A simple expression for discrete Painlevé equations

By

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Abstract

A simple expression of discrete Painlevé equations and their Lax pair is obtained by using an interpolation problem. We discuss mainly the case of $q$-Painlevé equation of type $E_8^{(1)}$.

§1. Structure of discrete Painlevé equations

The second order discrete Painlevé equations were classified by Sakai [13] as follows:

- **Elliptic**: $E_8^{(1)} \rightarrow A_1^{(1)}$
- **Multiplicative**: $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow A_{2+1}^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)}$
- **Additive**: $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_4^{(1)} \rightarrow A_3^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)}$

Each discrete Painlevé equation, represented as a rational map on $\mathbb{P}^1 \times \mathbb{P}^1$,

$$T: (f,g) \mapsto (\overline{f},\overline{g}) = \left( \frac{\psi_1(f,g)}{\psi_0(f,g)}, \frac{\phi_1(f,g)}{\phi_0(f,g)} \right),$$

has eight singular points where $\psi_0 = \psi_1 = 0$ or $\phi_0 = \phi_1 = 0$. Conversely, a configuration of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ (or nine points on $\mathbb{P}^2$) characterize the Painlevé equation. Let us look at some examples.
Example 1.1. $q$-$D_{5}^{(1)}[3]$: $T(t, f, g) = (qt, \overline{f}, \overline{g})$, $q = \frac{a_{3}a_{4}b_{1}b_{2}}{a_{1}a_{2}b_{3}b_{4}}$.

\[
\begin{align*}
\overline{f} &= \frac{(\overline{g} - b_{1}t)(\overline{g} - b_{2}t)}{(\overline{g} - b_{3})(\overline{g} - b_{4})}a_{3}a_{4}, \\
\overline{g} &= \frac{(f - a_{1}t)(f - a_{2}t)}{(f - a_{3})(f - a_{4})}b_{3}b_{4}.
\end{align*}
\]

The 8 singular points are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$:

Example 1.2. $q$-$E_{6}^{(1)}[12][14][11]$: $T(t, f, g) = (qt, \overline{f}, \overline{g})$, $q = \frac{b_{5}b_{6}b_{7}b_{8}}{b_{1}b_{2}b_{3}b_{4}}$.

\[
\begin{align*}
\frac{(fg - 1)(\overline{f}g - 1)}{(fg - 1)(\overline{f}g - 1)} &= \frac{qt^2(b_{1}g - 1)(b_{2}g - 1)(b_{3}g - 1)(b_{4}g - 1)}{(b_{5}b_{6}b_{7}g - 1)(b_{8}g - 1)}.
\end{align*}
\]

The 8 singular points are on the two lines $f = 0, g = 0$ and one curve $fg = 1$:

Example 1.3. $q$-$E_{7}^{(1)}[1][11]$: $T(t, f, g) = (qt, \overline{f}, \overline{g})$, $q = \frac{b_{5}b_{6}b_{7}b_{8}}{b_{1}b_{2}b_{3}b_{4}}$.

\[
\begin{align*}
\frac{(fg - 1)(\overline{f}g - 1)}{(fgt^2 - 1)(\overline{f}g - 1)} &= \frac{(b_{1}g - 1)(b_{2}g - 1)(b_{3}g - 1)(b_{4}g - 1)}{(b_{5}g - 1)(b_{6}g - 1)(b_{7}g - 1)(b_{8}g - 1)}.
\end{align*}
\]

The 8 singular points are on the two curves $fg = 1$ and $fgt^2 = 1$:
Example 1.4. $q$-$E_8^{(1)}[11]: T(k, \ell, f, g) = (\frac{k}{q}, q\ell, \overline{f}, \overline{g})$, $q = \frac{k^2\ell^2}{u_1\cdots u_8}$.

$$\frac{(\overline{f} - g)(f - g) - (\frac{k}{q} - \ell)(k - \ell)(\frac{q}{k} - \frac{1}{\ell})}{(\frac{k}{q} - \frac{q}{k})(\ell - \frac{1}{\ell}) - (\frac{q}{k} - \frac{1}{\ell})(\frac{k}{q} - \frac{1}{\ell})} = \frac{k^2}{q} \frac{A(\ell, g)}{B(\ell, g)},$$

(1.5)

$$\frac{(\overline{f} - g)(\overline{f} - g) - (\frac{k}{q} - \ell q)(\frac{k}{q} - \ell)(\frac{q}{k} - \frac{1}{\ell})}{(\frac{k}{q} - \frac{q}{k})(\ell - \frac{1}{\ell}) - (\frac{q}{k} - \frac{1}{\ell})(\frac{k}{q} - \frac{1}{\ell})} = q\ell^2 \frac{A(\frac{k}{q}, \overline{f})}{B(\frac{k}{q}, \overline{f})},$$

(1.6)

where $A(h, x), B(h, x)$ are polynomials in $x$ of degree 4 given by

$$A(h, x) = \left(\frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8\right) + \left(\frac{m_1}{h^2} - m_5 + 2hm_7\right)x + \left(-\frac{m_0}{h^3} + m_6 - 3hm_8\right)x^2 - m_7x^3 + m_8x^4,$$

$$B(h, x) = \left(\frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8\right) + \left(\frac{2m_1}{h^2} - \frac{m_3}{h} + hm_7\right)x + \left(-\frac{3m_0}{h^3} + \frac{m_2}{h^2} - hm_8\right)x^2 - \frac{m_1}{h^3}x^3 + \frac{m_0}{h^4}x^4,$$

(1.7)

$$U(z) = \frac{1}{z^4} \prod_{i=1}^{8} (z - u_i) = \frac{1}{z^4} \sum_{i=0}^{8} (-1)^i m_i z^i.$$

Though it is not so obvious in this form, the singularity of this equation are given by $(f, g) = (F(u_i), G(u_i))$, $(i = 1, \cdots, 8)$ where

$$F(u) = u + \frac{k}{u}, \quad G(u) = u + \frac{\ell}{u}.$$

(1.8)

These points are on the curve of bi-degree $(2, 2)$

$$\frac{(f - g)(\frac{f}{k} - \frac{g}{\ell}) - (k - \ell)(\frac{k}{k} - \frac{1}{\ell})}{(\frac{k}{k} - \frac{1}{k})(\frac{f}{k} - \frac{1}{\ell}) - (\frac{k}{k} - \frac{1}{k})(\frac{g}{\ell} - \frac{1}{\ell})} = 0,$$

(1.9)

which has a node at $(\infty, \infty)$. 
In the next section, we will rewrite the $q$-$E_{8}^{(1)}$ equation in simpler form where the singularity structure is manifest (see Theorem.2.2).

**Example 1.5.** The most generic equation, the elliptic-$E_{8}^{(1)}$ [13][11], is more complicated than $q$-$E_{8}^{(1)}$ case. The 8 points are on a smooth bi-degree (2, 2) curve (i.e. an elliptic curve):

There have been many challenges to obtain an explicit expression of the elliptic $E_{8}^{(1)}$ equation (e.g.[4][7][8]). We will give one simple expression (Theorem.3.1) which was obtained in [10] by a similar method as the $q$-case discussed below.

§ 2. **An approach from the Padé interpolation**

There exists a simple method to derive a Lax pair for Painlevé equations[15]. Using a discrete version of it, we will derive a simple form of $q$-Painlevé equation of type $E_{8}^{(1)}$. Here, we use parameters $a_{1}, a_{2}, a_{3}, b_{1}, b_{2}, b_{3}, k, \ell \in \mathbb{C}$ and $m, n \in \mathbb{Z}_{\geq 0}$ with constraints

\begin{equation}
q^{-1}k\ell = q^{-n}a_{1}a_{2}a_{3} = q^{-m}b_{1}b_{2}b_{3}.
\end{equation}

The time evolution is $\overline{k} = \frac{k}{q}, \overline{\ell} = q\ell$ (and $\overline{x} = x$ for $x = a_{i}, b_{i}, m, n$). The parameters $u_{i}$ in Example.1.4 are related to the parameters $a_{i}, b_{i}, k, \ell, m, n$ by $(u_{1}, \cdots, u_{8}) = (a_{1}, a_{2}, a_{3}, q^{-m-n}, b_{1}, b_{2}, b_{3}, q)$. 
Our starting point is the following Padé interpolation
\[ Y_s = \frac{q^{ns}(\frac{k}{a_1}, \frac{k}{a_2}, \frac{k}{a_3}, b_1, b_2, b_3)_s}{q^{ms}(a_1, a_2, a_3, \frac{k}{b_1}, \frac{k}{b_2}, \frac{k}{b_3})_s} = \frac{P_m}{Q_n}, \quad x = x_s, \quad (s = 0, 1, \cdots, m + n), \]
\[ (a)_i = \prod_{j=0}^{i-1} (1 - aq^j), \quad (a, \cdots, b)_s = (a)_s \cdots (b)_s, \]
where \( P_m, Q_n \) are polynomials of degree \( m, n \) in variable \( x \). In the followings, we use variable \( z \) such that \( x = z + \frac{k}{qz} \), hence \( P_m, Q_n \) are Laurent polynomials of the form
\[ P_m(z) = \sum_{i=0}^{m} u_i (z + \frac{k}{qz})^i, \quad Q_n(z) = \sum_{i=0}^{n} v_i (z + \frac{k}{qz})^i. \]
The interpolating points are \( x_s = q^{-s} + kq^{s-1} \) (\( q \)-quadratic grid) in variable \( x \), and hence \( z_s = q^{-s} \) (\( q \)-grid) in variable \( z \).

The main ingredients are the contiguous relations satisfied by \( u(z) = P_m(z) \) and \( v(z) = Y(z)Q_n(z) \) where \( Y(z) \) is a function such as \( Y(q^{-s}) = Y_s \). For instance, the relation between \( y(z), y(\frac{z}{q}), \overline{y}(\frac{z}{q}) \) is obtained by evaluating the Casorati determinant
\[ \begin{vmatrix} y(z) & y(\frac{z}{q}) & \overline{y}(\frac{z}{q}) \\ u(z) & u(\frac{z}{q}) & \overline{u}(\frac{z}{q}) \\ v(z) & v(\frac{z}{q}) & \overline{v}(\frac{z}{q}) \end{vmatrix} = 0. \]

This determinant divided by \( Y(z) \) is a Laurent polynomial in \( z \) and has many known zeros due to the interpolating condition \( u(z_s) = v(z_s) \). Hence one can determine the structure of the contiguous relations without knowing the explicit form of \( P_m \) and \( Q_n \).

**Proposition 2.1.** The following relations hold for \( y(z) = P_m(z), Y(z)Q_n(z) \):
\[ L_2(z) : \quad B_2(z) \left\{ g - G(\frac{k}{z}) \right\} y(z) - B_2(\frac{k}{z}) \left\{ g - G(z) \right\} y(\frac{z}{q}) + c\left\{ f - F(z) \right\} (z - \frac{k}{z}) \overline{y}(\frac{z}{q}) = 0, \]
\[ L_3(z) : \quad B_1(z) \left\{ g - G(\frac{k}{qz}) \right\} \overline{y}(\frac{z}{q}) - B_1(\frac{k}{qz}) \left\{ g - G(z) \right\} \overline{y}(z) + \frac{w}{c} \left\{ \overline{f} - \overline{F}(z) \right\} (z - \frac{k}{qz}) y(z) = 0, \]
where
\[ B_1(z) = \frac{1}{z^2} \prod_{i=1}^{4} (z - u_i), \quad B_2(z) = \frac{1}{z^2} \prod_{i=5}^{8} (z - u_i), \]
\[ F(z) = z + \frac{k}{z}, \quad G(z) = z + \frac{\ell}{z}. \]
and \( f, g, c, w \) are some constants (independent of \( z \)).

Combining \( L_2 \) and \( L_3 \), one can obtain the three term relation \( L_1 \) between \( y(qz), y(z), y(z/q) \).

Though the explicit form of the \( L_1 \) equation is complicated, it can be characterized by the following properties [16]:

1. As a polynomial in \( (f, g) \), it is of bi-degree \((3, 2)\).
2. It vanishes when \( f = F(u) \), \( g = G(u) \) with \( u = u_1, \cdots, u_8, qz, \frac{k}{z} \), and \( f = F(u) \), \( g = G(u) \) with \( u = z, qz \). Hence, the \( L_1 \) equation is equivalent with the \( L_1 \) equation in [17] up to some gauge transformations, and the equations \( L_2, L_3 \) (or \( L_1 \)) can be considered as a Lax pair for \( q \)-E\(_8^{(1)} \).

**Theorem 2.2.** The compatibility of the equations \( L_2, L_3 \) (2.5)(2.6) is equivalent to the relations

\[
\frac{\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}{\{f - F(\frac{k}{z})\}\{\overline{f} - \overline{F}(\frac{k}{z})\}} = \frac{U(z)}{U(\frac{k}{z})}, \quad \text{for} \quad g = G(z),
\]

\[
\frac{\{g - G(z)\}\{\overline{g} - \overline{G}(z)\}}{\{g - G(\frac{k}{qz})\}\{\overline{g} - \overline{G}(\frac{k}{qz})\}} = \frac{U(z)}{U(\frac{k}{qz})}, \quad \text{for} \quad \overline{f} = \overline{F}(z),
\]

along with an additional relation

\[
w = \frac{(k - \ell)(k - q\ell)U(z)}{k^2\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}, \quad \text{for} \quad g = G(z),
\]

where \( U(z) = B_1(z)B_2(z) = \frac{1}{z^4} \prod_{i=1}^{8}\frac{1}{z - u_i} \).

**Proof.** Putting \( g = G(z) \) in equations \( L_2(z) \) and \( L_3(z) \), we have the relation (2.10). Since \( G(z) = G(\frac{k}{z}) \), the relation (2.10) holds also when \( z \) is replaced by \( \frac{k}{z} \). Taking the ratio of these two relations, we obtain the equation (2.8). Putting \( \overline{f} = \overline{F}(z) \) in equations \( L_2(z) \) and \( L_3(z) \), we get the equation (2.9). Sufficiency of the equations (2.8) (2.9) (2.10) for the compatibility can be checked by a direct computation. \( \square \)
The equations (2.8)-(2.9) are the desired simple expression for the $q$-$E_8^{(1)}$. In fact, by eliminating the variable $z$, they correctly reproduce the equations (1.5)-(1.6), where the polynomials $A, B$ are given by

$$A(h, z + \frac{h}{z}) = \frac{zU(z) - \frac{h}{z}U(\frac{h}{z})}{z - \frac{h}{z}}, \quad B(h, z + \frac{h}{z}) = \frac{zU(\frac{h}{z}) - \frac{h}{z}U(z)}{z - \frac{h}{z}}.$$  

Remark. Up to now, we used only the defining relation $Y_s = \frac{P_m(x_s)}{Q_n(x_s)}$ for $P_m(x)$ and $Q_n(x)$. If we know the explicit forms of $P_m(x), Q_n(x)$, then we can determine the Painlevé variables $f, g$ explicitly. For the interpolation problem with general $Y_s$ and $x_s$, the following formula has been classically known by Cauchy and Jacobi

$$P_m(x) = f(x) \det \left( W_{i,j}^{(-)} \right)_{i,j=0}^{n}, \quad Q_n(x) = \det \left( W_{i,j}^{(+)} \right)_{i,j=0}^{n-1},$$  

(2.12)

$$W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{Y_s}{f'(x_s)} x_s^{i+j} (x - x_s) \pm 1, \quad f(x) = \prod_{s=0}^{m+n} (x - x_s).$$  

Applying this for the $q$-quadratic grid: $x = z + \frac{k}{qz}, \quad x_s = q^{-s} + kq^{s-1}$, we have

$$W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{(1 - kq^{2s-1})}{(1 - kq^{-1})} \frac{(k/q, q^{m+n})_{s}}{(q, kq^{m+n})_{s}} Y_s q^{(n-m)s} x_s^{i+j} (x - x_s) \pm 1.$$  

These expressions give the special solutions for $q$-$E_8^{(1)}$ Painlevé equation in terms of the $10W_9$ hypergeometric functions and their determinants (c.f.[5][6]).

Remark. The Padé approach to the degenerate cases were studied in [2][9]. The corresponding Padé problems are given by

<table>
<thead>
<tr>
<th>$q$-$P$</th>
<th>$q$-$E_7^{(1)}$</th>
<th>$q$-$E_6^{(1)}$</th>
<th>$q$-$D_5^{(1)}$</th>
<th>$q$-$A_4^{(1)}$</th>
<th>$q$-$A_{2+1}^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_s$</td>
<td>$(b_1, b_2, b_3)_s$</td>
<td>$(b_1, b_2)_s$</td>
<td>$c^s(b)_s$</td>
<td>$c^s(a)_s$</td>
<td>$c^s q^{(s-1)/2}$</td>
</tr>
</tbody>
</table>

with the grid $x_s = q^s$. There is a constraint $a_1 a_2 a_3 q^m = b_1 b_2 b_3 q^n$ for $q$-$E_7^{(1)}$ case.

§ 3. Elliptic case

As before, we use multiplicative parameters $k, \ell, u_1, \ldots, u_8, q$ ($k^2\ell^2 = q u_1 \cdots u_8$) and $p$, where $q$ is the base for the $q$-difference and $p$ is the period of the elliptic functions. Let $[z]$ be a theta function such that $[pz] = [z^{-1}] = z^{-1}[z]$, and define

$$a(z) = \left[ \frac{k}{z} \right], \quad b(z) = \left[ \frac{\beta}{z} \right],$$  

$$c(z) = \left[ \frac{\ell}{z} \right], \quad d(z) = \left[ \frac{\beta}{z} \right].$$  

(3.1)
Then the functions

\[
F(z) = \frac{b(z)}{a(z)}, \quad G(z) = \frac{d(z)}{c(z)},
\]

are elliptic functions such that

\[
F(pz) = F(z) = F\left(\frac{k}{z}\right), \quad G(pz) = G(z) = G\left(\frac{\ell}{z}\right),
\]

which gives the parametrization \( f = F(z), \ g = G(z) \) of the elliptic curve in Example 1.5. By the same method as the \( q \)-case in previous section, we have [10]

**Theorem 3.1.** The elliptic difference Painlevé equation of type \( E_{8}^{(1)} \) can be written in the form \((k, \ell, f, g) \mapsto (k/q, q\ell, \overline{f}, \overline{g})\), where \( \overline{f}, \overline{g} \) are given by

\[
\begin{align*}
\{a(\tilde{z})f - b(\tilde{z})\}\{\overline{a}(\tilde{z})\overline{f} - \overline{b}(\tilde{z})\} &= \frac{z^2}{2} \prod_{i=1}^{8} \left\{\frac{u_i}{\zbar}\right\}, \quad \text{for} \quad g = G(z), \quad \zbar = \frac{\ell}{z}, \\
\{c(\tilde{z})g - d(\tilde{z})\}\{\overline{c}(\tilde{z})\overline{g} - \overline{d}(\tilde{z})\} &= \frac{z^2}{2} \prod_{i=1}^{8} \left\{\frac{u_i}{\zbar}\right\}, \quad \text{for} \quad f = \overline{F}(z), \quad \zbar = \frac{k}{qz}.
\end{align*}
\]

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