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A simple expression for discrete Painlevé equations

By

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Abstract

A simple expression of discrete Painlevé equations and their Lax pair is obtained by using an interpolation problem. We discuss mainly the case of $q$-Painlevé equation of type $E_8^{(1)}$.

§ 1. Structure of discrete Painlevé equations

The second order discrete Painlevé equations were classified by Sakai [13] as follows:

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<th>elliptic $E_8^{(1)}$</th>
<th>multiplicative $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow A_{2+1}^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)}$</th>
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<td>additive $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_4^{(1)} \rightarrow A_3^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)}$</td>
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Each discrete Painlevé equation, represented as a rational map on $\mathbb{P}^1 \times \mathbb{P}^1$, 

\[
T : (f, g) \mapsto (\overline{f}, \overline{g}) = \left( \frac{\psi_1(f, g)}{\psi_0(f, g)}, \frac{\phi_1(f, g)}{\phi_0(f, g)} \right),
\]

has eight singular points where $\psi_0 = \psi_1 = 0$ or $\phi_0 = \phi_1 = 0$. Conversely, a configuration of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ (or nine points on $\mathbb{P}^2$) characterize the Painlevé equation. Let us look at some examples.

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Example 1.1. \( q-D_5^{(1)}[3] \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{a_3a_4b_1b_2}{a_1a_2b_3b_4} \).

\[ \begin{align*}
\overline{f} & = \frac{(\overline{g} - b_1t)(\overline{g} - b_2t)}{(\overline{g} - b_3)(\overline{g} - b_4)}a_3a_4, \\
\overline{g} & = \frac{(f - a_1t)(f - a_2t)}{(f - a_3)(f - a_4)}b_3b_4.
\end{align*} \]

(1.2)

The 8 singular points are on the four lines \( f = 0, f = \infty, g = 0 \) and \( g = \infty \):

![Diagram of four lines intersecting at 8 points]

Example 1.2. \( q-E_6^{(1)}[12][14][11] \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{b_5b_6b_7b_8}{b_1b_2b_3b_4} \).

\[ \begin{align*}
\frac{f\overline{f}}{(fg-1)(\overline{fg}-1)} & = \frac{qt^2(b_1g-1)(b_2g-1)(b_3g-1)(b_4g-1)}{b_5b_6(b_7gt-1)(b_8gt-1)}, \\
\frac{g\overline{g}}{(fg-1)(\overline{fg}-1)} & = \frac{q^2t^2(b_1-\overline{f})(b_2-\overline{f})(b_3-\overline{f})(b_4-\overline{f})}{(b_5-\overline{f}qt)(b_6-\overline{f}qt)}.
\end{align*} \]

(1.3)

The 8 singular points are on the two lines \( f = 0, g = 0 \) and one curve \( fg = 1 \):

![Diagram of two lines and curve intersecting at 8 points]

Example 1.3. \( q-E_7^{(1)}[1][11] \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{b_5b_6b_7b_8}{b_1b_2b_3b_4} \).

\[ \begin{align*}
\frac{(fg-1)(\overline{fg}-1)}{(fgt^2-1)(\overline{fgt^2}-1)} & = \frac{(b_1g-1)(b_2g-1)(b_3g-1)(b_4g-1)}{(b_5gt-1)(b_6gt-1)(b_7gt-1)(b_8gt-1)}, \\
\frac{(fgt^2-1)(\overline{fgt^2}-1)}{(fgq^2t^2-1)} & = \frac{(b_1-\overline{f})(b_2-\overline{f})(b_3-\overline{f})(b_4-\overline{f})q}{(b_5-\overline{f}qt)(b_6-\overline{f}qt)(b_7-\overline{f}qt)(b_8-\overline{f}qt)}.
\end{align*} \]

(1.4)

The 8 singular points are on the two curves \( fg = 1 \) and \( fgt^2 = 1 \):
Example 1.4. $q$-$E_8^{(1)}$ [11]: $T(k, \ell, f, g) = \left(\frac{k}{q}, q\ell, \overline{f}, \overline{g}\right), \quad q = \frac{k^2\ell^2}{u_1 \cdots u_8}$.

\begin{equation}
\frac{(\overline{f} - g)(f - g) - (\frac{k}{q} - \ell)(k - \ell)\frac{1}{k}}{(\frac{k}{q} - \frac{f}{\ell})(\frac{k}{q} - \frac{g}{\ell}) - (\frac{q}{k} - \frac{1}{\ell})(\frac{1}{k} - \frac{1}{\ell})\ell} = \frac{k^2}{q} \frac{A(\ell, g)}{B(\ell, g)},
\end{equation}

\begin{equation}
\frac{(\overline{f} - g)(\overline{f} - g) - (\frac{k}{q} - \ell q)(\frac{k}{q} - \ell)\frac{q}{k}}{(\frac{k}{q} - \frac{f}{\ell q})(\frac{k}{q} - \frac{g}{\ell q}) - (\frac{q}{k} - \frac{1}{\ell q})(\frac{1}{k} - \frac{1}{\ell q})\frac{k}{q}} = q\ell^2 \frac{A(\frac{k}{q}, \overline{f})}{B(\frac{k}{q}, \overline{f})},
\end{equation}

where $A(h, x), B(h, x)$ are polynomials in $x$ of degree 4 given by

\begin{equation}
A(h, x) = \left(\frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8\right) + \left(\frac{m_1}{h^2} - m_5 + 2hm_7\right)x + \left(-\frac{m_0}{h^3} + m_6 - 3hm_8\right)x^2 - m_7x^3 + m_8x^4,
\end{equation}

\begin{equation}
B(h, x) = \left(\frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8\right) + \left(\frac{2m_1}{h^2} - \frac{m_3}{h} + hm_7\right)x + \left(-\frac{3m_0}{h^3} + \frac{m_2}{h^2} - hm_8\right)x^2 - \frac{m_1}{h^3}x^3 + \frac{m_0}{h^4}x^4,
\end{equation}

\begin{equation}
U(z) = \frac{1}{z^4} \prod_{i=1}^{8} (z - u_i) = \frac{1}{z^4} \sum_{i=0}^{8} (-1)^i m_i z^i.
\end{equation}

Though it is not so obvious in this form, the singularity of this equation are given by $(f, g) = (F(u_i), G(u_i)), \quad (i = 1, \cdots, 8)$ where

\begin{equation}
F(u) = u + \frac{k}{u}, \quad G(u) = u + \frac{\ell}{u}.
\end{equation}

These points are on the curve of bi-degree $(2, 2)$

\begin{equation}
(f - g)(\frac{f}{k} - \frac{g}{\ell}) - (k - \ell)(\frac{1}{k} - \frac{1}{\ell}) = 0,
\end{equation}

which has a node at $(\infty, \infty)$. 

In the next section, we will rewrite the $q$-$E_8^{(1)}$ equation in simpler form where the singularity structure is manifest (see Theorem 2.2).

**Example 1.5.** The most generic equation, the elliptic-$E_8^{(1)}$ [13][11], is more complicated than $q$-$E_8^{(1)}$ case. The 8 points are on a smooth bi-degree $(2, 2)$ curve (i.e. an elliptic curve):

There have been many challenges to obtain an explicit expression of the elliptic $E_8^{(1)}$ equation (e.g.[4][7][8]). We will give one simple expression (Theorem 3.1) which was obtained in [10] by a similar method as the $q$-case discussed below.

§ 2. An approach from the Padé interpolation

There exists a simple method to derive a Lax pair for Painlevé equations[15]. Using a discrete version of it, we will derive a simple form of $q$-Painlevé equation of type $E_8^{(1)}$. Here, we use parameters $a_1, a_2, a_3, b_1, b_2, b_3, k, \ell \in \mathbb{C}$ and $m, n \in \mathbb{Z}_{\geq 0}$ with constraints

\begin{equation}
q^{-1}k\ell = q^{-n}a_1a_2a_3 = q^{-m}b_1b_2b_3.
\end{equation}

The time evolution is $\bar{k} = \frac{k}{q}$, $\bar{\ell} = q\ell$ (and $\bar{x} = x$ for $x = a_i, b_i, m, n$). The parameters $u_i$ in Example 1.4 are related to the parameters $a_i, b_i, k, \ell, m, n$ by $(u_1, \cdots, u_8) = (a_1, a_2, a_3, q^{-m-n}, b_1, b_2, b_3, q)$. 

Our starting point is the following Padé interpolation

\[ Y_s = \frac{q^{ns}(\frac{k}{a_1}, \frac{k}{a_2}, \frac{k}{a_3}, b_1, b_2, b_3)_s}{q^{ms}(a_1, a_2, a_3, \frac{k}{b_1}, \frac{k}{b_2}, \frac{k}{b_3})_s} = \frac{P_m}{Q_n}, \quad x = x_s, \quad (s = 0, 1, \ldots, m+n), \]

(2.2)

\[(a)_i = \prod_{j=0}^{i-1} (1 - aq^j), \quad (a, \cdots, b)_s = (a)_{s} \cdots (b)_{s},\]

where \(P_m, Q_n\) are polynomials of degree \(m, n\) in variable \(x\). In the followings, we use variable \(z\) such that \(x = z + \frac{k}{qz}\), hence \(P_m, Q_n\) are Laurent polynomials of the form

\[ P_m(z) = \sum_{i=0}^{m} u_i (z + \frac{k}{qz})^i, \quad Q_n(z) = \sum_{i=0}^{n} v_i (z + \frac{k}{qz})^i. \]

The interpolating points are \(x_s = q^{-s} + kq^{s-1}\) \((q\text{-quadratic grid})\) in variable \(x\), and hence \(z_s = q^{-s}\) \((q\text{-grid})\) in variable \(z\).

The main ingredients are the contiguous relations satisfied by \(u(z) = P_m(z)\) and \(v(z) = Y(z)Q_n(z)\) where \(Y(z)\) is a function such as \(Y(q^{-s}) = Y_s\). For instance, the relation between \(y(z), y(\frac{z}{q}), \overline{y}(\frac{z}{q})\) is obtained by evaluating the Casorati determinant

\[ |u(z) v(\frac{z}{q}) \overline{v}(\frac{z}{q}) y(z) y(\frac{z}{q}) \overline{y}(\frac{z}{q})| = 0. \]

This determinant divided by \(Y(z)\) is a Laurent polynomial in \(z\) and has many known zeros due to the interpolating condition \(u(z_s) = v(z_s)\). Hence one can determine the structure of the contiguous relations without knowing the explicit form of \(P_m\) and \(Q_n\).

**Proposition 2.1.** The following relations hold for \(y(z) = P_m(z), Y(z)Q_n(z)\):

\[ L_2(z) : \quad B_2(z) \left\{ g - G(\frac{k}{z}) \right\} y(z) - B_2(\frac{k}{z}) \left\{ g - G(z) \right\} \overline{y}(\frac{z}{q}) = 0, \]

(2.5)

\[ L_3(z) : \quad B_1(z) \left\{ g - G(\frac{k}{qz}) \right\} \overline{y}(\frac{z}{q}) - B_1(\frac{k}{qz}) \left\{ g - G(z) \right\} y(z) \]

(2.6)

\[ + \frac{w}{c} \left\{ f - \overline{F}(z) \right\} (z - \frac{k}{qz}) y(z) = 0, \]

where

\[ B_1(z) = \frac{1}{z^2} \prod_{i=1}^{4} (z - u_i), \quad B_2(z) = \frac{1}{z^2} \prod_{i=5}^{8} (z - u_i), \]

(2.7)

\[ F(z) = z + \frac{k}{z}, \quad G(z) = z + \frac{\ell}{z}, \]
and $f, g, c, w$ are some constants (independent of $z$).

Combining $L_2$ and $L_3$, one can obtain the three term relation $L_1$ between $y(qz), y(z), y(z/q)$.

Though the explicit form of the $L_1$ equation is complicated, it can be characterized by the following properties [16]: (1) As a polynomial in $(f, g)$, it is of bi-degree $(3,2)$. (2) It vanishes when $f = F(u)$, $g = G(u)$ with $u = u_1, \cdots, u_8, qz, \frac{k}{z},$ and $f = F(u)$, $\frac{(g - G(\frac{k}{qz})y(u)}{(g - G(u))y(\frac{u}{q})} = \frac{B_2(\frac{k}{u})}{B_2(u)}$ with $u = z, qz$. Hence, the $L_1$ equation is equivalent with the $L_1$ equation in [17] up to some gauge transformations, and the equations $L_2, L_3$ (or $L_1$) can be considered as a Lax pair for $q$-$E_8^{(1)}$.

**Theorem 2.2.** The compatibility of the equations $L_2, L_3$ (2.5)(2.6) is equivalent to the relations

(2.8) \[ \frac{\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}{\{f - F(\frac{k}{z})\}\{\overline{f} - \overline{F}(\frac{k}{z})\}} = \frac{U(z)}{U(\frac{k}{z})}, \quad \text{for} \quad g = G(z), \]

(2.9) \[ \frac{\{g - G(z)\}\{\overline{g} - \overline{G}(z)\}}{\{g - G(\frac{k}{qz})\}\{\overline{g} - \overline{G}(\frac{k}{qz})\}} = \frac{U(z)}{U(\frac{k}{qz})}, \quad \text{for} \quad \overline{f} = \overline{F}(z), \]

along with an additional relation

(2.10) \[ w = \frac{(k - \ell)(k - q\ell)U(z)}{k^2\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}, \quad \text{for} \quad g = G(z), \]

where $U(z) = B_1(z)B_2(z) = \frac{1}{z^4} \prod_{i=1}^{8} (z - u_i)$.

**Proof.** Putting $g = G(z)$ in equations $L_2(z)$ and $L_3(z)$, we have the relation (2.10). Since $G(z) = G(\frac{k}{z})$, the relation (2.10) holds also when $z$ is replaced by $\frac{k}{z}$. Taking the ratio of these two relations, we obtain the equation (2.8). Putting $\overline{f} = \overline{F}(z)$ in equations $L_2(z)$ and $L_3(z)$, we get the equation (2.9). Sufficiency of the equations (2.8) (2.9) (2.10) for the compatibility can be checked by a direct computation. \qed
The equations (2.8)(2.9) are the desired simple expression for the $q$-$E_{8}^{(1)}$. In fact, by eliminating the variable $z$, they correctly reproduce the equations (1.5)(1.6), where the polynomials $A, B$ are given by

\begin{equation}
A(h, z + \frac{h}{z}) = \frac{zU(z) - \frac{h}{z}U(\frac{h}{z})}{z - \frac{h}{z}}, \quad B(h, z + \frac{h}{z}) = \frac{zU(\frac{h}{z}) - \frac{h}{z}U(z)}{z - \frac{h}{z}}.
\end{equation}

**Remark.** Up to now, we used only the defining relation $Y_{s} = \frac{P_{m}(x_{s})}{Q_{n}(x_{s})}$ for $P_{m}(x)$ and $Q_{n}(x)$. If we know the explicit forms of $P_{m}(x), Q_{n}(x)$, then we can determine the Painlevé variables $f, g$ explicitly. For the interpolation problem with general $Y_{s}$ and $x_{s}$, the following formula has been classically known by Cauchy and Jacobi

\begin{equation}
P_{m}(x) = f(x) \det \left( W_{i,j}^{(-)} \right)_{i,j=0}^{n}, \quad Q_{n}(x) = \det \left( W_{i,j}^{(+)n-1} \right)_{i,j=0}^{n-1},
\end{equation}

where

\begin{equation}
W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{Y_{s}}{f'(x_{s})} x_{s}^{i+j}(x-x_{s})^{\pm 1}, \quad f(x) = \prod_{s=0}^{m+n} (x-x_{s}).
\end{equation}

Applying this for the $q$-quadratic grid: $x = z + \frac{k}{qz}, x_{s} = q^{-s} + kq^{s-1}$, we have

\begin{equation}
W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{(1-kq^{2s-1})}{(1-kq-1)} \frac{(k/q,q^{-m-n})_{s}}{(q,kq^{m+n})_{s}} Y_{s} q^{(n-m)s} x_{s}^{i+j}(x-x_{s})^{\pm 1}.
\end{equation}

These expressions give the special solutions for $q$-$E_{8}^{(1)}$ Painlevé equation in terms of the $10W_{9}$ hypergeometric functions and their determinants (c.f.[5][6]).

**Remark.** The Padé approach to the degenerate cases were studied in [2][9]. The corresponding Padé problems are

\begin{align*}
| & q-P \quad \text{q-$E_{7}^{(1)}$} \quad \text{q-$E_{6}^{(1)}$} \quad \text{q-$D_{5}^{(1)}$} \quad \text{q-$A_{4}^{(1)}$} \quad \text{q-$A_{2+1}^{(1)}$} \\
Y_{s} & \begin{array}{c}
(b_{1}, b_{2}, b_{3})_{s} \\
(a_{1}, a_{2}, a_{3})_{s}
\end{array} \quad \begin{array}{c}
(b_{1}, b_{2})_{s} \\
(a_{1}, a_{2})_{s}
\end{array} \quad \begin{array}{c}
c^{s}(b)_{s} \\
c^{s}(a)_{s}
\end{array} \quad \begin{array}{c}
c^{s}q^{\frac{s(s-1)}{2}}
\end{array} \quad \begin{array}{c}
c^{s}q^{\frac{s(s-1)}{2}}
\end{array}
\end{align*}

with the grid $x_{s} = q^{s}$. There is a constraint $a_{1}a_{2}a_{3}q^{m} = b_{1}b_{2}b_{3}q^{n}$ for q-$E_{7}^{(1)}$ case.

\section{Elliptic case}

As before, we use multiplicative parameters $k, \ell, u_{1}, \cdots, u_{8}, q (k^{2}\ell^{2} = qu_{1} \cdots u_{8})$ and $p$, where $q$ is the base for the $q$-difference and $p$ is the period of the elliptic functions. Let $[z]$ be a theta function such that $[pz] = [z^{-1}] = -z^{-1}[z]$, and define

\begin{equation}
\begin{align*}
a(z) &= \frac{\alpha}{z} \left[ \frac{k}{\alpha z} \right], \\
b(z) &= \frac{\beta}{z} \left[ \frac{k}{\beta z} \right], \\
c(z) &= \frac{\alpha}{z} \left[ \frac{\ell}{\alpha z} \right], \\
d(z) &= \frac{\beta}{z} \left[ \frac{\ell}{\beta z} \right],
\end{align*}
\end{equation}

with $q = \exp(2\pi i \tau) = \frac{\alpha \beta}{\ell \alpha z}$ and $\tau = \frac{\alpha \beta}{k \ell z}$. The functions $a, b, c, d$ are entire functions of $z$ and are multipliers of the theta functions $\theta(z)$.
Then the functions

\begin{equation}
F(z) = \frac{b(z)}{a(z)}, \quad G(z) = \frac{d(z)}{c(z)},
\end{equation}

are elliptic functions such that

\begin{equation}
F(pz) = F(z) = F\left(\frac{k}{z}\right), \quad G(pz) = G(z) = G\left(\frac{\ell}{z}\right),
\end{equation}

which gives the parametrization \(f = F(z), g = G(z)\) of the elliptic curve in Example 1.5.

By the same method as the \(q\)-case in previous section, we have [10]

**Theorem 3.1.** The elliptic difference Painlevé equation of type \(E_{8}^{(1)}\) can be written in the form \((k, \ell, f, g) \mapsto (k/q, q\ell, \overline{f}, \overline{g})\), where \(\overline{f}, \overline{g}\) are given by

\begin{equation}
\frac{\{a(\bar{z})f - b(\bar{z})\}}{\{a(z)f - b(z)\}} \frac{\{\bar{a}(\bar{z})\bar{f} - \bar{b}(\bar{z})\}}{\{\bar{a}(z)\bar{f} - \bar{b}(z)\}} = \frac{z^2}{\bar{z}^2} \prod_{i=1}^{8} \left[ \frac{u_i}{z} \right], \quad \text{for } g = G(z), \quad \bar{z} = \frac{\ell}{z},
\end{equation}

\begin{equation}
\frac{\{c(\bar{z})g - d(\bar{z})\}}{\{c(z)g - d(z)\}} \frac{\{\bar{c}(\bar{z})\bar{g} - \bar{d}(\bar{z})\}}{\{\bar{c}(z)\bar{g} - \bar{d}(z)\}} = \frac{\bar{z}^2}{z^2} \prod_{i=1}^{8} \left[ \frac{u_i}{\bar{z}} \right], \quad \text{for } \bar{f} = \overline{F}(z), \quad \bar{z} = \frac{k}{q\bar{z}}.
\end{equation}

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