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Solutions to Discrete Soliton Equations

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Abstract

We discuss several aspects of discrete bilinear equations of the following two types.

1. Type A equation

$$\begin{aligned}&[z_1\exp(D_1)+z_2\exp(D_2)+z_3\exp(D_3)]\tau\cdot\tau=0,\\&z_1+z_2+z_3=0.\end{aligned}$$

2. Type B equation

$$\begin{aligned}&[z_1\exp(D_1)+z_2\exp(D_2)+z_3\exp(D_3)+z_0\exp(D_0)]\tau\cdot\tau=0,\\&z_1+z_2+z_3+z_0=0,\quad D_1+D_2+D_3+D_0=0.\end{aligned}$$

§ 1. Preliminaries

§ 1.1. Parameter Transformation

We have Type A equation

$$[z_1e^{D_1}+z_2e^{D_2}+z_3e^{D_3}]\tau\cdot\tau=0, \tag{1}$$

under the condition $z_1+z_2+z_3=0$.
Equation (1) is transformed into a bilinear form of the discrete KP equation type,

$$[a(b-c)e^{D_1}+b(c-a)e^{D_2}+c(a-b)e^{D_3}]\tau\cdot\tau=0, \tag{2}$$

through the transformation of the parameters,

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$$z_1 = \frac{1}{b} - \frac{1}{c}, \quad z_2 = \frac{1}{c} - \frac{1}{a}, \quad z_3 = \frac{1}{a} - \frac{1}{b},$$

and the discrete KP type equation is trasformed into Type A equation through the same transformation of the parameters.

On the other hand the discrete BKP equation by Miwa

$$\begin{aligned}
&(a+b)(a+c)(b-a)\tau(l+1,m,n)\tau(l,m+1,n+1)\\
&+(b+c)(b+a)(c-a)\tau(l,m+1,n)\tau(l+1,m,n+1)\\
&+(c+a)(c+b)(a-b)\tau(l,m,n+1)\tau(l+1,m+1,n)\\
&+(a-b)(b-c)(c-a)\tau(l,m,n)\tau(l+1,m+1,n+1) = 0,
\end{aligned} \tag{3}$$

can not be transformed, through any parameter transformation, into Type B equation

$$[z_1 e^{D_1} + z_2 e^{D_2} + z_3 e^{D_3} + z_0 e^{D_0}]\tau \cdot \tau = 0, \tag{4}$$

$$z_1 + z_2 + z_3 + z_0 = 0 \quad \text{and} \quad D_1 + D_2 + D_3 + D_0 = 0. \tag{5}$$

However Typ B equation is easily transformed into the discrete BKP equation through the parameter transformation,

$$z_1 = (a+b)(a+c)(b-a), \quad z_2 = (b+c)(b+a)(c-a), \tag{6}$$

$$z_3 = (c+a)(c+b)(a-b), \quad z_0 = (a-b)(b-c)(c-a) \tag{7}$$

and the coordinates transformation, which will be discussed in the next subsection.

§ 1.2. Coordinates transformation I

We define the bilinear operators $\exp\delta(D_j), j = 1, 2, \cdots$, operating on an ordered pair of f and g,

$$\begin{aligned}
&\exp\delta(D_j) f(k_1, k_2, \cdots, k_j, \cdots) \cdot g(k_1, k_2, \cdots, k_j, \cdots)\\
&= f(k_1, k_2, \cdots, k_j + 1, \cdots) \cdot g(k_1, k_2, \cdots, k_j - 1, \cdots), \quad \text{for } j = 1, 2, \cdots.
\end{aligned} \tag{8}$$

where δ being a parameter.

Equation (8) is reduced, in the small limit of δ to,

$$D_j f \cdot g = \frac{\partial f}{\partial k_j} g - f \frac{\partial g}{\partial k_j}, \quad j = 1, 2, \cdots. \tag{9}$$

We consider a coordinates transformation among discrete variables, $\{l_1, l_2, l_3, \ldots\}$ and $\{k_1, k_2, k_3, \ldots\}$,

$$\begin{aligned}
l_1 &= a_1 k_1 + b_1 k_2 + c_1 k_3,\\
l_2 &= a_2 k_1 + b_2 k_2 + c_2 k_3,\\
l_3 &= a_3 k_1 + b_3 k_2 + c_3 k_3,
\end{aligned}$$

where a_i, b_i, c_i for $i = 1, 2, 3$ are constant.

The coordinates transformation gives

$$\frac{\partial}{\partial k_1} = \sum_{j=1}^{3} a_j \frac{\partial}{\partial l_j},$$
$$\frac{\partial}{\partial k_2} = \sum_{j=1}^{3} b_j \frac{\partial}{\partial l_j},$$
$$\frac{\partial}{\partial k_3} = \sum_{j=1}^{3} c_j \frac{\partial}{\partial l_j}.$$

Accordingly we find the bilinear operators are transformed into the following form

$$D_1 \equiv D_{k_1} = a_1 D_{l_1} + a_2 D_{l_2} + a_3 D_{l_3},$$
$$D_2 \equiv D_{k_2} = b_1 D_{l_1} + b_2 D_{l_2} + b_3 D_{l_3},$$
$$D_3 \equiv D_{k_3} = c_1 D_{l_1} + c_2 D_{l_2} + c_3 D_{l_3}.$$

Hence Eq.(1),

$$[a(b-c)\exp(D_1) + b(c-a)\exp(D_2) + c(a-b)\exp(D_3)]\tau \cdot \tau = 0. \tag{10}$$

is transformed into the discrete KP equation,

$$\begin{aligned} &[a(b-c)\exp[\frac{1}{2}(D_l - D_m - D_n)] + b(c-a)\exp[\frac{1}{2}(-D_l + D_m - D_n)] \\ &+c(a-b)\exp[\frac{1}{2}(-D_l - D_m + D_n)]\tau \cdot \tau = 0, \end{aligned} \tag{11}$$

where we put $D_{l_1} = D_l, D_{l_2} = D_m, D_{l_3} = D_n$ and $a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_3 = -\frac{1}{2}, b_1 = -\frac{1}{2}, b_2 = \frac{1}{2}, b_3 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}, c_3 = \frac{1}{2}$.

Equation(11) is expressed without using the bilinear operators by

$$\begin{aligned} &a(b-c)\tau(l+1,m,n)\tau(l,m+1,n+1) + \\ &b(c-a)\tau(l,m+1,n)\tau(l+1,m,n+1) + \\ &c(a-b)\tau(l,m,n+1)\tau(l+1,m+1,n) = 0, \end{aligned} \tag{12}$$

which is the discrete KP equation.

§ 1.3. Gauge transformation I.

The following bilinear equation

$$[z_1 e^{D_1} + z_2 e^{D_2} + z_3 e^{D_3} + z_4 e^{D_4}]\tau(k_1,k_2,k_3,k_4) \cdot \tau(k_1,k_2,k_3,k_4) = 0, \tag{13}$$

$$z_1 + z_2 + z_3 + z_4 = 0, \tag{14}$$

is invariant under the gauge transformation,

$$\tau \to \tau e^{c_1k_1+c_2k_2+c_3k_3+c_4k_4}, \qquad c_1, c_2, c_3, c_4 \text{ being constant.} \tag{15}$$

It can be shown as follows.

$$\begin{aligned}
&e^{D_j}\tau e^{c_1k_1+c_2k_2+c_3k_3+c_4k_4}\cdot \tau e^{c_1k_1+c_2k_2+c_3k_3+c_4k_4}\\
&=[e^{D_j}\tau\cdot\tau]e^{D_j}e^{c_1k_1+c_2k_2+c_3k_3+c_4k_4}\cdot e^{c_1k_1+c_2k_2+c_3k_3+c_4k_4}\\
&=[e^{D_j}\tau\cdot\tau]e^{2(c_1k_1+c_2k_2+c_3k_3+c_4k_4)}.
\end{aligned} \tag{16}$$

A gauge-invariant bilinear equation exhibits 2-soliton solution of the following form,

$$\tau(k_1, k_2, k_3) = 1 + e^{\eta_1} + e^{\eta_2} + a(1,2)e^{\eta_1+\eta_2}, \tag{17}$$

where

$$\eta_j = p_1(j)k_1 + p_2(j)k_2 + p_3(j)k_3 + p_4(j)k_4, \quad j = 1, 2 \tag{18}$$

and $p_s(j)(s = 1, 2.3, 4)$ are constant and related to wave numbers of jth-soliton and $a(1,2)$ is a phase shift induced after colliding with each other.

§ 1.4. Gauge transformation II.

We consider a nonautonomous bilinear equation

$$\begin{aligned}
&[z_1(k_1)e^{D_1} + z_2(k_2)e^{D_2} + z_3(k_3)e^{D_3} + z_4(k_4)e^{D_4}]\\
&\qquad \tau(k_1, k_2, k_3, k_4)\cdot\tau(k_1, k_2, k_3, k_4) = 0.
\end{aligned} \tag{19}$$

We apply an extended gauge transformation to Eq.(19)

$$\tau \to \tau e^{\phi_1(k_1)}. \tag{20}$$

Using one of the properties of the bilinear operators,

$$\begin{aligned}
&e^{D_1}\tau e^{\phi_1(k_1)}\cdot\tau e^{\phi_1(k_1)}\\
&=[e^{D_1}\tau\cdot\tau][e^{D_1}e^{\phi_1(k_1)}\cdot e^{\phi_1(k_1)}]\\
&=[e^{D_1}\tau\cdot\tau]e^{\phi_1(k_1+1)+\phi_1(k_1-1)}
\end{aligned} \tag{21}$$

Eq.(19) is transformed into the bilinear equation

$$\begin{aligned}
&z_1(k_1)[e^{D_1}\tau\cdot\tau]e^{\phi_1(k_1+1)-2\phi_1(k_1)+\phi_1(k_1-1)}\\
&+z_2(k_2)e^{D_2}\tau\cdot\tau + z_3(k_3)e^{D_3}\tau\cdot\tau + z_4(k_4)e^{D_4}\tau\cdot\tau = 0.
\end{aligned} \tag{22}$$

Accordingly Eq.(19) is transformed into

$$[c_1 e^{D_1} + z_2(k_2)e^{D_2} + z_3(k_3)e^{D_3} + z_4(k_4)e^{D_4}]\tau \cdot \tau = 0, \tag{23}$$

where the gauge-function $\phi_1(k_1)$ is determined by the second order linear difference equation [1],

$$\phi_1(k_1+1) - 2\phi_1(k_1) + \phi_1(k_1-1) = h_1(k_1), \tag{24}$$

$$h_1(k_1) \equiv [\log c_1]/z_1(k_1), \quad c_1 \text{ being an arbitrary constant.} \tag{25}$$

Similarly we may tranform Eq.(19) into

$$[c_1 e^{D_1} + c_2 e^{D_2} + c_3 e^{D_3} + c_4 e^{D_4}]\tau \cdot \tau = 0, \tag{26}$$

introducing the gauge-functions $\phi_2(k_2), \phi_3(k_3)$ and $\phi_4(k_4)$. The nonautonomous bilinear equation (19) is transformed, through the extended gauge transformation (20), into the nonautonomous bilinear equation (26).
But we have to pay for it. The boundary condition on a usual soliton $u(k_1, k_2, \cdots)$ is

$$u(k_1, k_2, \cdots)|_{boundary} = \text{const.(independent of } k_1, k_2, \cdots). \tag{27}$$

After the extended gauge transformation, $u(k_1, k_2, \cdots)|_{boundary}$ is not a constant any more but it depends on $\phi_1(k_1), \phi_2(k_2), \cdots$.
We shall show in the next section that the integrability of a discrete system depends on the boundary condition using the Toda equation as an example.

§ 1.5. Coordinates transformation II. From discrete to continuous

We have the KP hierarchy,

$$\begin{aligned}
&(D_1^4 - D_1 D_3 + 3D_2^3)\tau \cdot \tau = 0, \\
&[(D_1^3 + 2D_3)D_2 - 2D_1 D_4]\tau \cdot \tau = 0, \\
&\cdots
\end{aligned} \tag{28}$$

The two-soliton solution τ_2 of the KP hierarchy is written as

$$\tau_2(x_1, x_2, \cdots) = 1 + c_1 e^{\eta_1} + c_2 e^{\eta_2} + a(1,2)c_1 c_2 e^{\eta_1+\eta_2}, \tag{29}$$

$$a(1,2) = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}, \tag{30}$$

$$e^{\eta_j} = e^{(p_j - q_j)x_1 + (p_j^2 - q_j^2)x_2 + (p_j^3 - q_j^3)x_3 + \cdots} \quad \text{for} \quad j = 1, 2. \tag{31}$$

On the other hand we have the discrete KP hierarchy

$$
\begin{aligned}
&a_1(a_2-a_3)\tau(k_1+1,k_2,k_3)\tau(k_1,k_2+1,k_3+1)+\\
&a_1(a_2-a_3)\tau(k_1+1,k_2,k_3)\tau(k_1,k_2+1,k_3+1)+\\
&a_1(a_2-a_3)\tau(k_1+1,k_2,k_3)\tau(k_1,k_2+1,k_3+1)=0,\\
&\\
&z_1\tau(k_1+1,k_2,k_3,k_4)\tau(k_1,k_2+1,k_3+1,k_4+1)+\\
&z_2\tau(k_1,k_2+1,k_3,k_4)\tau(k_1+1,k_2,k_3+1,k_4+1)+\\
&z_3\tau(k_1,k_2,k_3+1,k_4)\tau(k_1+1,k_2+1,k_3,k_4+1)+\\
&z_4\tau(k_1,k_2,k_3,k_4+1)\tau(k_1+1,k_2+1,k_3+1,k_4)=0,\\
&\cdots.
\end{aligned}
\tag{32}
$$

where $a_1, a_2, a_3, \cdots$ are the intervals of the discrete variables $k_1, k_2, k_3, \cdots$ so that $x_1 = k_1a_1, x_2 = k_2a_2, x_3 = k_3a_3, \cdots$, and

$$z_1 = z_1(a_2, a_3, a_4), \quad z_2 = z_2(a_1, a_3, a_4), \tag{33}$$

$$z_3 = z_1(a_1, a_2, a_4), \quad z_4 = z_1(a_1, a_2, a_3). \tag{34}$$

The two-soliton solution τ_2 of the discrete KP hierarchy is written as

$$\tau_2(k_1, k_2, \cdots) = 1 + c_1e^{\eta_1} + c_2e^{\eta_2} + a(1,2)c_1c_2e^{\eta_1+\eta_2}, \tag{35}$$

$$a(1,2) = \frac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)}, \tag{36}$$

$$
\begin{aligned}
e^{\eta_j} &= (\frac{1-p_ja_1}{1-q_ja_1})^{-k_1}(\frac{1-p_ja_2}{1-q_ja_2})^{-k_2}(\frac{1-p_ja_3}{1-q_ja_3})^{-k_3}\cdots,\\
&\text{for} \quad j = 1, 2.
\end{aligned}
\tag{37}
$$

The coordinates transformation between the continuous variables $x_1, x_2, \cdots$ and the discrete variables $k_1, k_2, \cdots$ is based on the fact that

$$\tau(x_1, x_2, \cdots) = \tau(k_1, k_2, \cdots). \tag{38}$$

The identity (38) implies that η_j in Eq.(31) is equal to η_j in Eq.(37),namely

$$\sum_{\nu=1}^{\infty}(p_j^\nu - q_j^\nu)x_\nu = \sum_{n=1}^{\infty} -k_n[\log(1-p_ja_n) - \log(1-q_ja_n)]. \tag{39}$$

The *r.h.s* of Eq.(39) is expanded, using Taylor expansion formula,

$$-\log(1-x) = \sum_{\nu=1}^{\infty}\frac{1}{\nu}x^\nu, \tag{40}$$

in power series in a_n

$$r.h.s = \sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{k_n}{\nu}(p_j^\nu - q_j^\nu)a_n^\nu. \tag{41}$$

Comparing it with $l.h.s$, we obtain

$$x_\nu = \frac{1}{\nu}\sum_{n=1}^{\infty}k_n a_n^\nu, \tag{42}$$

by which the discrete KP hierarchy (32) is transformed into the continuous KP hierarchy (28).

§ 2. Integrability of a discrete system depends on the boundary condition

The Toda equation is known to be equivalent to the nonlinear LC circuit equation, which is expressed in the following forms.
(i) Continuous-time Toda equation ,

$$\frac{d}{dt}\log(V_n) = I_{n-1} - I_n, \quad \frac{d}{dt}I_n = V_n - V_{n+1}. \tag{43}$$

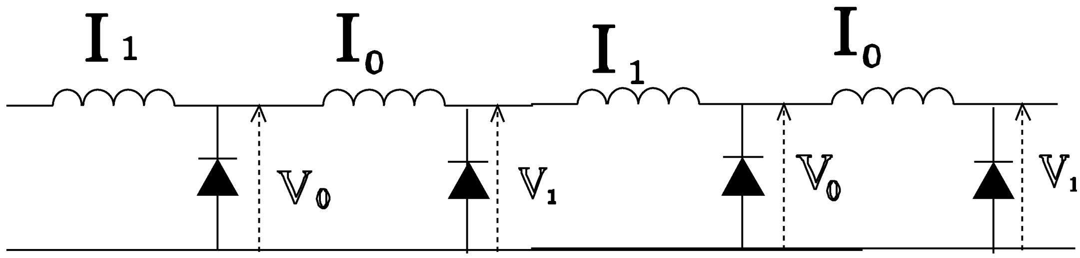
(ii) Discrete-time Toda eqaution,

$$\frac{V_n^{m+1}}{V_n^m} = \frac{W_{n-1}^m}{W_n^m}, \quad \frac{W_n^{m+1}}{W_n^m} = \frac{1+\hat{\delta}^2 V_n^{m+1}}{1+\hat{\delta}^2 V_{n+1}^{m+1}}, \tag{44}$$

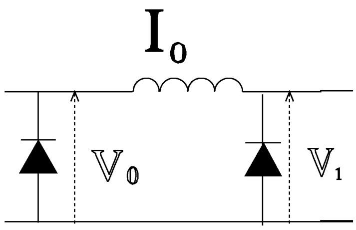
where $I_n^m = \log W_n^m$.

We show that the integrability of these equations depends on the boundary condition. We have three types of boundary condition on the Toda equation.

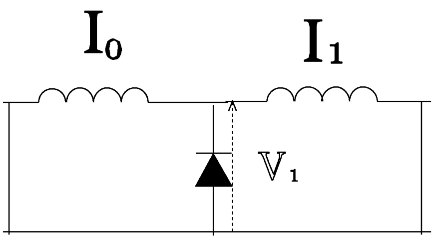
1. Periodic boundary conditions;
   $I_{n+2} = I_n, \quad V_{n+2} = V_n, \quad$ Period 2.



2. Π-type circuit: $I_{n-1} = I_{n+1} = 0$.



3. T-type circuit: $V_{n-1} = V_{n+1} = 0$.



The continuous equation (43) is expressed under these boundary conditions as follows.

1. For the boundary conditon of Period 2,we have

$$\frac{d}{dt}\log V_0 = I_1 - I_0, \quad \frac{d}{dt}I_0 = V_0 - V_1,$$
$$\frac{d}{dt}\log V_1 = I_0 - I_1, \quad \frac{d}{dt}I_1 = V_1 - V_0,$$

which exhibit three conserved quantities;

$$H_0 = I_0 + I_1,$$
$$H_1 = V_0 V_1,$$
$$H_2 = V_0 + V_1 + \frac{1}{2}(I_0^2 + I_1^2).$$

Hence the equation is integrable.

2. For the boundary condition of Π-type,we have

$$\frac{d}{dt}\log V_0 = -I_0,$$
$$\frac{d}{dt}I_0 = V_0 - V_1,$$
$$\frac{d}{dt}\log V_1 = I_0,$$

which exhibit two conserved quantities;

$$H_1 = V_0 V_1,$$
$$H_2 = V_0 + V_1 + \frac{1}{2}(I_0^2 + I_1^2).$$

Hence the equation is integrable.

3. For the boundary condition of T-type, we have

$$\frac{d}{dt}I_0 = -V_1,$$
$$\frac{d}{dt}\log V_1 = I_0 - I_1,$$
$$\frac{d}{dt}I_1 = V_1,$$

which exhibit two conserved quantities;

$$H_0 = I_0 + I_1,$$
$$H_2 = V_0 + \frac{1}{2}(I_0^2 + I_1^2).$$

Hence the equation is integrable.

We find the continuous equation (43) is integrable for all cases.
Next we consider the discrete equation (44), which is expressed under the boundary conditions as follows.

1. For the boundary conditon of Period 2,we have

$$\begin{cases} \frac{V_0^{m+1}}{V_0^m} = \frac{W_1^m}{W_0^m}, & \frac{W_0^{m+1}}{W_0^m} = \frac{1+\hat{\delta}^2 V_0^{m+1}}{1+\hat{\delta}^2 V_1^{m+1}} \\ \frac{V_1^{m+1}}{V_1^m} = \frac{W_0^m}{W_1^m}, & \frac{W_1^{m+1}}{W_1^m} = \frac{1+\hat{\delta}^2 V_1^{m+1}}{1+\hat{\delta}^2 V_0^{m+1}} \end{cases}. \tag{45}$$

We find two conserved quantities immediately:

$$H_0 = W_0^m W_1^m, \quad H_1 = V_0^m V_1^m.$$

Let $W_0^m = (H_0)^{(1/2)} y_m, \quad \hat{\delta}^2 V_0^m = x_m, \quad \hat{\delta}^4 H_1 = c$. Then Eq.(45) is reduced, with the help of the conserved quantities, to

$$\begin{cases} \frac{x_{m+1}}{x_m} = \frac{1}{y_m^2} \\ \frac{y_{m+1}}{y_m} = \frac{1+x_{m+1}}{1+c/x_{m+1}} \end{cases}, \tag{46}$$

whose conserved quantity H_2 is found to be

$$H_2 = 2x_m + 2\frac{1}{x_m} + y_m^2 + \frac{1}{y_m^2} + 2\frac{x_m}{y_m^2} + 2c\frac{x_m^2}{y_m} + \frac{x_m^2}{y_m^2} + c^2\frac{x_m^2}{y_m^2}. \tag{47}$$

which shows the discrete equation (46) is integrable.

2. For the boundary condition of Π-type, $(W_{n+1}^m = W_{n-1}^m = 1)$, we have

$$\begin{cases} \frac{V_0^{m+1}}{V_0^m} = \frac{1}{W_0^m} \\ \frac{W_0^{m+1}}{W_0^m} = \frac{1+\hat{\delta}^2 V_0^{m+1}}{1+\hat{\delta}^2 V_1^{m+1}} \\ \frac{V_1^{m+1}}{V_1^m} = W_0^m \end{cases}. \tag{48}$$

We find one conserved quantity immediately:

$$H_1 = V_0^m V_1^m.$$

Let $W_0^m = y_m, \quad \hat{\delta}^2 V_0^m = x_m, \quad \hat{\delta}^4 H_1 = c.$
Then Eq.(48) is reduced, with the help of the conserved quantity, to

$$\begin{cases} \frac{x_{m+1}}{x_m} = \frac{1}{y_m} \\ \frac{y_{m+1}}{y_m} = \frac{1+x_{m+1}}{1+c/x_{m+1}} \end{cases}, \tag{49}$$

whose conserved quantity H_2 is found to be

$$H_2 = \frac{x_m}{y_m} + x_m + c(\frac{y_m}{x_m} + \frac{1}{x_m}) + \frac{1}{y_m} + y_m. \tag{50}$$

Hence the discrete equation (49) is integrable.

3. For the boundary condition of T-type ($V_{n-1} = V_{n+1} = 1$), we have

$$\begin{cases} \frac{W_{-1}^{m+1}}{W_{-1}^m} = \frac{1+\hat{\delta}^2}{1+\hat{\delta}^2 V_0^{m+1}} \\ \frac{V_0^{m+1}}{V_0^m} = \frac{W_{-1}^m}{W_0^m} \\ \frac{W_0^{m+1}}{W_0^m} = \frac{1+\hat{\delta}^2 V_0^{m+1}}{1+\hat{\delta}^2} \end{cases}. \tag{51}$$

Equation (51),is reduced, using the conserved quantity, $H_0 = W_{-1}^m W_0^m$ to,

$$\begin{cases} \frac{V_0^{m+1}}{V_0^m} = \frac{H_0}{(W_0^m)^2} \\ \frac{W_0^{m+1}}{W_0^m} = \frac{1+\hat{\delta}^2 V_0^{m+1}}{1+\hat{\delta}^2} \end{cases}. \tag{52}$$

Let

$$\frac{\hat{\delta}^2}{1+\hat{\delta}^2} V_0^m = x_m, \quad W_0^m = (H_0)^{(1/2)} y_m, \quad \frac{1}{1+\hat{\delta}^2} = c.$$

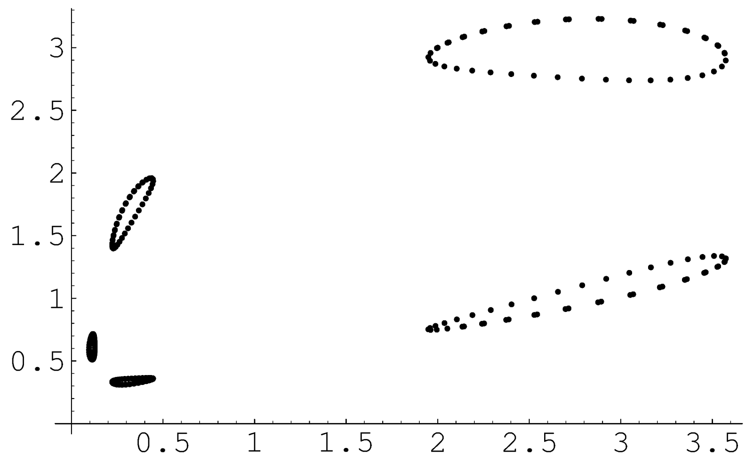
Then Eq.(52) becomes

$$\begin{cases} \frac{x_{m+1}}{x_m} = \frac{1}{y_m^2} \\ \frac{y_{m+1}}{y_m} = c + x_{m+1} \end{cases}. \tag{53}$$

I have concluded that Eq.(53) is not integrable by the following reasons.

(a) Numerical calculations of the algebraic entropy of Eq.(53) indicate that it approaches to $\log 2$, which implies nonintegrability of the system.

(b) A numerical mapping of Eq.(53) for $c = 1/4$ starting with $x_0 = 2$ and $y_0 = 3$ is given below



which indicates non-integrability of Eq.(53).

§ 3. Bilinear Form of the Non-autonomous Discrete KdV Equation

Matsuura[2] has succeeded in obtaining the non-autonomous discrete KdV equation

$$
\begin{aligned}
&(\frac{1}{b_{m+1}} - \frac{1}{c_{n+1}})\frac{1}{u_{n+1}^{m+1}} - (\frac{1}{b_m} - \frac{1}{c_n})\frac{1}{u_n^m} \\
&= (\frac{1}{b_{m+1}} + \frac{1}{c_n})u_n^{m+1} - (\frac{1}{b_m} + \frac{1}{c_{n+1}})u_{n+1}^m,
\end{aligned}
\tag{54}
$$

using "Discrete differential geometry".
We have the non-autonomous discrete KP equation [3],[4],

$$
\begin{aligned}
&a_l(b_m - c_n)\tau(l+1, m, n)\tau(l, m+1, n+1) + \\
&b_m(c_n - a_l)\tau(l, m+1, n)\tau(l+1, m, n+1) + \\
&c_n(a_l - b_m)\tau(l, m, n+1)\tau(l+1, m+1, n) = 0,
\end{aligned}
\tag{55}
$$

which exhibits N-soliton solution.
It is known that the discrete (autonomous) KP equation can be transformed into the discrete (autonomous) KdV equation by the reduction procedure so that solutions to the discrete KdV equation could be obtained without any difficulty.
However the usual reduction procedure can not be applied to the non-autonomous cases.
Kajiwara and Ohta [5] succeded in finding soliton solutions to the non-autonomous discrete KdV equation bypassing the bilinear form.

We shall show a new reduction procedure to find the bilinear form of the non-autonomous discrete KdV equation.

The procedure is based on a coordinates transformation II. Let a_l in Eq.(55) be a small parameter a (=const.). We write Eq.(55) as

$$
\begin{aligned}
& a(b_m - c_n)\tau(l+\frac{1}{2}, m, n)\cdot\tau(l-\frac{1}{2}, m+1, n+1) \\
& + b_m(c_n - a)\tau(l-\frac{1}{2}, m+1, n)\cdot\tau(l+\frac{1}{2}, m, n+1) \\
& + c_n(a - b_m))\tau(l-\frac{1}{2}, m, n+1)\cdot\tau(l+\frac{1}{2}, m+1, n) = 0. \qquad (56)
\end{aligned}
$$

We introduce a continuous variable x by the relation $l = ax$ so that

$$\frac{\partial}{\partial l} = a\frac{\partial}{\partial x} \quad \text{and } D_l = aD_x \qquad (57)$$

Then Eq.(56) is written as

$$
\begin{aligned}
& a(b_m - c_n)\exp[\frac{1}{2}aD_x]\tau(x, m, n)\cdot\tau(x, m+1, n+1) \\
& + b_m(c_n - a)\exp[-\frac{1}{2}aD_x]\tau(x, m+1, n)\cdot\tau(x, m, n+1) \\
& + c_n(a - b_m)\exp[-\frac{1}{2}aD_x]\tau(x, m, n+1)\cdot\tau(x, m+1, n) = 0. \qquad (58)
\end{aligned}
$$

Hereafter we put $\tau(x, m, n) = f_n^m$ for short. Expanding Eq.(58) in power series of a we obtain, from the coefficients of a and a^2, a coupled semi-discrete bilinear equations

$$(b_m - c_n)[f_n^m f_{n+1}^{m+1} - f_n^{m+1} f_{n+1}^m] - b_m c_n D_x f_n^{m+1}\cdot f_{n+1}^m = 0, \qquad (59)$$

$$(b_m - c_n)D_x f_n^m\cdot f_{n+1}^{m+1} + (b_m + c_n)D_x f_n^{m+1}\cdot f_{n+1}^m = 0, \qquad (60)$$

which are the bilinear form of the non-autonomous discrete KdV equation.

We transform the coupled bilinear equations into the non-autonomous discrete KdV equation of u_n^m.
We write Eq.(59) as

$$\frac{(b_m - c_n)}{b_m c_n}[f_n^m f_{n+1}^{m+1} - f_n^{m+1} f_{n+1}^m] = D_x f_n^{m+1}\cdot f_{n+1}^m. \qquad (61)$$

Substituting it into Eq.(60) we obtain

$$\frac{(b_m + c_n)}{b_m c_n}[f_n^m f_{n+1}^{m+1} - f_n^{m+1} f_{n+1}^m] = -D_x f_n^m\cdot f_{n+1}^{m+1}. \qquad (62)$$

We introduce a dependent variable u_n^m defined by

$$u_n^m = \frac{f_n^{m+1} f_{n+1}^m}{f_{n+1}^{m+1} f_n^m} \qquad (63)$$

and shift operators p and q operating on an arbitrary function h_n^m defined by

$$p\, h_n^m = h_n^{m+1} \quad \text{and} \quad q\, h_n^m = h_{n+1}^m. \tag{64}$$

Dividing Eqs.(61) and (62) by $f_n^{m+1} f_{n+1}^m$ and $f_n^m f_{n+1}^{m+1}$ respectively, we obtain

$$\frac{(b_m - c_n)}{b_m\, c_n}[\frac{1}{u_n^m} - 1] = \frac{\partial}{\partial x}[\log f_n^{m+1} - \log f_{n+1}^m] = (p-q)\frac{\partial}{\partial x}\log f_n^m$$

and

$$\frac{(b_m + c_n)}{b_m\, c_n}[1 - u_n^m] = -\frac{\partial}{\partial x}[\log f_n^m - \log f_{n+1}^{m+1}] = (pq-1)\frac{\partial}{\partial x}\log f_n^m.$$

Operating the shift operators on the above equations we eliminate the term $\frac{\partial}{\partial x}\log\, f_n^m$ and obtain a discrete equation of u_n^m

$$(p\, q - 1)(\frac{1}{b_m} - \frac{1}{c_n})(1 - \frac{1}{u_n^m}) = (p-q)(\frac{1}{b_m} + \frac{1}{c_n})(1 - u_n^m) \tag{65}$$

which is transformed into the non-autonomous discrete KdV equation,

$$\begin{aligned}
&(\frac{1}{b_{m+1}} - \frac{1}{c_{n+1}})\frac{1}{u_{n+1}^{m+1}} - (\frac{1}{b_m} - \frac{1}{c_n})\frac{1}{u_n^m} \\
&\qquad = (\frac{1}{b_{m+1}} + \frac{1}{c_n})u_n^{m+1} - (\frac{1}{b_m} + \frac{1}{c_{n+1}})u_{n+1}^m.
\end{aligned} \tag{66}$$

§ 4. Discrete Soliton Equations of Type B

We have so many papers discussing discrete soliton equations. However they are all concerned with Type A (discrete KP) equation except very few cases.

We have soliton equations of type B, which is expressed by

$$\begin{aligned}
&[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3) + z_0 \exp(D_0)]\tau \cdot \tau = 0, \\
&z_1 + z_2 + z_3 + z_0 = 0, \quad D_1 + D_2 + D_3 + D_0 = 0.
\end{aligned} \tag{67}$$

A special case ($z_0 = 0$) of type B equation is reduced to type A equation.
Type B equation is transformed into the discrete BKP equation by Miwa

$$\begin{aligned}
&(a+b)(a+c)(b-a)\tau(l+1,m,n)\tau(l,m+1,n+1) \\
&+(b+c)(b+a)(c-a)\tau(l,m+1,n)\tau(l+1,m,n+1) \\
&+(c+a)(c+b)(a-b)\tau(l,m,n+1)\tau(l+1,m+1,n) \\
&+(a-b)(b-c)(c-a)\tau(l,m,n)\tau(l+1,m+1,n+1) = 0.
\end{aligned} \tag{68}$$

Miwa obtained N-soliton solution [6] to it.
The structure of the higher order discrete BKP equations is clearified by an extended form of the addition formula for phaffian [7] which can also be applied to the non-autonomous BKP equations [8].

§ 5. Type B equation

We have several difficulties in solving Type B equation:

1. Type B equation can not be transformed into the discrete BKP equation by the parameter transformation.
2. Soliton solutions to Type B equation can not be expressed by determinants. So we can not use the determinant technique which is very useful in finding N-soliton solution to discrete KP equation.
3. It is not easy to prove N-soliton solution to Type B equation by the pfaffian technique up to now.

Only way to find soliton solutions to Type B equation is the perturbation method up to now.

§ 5.1. Find Multi-Soliton Solution to Type B Equation

We shall solve a Type B equation of the following form,

$$\begin{aligned}
&z_1 f(x_1+1,x_2,x_3)f(x_1,x_2+1,x_3+1)\\
&+z_2 f(x_1,x_2+1,x_3)f(x_1+1,x_2,x_3+1)\\
&+z_3 f(x_1,x_2,x_3+1)f(x_1+1,x_2+1,x_3)\\
&+z_0 f(x_1,x_2,x_3)f(x_1+1,x_2+1,x_3+1)=0,
\end{aligned} \tag{69}$$

where z_1, z_2, z_3, z_0 are arbtrary constant satisfying a condition $z_1+z_2+z_3+z_0=0$. We note that Eq.(69) is invariant under the gauge transformation,

$$f \to f\eta_0, \quad \eta_0 = c_0 c_1^{x_1} c_2^{x_2} c_3^{x_3},$$

where $c_j, j=0,1,2,3$ are arbitrary constants.

§ 5.2. One-soliton solution to Type B equation

We assume the following form of f for one-soliton solution to Eq.(69)

$$\begin{aligned}
&f = 1 + \epsilon s(j_1, x_1, x_2, x_3),\\
&s(j_1, x_1, x_2, x_3) = c(j_1)p_1(j_1)^{x_1}p_2(j_1)^{x_2}p_3(j_1)^{x_3},
\end{aligned} \tag{70}$$

for all j_1, where ϵ is a book-keeping apameter, $c(j_1)$ is an arbitrary parameter related to a position of j_1th-soliton and $p_1(j_1), p_2(j1), p_3(j_1)$ are parameters of j_1th-soliton related to the frequency and the wave numbers,namely

$$p_1(j_1) = \exp(\omega_1), \quad p_2(j_1) = \exp(q_1), \quad p_3(j_1) = \exp(k_1). \tag{71}$$

Substituting Eq.(70) into Eq.(69) gives the dispersion relation,

$$p_1(j_1) = -\frac{z_1p_2(j_1)p_3(j_1) + z_2p_2(j_1) + z_3p_3(j_1) + z_0}{z_1 + z_2p_3(j_1) + z_3p_2(j_1) + z_0p_2(j_1)p_3(j_1)}. \tag{72}$$

§ 5.3. Two-soliton solution to Type B equation

Two-soliton solution is given by assuming f_2 to be

$$\begin{aligned}
f_2 = 1 + \epsilon[s(j_1, x_1, x_2, x_3) + s(j_2, x_1, x_2, x_3)]&\\
+\epsilon^2a(j_1, j_2)s(j_1, x_1, x_2, x_3)s(j_2, x_1, x_2, x_3).&
\end{aligned} \tag{73}$$

Substituting f_2 into Eq.(69) we find the phase shift $a(j_1, j_2)$ is given by

$$a(j_1, j_2) = -A_n(j_1, j_2)/A_d(j_1, j_2), \tag{74}$$

$$\begin{aligned}
A_n(j_1, j_2) = &\, p_1(j_1)p_2(j_1)p_3(j_1)z_0 + p_1(j_1)p_2(j_1)p_3(j_2)z_3\\
&+p_1(j_1)p_2(j_2)p_3(j_1)z_2 + p_1(j_1)p_2(j_2)p_3(j_2)z_1\\
&+p_1(j_2)p_2(j_1)p_3(j_1)z_1 + p_1(j_2)p_2(j_1)p_3(j_2)z_2\\
&+p_1(j_2)p_2(j_2)p_3(j_1)z_3 + p_1(j_2)p_2(j_2)p_3(j_2)z_0,
\end{aligned} \tag{75}$$

$$\begin{aligned}
A_d(j_1, j_2) = &\, p_1(j_1)p_1(j_2)p_2(j_1)p_2(j_2)p_3(j_1)p_3(j_2)z_0\\
&+p_1(j_1)p_1(j_2)p_2(j_1)p_2(j_2)z_3\\
&+p_1(j_1)p_1(j_2)p_3(j_1)p_3(j_2)z_2\\
&+p_1(j_1)p_1(j_2)z_1 + p_2(j_1)p_2(j_2)p_3(j_1)p_3(j_2)z_1\\
&+p_2(j_1)p_2(j_2)z_2 + p_3(j_1)p_3(j_2)z_3 + z_0.
\end{aligned} \tag{76}$$

§ 5.4. Three-soliton solution

Three-soliton solution is given by assuming f_3 to be

$$\begin{aligned} f_3 &= 1+\epsilon[s(j_1,x_1,x_2,x_3)+s(j_2,x_1,x_2,x_3)+s(j_3,x_1,x_2,x_3)] \\ &+\epsilon^2[a(j_1,j_2)s(j_1,x_1,x_2,x_3)s(j_2,x_1,x_2,x_3) \\ &\quad +a(j_1,j_3)s(j_1,x_1,x_2,x_3)s(j_3,x_1,x_2,x_3) \\ &\quad +a(j_2,j_3)s(j_2,x_1,x_2,x_3)s(j_3,x_1,x_2,x_3)] \\ &+\epsilon^3 a(j_1,j_2,j_3)s(j_1,x_1,x_2,x_3)s(j_2,x_1,x_2,x_3)s(j_3,x_1,x_2,x_3). \end{aligned} \tag{77}$$

Substituting f_3 into Eq.(69) we find,in the order of ϵ^3, the phase shift $a(j_1,j_2,j_3)$ is given by a form

$$a(j_1,j_2,j_3) = -[a(j_1,j_2)b(j_3)+a(j_1,j_3)b(j_2)+a(j_2,j_3)b(j_1)] \tag{78}$$

where $b(j_s)$ for s=1,2,3 are rational polynomials of $p_1(j_1), p_2(j_3), \cdots$.
The three soliton solution f_3 solves Eq.(69) if the following condition holds

$$a(j_1,j_2,j_3) = a(j_1,j_2)a(j_1,j_3)a(j_2,j_3). \tag{79}$$

Note that the condition is a sufficient condition not a necessary condition [9]. We have Periodic Phase Soliton (PPS) equation whose 3-soliton solution does not satisfiy the condition but is an exact solution.
We note that the number of terms in A_n and A_d ($a(j_1,j_2) = -A_n/A_d)$) are

$$A_n = A_d = 124$$

so that the terms involved in the condition (79) are huge $> 100^3$.
We have proved that the condition holds using a computer algebra system REDUCE(Free CSL version,32bits).
Also we have checked it *numerically* that the condition (79) holds for 4-soliton solution,

$$a(j_1,j_2,j_3,j_4) = \prod_{1\le k<l\le 4} a(j_k,j_l).$$

These facts strongly support the conjecture that the integrability condition on N-soliton solution

$$a(j_1,j_2,\cdots,j_n) = \prod_{1\le k<l\le n} a(j_k,j_l)$$

holds for $n = 1,2,3,\cdots N$.
We have not succeeded in proving it until now.
Lately we have revealed that type B equation play an important role in soliton theory.

1. A extended discrete-time Toda equation

$$\cosh(\frac{1}{2}\alpha D_m)[\sinh^2(\frac{1}{2}D_m) - \delta^2 \sinh^2(\frac{1}{2}D_n)]f \cdot f = 0. \tag{80}$$

For $\alpha = 1$ it is a discrete-time Toda equation of type B [10], whose N-soliton solution is not obtained yet. We have obtained 3-soliton solution.
For $\alpha = 2$ it is *not integrable* but in the ultradiscrete limit, Eq.(80) for $\alpha = 2$ is transformed into an *integrable equation* by virtue of the *convexity of the ultradiscrete τ-functions.*

2. An extended Box and Ball system (BBs) equations [11],

$$\begin{aligned}[(1+\delta_2)\exp(D_m + \frac{1}{2}D_n) - (1-\delta_1)\exp(\frac{1}{2}D_n) \\ -\delta_1 \exp(D_m - \frac{1}{2}D_n) - \delta_2 \exp[(\alpha + \frac{1}{2})D_n]f \cdot f = 0,\end{aligned} \tag{81}$$

which is reduced, for $\delta_2 = 0$, to discrete KdV equation being a discrete form of Box and Ball system. Eq. (81) in general describes an interaction between solitons of KdV type and those of Toda type.

3. Periodic Phase Solitons(PPS).
A discrete bilinear equation,

$$f_n^{m+1} f_{n+1}^m - f_{n+1}^{m+1} f_n^m = \delta(f_{n+M+1}^{m+1} f_{n-M}^m - f_{n+M}^{m+1} f_{n-M+1}^m), \tag{82}$$

is introduced, where m and n are discrete time and space respectively, M and δ being a natural number and a time interval, respectively.
Equation (82) is invariant under the following gauge transformation,

$$f_n^m \to f_n^m \ \phi(n),$$

where $\phi(n)$ is a periodic phase function of n, $\phi(n+M) = \phi(n)$.

Hirota, Ohta and Nagai[9] have shown that the integrability condition $a(1,2,3) = a(1,2)a(1,3)a(2,3)$ of the phase shifts of solitons *does not hold* for the PPS solution because of a long range interaction of soliton due to the periodic phase function $\phi(n)$. Nevertheless Eq.(82) exhibits exact 3-soliton solutions.

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