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Generalized rounding radix systems and simultaneous radix systems

By

Péter Burcsi*

Abstract

In this short note we define generalized rounding radix systems which are a common generalization for shift radix systems (SRS), symmetric SRS, ε-SRS and matrix numeration systems. We investigate basic properties of such systems. As an application, we propose a possible generalization of simultaneous number systems to continuous parameter spaces.

§ 1. Introduction

The introduction and further research [1, 2, 3, 4] of shift radix systems (SRS) unified the study of canonical number systems (CNS) and β-expansions, as explained in [1], thus revealing deep connections and raising new questions. One of the main benefits of SRS is the extension of possible parameters to real values instead of integer coefficient polynomials as in the case of CNS or β-expansions. With a slight modification, SRS are also capable of describing number systems with symmetric or shifted digits sets: these are symmetric SRS [5, 9] and ε-SRS [15], respectively.

Another possible generalization of CNS is an extension to lattices, first introduced in the context of self-replicating plane tilings, see e.g. [11]. These are called matrix numeration systems in [6], and some algorithmic aspects were discussed in [8, 12].

Simultaneous number systems were introduced in [10], they were investigated and generalized e.g. in [13, 16]. In all cases considered so far, parameter values have been integers. We propose a possible generalization for continuous parameter values, using the definition of GRRS. We consider this as the main section of this paper, as it seems possible that a better understanding of the dynamics of simultaneous systems can be obtained by investigating this continuous version.

The paper is built up as follows: we define GRRS in section 2, and explain how they generalize other systems – most are straightforward observations. In section 3 we explain how

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*Eötvös Loránd University, Budapest, Hungary
e-mail: `peter.burcsi@compalg.inf.elte.hu`

known decision algorithms can be applied to GRRS, mainly based on ideas from other numeration systems. In section 4, we propose a definition for simultaneous SRS, using GRRS. We present definitions and formulate some open problems about simultaneous systems. In the final section, we propose further questions and research directions about GRRS.

§ 2. Generalized rounding radix systems

In this section we define generalized rounding radix systems and consider how they relate to other concepts of numeration. First we define genaralized rounding functions – for a motivation recall that symmetric SRS and ε-SRS only differ from SRS in the way that rounding of non-integer values is performed. We denote by $\{x\}$ the fractional part $(x - \lfloor x \rfloor)$ of x. When $\mathbf{x}$ is a vector, then $\lfloor \mathbf{x} \rfloor$ and $\{\mathbf{x}\}$ are the vectors where we take the integer, resp. fractional parts coordinate-wise.

Definition 2.1. Let d be a positive integer. For any function $K : [0,1)^d \to \mathbb{Z}^d$, we define a generalized rounding function

$$\begin{aligned} &p_K : \mathbb{R}^d \to \mathbb{Z}^d \\ &p_K(\mathbf{x}) = \lfloor \mathbf{x} \rfloor + K(\{\mathbf{x}\}) \quad . \end{aligned}$$

We say that p_K is a bounded generalized rounding function if the image of K is bounded (or equiavalently, finite).

Example 2.2. If $d = 1$, then the floor function is obtained by taking $K(u) = 0$ for all $u \in [0,1)$. The ceiling function (rounding upwards) is obtained by taking $K(u) = 1$ for $u \neq 0$ and $K(0) = 0$. Rounding to the nearest integer is obtained by taking $K(u) = 0$ for $0 \leq u < 1/2$, $K(u) = 1$ for $1/2 \leq u < 1$.

Remark. Note that the rounding direction only depends on the fractional part, thus rounding towards or from zero is not captured by this concept. Also note that the rounding function is not defined coordinate-wise, it is thus possible that the rounded value of one coordinate is influenced by the fractional parts of other coordinates.

In the following, we use column vectors when we multiply by matrices.

Definition 2.3. Let d be a positive integer, $M \in \mathbb{R}^{d\times d}$, $K : [0,1)^d \to \mathbb{Z}^d$ such that $K(0) = 0$, and p_K the associated rounding function. We define a mapping $\tau_{M,K} : \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$\tau_{M,K}(a_1, a_2, \ldots, a_d) = p_K(M(a_1, a_2, \ldots, a_d))$$

We say that the map $\tau_{M,K}$ is a generalized rounding radix system (GRRS for short). If for every $\mathbf{a} \in \mathbb{Z}^d$, there exists $n \in \mathbb{N}$ with $\tau^n_{M,K}(\mathbf{a}) = 0$, then we say that the GRRS is a GRRS with finiteness property.

Remark. Note that it is possible for $K_1 \neq K_2$ that $\tau_{M,K_1} = \tau_{M,K_2}$. For this to be the case, it is necessary and sufficient for K_1 and K_2 to coincide on the image $I \subseteq [0,1)^d$ of $\mathbb{Z}^d$ by the map $\mathbf{a} \mapsto \{M\mathbf{a}\}$. Thus the values taken by K outside I are irrelevant for $\tau_{M,K}$. Note further that $K(0) = 0$ is equivalent to $\tau_{M,K}(0) = 0$.

The following observations are immediate from the definition.

Proposition 2.4. *Every SRS is a GRRS.*

Proof. Recall from [1] that for each $\mathbf{r} = (r_1, r_2, \ldots, r_d) \in \mathbb{R}^d$ we assign a map $\tau_{\mathbf{r}}(\mathbf{a}) = \tau_{\mathbf{r}}(a_1, a_2, \ldots, a_d) = (a_2, a_3, \ldots, a_d, -\lfloor \mathbf{r} \cdot \mathbf{a} \rfloor)$, where $\mathbf{r} \cdot \mathbf{a}$ denotes the scalar product. It is easy to check that $\tau_{\mathbf{r}} = \tau_{M,K}$, where M and K can be chosen as follows:

$$M(x_1, x_2, \ldots, x_d) = (x_2, x_3, \ldots, x_d, -(r_1 x_1 + r_2 x_2 + \cdots + r_d x_d)) \tag{2.1}$$

$$K(u_1, u_2, \ldots, u_{d-1}, u_d) = (0, 0, \ldots, 0, \lceil u_d \rceil) \quad . \tag{2.2}$$

Thus $\tau_{\mathbf{r}}$ forms an SRS if and only if $\tau_{M,K}$ forms a GRRS, and the claim follows. □

Proposition 2.5. *Every ε-SRS is a GRRS.*

Proof. As defined in [15], now $\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, a_3, \ldots, a_d, -\lfloor \mathbf{r} \cdot \mathbf{a} + \varepsilon \rfloor)$. The proof is essentially identical to the SRS case, but with $K(u_1, u_2, \ldots, u_d) = (0, 0, \ldots, \lceil u_d - \varepsilon \rceil)$. □

Corollary 2.6. *Every symmetric SRS is a GRRS.*

Proof. As defined in [5], now $\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, a_3, \ldots, a_d, -\lfloor \mathbf{r} \cdot \mathbf{a} + 1/2 \rfloor)$. This is thus a special case of ε-SRS with $\varepsilon = 1/2$. □

We now turn our attention to number systems with special digit sets. They have previously been defined for number fields, but we use the following alternative equivalent form of the general definition used in [12, 8]. They are also special cases of digits systems of [16].

Definition 2.7. Let $L \in \mathbb{Z}^{d \times d}$ be a matrix with $n = |detL| \geq 2$. Let $D = \{\mathbf{D}_1 = 0, \mathbf{D}_2, \ldots, \mathbf{D}_n\} \subseteq \mathbb{Z}^d$ a complete set of representatives of $\mathbb{Z}^d / L\mathbb{Z}^d$. We define $\varphi_{L,D}$ by letting

$$\varphi_{L,D}(\mathbf{a}) = L^{-1}(\mathbf{a} - \mathbf{D}_i) \quad ,$$

where $\mathbf{D}_i = \mathbf{D}_i(\mathbf{a})$ is the unique element of D such that $\varphi_{L,D}(\mathbf{a}) \in \mathbb{Z}^d$. We call the arising numeration system a matrix numeration system.

Proposition 2.8. *Every matrix numeration system is a GRRS.*

Proof. Define $\mathbf{u}(\mathbf{a}) = \{L^{-1}\mathbf{a}\} \in [0,1)^d$. The proof relies on the fact that $\mathbf{D}_i(\mathbf{a})$ is determined by $\mathbf{u}(\mathbf{a})$. On each coset of $\mathbb{Z}^d / L\mathbb{Z}^d$, the mapping $\mathbf{a} \mapsto \mathbf{u}(\mathbf{a})$ is constant, and there is exactly one element of D in this coset, thus it is sufficient to define K on each $\mathbf{u}(\mathbf{D}_i)$. Define $M = L^{-1}$ and $K(\mathbf{u}(\mathbf{D}_i)) = -M\mathbf{D_i}$. This gives $\varphi_{L,D} = \tau_{M,K}$. □

Remark. Note that SRS, ε-SRS and matrix numeration systems all give rise to GRRS with bounded rounding functions.

§ 3. Decision algorithms for the contractive bounded case

First we show a result that explains why in the subsequent results we require boundedness of the rounding function. Whenever d is clear from the context, M denotes a d by d real matrix and K a function from $[0,1)^d$ to $\mathbb{Z}^d$ with $K(0)=0$, and $\tau_{M,K}$ is the associated function as in definition 2.3.

The following lemma is trivial.

Lemma 3.1. *Let $M \in \mathbb{Z}^{d\times d}$ be a diagonal matrix with irrational entries in the diagonal. Then the map $\mathbf{u}(\mathbf{a}) = \{M\mathbf{a}\}$ is injective on $\mathbb{Z}^d$.*

Theorem 3.2. *For any function $f : \mathbb{Z}^d \to \mathbb{Z}^d$ with $f(0) = 0$, there exists M and K such that $f = \tau_{M,K}$.*

Proof. Let M be an arbitrary diagonal matrix of irrational elements. Define $K(\{M\mathbf{a}\}) = f(\mathbf{a}) - \lfloor M\mathbf{a} \rfloor$. By the lemma, K is well defined on the range of $\mathbf{a} \mapsto \{M\mathbf{a}\}$. Extend K arbitrarily to $[0,1)^d$. Clearly $f(\mathbf{a}) = \tau_{M,K}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{Z}^d$. □

In what follows, K will always be a bounded rounding function. The following two theorems are stated without proof. The proofs are essentially identical to the case of SRS, found in [1].

Theorem 3.3. *If K is a bounded rounding function and $\tau_{M,K}$ forms a GRRS, then the spectral radius of M is less than or equal to 1.*

Theorem 3.4. *If K is a bounded rounding function and the spectral radius of M is less than 1 (i.e. M is contractive), then the orbit of all $\mathbf{a} \in \mathbb{Z}^d$ under $\tau_{M,K}$ is eventually periodic. Furthermore, an explicit upper bound (depending on M and K) can be given for the coordinates of a periodic element.*

The latter theorem can be applied for a contractive M to decide if the GRRS property holds.

Remark. When the spectral radius of M is exactly 1, the dynamical properties depend on K. If, for example if M is the rotation by $\pi/3$ on the plane, then an analogue of lemma 3.1 applies, so we can define K to obtain arbitrary dynamics. Even if we restrict the range of K to $\{(0,0),(0,1),(1,0),(1,1)\}$, we can obtain different behavior: GRRS, periodic orbits for all $\mathbf{a} \in \mathbb{Z}^2$, or infinite orbits for some $\mathbf{a}$.

For the contractive case, another algorithm, the construction of the set of witnesses, is also applicable for the decision of the GRRS property. This algorithm was invented independently in [7] and [14]. For the SRS version, see [1], and for the matrix numeration system version, [8]. .

Theorem 3.5. *Let M be contractive, K bounded and $\tau = \tau_{M,K}$. Let $V_0 \in \mathbb{Z}^d$ be the set consisting of* 0 *and all vectors of the form* $(0, 0, \dots, 0, \pm 1, 0, \dots, 0)$. *Define $T(\mathbf{a})$ to be the set* $\{\tau(\mathbf{a}+\mathbf{v}) - \tau(\mathbf{a}) \mid \mathbf{a} \in \mathbb{Z}^d\}$. *Let $V_{j+1} = V_j \cup \{T(\mathbf{v}) \mid \mathbf{v} \in V_j\}$ for $j > 0$. Then the sequence $V_0, V_1, \dots$ stabilizes with a value V, and we have a GRRS if and only if the only period of τ in V is the fixed point 0.*

Proof. The algorithm terminates because M is contractive and K is bounded. First note that by construction, for any $\mathbf{a} \in \mathbb{Z}^d$ and $\mathbf{v} \in V$, there exists $\mathbf{v}' \in V$ such that $\tau(\mathbf{a}+\mathbf{v}) = \tau(\mathbf{a})+\mathbf{v}'$. Using double induction, it follows that for all positive integers n, p, and all $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$,

$$\tau^n(\mathbf{a} + \sum_{i=1}^{p} \mathbf{v_i}) = \tau^n(\mathbf{a}) + \sum_{i=1}^{p} \mathbf{v}'_i$$

for some $\mathbf{v}'_1, \dots, \mathbf{v}'_p \in V$. That means that if a lattice point can be written as the sum of $p+1$ elements of V, then some iterate of it is the sum of p elements of V. Repeating the argument p more times, we reach an iterate that equals 0.

For detailed computation, see the proofs found in the cited papers, which are easily modified to the present situation. □

Remark. Note that the calculation of the set $T(\mathbf{v})$ is not trivial, but it can be described in the case of SRS or matrix numeration systems. For the general case and for practical purposes, the exact calculation is not neceesary, any set $T'(\mathbf{v}) \supseteq T(\mathbf{v})$ is applicable if the diameter of T' is uniformly bounded in $\mathbf{v}$. If K is bounded, we have

$$\begin{aligned}
&\tau(\mathbf{a}+\mathbf{v}) - \tau(\mathbf{a}) = p_K(M(\mathbf{a}+\mathbf{v})) - p_K(M\mathbf{a}) = \\
&M(\mathbf{a}+\mathbf{v}) - \{M(\mathbf{a}+\mathbf{v})\} + K(\{M(\mathbf{a}+\mathbf{v})\}) - (M\mathbf{a} - \{M\mathbf{a}\} + K(\{M\mathbf{a}\})) = \\
&M\mathbf{v} - \{M(\mathbf{a}+\mathbf{v})\} + K(\{M(\mathbf{a}+\mathbf{v})\}) + \{M\mathbf{a}\} - K(\{M\mathbf{a}\})) = \\
&\tau(\mathbf{v}) + \{M\mathbf{v}\} - K(\{M\mathbf{v}\}) - \{M(\mathbf{a}+\mathbf{v})\} + K(\{M(\mathbf{a}+\mathbf{v})\}) + \{M\mathbf{a}\} - K(\{M\mathbf{a}\})) \in \\
&\in \tau(\mathbf{v}) + (\operatorname{rng}(K) - \operatorname{rng}(K) - \operatorname{rng}(K) + \{0,1\}^d)
\end{aligned}$$

where the last sum is the Minkovski sum of the summands. Intuitively, $\tau(\mathbf{a}+\mathbf{v}) - \tau(\mathbf{a})$ is near $\tau(\mathbf{v})$, where the error comes from rounding.

§ 4. Simultaneous systems

In this section we propose a definition for simultaneous SRS.

Simultaneous CNS were defined in [10]. We recall the definition, which we use in a broader sense for the purposes of the present paper.

Definition 4.1. Let $N_1, N_2 \in \mathbb{N}$ relatively prime, $2 \le N_1 < N_2$, $D = \{0, 1, \dots, N_1N_2 - 1\}$. Then the triple $(-N_1, -N_2, D)$ is a simultaneous number system with finiteness property if for all $z_1, z_2 \in \mathbb{Z}$, there exists n and $b_0, b_1, \dots, b_n \in D$ such that

$$z_1 = \sum_{j=0}^{n} b_j(-N_1)^j \quad , \quad z_2 = \sum_{j=0}^{n} b_j(-N_2)^j \quad ,$$

It is shown in [10] that N_1 and N_2 give a simultaneous number system with finiteness property if and only if $N_2 = N_1 + 1$. If we reformulate the definition of simultaneous systems from a dynamical perspective, we can define $\varphi_{-N_1,-N_2}(z_1, z_2) = ((z_1 - b)/(-N_1), (z_2 - b)/(-N_2))$ for the unique $b \in D$ that gives an integer result. The finiteness property means that φ eventually takes every point to 0. We might still be interested in the dynamics of $\varphi_{-N_1,-N_2}$ even if this finiteness property fails. In the present context we will use the simultaneous number system to mean any $\varphi_{-N_1,-N_2}$.

Remark. Note that a matrix numeration system constructed from a diagonal matrix with entries $-N_1$ and $-N_2$, and "diagonal" digits $\{(0,0), (1,1), \ldots, (N_1N_2 - 1, N_1N_2 - 1)\}$ can also be used to define the dynamics of a simultaneous system.

Example 4.2. Let $N_1 = 3$, $N_2 = 4$ and let $z_1 = 38$, $z_2 = 27$. Then the orbit of (z_1, z_2) at φ is $(38, 27) \xrightarrow{11} (-9, -4) \xrightarrow{0} (3, 1) \xrightarrow{9} (2, 2) \xrightarrow{2} (0, 0)$ (we put the appropriate values of b on the arrows). Equivalently

$$38 = 11 + 0 \cdot (-3) + 9 \cdot (-3)^2 + 2 \cdot (-3)^3$$
$$27 = 11 + 0 \cdot (-4) + 9 \cdot (-4)^2 + 2 \cdot (-4)^3$$

It is tempting to generalize the situation to arbitrary *real* N_1, N_2 by requiring that for any $z_1, z_2 \in \mathbb{Z}$, there should exist $b \in \mathbb{R}$ such that $((z_1 - b)/(-N_1), (z_2 - b)/(-N_2)) \in \mathbb{Z}^2$. Unfortunately, as it is easily verified, whenever at least one of N_1 or N_2 is irrational, this is impossible (in the integer case we can use the Chinese Remainder to obtain such a b). Instead, we proceed by noting that (classical) simultaneous number systems are GRRS. By the definitions we have the following proposition.

Proposition 4.3. *Let $N_1, N_2 \leq 2$ integers, $D = \{0, 1, \ldots, N_1N_2 - 1\}$. The dynamics of the simultaneous number system $(-N_1, -N_2, D)$ is obtained as a GRRS, specifically $\varphi_{-N_1,-N_2} = \tau_{M,K}$ where*

$$M = \begin{pmatrix} -1/N_1 & 0 \\ 0 & -1/(N_2 \end{pmatrix}$$
$$K(\{-j/N_1\}, \{-j/(N_2)\}) = -(\lceil -j/N_1 \rfloor, \lceil -j/(N_2) \rceil),\ j = 0, 1, \ldots, N_1N_2 - 1$$

and K is defined arbitrarily elsewhere.

In order to obtain nicer formulas, we will work with an alternative equivalent formulation of the rounding function, using $\overline{K} : [0, 1)^d \to \mathbb{Z}^d$ defined by

$$\tau(\mathbf{a}) = \lfloor M\mathbf{a} \rfloor + K(\{M\mathbf{a}\}) = -\lfloor -M\mathbf{a} \rfloor + \overline{K}(\{-M\mathbf{a}\})$$

In figure 1, $\overline{K}$ is extended in such a way that it is constant on stripes whose borders have slope N_1/N_2. This is not the only possible extension, but it is relatively easy to generalize to non-integer N_1 and N_2.

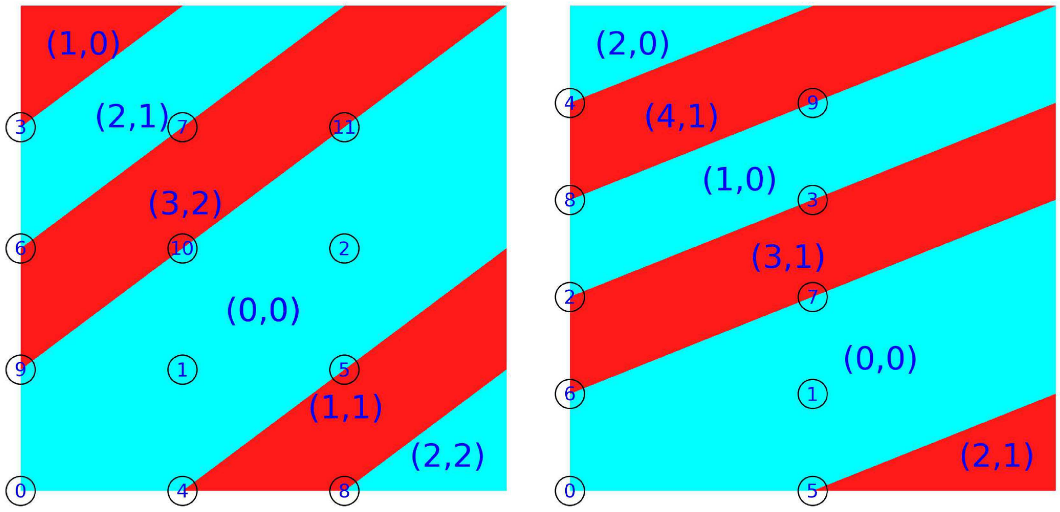

Figure 1. This figure illustrates a possible definition of $\overline{K}$ for which $\tau_{M,K}$ is the GRRS associated with the simultaneous number system with bases $N_1 = -3$, $N_2 = -4$ (left), and $N_1 = -2$, $N_2 = -5$ (right).

Definition 4.4. Let $N_1, N_2 > 1$. Let $V = \{\{kN_1/N_2\} \mid k = 1, \dots, \lceil N_2 \rceil - 1\}$ and $H = \{\{kN_2/N_1\} \mid k = 1, \dots, \lceil N_1 \rceil - 1\}$. Suppose that $(V \cup H) \cap \mathbb{Z} = \emptyset$. The generalized simultaneous number system associated to $(-N_1, -N_2)$ is defined through the following function $\overline{K}$. Draw segments in the unit square with slope N_1/N_2 that intersect the left side of the unit square at the elements of V, and segments that intersect the bottom side of the square at the elements of H (see figure 1). These segments divide the square into stripes. We call the open stripe containing the origin the central stripe, $\overline{K}(x, y) = 0$ on the central stripe. For points above the central stripe, we define $\overline{K}(x, y) = (k, \lfloor kN_1/N_2 \rfloor)$ if (x, y) is inside or on the lower boundary of the stripe bounded below by the segment starting from $(0, \{kN_1/N_2\})$. For points below the central stripe, we define $\overline{K}(x, y) = (\lfloor kN_2/N_1 \rfloor, k)$ if (x, y) is inside or on the upper boundary of the stripe bounded above by the segment starting from $(\{kN_2/N_1\}, 0)$.

Remark. The condition that the values kN_2/N_1 and kN_1/N_2 are never integers ensures that the stripes are non-degenerate, and degenerate cases contain the case when N_1 and N_2 are integers, but not coprime.

We formulate a few conjectures about generalized simultaneous systems. All of them are supported by empirical evidence.

Conjecture 4.5. *Let $N_1 > 1$. The generalized simultaneous number system associated to $(-N_1, -N_1 - 1)$ has the finiteness property for every N_1. When N_1 is an integer, this is known to hold, see [10].*

Conjecture 4.6. *Let $r > 1$ and $0 < s < r$. Let $N_1 = N_1(k) = kr - s$ and $N_2 = N_2(k) = (k + 1)r - s$ for $k = 1, 2, \dots$ Then the structure of periodic elements (the number and lengths of periods) stabilizes as $k \to \infty$.*

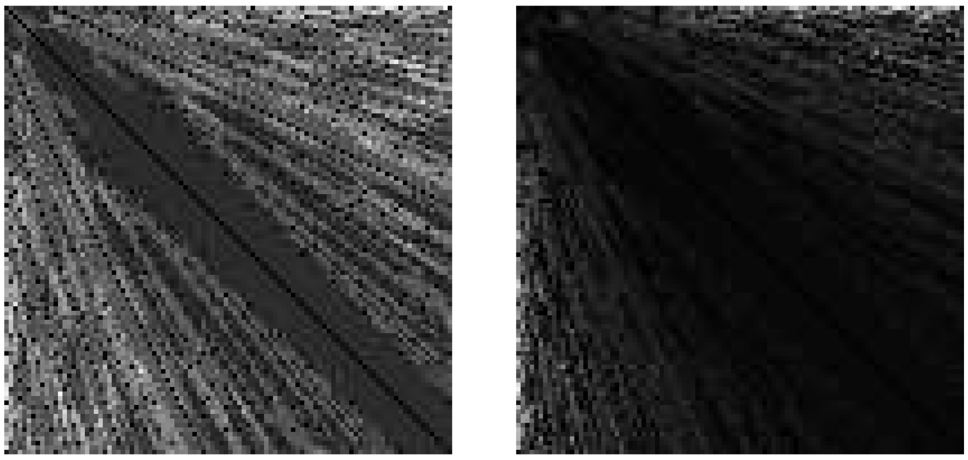

Figure 2. The two figures illustrate the number of periods and the number of periodic elements in some simultaneous systems. The base pair components range from 1.1 to 11 by steps of 0.1 on both axes. In the first figure, the number of periods ranges from 1 to 6 (from dark to light), and in the second one, from 1 to 34. Bases where we do not define systems (e.g. not coprime integer pairs) are the darkest black.

As a special case, we have a conjecture about the period structure for bases of the form $(-N_1, -N_1 - 2)$. Empirical results indicate the following.

Conjecture 4.7. *If k is large enough, then the simultaneous number system associated to $(2k-1, 2k+1)$ has one-non zero fixed point $(k+1, k)$ and no other non-zero periodic elements.*

A graphical interpretation of the period structure for some values of (N_1, N_2) is shown in figure 2.

§ 5. Summary

The investigation of GRRS and simultaneous radix systems is subject of our current research. Apart from the conjectures stated above, we propose the following research directions.

The first natural question to investigate is how simultaneous SRS behave with these definition in higher dimensions. It is observed in [13] that Gaussian integers with canonical digit sets fail to yield simultaneous number systems. Nagy therefore proposed alternative digit sets. Following these ideas, an important question is how one should define rounding in 4 dimensional space in order to obtain interesting 2 dimensional simultaneous systems.

Another direction would be to initially only fix M, and look for appropriate rounding, i.e. a function K that yield a GRRS. It would also be interesting to know how omitting the requirement $K(0) = 0$ in the definition of GRRS would change the dynamics.

Finally, matrix numeration systems have the property that the iterates of a system are also matrix numeration systems. SRS and GRRS do not share this property. Is this inherent in the nature of SRS, or could it be possible to define a generalization that is "closed under taking iterates"?

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