

As a result, we obtain the following result.

Lemma B.2 *Under Assumption B.1, the period* d *is the largest positive integer such that*

$$[\mathbf{\Gamma}(k)]_{i,j} > 0 \textit{ only if } k \equiv q(j) - q(i) \pmod{d}, \tag{B.3}$$

where q *is some injective function from* $\mathbb{J}$ *to* $\{0, 1, \dots, d-1\}$*. Furthermore,* $\mathbb{J}_m \triangleq \{j \in \mathbb{J}; q(j) = m\}$ *(*$m = 0, 1, \dots, d-1$*) are disjoint each other and* $\mathbb{J}_0 + \mathbb{J}_1 + \cdots + \mathbb{J}_{d-1} = \mathbb{J}$.

In the rest of this section, we discuss the relationship between the period d of MAdP $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$ and the eigenvalues of the generating function $\widehat{\mathbf{\Gamma}}(z)$ defined by $\sum_{k\in\mathbb{Z}} z^k \mathbf{\Gamma}(k)$. Let $\mathbf{\Delta}(z)$ denote a $J \times J$ diagonal matrix whose jth diagonal element is equal to $z^{-q(j)}$. It then follows from (B.3) that

$$\widehat{\mathbf{\Gamma}}(z) = \mathbf{\Delta}(z)\mathbf{\Lambda}^*(z^d)\mathbf{\Delta}(z)^{-1} = \mathbf{\Delta}(z/|z|)\mathbf{\Lambda}^*(z^d)\mathbf{\Delta}(z/|z|)^{-1}, \tag{B.4}$$

where $\mathbf{\Lambda}^*(z)$ denotes a $J \times J$ matrix whose (i, j)th element is given by

$$[\mathbf{\Lambda}^*(z)]_{i,j} = \sum_{n\in\mathbb{Z}} z^n [\mathbf{\Gamma}(nd + q(j) - q(i))]_{i,j}.$$

Let $\boldsymbol{\gamma}(z)$ and $\boldsymbol{g}(z)$ denote left- and right-eigenvectors of $\widehat{\mathbf{\Gamma}}(z)$ corresponding to eigenvalue $\delta(\widehat{\mathbf{\Gamma}}(z))$, normalized such that

$$\boldsymbol{\gamma}(z)\mathbf{\Delta}(z/|z|)\boldsymbol{e} = 1, \qquad \boldsymbol{\gamma}(z)\boldsymbol{g}(z) = 1. \tag{B.5}$$

We then have the following lemma.

Lemma B.3 *Let $I_\gamma = \{y > 0; \sum_{k\in\mathbb{Z}} y^k \mathbf{\Gamma}(k) < \infty\}$ and $\omega_x = \exp(2\pi \mathrm{i}/x)$ ($x \ge 1$). If Assumption B.1 holds, then the following hold for any $y \in I_\gamma$ and $\nu = 0, 1, \dots, d-1$.*

(a) $\delta(\widehat{\mathbf{\Gamma}}(y\omega_d^\nu)) = \delta(\widehat{\mathbf{\Gamma}}(y))$, both of which are simple eigenvalues.

(b) $\boldsymbol{\gamma}(y\omega_d^\nu) = \boldsymbol{\gamma}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1}$ and $\boldsymbol{g}(y\omega_d^\nu) = \mathbf{\Delta}(\omega_d^\nu)\boldsymbol{g}(y)$.

Proof. It follows from (B.4) that for $\nu = 0, 1, \dots, d-1$,

$$\begin{aligned}
\widehat{\mathbf{\Gamma}}(y\omega_d^\nu) &= \mathbf{\Delta}(y\omega_d^\nu)\mathbf{\Lambda}^*(y^d)\mathbf{\Delta}(y\omega_d^\nu)^{-1}, \\
&= \mathbf{\Delta}(\omega_d^\nu)[\mathbf{\Delta}(y)\mathbf{\Lambda}^*(y^d)\mathbf{\Delta}(y)^{-1}]\mathbf{\Delta}(\omega_d^\nu)^{-1} \\
&= \mathbf{\Delta}(\omega_d^\nu)\widehat{\mathbf{\Gamma}}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1},
\end{aligned} \tag{B.6}$$

which implies the statement (a) because $\widehat{\mathbf{\Gamma}}(y)$ is nonnegative and irreducible. Next we prove the statement (b). Pre-multiplying both sides of (B.6) by $\boldsymbol{\gamma}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1}$ and using $\delta(\widehat{\mathbf{\Gamma}}(y\omega_d^\nu)) = \delta(\widehat{\mathbf{\Gamma}}(y))$, we have

$$\begin{aligned}
\left[\boldsymbol{\gamma}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1}\right]\widehat{\mathbf{\Gamma}}(y\omega_d^\nu) &= \delta(\widehat{\mathbf{\Gamma}}(y))\left[\boldsymbol{\gamma}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1}\right] \\
&= \delta(\widehat{\mathbf{\Gamma}}(y\omega_d^\nu))\left[\boldsymbol{\gamma}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1}\right].
\end{aligned} \tag{B.7}$$

Similarly, we obtain

$$\widehat{\mathbf{\Gamma}}(y\omega_d^\nu)\left[\mathbf{\Delta}(\omega_d^\nu)\boldsymbol{g}(y)\right] = \delta(\widehat{\mathbf{\Gamma}}(y\omega_d^\nu))\left[\mathbf{\Delta}(\omega_d^\nu)\boldsymbol{g}(y)\right]. \tag{B.8}$$

It follows from (B.7) and (B.8) that there exist some constants φ_1 and φ_2 such that

$$\boldsymbol{\gamma}(y\omega_d^\nu) = \varphi_1\boldsymbol{\gamma}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1}, \quad \boldsymbol{g}(y\omega_d^\nu) = \varphi_2\mathbf{\Delta}(\omega_d^\nu)\boldsymbol{g}(y).$$

We can easily confirm that $\varphi_1 = \varphi_2 = 1$ satisfies the normalizing condition (B.5). □

Theorem B.1 *Suppose Assumption B.1 holds, and let ω denote a complex number such that $|\omega| = 1$. If $\delta(\widehat{\boldsymbol{\Gamma}}(y)) = 1$ for some $y \in I_\gamma$, then $\delta(\widehat{\boldsymbol{\Gamma}}(y\omega)) = 1$ if and only if $\omega^d = 1$. Therefore*

$$d = \max\{n \in \mathbb{N}; \delta(\widehat{\boldsymbol{\Gamma}}(y\omega_n)) = 1\}. \tag{B.9}$$

Furthermore, if $\delta(\widehat{\boldsymbol{\Gamma}}(y\omega)) = 1$, the eigenvalue is simple.

Proof. Although Theorem B.1 can be proved in a similar way to Proposition 14 in [21], we give a complete proof for completeness.

Since the if-part follows from Lemma B.3, we prove the only-if part. Let $\boldsymbol{V}(\omega)$ denote a $J \times J$ matrix such that

$$\boldsymbol{V}(\omega) = \mathrm{diag}(\boldsymbol{g}(y))^{-1}\boldsymbol{\Gamma}^*(y\omega)\mathrm{diag}(\boldsymbol{g}(y)), \qquad |\omega| = 1,$$

where $\mathrm{diag}(\boldsymbol{x})$ denotes a diagonal matrix whose jth diagonal element is equal to $[\boldsymbol{x}]_j$ for a vector $\boldsymbol{x}$. It is easy to see that $\boldsymbol{V}(1)$ is irreducible and stochastic and $\delta(\boldsymbol{V}(\omega)) = \delta(\widehat{\boldsymbol{\Gamma}}(y\omega)) = 1$. Let $\boldsymbol{f} = (f_j; j \in \mathbb{J})$ denote a right eigenvector of $\boldsymbol{V}(\omega)$ corresponding to $\delta(\boldsymbol{V}(\omega)) = 1$. We then have for any $n \in \mathbb{N}$, $(\boldsymbol{V}(\omega))^n\boldsymbol{f} = \boldsymbol{f}$ and thus

$$f_i = \sum_{j\in\mathbb{J}}[(\boldsymbol{V}(\omega))^n]_{i,j}f_j = \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{Z}} y^k[\boldsymbol{\Gamma}^{n*}(k)]_{i,j}\frac{[\boldsymbol{g}(y)]_j}{[\boldsymbol{g}(y)]_i}\cdot\omega^k f_j, \qquad i,j \in \mathbb{J}, \tag{B.10}$$

where $\{\boldsymbol{\Gamma}^{n*}(k); k \in \mathbb{Z}\}$ is the nth-fold convolution of $\{\boldsymbol{\Gamma}(k); k \in \mathbb{Z}\}$ with itself. Note that

$$\sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{Z}} y^k[\boldsymbol{\Gamma}^{n*}(k)]_{i,j}[\boldsymbol{g}(y)]_j/[\boldsymbol{g}(y)]_i = 1,$$

because $(\boldsymbol{V}(1))^n\boldsymbol{e} = \boldsymbol{e}$. Let i' denote an element of $\mathbb{J}$ such that $|f_{i'}| \ge |f_j|$ for all $j \in \mathbb{J}$. It then follows from (B.10) that for any $j \in \mathbb{J}$,

$$\omega^k\frac{f_j}{f_{i'}} = 1 \text{ if } [\boldsymbol{\Gamma}^{n*}(k)]_{i',j} > 0. \tag{B.11}$$

Since $\widehat{\boldsymbol{\Gamma}}(1)$ is irreducible, for any $j \in \mathbb{J}$ there exist some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $[\boldsymbol{\Gamma}^{n*}(k)]_{i',j} > 0$. Thus (B.11) implies that $|f_j|$'s are all equal, because $|\omega| = 1$. We now consider a path from phase i to phase i such that

$$(0,i) \to (k_1,i_1) \to (k_2,i_2) \to (k_m,i_m) \triangleq (k_m,i),$$

where $(k_l,i_l) \in \mathbb{Z}\times\mathbb{J}$ for $l = 1,2,\dots,m$ and $m \in \mathbb{N}$. Since the period of MAdP $\{\boldsymbol{\Gamma}(k)\}$ is equal to d, $k_1 + k_2 + \cdots + k_m$ is a multiple of d. From (B.11), we have $\omega^{k_1+k_2+\cdots+k_m} = 1$ and thus $\omega^d = 1$. The proof of the only-if part is completed.

As for the remaining statements, (B.9) is obvious, and it follows from Lemma B.3 (a) that if $\delta(\widehat{\boldsymbol{\Gamma}}(y\omega)) = 1$, then the eigenvalue $\delta(\widehat{\boldsymbol{\Gamma}}(y\omega_d^\nu)) = 1$ is simple for $\nu = 0,1,\dots,d-1$. □

Remark B.1 *Theorem B.1 provides a definition of the period of MAdP* $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$*. In a very similar way, Shurenkov [59] defined the period of MAdPs with proper kernels. In the context of this thesis, his definition is as follows:*

$$d = \max\left\{n \in \mathbb{N}; \widehat{\mathbf{\Gamma}}(\omega_n)\boldsymbol{f} = \boldsymbol{f} \textit{ for some } \boldsymbol{f} \in \mathbb{C}^J \textit{ such that } |[\boldsymbol{f}]_j| = 1\ (j \in \mathbb{J})\right\}. \tag{B.12}$$

We can confirm that (B.12) is equivalent to Theorem B.1 if $\widehat{\mathbf{\Gamma}}(1)$ *is stochastic. Shurenkov [59] also implied that the statement of Lemma B.2 holds, based on which Alsmeyer [2] defined the period of MAdPs.*

Appendix C

Subexponential Distributions

This appendix provides a brief overview of two classes of subexponential distributions on $\mathbb{Z}_+$ and show some related results, which are required in Chapter 2. One of the classes is the class of "ordinal" subexponential distributions introduced by Chistyakov [14], and the other one is the class of "locally" subexponential distributions introduced by Chover et al. [15] and generalized by Asmussen et al. [8].

In what follows, let U denote a random variable in $\mathbb{Z}_+$ and U_j ($j \in \mathbb{Z}_+$) denote independent copies of U. Let $U_{\rm e}$ denote the discrete equilibrium random variable of U, distributed with $\mathsf{P}(U_{\rm e} = k) = \mathsf{P}(U > k)/\mathsf{E}[U]$ ($k \in \mathbb{Z}_+$). Furthermore, for any $h \in \mathbb{N} \cup \{\infty\}$, let $\Delta_h = (0, h]$ and $k + \Delta_h = \{x \ge 0; k < x \le k + h\}$ for $k \in \mathbb{Z}_+$.

C.1 Ordinal subexponential class

We begin with the definition of the long-tailed class, which covers the subexponential class.

Definition C.1 ([8, 17, 60]) *A random variable U in $\mathbb{Z}_+$ and its distribution are said to be long-tailed if $\mathsf{P}(U > k) > 0$ for all $k \in \mathbb{Z}_+$ and*

$$\lim_{k\to\infty} \frac{\mathsf{P}(U > k + 1)}{\mathsf{P}(U > k)} = 1.$$

The class of long-tailed distributions is denoted by $\mathcal{L}$.

The following result is used to derive some of the asymptotic results presented in Section 3.3.

Proposition C.1 (Proposition A.1 in [42]) *If $U_{\rm e} \in \mathcal{L}$, then for any $h \in \mathbb{N}$, $\ell_0 \in \mathbb{Z}_+$ and $\nu = 0, 1, \dots, h-1$,*

$$\frac{1}{\mathsf{E}[U]} \lim_{k\to\infty} \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U > k + \ell h + \nu)}{\mathsf{P}(U_{\rm e} > k)} = \frac{1}{h}.$$

We now introduce the definition of the subexponential class.

Definition C.2 ([14, 17, 60]) *A random variable U and its distribution are said to be subexponential if $\mathsf{P}(U > k) > 0$ for all $k \in \mathbb{Z}_+$ and*

$$\lim_{k\to\infty} \frac{\mathsf{P}(U_1 + U_2 > k)}{\mathsf{P}(U > k)} = 2.$$

The class of subexponential distributions is denoted by $\mathcal{S}$.

Remark C.1 *It is known that $\mathcal{S} \subsetneq \mathcal{L}$. Furthermore, if $Y \in \mathcal{L}$, then Y is heavy-tailed, i.e., $\lim_{k\to\infty} \mathrm{e}^{\varepsilon k}\mathsf{P}(Y > k) = \infty$ for all $\varepsilon > 0$. For details, see [20]. In addition, there exists an example of not subexponential but long-tailed distributions (see [53]).*

The following is a discrete analog of class $\mathcal{S}^*$ introduced by Klüppelberg [36].

Definition C.3 *A random variable U and its distribution belong to class $\mathcal{S}^*$ if $\mathsf{P}(U > k) > 0$ for all $k \in \mathbb{Z}_+$ and*

$$\lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{\mathsf{P}(U > k-\ell)\mathsf{P}(U > \ell)}{\mathsf{P}(U > k)} = 2\mathsf{E}[U] < \infty. \tag{C.1}$$

Proposition C.2 (Proposition A.2 in [42]) *If $Y \in \mathcal{S}^*$, then $Y \in \mathcal{S}$ and $Y_{\mathrm{e}} \in \mathcal{S}$.*

Proposition C.3 (Proposition A.3 in [42]) *For $d_i \in \mathbb{N}$ ($i = 0,1,2$), let $\{\boldsymbol{P}(k); k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{Q}(k); k \in \mathbb{Z}_+\}$ denote nonnegative $d_0 \times d_1$ and $d_1 \times d_2$ matrix sequences, respectively, such that $\boldsymbol{P} := \sum_{k=0}^{\infty} \boldsymbol{P}(k)$ and $\boldsymbol{Q} := \sum_{k=0}^{\infty} \boldsymbol{Q}(k)$ are finite. Suppose that for some $U \in \mathcal{S}$,*

$$\lim_{k\to\infty} \frac{\boldsymbol{P}(k)}{\mathsf{P}(U > k)} = \widetilde{\boldsymbol{P}} \ge \boldsymbol{O}, \quad \lim_{k\to\infty} \frac{\boldsymbol{Q}(k)}{\mathsf{P}(U > k)} = \widetilde{\boldsymbol{Q}} \ge \boldsymbol{O}.$$

We then have

$$\lim_{k\to\infty} \frac{\boldsymbol{P} * \boldsymbol{Q}(k)}{\mathsf{P}(U > k)} = \widetilde{\boldsymbol{P}}\boldsymbol{Q} + \boldsymbol{P}\widetilde{\boldsymbol{Q}}.$$

C.2 Locally subexponential class

We first introduce the locally long-tailed class, which is required by the definition of the locally subexponential class.

Definition C.4 (Definition 1 in [8]) *A random variable U and its distribution F are called locally long-tailed with span $h \in \mathbb{N} \cup \{\infty\}$ if $\mathsf{P}(U \in k + \Delta_h) > 0$ for all sufficiently large k and*

$$\lim_{k\to\infty} \frac{\mathsf{P}(U \in k + 1 + \Delta_h)}{\mathsf{P}(U \in k + \Delta_h)} = 1.$$

We denote by $\mathcal{L}_{\mathrm{loc}}(h)$ the class of locally long-tailed distributions with span h hereafter.

Remark C.2 *By definition, $\mathcal{L}_{\mathrm{loc}}(\infty) = \mathcal{L}$. Furthermore, if $U \in \mathcal{L}_{\mathrm{loc}}(1)$, then $U \in \mathcal{L}_{\mathrm{loc}}(n)$ for all $n = 2,3,\dots$ and $U \in \mathcal{L}$.*

The following proposition is a locally asymptotic version of Proposition C.1.

Proposition C.4 *If (i) $U \in \mathcal{L}_{\mathrm{loc}}(1)$; or (ii) $U \in \mathcal{L}$ and $\{\mathsf{P}(U = k)\}$ is eventually nonincreasing, then for any $h \in \mathbb{N}$, $\ell_0 \in \mathbb{Z}_+$ and $\nu = 0,1,\dots,h-1$,*

$$\lim_{k\to\infty} \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + \nu)}{\mathsf{P}(U > k)} = \frac{1}{h}. \tag{C.2}$$

Proof. We assume that condition (i) holds. It follows from $U \in \mathcal{L}_{\rm loc}(1)$ that for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ and $l \in \mathbb{Z}_+$,

$$1-\varepsilon \le \frac{\mathsf{P}(U = k + lh + \nu)}{\mathsf{P}(U = k + lh)} \le 1+\varepsilon, \qquad \nu = 0,1,\dots,h-1.$$

Thus for all $k \ge k_0$, we have

$$1-\varepsilon \le \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + \nu)}{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)} \le 1+\varepsilon, \qquad \nu = 0,1,\dots,h-1,$$

which leads to

$$\lim_{k\to\infty} \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + \nu)}{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)} = 1, \qquad \nu = 0,1,\dots,h-1. \tag{C.3}$$

Therefore (C.3) yields for $\nu = 0,1,\dots,h-1$,

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + \nu)}{\mathsf{P}(U > k + \ell_0 h - 1)} \\
&\quad = \lim_{k\to\infty} \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + \nu)}{\sum_{m=\ell_0 h}^{\infty} \mathsf{P}(U = k + m)} \\
&\quad = \lim_{k\to\infty} \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)}{\sum_{j=0}^{h-1}\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + j)} \cdot \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + \nu)}{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)} = \frac{1}{h}.
\end{aligned} \tag{C.4}$$

Note here that if $U \in \mathcal{L}_{\rm loc}(1)$, then $U \in \mathcal{L}$ and thus $\lim_{k\to\infty} \mathsf{P}(U > k + \ell_0 h - 1)/\mathsf{P}(U > k) = 1$. As a result, (C.4) implies (C.2).

Next we assume that condition (ii) holds. It then follows that for all sufficiently large k,

$$\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h) \ge \sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + j), \qquad j \in \mathbb{Z}_+. \tag{C.5}$$

Thus for any fixed (possibly negative) integer i,

$$\begin{aligned}
\lim_{k\to\infty} \frac{\mathsf{P}(U = k + \ell_0 h + i)}{h\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)} &\le \lim_{k\to\infty} \frac{\mathsf{P}(U = k + \ell_0 h + i)}{\sum_{j=0}^{h-1}\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + j)} \\
&= \lim_{k\to\infty} \frac{\mathsf{P}(U > k + \ell_0 h + i - 1) - \mathsf{P}(U > k + \ell_0 h + i)}{\mathsf{P}(U > k + \ell_0 h - 1)} = 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \frac{\mathsf{P}(U = k + \ell_0 h + i)}{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)} = 0. \tag{C.6}$$

Furthermore, (C.5) yields for all sufficiently large k,

$$\begin{aligned}
1 &\ge \frac{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h + \nu)}{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)} \\
&\ge 1 - \frac{\mathsf{P}(U = k + \ell_0 h)}{\sum_{\ell=\ell_0}^{\infty} \mathsf{P}(U = k + \ell h)}, \qquad \nu = 0,1,\dots,h-1,
\end{aligned}$$

from which and (C.6) it follows that (C.3) holds for $\nu = 0,1,\dots,h-1$. Therefore we can prove (C.2) in the same way as the case of condition (i). □

Definition C.5 (Definition 2 in [8]) *A random variable U and its distribution F are called locally subexponential with span $h \in \mathbb{N} \cup \{\infty\}$ if $U \in \mathcal{L}_{\rm loc}(h)$ and*

$$\lim_{k\to\infty} \frac{\mathsf{P}(U_1 + U_2 \in k + \Delta_h)}{\mathsf{P}(U \in k + \Delta_h)} = 2.$$

We denote by $\mathcal{S}_{\rm loc}(h)$ the class of locally subexponential distributions with span h. Obviously, $\mathcal{S}_{\rm loc}(\infty)$ is equivalent to (ordinal) subexponential class $\mathcal{S}$ (see Definition C.2). Furthermore, Definition C.5 shows that $\mathcal{S}_{\rm loc}(h) \subset \mathcal{L}_{\rm loc}(h)$.

Remark C.3 *If $U \in \mathcal{S}_{\rm loc}(h)$ for some $h \in \mathbb{N}$, then $U \in \mathcal{S}_{\rm loc}(nh)$ for all $n \in \mathbb{N}$ and $U \in \mathcal{S}$ (see Remark 2 in [8]).*

Proposition C.5 *$U \in \mathcal{S}^*$ if and only if $U_{\rm e} \in \mathcal{S}_{\rm loc}(1)$.*

Proof. The if-part is obvious. Indeed, since $\mathsf{P}(U_{\rm e} = k) = \mathsf{P}(U > k)/\mathsf{E}[U]$ for $k \in \mathbb{Z}_+$, it follows that if $U_{\rm e} \in \mathcal{S}_{\rm loc}(1)$, then (C.1) holds, i.e., $U \in \mathcal{S}^*$.

On the other hand, suppose (C.1) holds for $h = 1$. We then have

$$\lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{\mathsf{P}(U_{\rm e} = k - \ell)\mathsf{P}(U_{\rm e} = \ell)}{\mathsf{P}(U_{\rm e} = k)} = 2.$$

Furthermore, $U \in \mathcal{S} \subset \mathcal{L}$ (see Proposition C.2) and thus

$$\lim_{k\to\infty} \frac{\mathsf{P}(U > k+1)}{\mathsf{P}(U > k)} = \lim_{k\to\infty} \frac{\mathsf{P}(U_{\rm e} = k+1)}{\mathsf{P}(U_{\rm e} = k)} = 1.$$

As a result, $U_{\rm e} \in \mathcal{S}_{\rm loc}(1)$. □

Proposition C.6 (Proposition 3 in [8]) *Suppose $U \in \mathcal{S}_{\rm loc}(h)$ for some $h \in \mathbb{N} \cup \{\infty\}$ and let $U^{(j)}$ ($j \in \mathbb{N}$) denote independent random variables in $\mathbb{Z}_+$ such that*

$$\lim_{k\to\infty} \frac{\mathsf{P}(U^{(j)} \in k + \Delta_h)}{\mathsf{P}(U \in k + \Delta_h)} = c_j \in \mathbb{R}_+.$$

We then have, for $n \in \mathbb{N}$,

$$\lim_{k\to\infty} \frac{\mathsf{P}(U^{(1)} + U^{(2)} + \cdots + U^{(n)} \in k + \Delta_h)}{\mathsf{P}(U \in k + \Delta_h)} = \sum_{j=1}^{n} c_j.$$

Furthermore, if $\sum_{j=1}^{n} c_j > 0$, then $U^{(1)} + U^{(2)} + \cdots + U^{(n)} \in \mathcal{S}_{\rm loc}(h)$.

Proposition C.7 *Let $\{F(k); k \in \mathbb{Z}_+\}$ and $\{F_j(k); k \in \mathbb{Z}_+\}$ ($j = 1, 2, \dots, m$) denote probability mass functions. If (i) $F \in \mathcal{S}_{\rm loc}(1)$; and (ii) for $j = 1, 2, \dots, m$,*

$$\lim_{k\to\infty} \frac{F_j(k)}{F(k)} = c_j \in \mathbb{R}_+. \tag{C.7}$$

then, for any $\varepsilon > 0$ there exists some $C_\varepsilon \in (0, \infty)$ such that

$$F_1^{*n_1} * F_2^{*n_2} * \cdots * F_m^{*n_m}(k) \le C_\varepsilon (1 + \varepsilon)^{n_1 + n_2 + \cdots + n_m} F(k), \tag{C.8}$$

for all $k > \sup\{k \in \mathbb{Z}_+; F(k) = 0\}$ and $n_1, n_2, \dots, n_m \in \mathbb{N}$.

Proof. The techniques for the proof are based on Lemma 4.2 in [6] and Lemma 10 in [28], though some modifications are required. For the reader's convenience, we provide a complete proof of this proposition.

We first prove the statement under an additional condition that $c_j > 0$ for all $j = 1, 2, \dots, m$, and then remove the condition.

Let $C = \max\{1, c_1, \dots, c_m\}$, $d_0 = 1$ and $d_j = c_j/C \le 1$ for $j = 1, 2, \dots, m$. Let $F_0(k)$ ($k \in \mathbb{Z}_+$) denote a probability mass function such that $F_0(k) = CF(k)$ for all sufficiently large $k \ge k_0$, where k_0 is a positive integer such that $F(k) > 0$ for all $k \ge k_0$ (see Definitions C.4 and C.5).

From (C.7), we have

$$\lim_{k\to\infty} \frac{F_j(k)}{F_0(k)} = d_j \le 1, \qquad j = 0, 1, \dots, m. \tag{C.9}$$

Furthermore, since $F_j \in \mathcal{S}_{\mathrm{loc}}(1) \subset \mathcal{L}_{\mathrm{loc}}(1)$ (see Proposition C.6),

$$\begin{aligned} \lim_{n\to\infty}\lim_{k\to\infty} \frac{\sum_{\ell=0}^{n} F_i(\ell)F_j(k-\ell)}{F_j(k)} &= \lim_{n\to\infty}\sum_{\ell=0}^{n} F_i(\ell) = 1, & \text{(C.10)}\\ \lim_{k\to\infty} \frac{F_i * F_j(k)}{F_0(k)} &= d_i + d_j, & \text{(C.11)} \end{aligned}$$

for all $i, j = 0, 1, \dots, m$. Thus any $\varepsilon > 0$, there exist some positive integers k' and k'' such that $k'' > 2k' \ge 2k_0$, $F_0(k) = CF(k) \le 1$ for all $k \ge k'$ and for all $i, j = 0, 1, \dots, m$,

$$\frac{F_0(k+1)}{F_0(k)} \ge 1 - \varepsilon, \qquad \forall k \ge k', \tag{C.12}$$

$$d_j - \frac{\varepsilon}{8} \le \frac{F_j(k)}{F_0(k)} \le 1 + \frac{\varepsilon}{2}, \qquad \forall k \ge k', \tag{C.13}$$

$$\frac{\sum_{\ell=0}^{k'-1} F_i(\ell)F_j(k-\ell)}{F_j(k)} \ge 1 - \frac{\varepsilon}{8d_j}, \qquad \forall k \ge k'', \tag{C.14}$$

$$F_i * F_j(k) \le (d_i + d_j + \varepsilon/4)F_0(k), \qquad \forall k \ge k''. \tag{C.15}$$

Note here that (C.12), (C.13), (C.14) and (C.15) follow from $F_0 \in \mathcal{L}_{\mathrm{loc}}(1)$, (C.9), (C.10) and (C.11), respectively.

We now show (C.8) for the convolution of two mass functions F_i and F_j ($i, j = 0, 1, \dots, m$). Note that

$$F_i * F_j(k) = \sum_{\ell=0}^{k-k'} F_i(k-\ell)F_j(\ell) + \sum_{\ell=0}^{k'-1} F_i(\ell)F_j(k-\ell). \tag{C.16}$$

It then follows from (C.13), (C.14) and (C.15) that for $k \ge k'' > 2k'$,

$$\begin{aligned} \sum_{\ell=0}^{k-k'} F_i(k-\ell)F_j(\ell) &= F_i * F_j(k) - \sum_{\ell=0}^{k'-1} F_i(\ell)F_j(k-\ell)\\ &\le \left(d_i + d_j + \frac{\varepsilon}{4}\right)F_0(k) - \left(1 - \frac{\varepsilon}{8d_j}\right)F_j(k)\\ &\le \left[\left(d_i + d_j + \frac{\varepsilon}{4}\right) - \left(1 - \frac{\varepsilon}{8d_j}\right)\left(d_j - \frac{\varepsilon}{8}\right)\right]F_0(k)\\ &\le \left(d_i + \frac{\varepsilon}{2}\right)F_0(k) \le \left(1 + \frac{\varepsilon}{2}\right)CF(k), \end{aligned} \tag{C.17}$$

where the last inequality is due to $d_j \le 1$ and $F_0(k) = CF(k)$ for all $k \ge k'$. Applying (C.17) to (C.16), we have for $k \ge k'' > 2k'$,

$$\begin{aligned} F_i * F_j(k) &\le \left(1+\frac{\varepsilon}{2}\right) CF(k) + \sum_{\ell=0}^{k'-1} F_i(\ell)F_j(k-\ell) \\ &\le \left(1+\frac{\varepsilon}{2}\right) CF(k) + \sup_{k-k'+1\le\ell\le k} F_j(\ell). \end{aligned} \tag{C.18}$$

Furthermore, for $k \ge k'' > 2k'$, $k-k'+1 > k'+1$ and thus (C.12) and (C.13) yield

$$\begin{aligned} \sup_{k-k'+1\le\ell\le k} F_j(\ell) &\le \left(1+\frac{\varepsilon}{2}\right) \sup_{k-k'+1\le\ell\le k} F_0(\ell) \\ &= \left(1+\frac{\varepsilon}{2}\right) \sup_{k-k'+1\le\ell\le k} \frac{F_0(\ell)}{F_0(k)} \cdot CF(k) \\ &\le \left(1+\frac{\varepsilon}{2}\right) \frac{1}{(1-\varepsilon)^{k'-1}} \cdot CF(k) \\ &= \left(1+\frac{\varepsilon}{2}\right) C'_\varepsilon \cdot CF(k), \quad k \ge k'' > 2k', \end{aligned} \tag{C.19}$$

where $C'_\varepsilon = 1/(1-\varepsilon)^{k'-1}$. Substituting (C.19) into (C.18), we obtain

$$\begin{aligned} F_i * F_j(k) &\le \left(1+\frac{\varepsilon}{2}\right) \left(1+C'_\varepsilon\right) CF(k) \\ &\le (1+\varepsilon) \cdot 2C'_\varepsilon CF(k) \\ &\le 2C'_\varepsilon \cdot (1+\varepsilon)^2 CF(k), \quad k \ge k'', \end{aligned} \tag{C.20}$$

where we use $C'_\varepsilon \ge 1$. Note here that $F_i * F_j(k) \le 1$ for all $k \in \mathbb{Z}_+$ and

$$\sup_{k_0\le k\le k''-1} F(k)/F(k'') \in (0,\infty).$$

Therefore there exists some $C''_\varepsilon > 0$ such that

$$\begin{aligned} F_i * F_j(k) &\le C''_\varepsilon \frac{F(k)}{F(k'')} \\ &\le \frac{C''_\varepsilon}{CF(k'')} \cdot (1+\varepsilon)^2 CF(k), \quad k_0 \le k \le k''-1. \end{aligned} \tag{C.21}$$

We now define K_ε as

$$K_\varepsilon = \max\left(2C'_\varepsilon, \frac{C''_\varepsilon}{CF(k'')}, \frac{2+\varepsilon}{\varepsilon(1+\varepsilon)^2} C'_\varepsilon\right).$$

We then have the following inequality (which is used later).

$$\left(1+\frac{\varepsilon}{2}\right) C'_\varepsilon \le K_\varepsilon (1+\varepsilon)^2 \frac{\varepsilon}{2}. \tag{C.22}$$

Furthermore, combining (C.20) and (C.21) leads to

$$F_i * F_j(k) \le K_\varepsilon (1+\varepsilon)^2 CF(k), \qquad k \ge k_0. \tag{C.23}$$

Next we show (C.8) for the convolution of three mass functions F_i, F_j and F_ν ($i, j, \nu = 0, 1, \dots, m$). It follows from (C.23) and $F_0(k) = CF(k)$ for all $k \ge k'$ that

$$F_i * F_j(k) \le K_\varepsilon (1+\varepsilon)^2 F_0(k), \qquad k \ge k'.$$

From this and (C.19), we have for $k \ge k'' > 2k'$,

$$\begin{aligned} & F_i * F_j * F_\nu(k) \\ &\quad = \sum_{\ell=0}^{k-k'} F_i * F_j(k-\ell) F_\nu(\ell) + \sum_{\ell=0}^{k'-1} F_i * F_j(\ell) F_\nu(k-\ell) \\ &\quad \le \sum_{\ell=0}^{k-k'} F_i * F_j(k-\ell) F_\nu(\ell) + \sup_{k-k'+1 \le \ell \le k} F_\nu(\ell) \\ &\quad \le K_\varepsilon (1+\varepsilon)^2 \sum_{\ell=0}^{k-k'} F_0(k-\ell) F_\nu(\ell) + \left(1 + \frac{\varepsilon}{2}\right) C'_\varepsilon C F(k). \end{aligned} \tag{C.24}$$

Applying (C.17) and (C.22) to (C.24) yields for $k \ge k'' > 2k'$,

$$\begin{aligned} & F_i * F_j * F_\nu(k) \\ &\quad \le K_\varepsilon (1+\varepsilon)^2 \left(1 + \frac{\varepsilon}{2}\right) CF(k) + K_\varepsilon (1+\varepsilon)^2 \frac{\varepsilon}{2} CF(k) \\ &\quad = K_\varepsilon (1+\varepsilon)^2 \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) CF(k) \\ &\quad = K_\varepsilon (1+\varepsilon)^3 CF(k). \end{aligned}$$

Furthermore, using $C''_\varepsilon > 0$ such that (C.21) holds, we obtain

$$\begin{aligned} F_i * F_j * F_\nu(k) &\le C''_\varepsilon \frac{F(k)}{F(k'')} \\ &\le \frac{C''_\varepsilon}{CF(k'')} \cdot (1+\varepsilon)^3 CF(k), \quad k_0 \le k \le k'' - 1. \end{aligned}$$

Therefore $F_i * F_j * F_\nu(k) \le K_\varepsilon (1+\varepsilon)^3 CF(k)$ for $k \ge k_0$.

By repeating the above argument, we can prove that (C.8) holds under the additional condition that $c_j > 0$ for all $j = 1, 2, \dots, m$. In what follows, we remove this condition.

Without loss of generality, we assume that $c_j = 0$ for $j = 1, 2, \dots, m'$ ($1 \le m' \le m$) and $c_j > 0$ for $j = m'+1, m'+2, \dots, m$. Thus, for any $\delta > 0$, there exists some positive integer $k_* := k_*(\delta) \ge k_0$ such that for all $k \ge k_*$,

$$F_j(k) \le \delta F(k), \qquad j = 1, 2, \dots, m'.$$

Let $\{\tilde{F}_j(k); k \in \mathbb{Z}_+\}$ ($j = 1, 2, \dots, m'$) denote a probability mass function such that

$$\tilde{F}_j(k) = \begin{cases} F_j(k)/\Theta_j, & k < k_*, \\ \delta F(k)/\Theta_j, & k \ge k_*, \end{cases}$$

where $\Theta_j := \Theta_j(\delta) = \sum_{k=0}^{k_*-1} F_j(k) + \sum_{k=k_*}^{\infty} \delta F(k)$. It then follows that $F_j(k) \le \Theta_j \tilde{F}_j(k)$ for all $k \in \mathbb{Z}_+$ and $j = 1, 2, \dots, m'$. Thus we have

$$\begin{aligned} & F_1^{*n_1} * F_2^{*n_2} * \cdots * F_m^{*n_m}(k) \\ & \quad \le \prod_{j=1}^{m'} \Theta_j^{n_j} \cdot \tilde{F}_1^{*n_1} * \cdots * \tilde{F}_{m'}^{*n_{m'}} * F_{m'+1}^{*n_{m'+1}} * \cdots * F_m^{*n_m}(k). \end{aligned} \tag{C.25}$$

By definition,

$$\lim_{k\to\infty} \frac{\tilde{F}_j(k)}{F(k)} = \frac{\delta}{\Theta_j} > 0, \qquad j = 1, 2, \dots, m'.$$

Therefore for any $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that

$$\tilde{F}_1^{*n_1} * \cdots * \tilde{F}_{m'}^{*n_{m'}} * F_{m'+1}^{*n_{m'+1}} * \cdots * F_m^{*n_m}(k) \quad \le \quad C_\varepsilon (1+\varepsilon)^{n_1+n_2+\cdots+n_m} F(k). \tag{C.26}$$

Note here that $\lim_{\delta\downarrow 0} \Theta_j(\delta) = 1$ for all $j = 1, 2, \dots, m'$. Substituting (C.26) into (C.25) and letting $\delta \downarrow 0$ yields (C.8). □

Proposition C.8 *For $d_i \in \mathbb{N}$ ($i = 0, 1, 2$), let $\{\boldsymbol{P}(k); k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{Q}(k); k \in \mathbb{Z}_+\}$ denote nonnegative $d_0 \times d_1$ and $d_1 \times d_2$ matrix sequences, respectively, such that $\boldsymbol{P} := \sum_{k=0}^{\infty} \boldsymbol{P}(k)$ and $\boldsymbol{Q} := \sum_{k=0}^{\infty} \boldsymbol{Q}(k)$ are finite. Suppose that for some $U \in \mathcal{S}_{\rm loc}(1)$,*

$$\lim_{k\to\infty} \frac{\boldsymbol{P}(k)}{\mathsf{P}(U = k)} = \widetilde{\boldsymbol{P}} \ge \boldsymbol{O}, \quad \lim_{k\to\infty} \frac{\boldsymbol{Q}(k)}{\mathsf{P}(U = k)} = \widetilde{\boldsymbol{Q}} \ge \boldsymbol{O}.$$

We then have

$$\lim_{k\to\infty} \frac{\boldsymbol{P} * \boldsymbol{Q}(k)}{\mathsf{P}(U = k)} = \widetilde{\boldsymbol{P}}\boldsymbol{Q} + \boldsymbol{P}\widetilde{\boldsymbol{Q}}.$$

Proof. This proposition can be proved in the same way as Proposition A.3 in [42], and thus the proof is omitted. □

Proposition C.9 *Let $\{\boldsymbol{W}(k); k \in \mathbb{Z}_+\}$ denote a sequence of (finite dimensional) nonnegative square matrices such that $\sum_{n=0}^{\infty} \boldsymbol{W}^n = (\boldsymbol{I} - \boldsymbol{W})^{-1} < \infty$, where $\boldsymbol{W} = \sum_{k=0}^{\infty} \boldsymbol{W}(k)$. If there exists some $U \in \mathcal{S}_{\rm loc}(1)$ such that*

$$\lim_{k\to\infty} \frac{\boldsymbol{W}(k)}{\mathsf{P}(U = k)} = \widetilde{\boldsymbol{W}} \ge \boldsymbol{O},$$

then

$$\lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} \boldsymbol{W}^{*n}(k)}{\mathsf{P}(U = k)} = (\boldsymbol{I} - \boldsymbol{W})^{-1} \widetilde{\boldsymbol{W}} (\boldsymbol{I} - \boldsymbol{W})^{-1}.$$

Proof. Using Proposition C.8, we can readily prove, by induction, that

$$\lim_{k\to\infty} \frac{\boldsymbol{W}^{*n}(k)}{\mathsf{P}(U = k)} = \sum_{\ell=0}^{n-1} \boldsymbol{W}^{\ell} \widetilde{\boldsymbol{W}} \boldsymbol{W}^{n-\ell-1}. \tag{C.27}$$

Furthermore, it follows from Proposition C.7 that for any $\varepsilon > 0$ there exist some $k_0 \in \mathbb{Z}_+$ and some $C_\varepsilon \in (0,\infty)$ such that for all $k \ge k_0$ and $n \in \mathbb{N}$,

$$\frac{[\boldsymbol{W}^{*n}(k)]_{i,j}}{\mathsf{P}(U=k)} \le C_\varepsilon (1+\varepsilon)^n [\boldsymbol{W}^n]_{i,j}.$$

Note here that $\mathrm{sp}(\boldsymbol{W}) < 1$ and thus $\sum_{n=1}^{\infty}(1+\varepsilon)^n \boldsymbol{W}^n < \infty$ for any sufficiently small $\varepsilon > 0$. As a result, using the dominated convergence theorem and (C.27), we obtain

$$\begin{aligned}
\lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} \boldsymbol{W}^{*n}(k)}{\mathsf{P}(U=k)} &= \lim_{k\to\infty} \frac{\boldsymbol{W}^{*0}(k)}{\mathsf{P}(U=k)} + \sum_{n=1}^{\infty} \lim_{k\to\infty} \frac{\boldsymbol{W}^{*n}(k)}{\mathsf{P}(U=k)} \\
&= \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \boldsymbol{W}^{\ell} \widetilde{\boldsymbol{W}} \boldsymbol{W}^{n-\ell-1} \\
&= (\boldsymbol{I} - \boldsymbol{W})^{-1} \widetilde{\boldsymbol{W}} (\boldsymbol{I} - \boldsymbol{W})^{-1}.
\end{aligned}$$

□

C.3 Examples of locally subexponential case

C.3.1 M/GI/1 queue with Pareto service-time distribution

We consider a stable M/GI/1 queue with a Pareto service-time distribution. Let λ denote the arrival rate of customers. Let H denote the service time distribution, which is given by

$$H(x) = 1 - (x+1)^{-\gamma}, \quad x \ge 0,$$

with $\gamma > 1$ and $\gamma \notin \mathbb{N}$. Note here that the mean service time is equal to $1/(\gamma-1)$ and thus the traffic intensity, denoted by ρ, is equal to $\lambda/(\gamma-1) < 1$. Let $\widetilde{H}(s)$ denote the Laplace-Stieltjes transform (LST) of the service time distribution H. It then follows from Theorem 8.1.6 in [12] that

$$\widetilde{H}(s) = \sum_{j=0}^{\lfloor\gamma\rfloor} h_j \frac{(-s)^j}{j!} - \Gamma(1-\gamma)s^{\gamma} + o(s^{\gamma}), \tag{C.28}$$

where $h_j = \int_0^\infty x^j \mathrm{d}H(x)$ $(j = 1,2,\dots)$, $f(x) = o(g(x))$ represents $\lim_{x\to 0} f(x)/g(x) = 0$ and Γ denotes the Gamma function. Equation (C.28) yields

$$\widetilde{H}(\lambda - \lambda z) = \sum_{j=0}^{\lfloor\gamma\rfloor} h_j \frac{(-\lambda)^j (1-z)^j}{j!} - \Gamma(1-\gamma)\lambda^{\gamma}(1-z)^{\gamma} + o((1-z)^{\gamma}). \tag{C.29}$$

It is well-known that the stationary queue length distribution of the M/GI/1 queue, denoted by $\{x(k); k \in \mathbb{Z}_+\}$, is identical with the stationary distribution of the following stochastic matrix:

$$\begin{pmatrix}
\alpha(0) & \alpha(1) & \alpha(2) & \alpha(3) & \cdots \\
\alpha(0) & \alpha(1) & \alpha(2) & \alpha(3) & \cdots \\
0 & \alpha(0) & \alpha(1) & \alpha(2) & \cdots \\
0 & 0 & \alpha(0) & \alpha(1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

where $\{\alpha(k); k \in \mathbb{Z}_+\}$ satisfies $\sum_{k=0}^{\infty} z^k \alpha(k) = \widetilde{H}(\lambda - \lambda z)$ and thus $\sum_{k=1}^{\infty} k\alpha(k) = \rho$.

Let $\overline{\alpha}(k) = \sum_{l=k+1}^{\infty} \alpha_l$ for $k \in \mathbb{Z}_+$. From (C.29), we then have

$$\begin{aligned}\sum_{k=0}^{\infty} z^k \overline{\alpha}(k) &= \frac{1 - \widetilde{H}(\lambda - \lambda z)}{1 - z} \\ &= -\sum_{j=1}^{\lfloor \gamma \rfloor} h_j \frac{(-\lambda)^j (1-z)^{j-1}}{j!} \\ &\quad + \Gamma(1-\gamma)\lambda^{\gamma}(1-z)^{\gamma-1} + o((1-z)^{\gamma-1}). \end{aligned} \tag{C.30}$$

Applying Lemma 5.3.2 in [65] to (C.29) and (C.30) yields

$$\alpha(k) \overset{k}{\sim} \gamma \lambda^{\gamma} k^{-\gamma-1}, \tag{C.31}$$

$$\overline{\alpha}(k) \overset{k}{\sim} \lambda^{\gamma} k^{-\gamma}, \tag{C.32}$$

where $f(x) \overset{x}{\sim} g(x)$ represents $\lim_{x\to\infty} f(x)/g(x) = 1$. Note that (C.31) shows that the discrete distribution $\{\alpha(k); k \in \mathbb{Z}_+\}$ is in the class $\mathcal{L}_{\text{loc}}$. In fact, as shown later, $\{\alpha(k)\} \in \mathcal{S}^*$, i.e., $\{\alpha_{\text{e}}(k)\} \in \mathcal{S}_{\text{loc}}(1)$, where $\alpha_{\text{e}}(k) = \overline{\alpha}(k)/\rho$ for $k = 0, 1, \dots$. Therefore it follows from Theorem 3.3 that

$$x(k) \overset{k}{\sim} \frac{\rho}{1-\rho} \cdot \alpha_{\text{e}}(k) = \frac{\rho}{1-\rho} \cdot \frac{\overline{\alpha}(k)}{\rho} \overset{k}{\sim} \frac{\lambda^{\gamma}}{1-\rho} k^{-\gamma}.$$

In what follows, we prove that $\{\alpha(k)\} \in \mathcal{S}^*$, i.e.,

$$\sum_{\ell=0}^{k} \overline{\alpha}(\ell)\overline{\alpha}(k-\ell) \overset{k}{\sim} 2\rho \cdot \overline{\alpha}(k).$$

Let $\nu := \nu(k)$ denote an integer such that $k/3 \le \nu(k) < k/2$. For $k \in \mathbb{Z}_+$, we have

$$\sum_{\ell=0}^{k} \frac{\overline{\alpha}(\ell)\overline{\alpha}(k-\ell)}{\overline{\alpha}(k)} = 2\sum_{\ell=0}^{\nu-1} \overline{\alpha}(\ell) \frac{\overline{\alpha}(k-\ell)}{\overline{\alpha}(k)} + \sum_{\ell=\nu}^{k-\nu} \overline{\alpha}(\ell) \frac{\overline{\alpha}(k-\ell)}{\overline{\alpha}(k)}. \tag{C.33}$$

From (C.32), we obtain

$$\lim_{k\to\infty} \sum_{\ell=0}^{\nu-1} \overline{\alpha}(\ell) \frac{\overline{\alpha}(k-\ell)}{\overline{\alpha}(k)} = \sum_{\ell=0}^{\nu-1} \overline{\alpha}(\ell) \lim_{k\to\infty} \frac{\overline{\alpha}(k-\ell)}{\overline{\alpha}(k)} = \sum_{\ell=0}^{\nu-1} \overline{\alpha}(\ell). \tag{C.34}$$

Furthermore, it follows from (C.32) that for any $\varepsilon > 0$ there exists some $k_* \in \mathbb{Z}_+$ such that for all $k \ge k_*/3$,

$$1 - \varepsilon < \frac{\overline{\alpha}(k)}{\lambda^{\gamma} k^{-\gamma}} < 1 + \varepsilon,$$

which implies that for $k \ge k_*$ and $k/3 \le \nu < k/2$,

$$\begin{aligned}\sum_{\ell=\nu}^{k-\nu} \overline{\alpha}(\ell) \frac{\overline{\alpha}(k-\ell)}{\overline{\alpha}(k)} &\le \frac{(1+\varepsilon)^2}{1-\varepsilon} \sum_{\ell=\nu}^{k-\nu} \lambda^{\gamma} \ell^{-\gamma} \left(\frac{k-\ell}{k}\right)^{-\gamma} \\ &\le \frac{(1+\varepsilon)^2}{1-\varepsilon} \lambda^{\gamma} (k - 2\nu + 1) \nu^{-\gamma} \left(\frac{\nu}{k}\right)^{-\gamma} \\ &\le \frac{(1+\varepsilon)^2}{1-\varepsilon} \lambda^{\gamma} k \left(\frac{k}{3}\right)^{-\gamma} 3^{\gamma} \\ &\le \frac{(1+\varepsilon)^2}{1-\varepsilon} (9\lambda)^{\gamma} k^{-\gamma+1} \to 0, \quad \text{as } k \to \infty. \end{aligned} \tag{C.35}$$

Finally, applying (C.34) and (C.35) to (C.33) and letting $\nu \to \infty$ yield

$$\lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{\overline{\alpha}(\ell)\overline{\alpha}(k-\ell)}{\overline{\alpha}(k)} = 2\sum_{\ell=0}^{\infty} \overline{\alpha}(\ell) = 2\rho.$$

C.3.2 Discrete-time queue with disasters and Pareto-distributed batch arrivals

This section considers a discrete-time single-server queue with disasters and Pareto-distributed batch arrivals. The time interval $[n, n+1)$ ($n \in \mathbb{Z}_+$) is called slot n. Customers and disasters can arrive at the beginnings of respective slots, whereas departures of served customers can occur at the ends of respective slots.

We assume that the numbers of customer arrivals in respective slots are independent and identically distributed (i.i.d.) with a discrete Pareto distribution, $\beta(k) = 1/(k+1)^\gamma - 1/(k+2)^\gamma$ ($k \in \mathbb{Z}_+$), where $\gamma > 1$. Service times are i.i.d. with a geometric distribution with mean $1/(1-q)$ ($0 < q < 1$). We also assume that at most one disaster occurs at one slot with probability ϕ ($0 < \phi < 1$), which are independent of the arrival process of customers. If a disaster occurs in a slot, then both customers arriving in the slot and all the ones in the system are removed.

Let L_n ($n \in \mathbb{Z}_+$) denote the number of customers at the middle of slot n. It then follows from Proposition 1.8 that $\{L_n; n \in \mathbb{Z}_+\}$ is an ergodic Markov chain whose transition probability matrix is given by

$$\begin{pmatrix}
b(0) & b(1) & b(2) & b(3) & b(4) & \cdots \\
\phi + a(0) & a(1) & a(2) & a(3) & a(4) & \cdots \\
\phi & a(0) & a(1) & a(2) & a(3) & \cdots \\
\phi & 0 & a(0) & a(1) & a(2) & \cdots \\
\phi & 0 & 0 & a(0) & a(1) & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},$$

where

$$\begin{aligned}
b(0) &= \phi + (1-\phi)\beta(0), \\
b(k) &= (1-\phi)\beta(k), && k = 1, 2, \dots, \\
a(0) &= (1-\phi)\beta(0)(1-q), \\
a(k) &= (1-\phi)[\beta(k-1)q + \beta(k)(1-q)], && k = 1, 2, \dots.
\end{aligned}$$

It is easy to see that $\sum_{k=0}^{\infty} a(k) = 1 - \phi < 1$ and

$$\lim_{k\to\infty} \frac{a(k)}{\beta(k)} = 1 - \phi, \quad \lim_{k\to\infty} \frac{b(k)}{\beta(k)} = 1 - \phi.$$

Note here that $\{\beta(k); k \in \mathbb{Z}_+\}$ is decreasing and

$$\beta(k) \overset{k}{\sim} \gamma k^{-\gamma-1}.$$

Thus as in subsection C.3.1, we can show that $\{\beta(k); k \in \mathbb{Z}_+\} \in \mathcal{S}_{\mathrm{loc}}(1)$. As a result, Theorem 3.5 yields

$$\lim_{n\to\infty} \mathsf{P}(L_n = k) \overset{k}{\sim} \frac{1-\phi}{\phi}\beta(k) \overset{k}{\sim} \frac{1-\phi}{\phi}\gamma k^{-\gamma-1}.$$

C.4 Related results on long-tailed distributions

In this subsection, we provide several results related to long-tailed distributions. These results are used in deriving light-tailed asymptotic formulae in Chapter 2, more specifically, in the case where θ_+ and θ_- do not exist (see e.g., Assumption 2.1). Proposition C.10 below is an extension of Lemma 10 of Jelenković and Lazar [28]. The proposition can be proved by using the techniques presented in Jelenković and Lazar [28] and Asmussen et al. [6] though many (minor) modifications are required. Thus, we provide a complete proof of this proposition.

Proposition C.10 *Let $\{F(k); k \in \mathbb{Z}_+\}$ and $\{F_j(k); k \in \mathbb{Z}_+\}$ ($j = 1, 2, \dots, m$) denote probability mass functions of random variables in $\mathbb{Z}_+$ such that $\sum_{k=0}^{\infty} \gamma^k F(k)$ and $\sum_{k=0}^{\infty} \gamma^k F_j(k)$ are finite for some $\gamma \ge 1$ and*

$$\lim_{k\to\infty} \frac{\overline{F}_j(k)}{\gamma^{-k}\overline{F}(k)} = c_j \ge 0, \qquad j = 1, 2, \dots, m. \tag{C.36}$$

If $\{F(k)\} \in \mathcal{S}$ and $\gamma = 1$, or if $\{F(k)\} \in \mathcal{S}^*$ and $\gamma > 1$, then the following hold:

(a) For all $n_1, n_2, \dots, n_m \in \mathbb{N}$,

$$\lim_{k\to\infty} \frac{\overline{F_1^{*n_1} * F_2^{*n_2} * \cdots * F_m^{*n_m}}(k)}{\gamma^{-k}\overline{F}(k)} = \sum_{i=1}^{m} c_i n_i \left(\widehat{F}_i(\gamma)\right)^{n_i-1} \prod_{\substack{j=1,2,\dots,m \\ j\neq i}} \left(\widehat{F}_j(\gamma)\right)^{n_j}, \tag{C.37}$$

where $\widehat{F}_i(z) = \sum_{\ell=0}^{\infty} z^{\ell} F_i(\ell)$ for $i = 1, 2, \dots, m$.

(b) For any $\varepsilon > 0$, there exists some $C_\varepsilon \in (0, \infty)$ such that

$$\overline{F_1^{*n_1} * F_2^{*n_2} * \cdots * F_m^{*n_m}}(k) \le C_\varepsilon (1+\varepsilon)^{n_1+n_2+\cdots+n_m} \gamma^{-k}\overline{F}(k), \tag{C.38}$$

for all $k \in \mathbb{Z}_+$ and $n_1, n_2, \dots, n_m \in \mathbb{N}$.

Proof. The case where $\{F(k)\} \in \mathcal{S}$ and $\gamma = 1$ is equivalent to Lemma 10 of Jelenković and Lazar [28]. Therefore, we prove only the case where $\{F(k)\} \in \mathcal{S}^*$ and $\gamma > 1$.

We first prove the statement (a). To this end, it suffices to show that

$$\lim_{k\to\infty} \frac{\overline{F_i * F_j}(k)}{\gamma^{-k}\overline{F}(k)} = c_i \widehat{F}_j(\gamma) + c_j \widehat{F}_i(\gamma), \quad i, j = 1, 2, \dots, m. \tag{C.39}$$

From (C.36), we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{\overline{F_i * F_j}(k)}{\gamma^{-k}\overline{F}(k)} &= \left[\lim_{k\to\infty} \frac{\overline{F}_i(k)}{\gamma^{-k}\overline{F}(k)} + \lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} \right] \\
&= \left[c_i + \lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} \right].
\end{aligned}$$

Therefore, (C.39) holds if

$$\lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} = c_i(\widehat{F}_j(\gamma) - 1) + c_j \widehat{F}_i(\gamma). \tag{C.40}$$

In what follows, we prove that (C.40) holds.

It follows from (C.36) and $\{F(k)\} \in \mathcal{S}^* \subset \mathcal{L}$ (see Definitions C.1 and C.3) that

$$\begin{aligned} \lim_{k\to\infty} \frac{F_i(k)}{\gamma^{-k}\overline{F}(k)} &= \gamma \lim_{k\to\infty} \frac{\overline{F}_i(k-1)}{\gamma^{-k+1}\overline{F}(k-1)} \frac{\overline{F}(k-1)}{\overline{F}(k)} - \lim_{k\to\infty} \frac{\overline{F}_i(k)}{\gamma^{-k}\overline{F}(k)} \\ &= (\gamma-1)c_i. \end{aligned} \tag{C.41}$$

Furthermore, (C.36) and (C.41) imply that for any $\varepsilon > 0$ there exists some $k_0 \in \mathbb{N}$ such that for all $k \ge k_0 - 1$,

$$c_j - \varepsilon \le \frac{\overline{F}_j(k)}{\gamma^{-k}\overline{F}(k)} \le c_j + \varepsilon, \qquad j = 1, 2, \dots, m, \tag{C.42}$$

$$(\gamma-1)(c_i - \varepsilon) \le \frac{F_i(k)}{\gamma^{-k}\overline{F}(k)} \le (\gamma-1)(c_i + \varepsilon), \qquad i = 1, 2, \dots, m. \tag{C.43}$$

We now fix $k > 2k_0$ and decompose the term

$$\sum_{\ell=0}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)}$$

as follows:

$$\begin{aligned} \sum_{\ell=0}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} &= \sum_{\ell=0}^{k_0-1} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} + \sum_{\ell=k_0}^{k-k_0} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} \\ &\quad + \sum_{\ell=k-k_0+1}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)}. \end{aligned} \tag{C.44}$$

From (C.36) and $\{F(k)\} \in \mathcal{L}$, we have

$$\begin{aligned} &\lim_{k_0\to\infty} \lim_{k\to\infty} \sum_{\ell=0}^{k_0-1} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} \\ &= \lim_{k_0\to\infty} \sum_{\ell=0}^{k_0-1} \gamma^{\ell} F_i(\ell) \lim_{k\to\infty} \frac{\overline{F}_j(k-\ell)}{\gamma^{-k+\ell}\overline{F}(k-\ell)} \frac{\overline{F}(k-\ell)}{\overline{F}(k)} \\ &= \sum_{\ell=0}^{\infty} \gamma^{\ell} c_j F_i(\ell) = c_j \widehat{F}_i(\gamma). \end{aligned} \tag{C.45}$$

From (C.42) and (C.43), we also obtain

$$\begin{aligned} \sum_{\ell=k_0}^{k-k_0} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} &= \sum_{\ell=k_0}^{k-k_0} \frac{F_i(\ell)}{\gamma^{-\ell}\overline{F}(\ell)} \frac{\overline{F}_j(k-\ell)}{\gamma^{-k+\ell}\overline{F}(k-\ell)} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} \\ &\le (\gamma-1)(c_i+\varepsilon)(c_j+\varepsilon) \sum_{\ell=k_0}^{k-k_0} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)}. \end{aligned} \tag{C.46}$$

Note here that

$$\sum_{\ell=0}^{k} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} = 2\sum_{\ell=0}^{k_0-1} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} + \sum_{\ell=k_0}^{k-k_0} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)}.$$

Note also that since $\{F(k)\} \in \mathcal{S}^* \subset \mathcal{L}$ (see Definition C.3),

$$\begin{aligned}
\lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} &= 2\sum_{\ell=0}^{\infty} \overline{F}(\ell), \\
\lim_{k_0\to\infty} \lim_{k\to\infty} \sum_{\ell=0}^{k_0-1} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} &= \lim_{k_0\to\infty} \sum_{\ell=0}^{k_0-1} \overline{F}(\ell) = \sum_{\ell=0}^{\infty} \overline{F}(\ell).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\lim_{k_0\to\infty} \lim_{k\to\infty} \sum_{\ell=k_0}^{k-k_0} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} &= \lim_{k\to\infty} \sum_{\ell=0}^{k} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} \\
&\quad - 2 \lim_{k_0\to\infty} \lim_{k\to\infty} \sum_{\ell=0}^{k_0-1} \frac{\overline{F}(\ell)\overline{F}(k-\ell)}{\overline{F}(k)} \\
&= 2\sum_{\ell=0}^{\infty} \overline{F}(\ell) - 2\sum_{\ell=0}^{\infty} \overline{F}(\ell) = 0.
\end{aligned}$$

Substituting this into (C.46) yields

$$\lim_{k_0\to\infty} \limsup_{k\to\infty} \sum_{\ell=k_0}^{k-k_0} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} = 0. \tag{C.47}$$

Furthermore, it follows from (C.43) that

$$\begin{aligned}
\sum_{\ell=k-k_0+1}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} &= \sum_{\ell=k-k_0+1}^{k} \frac{\overline{F}(\ell)}{\overline{F}(k)} \frac{\overline{F}_j(k-\ell)}{\gamma^{-k+\ell}} \frac{F_i(\ell)}{\gamma^{-\ell}\overline{F}(\ell)} \\
&\le (\gamma-1)(c_i+\varepsilon) \sum_{\ell=k-k_0+1}^{k} \frac{\overline{F}(\ell)}{\overline{F}(k)} \frac{\overline{F}_j(k-\ell)}{\gamma^{-k+\ell}} \\
&= (\gamma-1)(c_i+\varepsilon) \sum_{\ell=0}^{k_0-1} \frac{\overline{F}(k-l)}{\overline{F}(k)} \gamma^{\ell}\overline{F}_j(\ell),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\lim_{k_0\to\infty} \limsup_{k\to\infty} \sum_{\ell=k-k_0+1}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} \\
&\le (\gamma-1)(c_i+\varepsilon) \lim_{k_0\to\infty} \sum_{\ell=0}^{k_0-1} \limsup_{k\to\infty} \frac{\overline{F}(k-\ell)}{\overline{F}(k)} \gamma^{\ell}\overline{F}_j(\ell) \\
&= (\gamma-1)(c_i+\varepsilon) \sum_{\ell=0}^{\infty} \gamma^{\ell}\overline{F}_j(\ell) \\
&= (c_i+\varepsilon)(\widehat{F}_j(\gamma)-1),
\end{aligned} \tag{C.48}$$

where the second last equality follows from $\{F(k)\} \in \mathcal{L}$; and the last one follows from $\sum_{\ell=0}^{\infty} \gamma^{\ell}\overline{F}_j(\ell) = (\widehat{F}_j(\gamma)-1)/(\gamma-1)$. Thus, combining (C.44) with (C.45), (C.47) and (C.48) and letting $\varepsilon \to 0$, we have

$$\limsup_{k\to\infty} \sum_{\ell=0}^{k} \frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} \le c_i\widehat{F}_j(\gamma) + \widehat{F}_i(\gamma)c_j.$$

Similarly, we can show that

$$\liminf_{k\to\infty}\sum_{\ell=0}^{k}\frac{F_i(\ell)\overline{F}_j(k-\ell)}{\gamma^{-k}\overline{F}(k)} \ge c_i\widehat{F}_j(\gamma)+\widehat{F}_i(\gamma)c_j.$$

As a result, we obtain (C.40).

We move on to the proof of the statement (b). For this purpose, we assume that $c_j > 0$ for all $j = 1,2,\dots,m$ (which is removed later). Let $C = \max\{1,c_1,\dots,c_m\}$, $d_0 = 1$ and $d_j = c_j/C \le 1$, for $j = 1,2,\dots,m$. Let

$$\overline{F}_0(k) = \min(1, C\overline{F}(k)), \qquad k \in \mathbb{Z}_+. \tag{C.49}$$

From (C.36) and (C.49), we have

$$\lim_{k\to\infty}\frac{\overline{F}_j(k)}{\gamma^{-k}\overline{F}_0(k)} = d_j \le 1, \qquad j = 0,1,\dots,m, \tag{C.50}$$

and thus, for $i,j = 0,1,\dots,m$ and $n \in \mathbb{Z}_+$,

$$\begin{aligned}\lim_{k\to\infty}\sum_{\ell=0}^{n}\frac{\overline{F}_j(k-\ell)F_i(\ell)}{\overline{F}_j(k)} &= \sum_{\ell=0}^{n}\lim_{k\to\infty}\frac{\overline{F}_j(k-\ell)}{\gamma^{-k+\ell}\overline{F}_0(k-\ell)}\frac{\gamma^{-k}\overline{F}_0(k)}{\overline{F}_j(k)}\frac{\overline{F}_0(k-\ell)}{\overline{F}_0(k)}\gamma^{\ell}F_i(\ell)\\ &= d_j\cdot d_j^{-1}\cdot 1\cdot\sum_{\ell=0}^{n}\gamma^{\ell}F_i(\ell) = \sum_{\ell=0}^{n}\gamma^{\ell}F_i(\ell),\end{aligned}$$

where we use $\{F_0(k)\} \in \mathcal{L}$ in the first equality. The above equation leads to

$$\lim_{n\to\infty}\lim_{k\to\infty}\sum_{\ell=0}^{n}\frac{\overline{F}_j(k-\ell)F_i(\ell)}{\overline{F}_j(k)} = \sum_{\ell=0}^{\infty}\gamma^{\ell}F_i(\ell) = \widehat{F}_i(\gamma) \ge 1. \tag{C.51}$$

Therefore, it follows from (C.49), (C.50), (C.51) and (C.39) that for any $\varepsilon > 0$ there exist some positive integers k' and $k'' > 2k'$ such that

$$\overline{F}_0(k) = C\overline{F}(k) \le 1, \qquad k \ge k', \tag{C.52}$$

and for all $i,j = 0,1,\dots,m$,

$$d_j - \frac{\varepsilon}{8} \le \frac{\overline{F}_j(k)}{\gamma^{-k}\overline{F}_0(k)} \le 1 + \frac{\varepsilon}{2}, \qquad k \ge k', \tag{C.53}$$

$$\sum_{\ell=0}^{k'-1}\frac{\overline{F}_j(k-\ell)F_i(\ell)}{\overline{F}_j(k)} \ge 1 - \frac{\varepsilon}{8d_j}, \qquad k \ge k'', \tag{C.54}$$

$$\overline{F_i * F_j}(k) \le (d_i + d_j + \varepsilon/4)\gamma^{-k}\overline{F}_0(k), \qquad k \ge k''. \tag{C.55}$$

Using the above inequalities (C.52)–(C.55), we first prove the statement (b) for the convolution of two probability mass functions $\{F_i(k)\}$ and $\{F_j(k)\}$ ($i,j \in \{0,1,\dots,m\}$). From (C.53)–(C.55), we obtain for

$k \geq k'' > 2k'$,

$$\begin{aligned}
&\overline{F_i * F_j}(k) - \sum_{\ell=0}^{k'} F_i(\ell)\overline{F}_j(k-\ell) \\
&\leq \left(d_i + d_j + \frac{\varepsilon}{4}\right)\gamma^{-k}\overline{F}_0(k) - \left(1 - \frac{\varepsilon}{8d_j}\right)\overline{F}_j(k) \\
&\leq \left[\left(d_i + d_j + \frac{\varepsilon}{4}\right) - \left(1 - \frac{\varepsilon}{8d_j}\right)\left(d_j - \frac{\varepsilon}{8}\right)\right]\gamma^{-k}\overline{F}_0(k) \\
&\leq \left(d_i + \frac{\varepsilon}{2}\right)\gamma^{-k}\overline{F}_0(k) \leq \left(1 + \frac{\varepsilon}{2}\right)\gamma^{-k}\overline{F}_0(k), \qquad \text{(C.56)}
\end{aligned}$$

where the last inequality holds because $d_j \leq 1$ for $j = 0, 1, \ldots, m$. From (C.56), we have

$$\begin{aligned}
\overline{F_i * F_j}(k) &\leq \left(1 + \frac{\varepsilon}{2}\right)\gamma^{-k}\overline{F}_0(k) + \sum_{\ell=0}^{k'} F_i(\ell)\overline{F}_j(k-\ell) \\
&\leq \left(1 + \frac{\varepsilon}{2}\right)\gamma^{-k}\overline{F}_0(k) + \sum_{\ell=0}^{k'} F_i(\ell)\overline{F}_j(k-k') \\
&\leq \left(1 + \frac{\varepsilon}{2}\right)\gamma^{-k}\overline{F}_0(k) + \overline{F}_j(k-k'), \qquad k \geq k'' > 2k'.
\end{aligned}$$

Applying (C.53) to the right hand side of the above inequality yields

$$\begin{aligned}
\overline{F_i * F_j}(k) &\leq \left[\left(1 + \frac{\varepsilon}{2}\right) + \left(1 + \frac{\varepsilon}{2}\right)\gamma^{k'}\frac{\overline{F}_0(k-k')}{\overline{F}_0(k)}\right]\gamma^{-k}\overline{F}_0(k) \\
&\leq \left(1 + \gamma^{k'}\sup_{k \geq k''}\frac{\overline{F}_0(k-k')}{\overline{F}_0(k)}\right)\left(1 + \frac{\varepsilon}{2}\right)\gamma^{-k}\overline{F}_0(k), \quad k \geq k'' > 2k'. \qquad \text{(C.57)}
\end{aligned}$$

Note here that since $\gamma > 1$ and $\{F_0(k)\} \in \mathcal{L}$,

$$1 \leq C^{(1)} := \gamma^{k'}\sup_{k \geq k''}\frac{\overline{F}_0(k-k')}{\overline{F}_0(k)} < \infty. \qquad \text{(C.58)}$$

Substituting this into (C.57), we obtain for $k \geq k''$,

$$\begin{aligned}
\overline{F_i * F_j}(k) &\leq (1 + C^{(1)})\left(1 + \frac{\varepsilon}{2}\right)\gamma^{-k}\overline{F}_0(k) \\
&\leq 2C^{(1)}(1+\varepsilon)^2\gamma^{-k}\overline{F}_0(k) \\
&= 2C^{(1)}(1+\varepsilon)^2\gamma^{-k}C\overline{F}(k), \qquad \text{(C.59)}
\end{aligned}$$

where the last equality follows from (C.52). On the other hand, for $k < k''$,

$$\overline{F_i * F_j}(k) \leq 1 \leq \frac{\gamma^{-k}\overline{F}(k)}{\gamma^{-k''}\overline{F}(k'')} \leq \frac{1}{\gamma^{-k''}\overline{F}(k'')} \cdot (1+\varepsilon)^2\gamma^{-k}\overline{F}(k). \qquad \text{(C.60)}$$

We now define K_ε as

$$K_\varepsilon = \max\left(2C^{(1)}, \frac{1}{C\gamma^{-k''}\overline{F}(k'')}, \frac{2C^{(1)}}{\varepsilon}\right), \qquad \text{(C.61)}$$

where we fix $\varepsilon > 0$ such that $K_\varepsilon \ge 2C^{(1)}/\varepsilon$ for later use. It follows from (C.59) and (C.60) that

$$\overline{F_i * F_j}(k) \le K_\varepsilon (1+\varepsilon)^2 \gamma^{-k} C \overline{F}(k), \qquad k \in \mathbb{Z}_+. \tag{C.62}$$

The inequality (C.62) shows that the statement (b) holds for the convolution of two probability mass functions $\{F_i(k)\}$ and $\{F_j(k)\}$ $(i,j \in \{0,1,\dots,m\})$.

We next consider the convolution of three probability mass functions $\{F_i(k)\}$, $\{F_j(k)\}$ and $\{F_\nu(k)\}$ $(i,j,\nu \in \{0,1,\dots,m\})$. Using (C.62), (C.49) and $\overline{F_i * F_j}(k') \le 1$, we have for $k \ge k'' > 2k'$,

$$\begin{aligned}
&\overline{F_i * F_j * F_\nu}(k) \\
&\quad = \sum_{\ell=0}^{k-k'} \overline{F_i * F_j}(k-\ell) F_\nu(\ell) + \overline{F_i * F_j}(k') \overline{F}_\nu(k-k') + \sum_{\ell=0}^{k'} F_i * F_j(\ell) \overline{F}_\nu(k-\ell) \\
&\quad \le \sum_{\ell=0}^{k-k'} \overline{F_i * F_j}(k-\ell) F_\nu(\ell) + \overline{F_i * F_j}(k') \overline{F}_\nu(k-k') + \overline{F}_\nu(k-k') \\
&\quad \le K_\varepsilon (1+\varepsilon)^2 \left(\sum_{\ell=0}^{k-k'} \gamma^{-k+\ell} \overline{F}_0(k-\ell) F_\nu(\ell) + \gamma^{-k'} \overline{F}_0(k') \overline{F}_\nu(k-k') \right) + \overline{F}_\nu(k-k') \\
&\quad \le K_\varepsilon (1+\varepsilon)^2 \left(\sum_{\ell=0}^{k-k'} \overline{F}_0(k-\ell) F_\nu(\ell) + \overline{F}_0(k') \overline{F}_\nu(k-k') \right) + \overline{F}_\nu(k-k'),
\end{aligned} \tag{C.63}$$

where the last inequality holds because of $\gamma \ge 1$. Note here that for $k \in \mathbb{Z}_+$,

$$\overline{F_i * F_j}(k) = \sum_{\ell=0}^{k-k'} \overline{F}_i(k-\ell) F_j(\ell) + \sum_{\ell=0}^{k'} F_i(\ell) \overline{F}_j(k-\ell) + \overline{F}_i(k') \overline{F}_j(k-k'),$$

from which and (C.56), we have for $i,j \in \{0,1,\dots,m\}$,

$$\begin{aligned}
&\sum_{\ell=0}^{k-k'} \overline{F}_i(k-\ell) F_j(\ell) + \overline{F}_i(k') \overline{F}_j(k-k') \\
&\quad = \overline{F_i * F_j}(k) - \sum_{\ell=0}^{k'} F_i(\ell) \overline{F}_j(k-\ell) \\
&\quad \le \left(1 + \frac{\varepsilon}{2}\right) \gamma^{-k} \overline{F}_0(k), \qquad k \ge k'' > 2k'.
\end{aligned} \tag{C.64}$$

Applying (C.64) and (C.53) to (C.63), we obtain for $k \ge k'' > 2k'$,

$$\begin{aligned}
&\overline{F_i * F_j * F_\nu}(k) \\
&\quad \le K_\varepsilon (1+\varepsilon)^2 \left(1 + \frac{\varepsilon}{2}\right) \gamma^{-k} \overline{F}_0(k) + \left(1 + \frac{\varepsilon}{2}\right) \gamma^{-k+k'} \overline{F}_0(k-k') \\
&\quad \le K_\varepsilon (1+\varepsilon)^2 \left(1 + \frac{\varepsilon}{2}\right) \gamma^{-k} \overline{F}_0(k) + \left(1 + \frac{\varepsilon}{2}\right) C^{(1)} \gamma^{-k} \overline{F}_0(k),
\end{aligned} \tag{C.65}$$

where the second inequality follows from (C.58). Note here that since $K_\varepsilon \ge 2C^{(1)}/\varepsilon$ (due to (C.61)),

$$\left(1 + \frac{\varepsilon}{2}\right) C^{(1)} \le (1+\varepsilon)^2 C^{(1)} \le K_\varepsilon (1+\varepsilon)^2 \frac{\varepsilon}{2}.$$

Substituting this inequality and (C.49) into (C.65) leads to for $k \ge k''$,

$$\begin{aligned}\overline{F_i * F_j * F_\nu}(k) &\le K_\varepsilon (1+\varepsilon)^2 \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) \gamma^{-k}\overline{F}_0(k) \\ &= K_\varepsilon (1+\varepsilon)^3 \gamma^{-k}\overline{F}_0(k) \\ &= K_\varepsilon (1+\varepsilon)^3 \gamma^{-k} C\overline{F}(k).\end{aligned}$$

Similarly to (C.60), we have for $k < k''$,

$$\begin{aligned}\overline{F_i * F_j * F_\nu}(k) \le 1 &\le \frac{\gamma^{-k}\overline{F}(k)}{\gamma^{-k''}\overline{F}(k'')} \le \frac{1}{\gamma^{-k''}\overline{F}(k'')}(1+\varepsilon)^3\gamma^{-k}\overline{F}(k) \\ &\le K_\varepsilon (1+\varepsilon)^3 \gamma^{-k} C\overline{F}(k).\end{aligned}$$

Therefore, the statement (b) holds for the convolution of three probability mass functions $\{F_i(k)\}$, $\{F_j(k)\}$ and $\{F_\nu(k)\}$ $(i, j, \nu \in \{0, 1, \ldots, m\})$. Recall that it is assumed in the above argument that $c_j > 0$ for all $j = 1, 2, \ldots, m$. Thus, by repeating the above argument, we can show that (C.38) holds under this additional condition.

In what follows, we remove the additional condition, i.e., $c_j > 0$ $(j = 1, 2, \ldots, m)$. To this end, without loss of generality, we assume that $c_j = 0$ for $j = 1, 2, \ldots, m'$ $(1 \le m' \le m)$ and $c_j > 0$ for $j = m'+1, m'+2, \ldots, m$. Under this assumption, for any $\delta > 0$ there exists some positive integer k_* such that for all $k \ge k_*$,

$$\overline{F}_j(k) \le \delta\gamma^{-k}\overline{F}(k), \qquad j = 1, 2, \ldots, m'.$$

Thus, let $\{H_\delta(k); k \in \mathbb{Z}_+\}$ denote a probability mass function such that

$$\overline{H}_\delta(k) = \min(1, \delta\gamma^{-k}\overline{F}(k)), \qquad k \in \mathbb{Z}_+.$$

We then have for $j = 1, 2, \ldots, m'$,

$$\overline{F}_j(k) \le \overline{H}_\delta(k), \qquad k \in \mathbb{Z}_+.$$

Therefore, we obtain

$$\begin{aligned}&\overline{F^{*n_1} * F^{*n_2} * \cdots * F^{*n_m}}(k) \\ &\quad \le \overline{H_\delta^{*(n_1+\cdots+n_{m'})} * F_{m'+1}^{*n_{m'+1}} * \cdots * F_m^{*n_m}}(k).\end{aligned}$$

Note here that

$$\lim_{k\to\infty} \frac{\overline{H}_\delta(k)}{\gamma^{-k}\overline{F}(k)} = \delta > 0.$$

Note also that the statement (b) have been proved under the additional condition that $c_j > 0$ for all $j = 1, 2, \ldots, m$. Therefore, for any $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that

$$\begin{aligned}&\overline{H_\delta^{*(n_1+\cdots+n_{m'})} * F_{m'+1}^{*n_{m'+1}} * \cdots * F_m^{*n_m}}(k) \\ &\quad \le C_\varepsilon (1+\varepsilon)^{n_1+n_2+\cdots+n_m}\gamma^{-k}\overline{F}(k).\end{aligned}$$

We have completed the proof of the statement (b). □

Using Proposition C.10, we obtain the following proposition, which is an extension of Proposition C.3, i.e., Proposition A.3 of Masuyama [42].

Proposition C.11 *Let* $\{\boldsymbol{P}(k); k \in \mathbb{Z}_+\}$ *and* $\{\boldsymbol{Q}(k); k \in \mathbb{Z}_+\}$ *denote nonnegative* $d_0 \times d_1$ *and* $d_1 \times d_2$ *matrix sequences, respectively, such that* $\widehat{\boldsymbol{P}}(\gamma)$ *and* $\widehat{\boldsymbol{Q}}(\gamma)$ *are finite for some* $\gamma \ge 1$, *where* $\widehat{\boldsymbol{P}}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{P}(k)$ *and* $\widehat{\boldsymbol{Q}}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{Q}(k)$. *Suppose that for some random variable* U *in* $\mathbb{Z}_+$,

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{P}}(k)}{\gamma^{-k}\mathsf{P}(U > k)} = \widetilde{\boldsymbol{P}} \ge \boldsymbol{O}, \tag{C.66}$$

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{Q}}(k)}{\gamma^{-k}\mathsf{P}(U > k)} = \widetilde{\boldsymbol{Q}} \ge \boldsymbol{O}, \tag{C.67}$$

where $\widetilde{\boldsymbol{P}} = \widetilde{\boldsymbol{Q}} = \boldsymbol{O}$ *is allowed. If* $U \in \mathcal{S}$ *and* $\gamma = 1$, *or if* $U \in \mathcal{S}^*$ *and* $\gamma > 1$, *then*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{P} * \boldsymbol{Q}}(k)}{\gamma^{-k}\mathsf{P}(U > k)} = \widetilde{\boldsymbol{P}}\widehat{\boldsymbol{Q}}(\gamma) + \widehat{\boldsymbol{P}}(\gamma)\widetilde{\boldsymbol{Q}}. \tag{C.68}$$

Proof. Let $\mathbb{D}_1 = \{1, 2, \dots, d_1\}$ and $\mathbb{D}_1(i,j) = \{\nu \in \mathbb{D}_1; [\widehat{\boldsymbol{P}}(1)]_{i,\nu}[\widehat{\boldsymbol{Q}}(1)]_{\nu,j} > 0\}$ for $i \in \{1, 2, \dots, d_0\}$ and $j \in \{1, 2, \dots, d_2\}$. In what follows, we fix $i \in \{1, 2, \dots, d_0\}$ and $j \in \{1, 2, \dots, d_2\}$ arbitrarily.

Let $\underline{\boldsymbol{P}}(k) = \sum_{\ell=0}^{k} \boldsymbol{P}(\ell)$ for $k \in \mathbb{Z}_+$. We then have

$$\overline{\boldsymbol{P} * \boldsymbol{Q}}(k) = \widehat{\boldsymbol{P}}(1)\widehat{\boldsymbol{Q}}(1) - \sum_{\ell=0}^{k} \underline{\boldsymbol{P}}(k-\ell)\boldsymbol{Q}(\ell), \qquad k \in \mathbb{Z}_+,$$

and thus

$$\begin{aligned}
[\overline{\boldsymbol{P} * \boldsymbol{Q}}(k)]_{i,j} &= \sum_{\nu\in\mathbb{D}_1(i,j)} \left([\widehat{\boldsymbol{P}}(1)]_{i,\nu}[\widehat{\boldsymbol{Q}}(1)]_{\nu,j} - \sum_{\ell=0}^{k} [\underline{\boldsymbol{P}}(k-\ell)]_{i,\nu}[\boldsymbol{Q}(\ell)]_{\nu,j} \right) \\
&= \sum_{\nu\in\mathbb{D}_1(i,j)} [\widehat{\boldsymbol{P}}(1)]_{i,\nu}[\widehat{\boldsymbol{Q}}(1)]_{\nu,j} \left(1 - \sum_{\ell=0}^{k} \frac{[\underline{\boldsymbol{P}}(k-\ell)]_{i,\nu}}{[\widehat{\boldsymbol{P}}(1)]_{i,\nu}} \frac{[\boldsymbol{Q}(\ell)]_{\nu,j}}{[\widehat{\boldsymbol{Q}}(1)]_{\nu,j}} \right).
\end{aligned} \tag{C.69}$$

We now define $P_{i,\nu}$ and $Q_{\nu,j}$ ($\nu \in \mathbb{D}_1(i,j)$) as random variables in $\mathbb{Z}_+$ such that for all $k \in \mathbb{Z}_+$,

$$\mathsf{P}(P_{i,\nu} = k) = \frac{[\boldsymbol{P}(k)]_{i,\nu}}{[\widehat{\boldsymbol{P}}(1)]_{i,\nu}}, \qquad \mathsf{P}(Q_{\nu,j} = k) = \frac{[\boldsymbol{Q}(k)]_{\nu,j}}{[\widehat{\boldsymbol{Q}}(1)]_{\nu,j}}.$$

Using these random variables, we rewrite (C.69) as

$$[\overline{\boldsymbol{P} * \boldsymbol{Q}}(k)]_{i,j} = \sum_{\nu\in\mathbb{D}_1(i,j)} [\widehat{\boldsymbol{P}}(1)]_{i,\nu}[\widehat{\boldsymbol{Q}}(1)]_{\nu,j}\mathsf{P}(P_{i,\nu} + Q_{\nu,j} > k). \tag{C.70}$$

Note here that (C.66) and (C.67) imply that, for $\nu \in \mathbb{D}_1(i,j)$,

$$\lim_{k\to\infty} \frac{\mathsf{P}(P_{i,\nu} > k)}{\gamma^{-k}\mathsf{P}(U > k)} = \frac{[\widetilde{\boldsymbol{P}}]_{i,\nu}}{[\widehat{\boldsymbol{P}}(1)]_{i,\nu}}, \qquad \lim_{k\to\infty} \frac{\mathsf{P}(Q_{\nu,j} > k)}{\gamma^{-k}\mathsf{P}(U > k)} = \frac{[\widetilde{\boldsymbol{Q}}]_{\nu,j}}{[\widehat{\boldsymbol{Q}}(1)]_{\nu,j}}. \tag{C.71}$$

Note also that if $[\widehat{\boldsymbol{P}}(1)]_{i,\nu} = 0$ (resp. $[\widehat{\boldsymbol{Q}}(1)]_{\nu,j} = 0$) then $[\widetilde{\boldsymbol{P}}]_{i,\nu} = 0$ (resp. $[\widetilde{\boldsymbol{Q}}]_{\nu,j} = 0$), and thus

$$[\widetilde{\boldsymbol{P}}]_{i,\nu}[\widehat{\boldsymbol{Q}}(1)]_{\nu,j} + [\widehat{\boldsymbol{P}}(1)]_{i,\nu}[\widetilde{\boldsymbol{Q}}]_{\nu,j} = 0, \qquad \nu \in \mathbb{D}_1 \backslash \mathbb{D}_1(i,j). \tag{C.72}$$

By applying Proposition C.10 (a) to (C.70) and using (C.71) and (C.72), we obtain

$$\begin{aligned}\lim_{k\to\infty}\frac{[\overline{\boldsymbol{P}*\boldsymbol{Q}}(k)]_{i,j}}{\gamma^{-k}\mathsf{P}(U>k)} &= \sum_{\nu\in\mathbb{D}_1(i,j)}[\widehat{\boldsymbol{P}}(1)]_{i,\nu}[\widehat{\boldsymbol{Q}}(1)]_{\nu,j} \\ &\quad\times\left(\frac{[\widetilde{\boldsymbol{P}}]_{i,\nu}}{[\widehat{\boldsymbol{P}}(1)]_{i,\nu}}\frac{[\widehat{\boldsymbol{Q}}(\gamma)]_{\nu,j}}{[\widehat{\boldsymbol{Q}}(1)]_{\nu,j}}+\frac{[\widehat{\boldsymbol{P}}(\gamma)]_{i,\nu}}{[\widehat{\boldsymbol{P}}(1)]_{i,\nu}}\frac{[\widetilde{\boldsymbol{Q}}]_{\nu,j}}{[\widehat{\boldsymbol{Q}}(1)]_{\nu,j}}\right) \\ &= \sum_{\nu\in\mathbb{D}_1(i,j)}\left([\widetilde{\boldsymbol{P}}]_{i,\nu}[\widehat{\boldsymbol{Q}}(\gamma)]_{\nu,j}+[\widehat{\boldsymbol{P}}(\gamma)]_{i,\nu}[\widetilde{\boldsymbol{Q}}]_{\nu,j}\right) \\ &= \sum_{\nu\in\mathbb{D}_1}\left([\widetilde{\boldsymbol{P}}]_{i,\nu}[\widehat{\boldsymbol{Q}}(\gamma)]_{\nu,j}+[\widehat{\boldsymbol{P}}(\gamma)]_{i,\nu}[\widetilde{\boldsymbol{Q}}]_{\nu,j}\right),\end{aligned}$$

which leads to (C.68). □

From Propositions C.10 and C.11, we obtain the following result, which is an extension of Lemma 6 of Jelenković and Lazar [28].

Proposition C.12 *Let* $\{\boldsymbol{W}(k);k\in\mathbb{Z}_+\}$ *denote a nonnegative* $d\times d$ *matrix sequences such that* $\delta(\widehat{\boldsymbol{W}}(\gamma))<1$ *for some* $\gamma\ge 1$*, where* $\widehat{\boldsymbol{W}}(z)=\sum_{k=0}^{\infty}z^k\boldsymbol{W}(k)$*. Suppose that, for some random variable* $U\in\mathbb{Z}_+$*,*

$$\lim_{k\to\infty}\frac{\overline{\boldsymbol{W}}(k)}{\gamma^{-k}\mathsf{P}(U>k)}=\widetilde{\boldsymbol{W}}\ge\boldsymbol{O}. \tag{C.73}$$

If $U\in\mathcal{S}$ *and* $\gamma=1$*, or if* $U\in\mathcal{S}^*$ *and* $\gamma>1$*, then*

$$\lim_{k\to\infty}\frac{\overline{\sum_{n=0}^{\infty}\boldsymbol{W}^{*n}}(k)}{\gamma^{-k}\mathsf{P}(U>k)}=\left(\boldsymbol{I}-\widehat{\boldsymbol{W}}(\gamma)\right)^{-1}\widetilde{\boldsymbol{W}}\left(\boldsymbol{I}-\widehat{\boldsymbol{W}}(\gamma)\right)^{-1}.$$

Proof. Using Proposition C.10 (b) and proceeding as in the proof of Lemma 3.4 of Masuyama et al. [45], we can readily prove that for any $\varepsilon>0$ there exists some $C_\varepsilon\in(0,\infty)$ such that for all $k\in\mathbb{Z}_+$,

$$\overline{\boldsymbol{W}^{n*}}(k)\le C_\varepsilon(1+\varepsilon)^n\gamma^{-k}\mathsf{P}(U>k)\left(\widehat{\boldsymbol{W}}(\gamma)\right)^n,$$

which yields for all sufficiently small $\varepsilon>0$,

$$\begin{aligned}\frac{\overline{\sum_{n=0}^{\infty}\boldsymbol{W}^{n*}}(k)}{\gamma^{-k}\mathsf{P}(U>k)} &= \sum_{n=0}^{\infty}\frac{\overline{\boldsymbol{W}^{n*}}(k)}{\gamma^{-k}\mathsf{P}(U>k)}\le C_\varepsilon\sum_{n=0}^{\infty}(1+\varepsilon)^n\left(\widehat{\boldsymbol{W}}(\gamma)\right)^n \\ &= C_\varepsilon\left[\boldsymbol{I}-(1+\varepsilon)\widehat{\boldsymbol{W}}(\gamma)\right]^{-1}<\infty.\end{aligned}$$

Therefore, using the dominated convergence theorem and Proposition C.11, we obtain

$$\begin{aligned}\lim_{k\to\infty}\frac{\overline{\sum_{n=0}^{\infty}\boldsymbol{W}^{n*}}(k)}{\gamma^{-k}\mathsf{P}(U>k)} &= \sum_{n=0}^{\infty}\lim_{k\to\infty}\frac{\overline{\boldsymbol{W}^{n*}}(k)}{\gamma^{-k}\mathsf{P}(U>k)} \\ &= \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\left(\widehat{\boldsymbol{W}}(\gamma)\right)^{\ell}\widetilde{\boldsymbol{W}}\left(\widehat{\boldsymbol{W}}(\gamma)\right)^{n-\ell-1} \\ &= \left(\boldsymbol{I}-\widehat{\boldsymbol{W}}(\gamma)\right)^{-1}\widetilde{\boldsymbol{W}}\left(\boldsymbol{I}-\widehat{\boldsymbol{W}}(\gamma)\right)^{-1}.\end{aligned}$$

□

Appendix D

Continuity and Differentiability of Eigenvalues and Eigenvectors

In this appendix, we summarize the basic results on the continuity and differentiability of the eigenvalues and eigenvectors of matrix-valued functions. For this purpose, we define $\boldsymbol{P}(\,\cdot\,)$ as an $n \times n$ matrix function on $\mathbb{R}$.

Proposition D.1 *Suppose that, for some $m \in \mathbb{N}$, $\boldsymbol{P}(\xi)$ is m-times differentiable with respect to ξ. If $\boldsymbol{P}(\xi_0)$, $\xi_0 \in \mathbb{R}$, has a simple eigenvalue λ_0, then, in the neighbor of $\xi = \xi_0$, $\boldsymbol{P}(\xi)$ has an m-times differentiable eigenvalue $\lambda(\xi)$, which is equal to λ_0 at $\xi = \xi_0$.*

Remark D.1 *Theorem 7 of Chapter 9 in [38] states the result on a special case where $\boldsymbol{P}(\xi)$ is once differentiable at $\xi = \xi_0$.*

Proof. Let $f(z, \xi)$ denote the characteristic polynomial of $\boldsymbol{P}(\xi)$, i.e.,

$$f(z, \xi) = \det(zI - \boldsymbol{P}(\xi)), \qquad z \in \mathbb{C},\ \xi \in \mathbb{R}.$$

Clearly, $f(z, \xi)$ is an nth-degree polynomial function of z and is m-times differentiable with respect to ξ. Since λ_0 is a simple eigenvalue of $\boldsymbol{P}(\xi_0)$, we have

$$f(\lambda_0, \xi_0) = 0, \qquad \lim_{z \to \lambda_0} \frac{\partial}{\partial z} f(z, \xi_0) \neq 0.$$

It thus follows from the implicit function theorem that there exists a neighborhood U of ξ_0 and an m-times differentiable function $\lambda : U \to V$ such that $\lambda(\xi_0) = \lambda_0$ and

$$f(z, \xi) = 0 \text{ if and only if } z = \lambda(\xi) \text{ for all } (z, \xi) \in (V, U),$$

which completes the proof. □

Proposition D.2 *Suppose that, for some $m \in \mathbb{N}$, $\boldsymbol{P}(\xi)$ is m-times differentiable with respect to ξ and has an eigenvalue $\lambda(\xi)$ in the domain $U \subseteq \mathbb{R}$ whose corresponding left and right eigenvectors $\boldsymbol{\varphi}(\xi)$ and $\boldsymbol{h}(\xi)$ are normalized such that $\boldsymbol{\varphi}(\xi)\boldsymbol{e}_* = 1$ and $\boldsymbol{\varphi}(\xi)\boldsymbol{h}(\xi) = 1$, where $\boldsymbol{e}_*$ is an arbitrary n dimensional vector whose elements satisfies $|[e_*]_j| = 1$, $j \in \{1, \dots, N\}$. If $\lambda(\xi)$ is a simple eigenvalue, then we can choose $\boldsymbol{\varphi}(\xi)$ and $\boldsymbol{h}(\xi)$ so that they are m-times differentiable with respect to ξ in U.*

Proof. Let $\boldsymbol{P}_{-ii}(\xi)$, $i \in \{1, 2, \ldots, n\}$, denote an $(n-1) \times (n-1)$ matrix obtained by removing the ith row and column of $\boldsymbol{P}(\xi)$. It then follows from Lemma 9 of Chapter 9 in [38] that there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\det(\lambda(\xi_0)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi_0)) \neq 0$ for any fixed $\xi_0 \in U$. It also follows from Proposition D.1 that $\det(\lambda(\xi)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi))$ is continuous at $\xi = \xi_0$. Thus, $\det(\lambda(\xi)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi)) \neq 0$ holds in some open interval $S \subseteq U$ including ξ_0.

We now define $\boldsymbol{\varphi}_{-i}(\xi)$ as the vector obtained by removing the ith element of $\boldsymbol{\varphi}(\xi)$. We also define $\boldsymbol{p}_{-i}(\xi)$ as the ith row vector of $\boldsymbol{P}(\xi)$ whose ith element is removed. We then have, for $\xi \in S$ and $i \in \{1, \ldots, n\}$,

$$\lambda(\xi)\boldsymbol{\varphi}_{-i}(\xi) = \boldsymbol{\varphi}_{-i}(\xi)\boldsymbol{P}_{-ii}(\xi) + \varphi_i(\xi)\boldsymbol{p}_{-i}(\xi). \tag{D.1}$$

It can be easily confirmed that $\varphi_{i_0}(\xi) \neq 0$ for $\xi \in S$. Indeed, if $\varphi_{i_0}(\xi) = 0$, then (D.1) leads to

$$\boldsymbol{\varphi}_{-i_0}(\xi)(\lambda(\xi)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi)) = \boldsymbol{0}.$$

Since $\det(\lambda(\xi)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi)) \neq 0$, the above equation implies that $\boldsymbol{\varphi}(\xi) = \boldsymbol{0}$, which yields a contradiction. As a result, $\varphi_{i_0}(\xi) \neq 0$ for $\xi \in S$. It thus follows from (D.1) that

$$\boldsymbol{\varphi}_{-i_0}(\xi) = \varphi_{i_0}(\xi)\boldsymbol{p}_{-i_0}(\xi)(\lambda(\xi)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi))^{-1}. \tag{D.2}$$

Therefore, by choosing $\varphi_{i_0}(\xi)$ as

$$\varphi_{i_0}(\xi) = \frac{1}{\sum_j \left[\boldsymbol{p}_{-i_0}(\xi)(\lambda(\xi)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi))^{-1}\right]_j [e_*]_j + [e_*]_{i_0}}, \tag{D.3}$$

$\boldsymbol{\varphi}(\xi)$ satisfies the normalized condition. Note that $\boldsymbol{P}_{-ii}(\xi)$, $\boldsymbol{p}_{-i}(\xi)$ and $\lambda(\xi)$ are m-times differentiable in the domain S due to Proposition D.1. Thus, (D.3) implies that $\varphi_{i_0}(\xi)$ is also m-times differentiable in the same domain. Combining this with (D.2), it follows that $\boldsymbol{\varphi}(\xi)$ is $m-$differentiable in the domain S. Finally, by applying the above argument to any ξ in U, we can choose am m-times differentiable eigenvector $\boldsymbol{\varphi}(\xi)$ in the domain U.

Next, we prove that $\boldsymbol{h}(\xi)$ is m-times differentiable in the domain U. Similarly to the case of $\boldsymbol{\varphi}(\xi)$, we have, for $\xi \in S$,

$$\boldsymbol{h}_{-i_0}(\xi) = h_{i_0}(\xi)(\lambda(\xi)(\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi))^{-1}\boldsymbol{q}_{-i_0}(\xi), \tag{D.4}$$

where $\boldsymbol{h}_{-i}(\xi)$ and $\boldsymbol{q}_{-i}(\xi)$ represent the vector obtained by removing the ith element of $\boldsymbol{h}(\xi)$ and the ith column vector of $\boldsymbol{P}(\xi)$ whose ith element is removed. The normalized condition and (D.4) yield

$$h_{i_0}(\xi) = \frac{1}{\sum_j \left[(\lambda(\xi)\boldsymbol{I} - \boldsymbol{P}_{-i_0 i_0}(\xi))^{-1}\boldsymbol{q}_{-i_0}(\xi)\right]_j [\varphi(\xi)]_j + [\varphi(\xi)]_{i_0}}. \tag{D.5}$$

Since $\boldsymbol{P}_{-ii}(\xi)$, $\boldsymbol{p}_{-i}(\xi)$, $\lambda(\xi)$, and $\boldsymbol{\varphi}(\xi)$ are m-times differentiable in the domain S, (D.5) shows that $h_{i_0}(\xi)$ is also m-times differentiable in the same domain. Therefore, from (D.4), $\boldsymbol{h}(\xi)$ is also m-times differentiable in the domain S. Finally, choosing any ξ in U completes the proof.

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