<table>
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<th>A simple expression for discrete Painleve equations (Novel Development of Nonlinear Discrete Integrable Systems)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>YAMADA, YASUHIKO</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2014), B47: 87-95</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/226214">http://hdl.handle.net/2433/226214</a></td>
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<tr>
<td>Rights</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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A simple expression for discrete Painlevé equations

By

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Abstract

A simple expression of discrete Painlevé equations and their Lax pair is obtained by using an interpolation problem. We discuss mainly the case of q-Painlevé equation of type $E_8^{(1)}$.

§ 1. Structure of discrete Painlevé equations

The second order discrete Painlevé equations were classified by Sakai [13] as follows:

elliptic 

$E_8^{(1)} \rightarrow \cdots \rightarrow A_1^{(1)}$

multiplicative 

$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow A_2^{(1)} \rightarrow A_1^{(1)}$

additive 

$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_4^{(1)} \rightarrow A_3^{(1)} \rightarrow A_1^{(1)}$

Each discrete Painlevé equation, represented as a rational map on $\mathbb{P}^1 \times \mathbb{P}^1$,

$T : (f, g) \mapsto (\overline{f}, \overline{g}) = \left( \frac{\psi_1(f, g)}{\psi_0(f, g)}, \frac{\phi_1(f, g)}{\phi_0(f, g)} \right)$,

has eight singular points where $\psi_0 = \psi_1 = 0$ or $\phi_0 = \phi_1 = 0$. Conversely, a configuration of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ (or nine points on $\mathbb{P}^2$) characterize the Painlevé equation. Let us look at some examples.

Received November 26, 2013. Revised March 20, 2014.
2010 Mathematics Subject Classification(s): 34M55, 39A13.
This work is partially supported by KAKENHI24340029 and KAKENHI21340036.
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Example 1.1. \( q-D_{5}^{(1)} \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{a_{3}a_{4}b_{1}b_{2}}{a_{1}a_{2}b_{3}b_{4}} \).

\[ \overline{f} = \frac{(\overline{g} - b_{1}t)(\overline{g} - b_{2}t)}{(\overline{g} - b_{3})(\overline{g} - b_{4})}a_{3}a_{4}, \]

\[ \overline{g}g = \frac{(f - a_{1}t)(f - a_{2}t)}{(f - a_{3})(f - a_{4})}b_{3}b_{4}. \]

(1.2)

The 8 singular points are on the four lines \( f = 0, f = \infty, g = 0 \) and \( g = \infty \):

Example 1.2. \( q-E_{6}^{(1)} \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{b_{5}b_{6}b_{7}b_{8}}{b_{1}b_{2}b_{3}b_{4}} \).

\[ \frac{(fg - 1)(\overline{f}g - 1)}{(fgt^{2} - 1)(\overline{f}gqt^{2} - 1)} = \frac{qt^{2}(b_{1}g - 1)(b_{2}g - 1)(b_{3}g - 1)(b_{4}g - 1)}{(b_{5}gt - 1)(b_{6}gt - 1)(b_{7}gt - 1)(b_{8}gt - 1)}, \]

(1.3)

\[ \frac{(fg - 1)(\overline{f}g - 1)}{(fgt^{2} - 1)(\overline{f}gqt^{2} - 1)} = \frac{q^{2}t^{2}(b_{1} - \overline{f})(b_{2} - \overline{f})(b_{3} - \overline{f})(b_{4} - \overline{f})}{(b_{5} - \overline{f}qt)(b_{6} - \overline{f}qt)}. \]

The 8 singular points are on the two lines \( f = 0, g = 0 \) and one curve \( fg = 1 \):

Example 1.3. \( q-E_{7}^{(1)} \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{b_{5}b_{6}b_{7}b_{8}}{b_{1}b_{2}b_{3}b_{4}} \).

\[ \frac{(fg - 1)(\overline{f}g - 1)}{(fgt^{2} - 1)(\overline{f}gqt^{2} - 1)} = \frac{(b_{1}g - 1)(b_{2}g - 1)(b_{3}g - 1)(b_{4}g - 1)}{(b_{5}gt - 1)(b_{6}gt - 1)(b_{7}gt - 1)(b_{8}gt - 1)}, \]

(1.4)

\[ \frac{(fgt^{2} - 1)(\overline{f}gqt^{2} - 1)}{(fg - 1)(\overline{f}g - 1)} = \frac{(b_{1} - \overline{f})(b_{2} - \overline{f})(b_{3} - \overline{f})(b_{4} - \overline{f})q}{(b_{5} - \overline{f}qt)(b_{6} - \overline{f}qt)(b_{7} - \overline{f}qt)(b_{8} - \overline{f}qt)}. \]

The 8 singular points are on the two curves \( fg = 1 \) and \( fgt^{2} = 1 \):
Example 1.4. \( qE_{8}^{(1)} \) \cite{11}: \( T(k, \ell, f, g) = (\frac{k}{q}, q\ell, \overline{f}, \overline{g}) \), \( q = \frac{k^2\ell^2}{u_1 \cdots u_8} \).

\[
\frac{1}{(\frac{2}{k} - \frac{q}{\ell})(\frac{k}{q} - \frac{q}{\ell}) - (\frac{k}{q} - \frac{q}{\ell})(\frac{1}{k} - \frac{1}{\ell})\ell} = \frac{k^2}{q} \frac{A(\ell, g)}{B(\ell, g)},
\]

and

\[
\frac{1}{(\frac{2}{k} - \frac{q}{\ell})(\frac{k}{q} - \frac{q}{\ell}) - (\frac{k}{q} - \frac{q}{\ell})(\frac{1}{k} - \frac{1}{\ell})\ell} = q\ell^2 \frac{A(\frac{k}{q}, \overline{f})}{B(\frac{k}{q}, \overline{f})},
\]

where \( A(h, x), B(h, x) \) are polynomials in \( x \) of degree 4 given by

\[
A(h, x) = \left( \frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8 \right) + \left( \frac{m_1}{h^2} - m_5 + 2hm_7 \right)x + \left( \frac{-m_0}{h^3} + m_6 - 3hm_8 \right)x^2 - m_7x^3 + m_8x^4,
\]

\[
B(h, x) = \left( \frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8 \right) + \left( \frac{2m_1}{h^2} - \frac{m_3}{h} + hm_7 \right)x + \left( \frac{-3m_0}{h^3} + m_2 - hm_8 \right)x^2 - \frac{m_1}{h^3}x^3 + \frac{m_0}{h^4}x^4,
\]

\[
U(z) = \frac{1}{z^4} \prod_{i=1}^{8} (z - u_i) = \frac{1}{z^4} \sum_{i=0}^{8} (-1)^i m_i z^i.
\]

Though it is not so obvious in this form, the singularity of this equation are given by \( (f, g) = (F(u_i), G(u_i)) \), \( (i = 1, \cdots, 8) \) where

\[
F(u) = u + \frac{k}{u}, \quad G(u) = u + \frac{\ell}{u}.
\]

These points are on the curve of bi-degree \( (2, 2) \)

\[
(f - g)(\frac{f}{k} - \frac{g}{\ell}) - (k - \ell)(\frac{1}{k} - \frac{1}{\ell}) = 0,
\]

which has a node at \( (\infty, \infty) \).
In the next section, we will rewrite the $q$-$E_{8}^{(1)}$ equation in simpler form where the singularity structure is manifest (see Theorem 2.2).

**Example 1.5.** The most generic equation, the elliptic-$E_{8}^{(1)}$ [13][11], is more complicated than $q$-$E_{8}^{(1)}$ case. The 8 points are on a smooth bi-degree $(2, 2)$ curve (i.e. an elliptic curve):

There have been many challenges to obtain an explicit expression of the elliptic $E_{8}^{(1)}$ equation (e.g. [4][7][8]). We will give one simple expression (Theorem 3.1) which was obtained in [10] by a similar method as the $q$-case discussed below.

§ 2. **An approach from the Padé interpolation**

There exists a simple method to derive a Lax pair for Painlevé equations[15]. Using a discrete version of it, we will derive a simple form of $q$-Painlevé equation of type $E_{8}^{(1)}$. Here, we use parameters $a_1, a_2, a_3, b_1, b_2, b_3, k, \ell \in \mathbb{C}$ and $m, n \in \mathbb{Z}_{\geq 0}$ with constraints

\[(2.1) \quad q^{-1}k\ell = q^{-n}a_1a_2a_3 = q^{-m}b_1b_2b_3.\]

The time evolution is $\overline{k} = \frac{k}{q}$, $\overline{\ell} = q\ell$ (and $\overline{x} = x$ for $x = a_i, b_i, m, n$). The parameters $u_i$ in Example 1.4 are related to the parameters $a_i, b_i, k, \ell, m, n$ by $(u_1, \cdots, u_8) = (a_1, a_2, a_3, q^{-m-n}, b_1, b_2, b_3, q)$. 
Our starting point is the following Padé interpolation

\[ Y_s = \frac{q^{ns}(k/a_1, k/a_2, k/a_3, b_1, b_2, b_3)_s}{q^{ms}(a_1, a_2, a_3, b_1, b_2, b_3)_s} = \frac{P_m}{Q_n}, \quad x = x_s, \quad (s = 0, 1, \cdots, m+n), \]

where \( P_m, Q_n \) are polynomials of degree \( m \), \( n \) in variable \( x \). In the followings, we use variable \( z \) such that \( x = z + \frac{k}{qz} \), hence \( P_m, Q_n \) are Laurent polynomials of the form

\[ P_m(z) = \sum_{i=0}^{m} u_i(z + \frac{k}{qz})^i, \quad Q_n(z) = \sum_{i=0}^{n} v_i(z + \frac{k}{qz})^i. \]

The interpolating points are \( x_s = q^{-s} + kq^{s-1} \) (\( q \)-quadratic grid) in variable \( x \), and hence \( z_s = q^{-s} \) (\( q \)-grid) in variable \( z \).

The main ingredients are the contiguous relations satisfied by \( u(z) = P_m(z) \) and \( v(z) = Y(z)Q_n(z) \) where \( Y(z) \) is a function such as \( Y(q^{-s}) = Y_s \). For instance, the relation between \( y(z), y(\frac{z}{q}), \overline{y}(\frac{z}{q}) \) is obtained by evaluating the Casorati determinant

\[ \begin{vmatrix} y(z) & y(\frac{z}{q}) & \overline{y}(\frac{z}{q}) \\ u(z) & u(\frac{z}{q}) & \overline{u}(\frac{z}{q}) \\ v(z) & v(\frac{z}{q}) & \overline{v}(\frac{z}{q}) \end{vmatrix} = 0. \]

This determinant divided by \( Y(z) \) is a Laurent polynomial in \( z \) and has many known zeros due to the interpolating condition \( u(z_s) = v(z_s) \). Hence one can determine the structure of the contiguous relations without knowing the explicit form of \( P_m \) and \( Q_n \).

**Proposition 2.1.** The following relations hold for \( y(z) = P_m(z), Y(z)Q_n(z) \):

\[ L_2(z) : \quad B_2(z) \left\{ g - G(z) \right\} y(z) - B_2(\frac{k}{qz}) \left\{ g - G(\frac{z}{q}) \right\} y(\frac{z}{q}) \]
\[ + c \left\{ f - F(z) \right\} (z - \frac{k}{z}) \overline{y}(\frac{z}{q}) = 0, \]

\[ L_3(z) : \quad B_1(z) \left\{ g - G(\frac{z}{q}) \right\} \overline{y}(\frac{z}{q}) - B_1(\frac{k}{qz}) \left\{ g - G(z) \right\} \overline{y}(z) \]
\[ + \frac{w}{c} \left\{ \overline{f} - \overline{F}(z) \right\} (z - \frac{k}{qz}) y(z) = 0, \]

where

\[ B_1(z) = \frac{1}{z^2} \prod_{i=1}^{4} (z - u_i), \quad B_2(z) = \frac{1}{z^2} \prod_{i=5}^{8} (z - u_i), \]

\[ F(z) = z + \frac{k}{z}, \quad G(z) = z + \frac{\ell}{z}. \]
and \(f, g, c, w\) are some constants (independent of \(z\)).

Combining \(L_2\) and \(L_3\), one can obtain the three term relation \(L_1\) between \(y(qz), y(z), y(z/q)\).

\[
\begin{array}{c}
\frac{y(qz)}{y(z)} = \frac{B_2(k/q)}{B_2}(k/q) = \frac{B_2(k/z)}{B_2}(k/z)
\end{array}
\]

Though the explicit form of the \(L_1\) equation is complicated, it can be characterized by the following properties [16]:

1. As a polynomial in \((f, g)\), it is of bi-degree \((3, 2)\).
2. It vanishes when \(f = F(u)\), \(g = G(u)\) with \(u = u_1, \ldots , u_8, qz, \frac{k}{z}\), and \(f = F(u)\), \(g = G(u)\) with \(u = z, qz\). Hence, the \(L_1\) equation is equivalent with the \(L_1\) equation in [17] up to some gauge transformations, and the equations \(L_2, L_3\) (or \(L_1\)) can be considered as a Lax pair for \(qE_8^{(1)}\).

**Theorem 2.2.** The compatibility of the equations \(L_2, L_3\) (2.5)(2.6) is equivalent to the relations

\[(2.8) \quad \frac{\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}{\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}} = \frac{U(z)}{U(\frac{k}{z})}, \quad \text{for} \quad g = G(z),
\]

\[(2.9) \quad \frac{\{g - G(z)\}\{\overline{g} - \overline{G}(z)\}}{\{g - G(z)\}\{\overline{g} - \overline{G}(z)\}} = \frac{U(z)}{U(\frac{k}{qz})}, \quad \text{for} \quad \overline{f} = \overline{F}(z),
\]

along with an additional relation

\[(2.10) \quad w = \frac{(k - \ell)(k - q\ell)U(z)}{k^2\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}, \quad \text{for} \quad g = G(z).
\]

where \(U(z) = B_1(z)B_2(z) = \frac{1}{z^4} \prod_{i=1}^{8}(z - u_i)\).

**Proof.** Putting \(g = G(z)\) in equations \(L_2(z)\) and \(L_3(z)\), we have the relation (2.10). Since \(G(z) = G(\frac{k}{z})\), the relation (2.10) holds also when \(z\) is replaced by \(\frac{k}{z}\). Taking the ratio of these two relations, we obtain the equation (2.8). Putting \(\overline{f} = \overline{F}(z)\) in equations \(L_2(z)\) and \(L_3(z)\), we get the equation (2.9). Sufficiency of the equations (2.8) (2.9) (2.10) for the compatibility can be checked by a direct computation. \(\square\)
The equations (2.8)(2.9) are the desired simple expression for the $q$-$E_{8}^{(1)}$. In fact, by eliminating the variable $z$, they correctly reproduce the equations (1.5)(1.6), where the polynomials $A, B$ are given by

$$A(h, z + \frac{h}{z}) = \frac{zU(z) - \frac{h}{z}U(\frac{h}{z})}{z - \frac{h}{z}}, \quad B(h, z + \frac{h}{z}) = \frac{zU(\frac{h}{z}) - \frac{h}{z}U(z)}{z - \frac{h}{z}}.$$  

Remark. Up to now, we used only the defining relation $Y_{s} = \frac{P_{m}(x_{s})}{Q_{n}(x_{s})}$ for $P_{m}(x)$ and $Q_{n}(x)$. If we know the explicit forms of $P_{m}(x), Q_{n}(x)$, then we can determine the Painlevé variables $f, g$ explicitly. For the interpolation problem with general $Y_{s}$ and $x_{s}$, the following formula has been classically known by Cauchy and Jacobi

$$P_{m}(x) = f(x) \det \left( W_{i,j}^{(-)} \right)_{i,j=0}^{n}, \quad Q_{n}(x) = \det \left( W_{i,j}^{(+)} \right)_{i,j=0}^{n-1},$$  

$$W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{Y_{s}}{f'(x_{s})}x_{s}^{i+j}(x-x_{s})^{\pm 1}, \quad f(x) = \prod_{s=0}^{m+n} (x-x_{s}).$$  

Applying this for the $q$-quadratic grid: $x = z + \frac{k}{qz}, \quad x_{s} = q^{-s} + kq^{s-1}$, we have

$$W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{(1-kq^{2s-1})}{(1-kq^{-1})} \frac{(k/q, q^{-m-n})_{s}}{(q, kq^{m+n})_{s}}Y_{s}q^{(n-m)s}x_{s}^{i+j}(x-x_{s})^{\pm 1}.$$  

These expressions give the special solutions for $q$-$E_{8}^{(1)}$ Painlevé equation in terms of the $10W_{9}$ hypergeometric functions and their determinants (c.f.[5][6]).

Remark. The Padé approach to the degenerate cases were studied in [2][9]. The corresponding Padé problems are

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<th>$q$-$E_{6}^{(1)}$</th>
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<td>$Y_{s}$</td>
<td>$k_{1, b_{2}, b_{3}}$</td>
<td>$k_{1, b_{2}}$</td>
<td>$c^{a}(b)_{s}$</td>
<td>$c^{a}(a)$</td>
<td>$c^{s}q^{s(\frac{s-1}{2})}$</td>
</tr>
<tr>
<td>$a_{1, a_{2}, a_{3}}$</td>
<td>$a_{1, a_{2}}$</td>
<td>$a_{1, a_{2}}$</td>
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with the grid $x_{s} = q^{s}$. There is a constraint $a_{1}a_{2}q^{m} = b_{1}b_{2}q^{n}$ for $q$-$E_{7}^{(1)}$ case.

§ 3. Elliptic case

As before, we use multiplicative parameters $k, \ell, u_{1}, \cdots, u_{8}, q$ ($k^{2}\ell^{2} = qu_{1} \cdots u_{8}$) and $p$, where $q$ is the base for the $q$-difference and $p$ is the period of the elliptic functions. Let $[z]$ be a theta function such that $[pz] = [z^{-1}] = -z^{-1}[z]$, and define

$$a(z) = \left[ \frac{k}{z} \right][\frac{k}{\alpha z}], \quad b(z) = \left[ \frac{\beta}{z} \right][\frac{k}{\beta z}],$$  

$$c(z) = \left[ \frac{\ell}{z} \right][\frac{\ell}{\alpha z}], \quad d(z) = \left[ \frac{\beta}{z} \right][\frac{\ell}{\beta z}],$$  

$$
\frac{A(h, z + \frac{h}{z})}{z - \frac{h}{z}}, \quad \frac{B(h, z + \frac{h}{z})}{z - \frac{h}{z}}.
$$
(\alpha \neq \beta). Then the functions

\[ F(z) = \frac{b(z)}{a(z)}, \quad G(z) = \frac{d(z)}{c(z)}, \]

are elliptic functions such that

\[ F(pz) = F(z) = F\left(\frac{k}{z}\right), \quad G(pz) = G(z) = G\left(\frac{\ell}{z}\right), \]

which gives the parametrization \( f = F(z), g = G(z) \) of the elliptic curve in Example 1.5. By the same method as the \( q \)-case in previous section, we have [10]

**Theorem 3.1.** The elliptic difference Painlevé equation of type \( E_{8}^{(1)} \) can be written in the form \((k, \ell, f, g) \mapsto (k/q, q\ell, \overline{f}, \overline{g})\), where \( \overline{f}, \overline{g} \) are given by

\[ \frac{\{a(\tilde{z})f - b(\tilde{z})\}\{\overline{a}(\tilde{z})\overline{f} - \overline{b}(\tilde{z})\}}{\{a(z)f - b(z)\}\{\overline{a}(z)\overline{f} - \overline{b}(z)\}} = \frac{z^2}{\tilde{z}^2} \prod_{i=1}^{8} \left[ \frac{u_i}{z} \right], \quad \text{for} \quad g = G(z), \quad \tilde{z} = \frac{\ell}{z}, \]

\[ \frac{\{c(\tilde{z})g - d(\tilde{z})\}\{\overline{c}(\tilde{z})\overline{g} - \overline{d}(\tilde{z})\}}{\{c(z)g - d(z)\}\{\overline{c}(z)\overline{g} - \overline{d}(z)\}} = \frac{z^2}{\tilde{z}^2} \prod_{i=1}^{8} \left[ \frac{u_i}{\tilde{z}} \right], \quad \text{for} \quad \overline{f} = \overline{F}(z), \quad \tilde{z} = \frac{k}{qz}. \]

**Acknowledgment.** The author thanks M.Noumi and S.Tsujimoto for discussions. He also thanks the organizers of the workshops "Discrete Integrable Systems - A Follow-up Meeting" (Newton Institute, July 2013) and "Novel Development of Nonlinear Discrete Integrable Systems" (RIMS, August 2013) where the results of this note were presented.

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