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Rényi-Parry germs of curves and dynamical zeta functions associated with real algebraic numbers

By

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Abstract

Let $\beta > 1$ be an algebraic number. The relations between the coefficient vector of its minimal polynomial and the digits of the Rényi β-expansion of unity are investigated in terms of the germ of curve associated with β, which is constructed from the Salem parametrization, and the Parry Upper function $f_\beta(z)$. If β is a Parry number, the Parry Upper function $f_\beta(z)$ is simply related to the dynamical zeta function $\zeta_\beta(z)$ of the dynamical system $([0,1], T_\beta)$ where T_β is the β-transformation. Using the theory of Puiseux several results on the zeros of $f_\beta(z)$ and a classification of βs off Parry numbers are suggested.

§ 1. Introduction: digits and algebraicity

The Rényi-Parry numeration system [Re] [Pa] [Fr] uses a real number $\beta > 1$ as base of numeration and inherits the properties of the dynamical system $([0,1], T_\beta)$, where $T_\beta : x \to \{\beta x\} = \beta x \mod 1$ is the β-transformation, for instance given by its dynamical zeta function $\zeta_\beta(z)$ [AM] [Bo2] [FLP] [PP] [Po] [V4] or by the Rényi β-expansion $\mathrm{d}_\beta(1) = 0.t_1 t_2 t_3 \ldots$ of 1 which controls the language in base β [B-T] [Bl] [Lo]. The analytic function $f_\beta(z) = -1 + \sum_{i \geq 1} t_i z^i$ is then fundamental and called the Parry Upper function (at β). When the base of numeration $\beta > 1$ is an algebraic number a basic question is then to find the relations between the coefficient vector of its minimal polynomial and the string of digits (t_i). The present study gives new solutions and directions for this study in the geometrical setting of germs of curves. A Parry number is by definition a real number $\beta > 1$ such that $\mathrm{d}_\beta(1)$ is finite (ends in infinitely many zeros), then called *simple*, or eventually periodic. Parry numbers are algebraic integers which are Perron numbers, and the collection of Parry numbers is dense in $(1, +\infty)$ [Pa]. To $\beta > 1$ an algebraic number, given by its minimal polynomial $P_\beta(X)$, assumed to be a Parry number, is associated its Parry polynomial $P_{\beta,P}(X) \in P_\beta(X)\mathbb{Z}[X]$ (with $P^*_{\beta,P}$ denoting its reciprocal

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polynomial), as

$$f_\beta(z) = -\frac{1}{\zeta_\beta(z)} = -\frac{P^*_{\beta,P}(z)}{(1 - z^{p+1})} \qquad \text{nonsimple } \beta \tag{1.1}$$

where $p + 1$ is the period length, and

$$f_\beta(z) = -\frac{1 - z^m}{\zeta_\beta(z)} = -P^*_{\beta,P}(z) \qquad \text{simple } \beta \tag{1.2}$$

where m is the length of $d_\beta(1)$. The Dynamical Norm Conjecture (S. Akiyama) states: that, if β is a nonsimple Parry number, then the algebraic norm $N(\beta)$ of β satisfies $|N(\beta)| = |t_m - t_{m+p+1}|$ if m the preperiod length and $p + 1$ the period length of $\mathrm{d}_\beta(1)$. Other relations are observed between the coefficients of $P_\beta(X)$ and (t_i), for instance for some Pisot numbers and Salem numbers [Bo2] [Bo3] [V2], but their origin remains still obscure in general, for instance with the distibution of palindromic motives, the asymptotic strings of zeros, the repetitions, in (t_i), with some Diophantine Approximation questions [AB] [Bu] [Ds] [S] and the Mahler measure of the base β [V1] (cf Akiyama and Kwon [AD] for a review).

For nonParry numbers β, relations between $f_\beta(z)$ and $\zeta_\beta(z)$ are obscure; and the unit circle is the natural boundary of $f_\beta(z)$ by the Szegő-Carlson-Polya Theorem. The approach which is followed here was introduced in [V4] and overcomes this difficulty. It is addressed to noninteger algebraic numbers $\beta > 1$, Parry or nonParry: it amounts to write the Parry Upper function $f_\beta(z)$ as a two-variable Taylor series $G_\beta(U, Z) \in \mathbb{C}[[U, Z]]$ parametrized by the Salem parametrization $[P^*_\beta(z), z - 1/\beta]$, then to use the theory of Puiseux [C] to deduce its decomposition as a finite product of factors and the coefficients involved in the formal series and Puiseux series in them, relating (t_i) and the values of the derivatives of the minimal polynomial of β. The adding of a second variable, most notably introduced differently by Boyd in several articles, is typical of studies on moduli of curves in algebraic geometry (Lefshetz [Lf], Duval [Dl]). The theory of Puiseux is used for desingularizing curves locally, for instance for algebraic functions (i.e. polynomiality with the two variables). Here, the canonical method we follow gives rise to a germ of curve (the "Rényi-Parry germ of curve associated with β") whose equation G_β is analytic in U and polynomial in Z; this method uses the Salem parametrization introduced by Salem in his 1945 article, inthere as a basic ingredient in the so-called "Salem construction" for convergent families of Salem numbers (M.J. Bertin, M. Pathiaux-Delefosse [BPD]).

This note is conceived as a short introduction, without proofs, to [V5].

§ 2. Rényi-Parry germ of curve

§ 2.1. Equation

Let $\beta > 1$ be an algebraic number and $P_\beta(X) = a_d(X - \beta^{(0)})(X - \beta^{(1)}) \ldots (X - \beta^{(d-1)}) = \sum_{j=0}^{d} a_j X^j$ its minimal polynomial, with $\beta = \beta^{(0)}$, $P^*_\beta(X) = X^{\deg \beta} P_\beta(1/X)$, its reciprocal polynomial, d := $\deg \beta$ assumed ≥ 2, $\mathrm{d}_\beta(1) = 0.t_1 t_2 t_3 \ldots$ the Rényi β-expansion of unity, equivalently $\beta = t_1 + \sum_{i \geq 2} t_i \beta^{-i+1}$ with $t_1 = \lceil \beta - 1 \rceil = \lfloor \beta \rfloor$, $t_i := \lfloor \beta T^{i-1}_\beta(1) \rfloor, i \geq 2$, and $T^i_\beta := T_\beta(T^{i-1}_\beta), i \geq 1$, T^0_β :=Id. The digits t_i belong to $\{0, 1, \ldots, \lfloor \beta \rfloor\}$. The subrings

$\mathbb{C}\{U\}[Z] \subset \mathbb{C}\{U,Z\} \subset \mathbb{C}[[U,Z]]$ denote the sets of convergent formal series, the first one with polynomiality in Z. For $g = \sum_{n\geq 0, m\geq 0} c_{n,m} U^n Z^m \in \mathbb{C}[[U,Z]]$, $\mathrm{ord}_U g$ denotes the greatest integer $j \geq 0$ such that $g = \sum_{n\geq j, m\geq 0} c_{n,m} U^n Z^m$ (i.e. with no nonzero term $c_{n,m} U^n Z^m$ indexed by $n < j$).

Theorem 2.1. *There exists a unique $G_\beta(U,Z) \in \mathbb{C}\{U\}[Z]$ such that*

$$G_\beta(P^*_\beta(z), z - \frac{1}{\beta}) \;=\; f_\beta(z)$$

for z in a neighbourhood of $1/\beta$, with $\deg_Z G_\beta(U,Z) < \deg \beta$. $G_\beta(U,Z)$ decomposes into one of the four following possibilities : (i) either $G_\beta(U,Z) = U \times e$, where $e = e(U,Z)$ is a unit in $\mathbb{C}\{U,Z\}$, (ii) or it is equal to $e \times W$, where $e = e(U,Z)$ is a unit in $\mathbb{C}\{U,Z\}$, and (ii-1) $deg_Z W(U,Z) = 1, \mathrm{ord}_U W(U,Z) > 1$, or (ii-2) $deg_Z W(U,Z) = 1, \mathrm{ord}_U W(U,Z) = 1$, or (ii-3) $deg_Z W(U,Z) > 1, \mathrm{ord}_U W(U,Z) = 1$ and W is an irreducible Weierstrass polynomial.

Denote

$$G_\beta(U,Z) := b_{d-1}(U) Z^{d-1} + b_{d-2}(U) Z^{d-2} + \ldots + b_1(U) Z + b_0(U), \tag{2.1}$$

with $b_j(U) := \sum_{r\geq 0} b_{j,r}(U) U^s$. The four possibilities define four types of Newton polygon of G_β. The last three cases come from the Weierstrass preparation theorem applied to G_β and the fact that $1/\beta$ is a simple zero of f_β. In the last case, the theory of Puiseux applies to provide a unique decomposition of the Weierstrass polynomial W as a finite product

$$W = \prod_\xi \Big(Z - \sum_{i\geq 0} \alpha_i \xi^i U^{\frac{i}{w}} \Big)$$

over all the Puiseux factors forming a unique conjugacy class. The Puiseux series, involved in the Puiseux factors, deduced from the Newton polygon,

$$\sum_{i\geq 0} \alpha_i U^{\frac{i}{w}} \quad \text{with conjugates} \quad \sum_{i\geq 0} \alpha_i \xi^i U^{\frac{i}{w}}$$

are fractionary power series for which the exponents are rational integers with common denominator $w := \deg_Z W$, with ξ running over the wth-roots of unity. The polynomial G_β defines a plane affine curve

$$\mathcal{C}_\beta := \{(U,Z) \in \mathbb{C}^2 \mid G_\beta(U,Z) \;=\; \sum_{m,n\geq 0} A_{m,n}\, U^n Z^m \;=\; 0\}$$

with coefficients $A_{m,n}$ in a field extension of $\mathbb{Q}(\beta)$, along with a ramified covering $\pi_\beta : \mathcal{C}_\beta \to \mathbb{C}$, the first projection map, of $\mathbb{C}$ (i.e. the U-plane).

§ 2.2. Perron-Frobenius operator and Eigenvalues

Let $\mathbb{C}((U)) := \bigcup_{n\in\mathbb{N}^*} U^{-n}\, \mathbb{C}[[U]]$ be the field of formal Laurent series of the variable U, and denote $\mathbb{C}((U))^* := \bigcup_{m\in\mathbb{N}^*} \mathbb{C}((U^{1/m}))$ the field of Laurent-Puiseux series with coefficients in $\mathbb{C}$. The ring $\mathbb{C}[[U]]^* := \bigcup_{m\in\mathbb{N}^*} \mathbb{C}[[U^{1/m}]]$ of the Puiseux series contains the ring of formal series $\mathbb{C}[[U]]$. By the Theorem of Puiseux [C] $\mathbb{C}((U))^*$ is algebraically closed, and any polynomial in $\mathbb{C}[[U]][X]$ has at least one X-root in $\mathbb{C}[[U]]^*$. By the change of origin, with $\widetilde{f_\beta}(Z) := f_\beta(z)$ and $\widetilde{P^*_\beta}(Z) := P^*_\beta(z)$, we obtain new coefficients vectors. Denoting $\mathbb{K}_\beta := \mathbb{Q}(\beta)$, and by $\mathbb{K}_\beta^{\mathcal{G}}$ the smallest Galois extension containing $\mathbb{K}_\beta$, $\widetilde{f_\beta}(Z) = \sum_{j\geq 1} \lambda_j Z^j$ with $\lambda_j = \lambda_j(\beta) := \sum_{q\geq 0} t_{j+q}\binom{j+q}{j}\left(\frac{1}{\beta}\right)^q$, $\quad j \geq 1$, and $\widetilde{P^*_\beta}(Z) = Z\left(\gamma_1 + \gamma_2 Z + \ldots + \gamma_d Z^{d-1}\right)$, with $\gamma_q = \sum_{j=q}^{d} a_{d-j}\binom{j}{q}\left(\frac{1}{\beta}\right)^{j-q} \in \mathbb{K}_\beta$. Let

$$M = M_U := \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{U}{\gamma_d} \\ 1 & 0 & \ldots & 0 & -\frac{\gamma_1}{\gamma_d} \\ 0 & 1 & & & \\ & & & & \\ 0 & 0 & & 1 & -\frac{\gamma_{d-1}}{\gamma_d} \end{pmatrix}. \tag{2.2}$$

The $d \times d$ square matrix M, with coefficients in $\mathbb{Q}(\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \ldots, \gamma_d^{\pm 1})[U] = \mathbb{K}_\beta[U]$, is the matrix of an operator on the vector space $\left(\mathbb{C}[[U]]^*\right)^d$. This vector space splits into a direct sum of d Eigenspaces of dimension one, with a priori Puiseux series as Eigenvalues, since $F = F(U, X) = \det(X\mathrm{Id} - M_U) \in \mathbb{K}_\beta[U][X]$. In fact, only one Eigenvalue of M_U lies in the maximal ideal $U\mathbb{C}[[U]]^*$ (this Eigenvalue even belongs to $U\mathbb{K}_\beta[[U]]$).

Theorem 2.2. *With the above-mentioned notations, the characteristic polynomial*

$$F(U, X) = X^d + \frac{\gamma_{d-1}}{\gamma_d}X^{d-1} + \ldots + \frac{\gamma_2}{\gamma_d}X^2 + \frac{\gamma_1}{\gamma_d}X - \frac{1}{\gamma_d}U \tag{2.3}$$

is uniquely decomposed as $e_M \times W_M$, with $e_M \in \mathbb{K}_\beta[[U]][X]$ a unit and $W_M(U, X) := (X - \sigma_1(U))$ the corresponding Weierstrass polynomial, with $\sigma_1 \in \mathbb{K}_\beta[[U]]$,

$$\sigma_1(U) = \left(\frac{1}{\gamma_1}\right)U - \left(\frac{\gamma_2}{\gamma_1^3}\right)U^2 + \left(\frac{2\gamma_2^2 - \gamma_1\gamma_3}{\gamma_1^5}\right)U^3 + \ldots \tag{2.4}$$

The other X-roots $\sigma_2, \sigma_3, \ldots, \sigma_d \in \mathbb{C}[[U]]^*$ of F, as $e_M = (X - \sigma_2)(X - \sigma_3)(\ldots)(X - \sigma_d)$, are distinct, have the respective constant terms

$$c_{0,j} := \sigma_j(0) = \frac{1}{\beta^{(j-1)}} - \frac{1}{\beta} \qquad \text{for } 2 \leq j \leq d,$$

and are such that $\sigma_j(U) \in \mathbb{K}_\beta^{\mathcal{G}}[[U]]$, with coefficients in the algebra over $\mathbb{Q}$ generated by the derivatives of the polynomial $\widetilde{P^*_\beta}(X)$ at $c_{0,j}$, as

$$\sigma_j(U) = c_{0,j} + \frac{1}{\widetilde{P^*_\beta}'(c_{0,j})}U - \frac{\widetilde{P^*_\beta}''(c_{0,j})}{2\,(\widetilde{P^*_\beta}'(c_{0,j}))^3}U^2 + \ldots \tag{2.5}$$

§ 3. Main Theorems

Let us turn to the explicit computation of the Rényi-Parry germ of curve associated with β. let us consider ${}^t(0 \;\; p_{j,1} \;\; p_{j,2} \;\ldots\; p_{j,d-1})$, $j \geq 1$, the last column vector of the matrix $\gamma_d^j M_0^j$.

By convention we put: $p_{j,0} = 0$, for all $j \geq 1$. The polynomials $p_{j,i} \in \mathbb{Z}[\gamma_1, \gamma_2, \ldots, \gamma_d]$ are homogeneous, of degree j, and satisfy, for $i = 1, 2, \ldots, d-1$,

$$p_{1,i} = -\gamma_i,$$

$$p_{j+1,i} = -\gamma_i \, p_{j,d-1} + \gamma_d \, p_{j,i-1} \qquad j \geq 1.$$

Theorem 3.1. *The constant coefficients of the Rényi-Parry germ of curve given by* (2.1) *are:* $b_{0,0} = 0$ *and*

$$b_{j,0} = \lambda_{j-1} + \sum_{q \geq d} \lambda_q \frac{p_{q-d+1,j-1}}{\gamma_d^{q-d}}, \qquad \text{for all } 1 \leq j \leq d-1.$$

The other coefficients $b_{j,r}, j \geq 0, r \geq 1$, are deduced from the derivatives of the characteristic polynomial F and the formal series $\sigma_j(U)$.

The case (ii-3) of Newton polygon, in Theorem 2.1, corresponds to the geometrical steming of conjugated irreducible curves in a neighbourhood of the origin ($U = 0, z = 1/\beta$) parametrized by the conjugated Puiseux series involved in the decomposition of the germ of curve (2.1). When these curves cross the U-plane in $\mathbb{C}^2$ of equation $U = 0$, then their intersection with this plane is one point, for which the inverse is called a beta-conjugate of β. The conjugation over irreducibles curves transports onto the collection of the beta-conjugates of β. Another consequence of the decomposition of G_β as in Theorem 2.1 is the following.

Theorem 3.2. *If* $\beta > 1$ *is an algebraic number, which is not a Parry number, the analytic function* $f_\beta(z)$ *does not cancel at the (Galois-) conjugates of* $1/\beta$.

§ 4. Diophantine Approximation and a possible classification of nonParry numbers

Case (i) in Theorem 2.1 exactly corresponds to β being a Parry number, and this case can be expressed in terms of the constant coefficients $b_{j,0}$.

Theorem 4.1. *With the above-mentioned notations, the algebraic number* $\beta > 1$*, of degree* d *(assumed* ≥ 2*), is a Parry number if and only if:*

$$b_{j,0} = 0, \qquad \text{for all } 1 \leq j \leq d-1.$$

Since 0 is an algebraic number and that the constant coefficients $b_{j,0}$ are given by summations, it means that the number and asymptotic density of the "missing terms" in these summations, when equal to 0, is "weak", in some sense, by easy Liouville arguments. So that if all the coefficients $b_{j,0}$ are equal to 0, then the eventual periodicity of the sequence (t_i) is obtained: in this case, eventual periodicity is forced.

The other cases (ii-1), (ii-2), (ii-3) in Theorem 2.1 correspond to weaker arguments. It suggests to attribute the rational number

$$w_\beta := 1 - \frac{\delta}{d} \in [0, 1]$$

to the algebraic number β, where $\delta \in \{1, 2, \ldots, d-1\}$ is the degree of the Weierstrass polynomial associated with G_β (it is the greatest integer $\leq d-1$ such that $b_{\delta,0} \neq 0$ with $b_{m,0} = 0$ for $1 \leq m < \delta$). The rational integer w_β is close to 0 if G_β admits a Weierstrass polynomial of high degree, and close to 1 if β is at large departure off the set of Parry numbers, i.e. with degrees δ small, or equal to 1. By convention, say that $w_\beta = 0$ if and only if β is a Parry number. These conditions are not strong enough to force the eventual periodicity ot the sequence of digits (t_i), but certainly a possible ordering/correlation of (t_i).

The topology of the set $\{w_\beta \mid \beta > 1$ nonParry algebraic number $\}$ probably merits attention, in particular the subset of it formed when β runs over a neighbourhood of unity. The rational number w_β may be used as a classifying parameter defined on the set of the real algebraic numbers $\beta > 1$. Let u/v be a rational number in $[0, 1)$. Interestingly, the set $\Lambda_{u/v} := \{\beta > 1 \mid w_\beta = u/v\}$ would have to be characterized. It is just known that Λ_0 is dense in $(1, +\infty)$ [Pa].

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