A simple expression for discrete Painlevé equations

By

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Abstract

A simple expression of discrete Painlevé equations and their Lax pair is obtained by using an interpolation problem. We discuss mainly the case of $q$-Painlevé equation of type $E_8^{(1)}$.

§ 1. Structure of discrete Painlevé equations

The second order discrete Painlevé equations were classified by Sakai [13] as follows:

**elliptic**

$E_8^{(1)} \rightarrow A_1^{(1)}$

**multiplicative**

$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow A_{2+1}^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)}$

**additive**

$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_4^{(1)} \rightarrow A_3^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)}$,

Each discrete Painlevé equation, represented as a rational map on $\mathbb{P}^1 \times \mathbb{P}^1$,

\begin{equation}
T : (f, g) \mapsto (\overline{f}, \overline{g}) = \left( \frac{\psi_1(f, g)}{\psi_0(f, g)}, \frac{\phi_1(f, g)}{\phi_0(f, g)} \right),
\end{equation}

has eight singular points where $\psi_0 = \psi_1 = 0$ or $\phi_0 = \phi_1 = 0$. Conversely, a configuration of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ (or nine points on $\mathbb{P}^2$) characterize the Painlevé equation. Let us look at some examples.
Example 1.1. $q$-$D_5^{(1)}$ [3]: $T(t, f, g) = (qt, \bar{f}, \bar{g}), \quad q = \frac{a_3a_4b_1b_2}{a_1a_2b_3b_4}$.

\[ \begin{align*}
\bar{f} f &= \frac{(\bar{g} - b_1t)(\bar{g} - b_2t)}{(\bar{g} - b_3)(\bar{g} - b_4)} a_3a_4, \\
\bar{g} g &= \frac{(f - a_1t)(f - a_2t)}{(f - a_3)(f - a_4)} b_3b_4. 
\end{align*} \]

(1.2)

The 8 singular points are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$:

Example 1.2. $q$-$E_6^{(1)}$ [12][14][11]: $T(t, f, g) = (qt, \bar{f}, \bar{g}), \quad q = \frac{b_5b_6b_7b_8}{b_1b_2b_3b_4}$.

\[ \begin{align*}
\frac{(fg - 1)(\bar{f}g - 1)}{(fg - 1)(\bar{f}g - 1)} &= \frac{qt^2(b_1g - 1)(b_2g - 1)(b_3g - 1)(b_4g - 1)}{b_5b_6(b_7gt - 1)(b_8gt - 1)}, \\
\frac{(fg - 1)(\bar{f}g - 1)}{(fg - 1)(\bar{f}g - 1)} &= \frac{q^2t^2(b_1 - \bar{f})(b_2 - \bar{f})(b_3 - \bar{f})(b_4 - \bar{f})}{(b_5 - \bar{f}qt)(b_6 - \bar{f}qt)}.
\end{align*} \]

(1.3)

The 8 singular points are on the two lines $f = 0, g = 0$ and one curve $fg = 1$:

Example 1.3. $q$-$E_7^{(1)}$ [1][11]: $T(t, f, g) = (qt, \bar{f}, \bar{g}), \quad q = \frac{b_5b_6b_7b_8}{b_1b_2b_3b_4}$.

\[ \begin{align*}
\frac{(fg - 1)(\bar{f}g - 1)}{(fgt^2 - 1)(\bar{f}gqt^2 - 1)} &= \frac{(b_1g - 1)(b_2g - 1)(b_3g - 1)(b_4g - 1)}{(b_5gt - 1)(b_6gt - 1)(b_7gt - 1)(b_8gt - 1)}, \\
\frac{(fg - 1)(\bar{f}g - 1)}{(fgt^2 - 1)(\bar{f}gqt^2 - 1)} &= \frac{(b_1 - \bar{f})(b_2 - \bar{f})(b_3 - \bar{f})(b_4 - \bar{f})q}{(b_5 - \bar{f}qt)(b_6 - \bar{f}qt)(b_7 - \bar{f}qt)(b_8 - \bar{f}qt)}.
\end{align*} \]

(1.4)

The 8 singular points are on the two curves $fg = 1$ and $fgt^2 = 1$:
Example 1.4. $q$-E$_8^{(1)}$ \[ T(k, \ell, f, g) = \left( \frac{k}{q}, q\ell, f, g \right), \quad q = \frac{k^2\ell^2}{u_1 \cdots u_8}. \]

(1.5) \[
\frac{(f - g)(f - g) - (\frac{k}{q} - \ell)(k - \ell)\frac{1}{\ell}}{(\frac{f}{k} - \frac{g}{\ell})(\frac{k}{k} - \frac{q}{\ell}) - (\frac{q}{k} - \frac{1}{\ell})(\frac{1}{k} - \frac{1}{\ell})\ell} = \frac{k^2}{q} A(\ell, g) \quad B(\ell, g),
\]

(1.6) \[
\frac{(f - g)(f - g) - (\frac{k}{q} - \ell q)(\frac{k}{q} - \ell)\frac{q}{k}}{(\frac{f}{k} - \frac{g}{\ell q})(\frac{f}{k} - \frac{q}{\ell}) - (\frac{q}{k} - \frac{1}{\ell q})(\frac{q}{k} - \frac{1}{\ell})\frac{k}{q}} = q\ell^2 A(\frac{k}{q}, \overline{f}) \quad B(\frac{k}{q}, \overline{f}),
\]

where $A(h, x)$, $B(h, x)$ are polynomials in $x$ of degree 4 given by

\[
A(h, x) = \left( \frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8 \right) + \left( \frac{m_1}{h^2} - m_5 + 2hm_7 \right)x + \left( \frac{-m_0}{h^3} + m_6 - 3hm_8 \right)x^2 - m_7x^3 + m_8x^4,
\]

(1.7) \[
B(h, x) = \left( \frac{m_0}{h^2} - \frac{m_2}{h} + m_4 - hm_6 + h^2m_8 \right) + \left( \frac{2m_1}{h^2} - \frac{m_3}{h} + hm_7 \right)x + \left( \frac{3m_0}{h^3} + \frac{m_2}{h^2} - hm_8 \right)x^2 - \frac{m_1}{h^3}x^3 + \frac{m_0}{h^4}x^4,
\]

\[
U(z) = \frac{1}{z^4} \prod_{i=1}^{8} (z - u_i) = \frac{1}{z^4} \sum_{i=0}^{8} (-1)^i m_i z_i.
\]

Though it is not so obvious in this form, the singularity of this equation are given by

\[
(f, g) = (F(u_i), G(u_i)), \quad (i = 1, \cdots, 8)
\]

where

(1.8) \[
F(u) = u + \frac{k}{u}, \quad G(u) = u + \frac{\ell}{u}.
\]

These points are on the curve of bi-degree (2, 2)

(1.9) \[
(f - g)\left( \frac{f}{k} - \frac{g}{\ell} \right) - (k - \ell)(\frac{1}{k} - \frac{1}{\ell}) = 0,
\]

which has a node at $(\infty, \infty)$. 
In the next section, we will rewrite the $q$-$E_{8}^{(1)}$ equation in simpler form where the singularity structure is manifest (see Theorem 2.2).

Example 1.5. The most generic equation, the elliptic-$E_{8}^{(1)}$ [13][11], is more complicated than $q$-$E_{8}^{(1)}$ case. The 8 points are on a smooth bi-degree $(2, 2)$ curve (i.e. an elliptic curve):

There have been many challenges to obtain an explicit expression of the elliptic $E_{8}^{(1)}$ equation (e.g.[4][7][8]). We will give one simple expression (Theorem 3.1) which was obtained in [10] by a similar method as the $q$-case discussed below.

§ 2. An approach from the Padé interpolation

There exists a simple method to derive a Lax pair for Painlevé equations[15]. Using a discrete version of it, we will derive a simple form of $q$-Painlevé equation of type $E_{8}^{(1)}$. Here, we use parameters $a_{1}, a_{2}, a_{3}, b_{1}, b_{2}, b_{3}, k, \ell \in \mathbb{C}$ and $m, n \in \mathbb{Z}_{\geq 0}$ with constraints

\begin{equation}
q^{-1}k\ell = q^{-n}a_{1}a_{2}a_{3} = q^{-m}b_{1}b_{2}b_{3}.
\end{equation}

The time evolution is $\overline{k} = \frac{k}{q}$, $\overline{\ell} = q\ell$ (and $\overline{x} = x$ for $x = a_{i}, b_{i}, m, n$). The parameters $u_{i}$ in Example 1.4 are related to the parameters $a_{i}, b_{i}, k, \ell, m, n$ by $(u_{1}, \cdots, u_{8}) = (a_{1}, a_{2}, a_{3}, q^{-m-n}, b_{1}, b_{2}, b_{3}, q)$. 

A simple expression for discrete Painlevé equations

Our starting point is the following Padé interpolation

\[ Y_s = \frac{q^{ns}(k_{a_1}, k_{a_2}, k_{a_3}, b_{1}, b_{2}, b_{3})}{q^{ms}(a_{1}, a_{2}, a_{3}, k_{b_{1}}, k_{b_{2}}, k_{b_{3}})} = \frac{P_m}{Q_n}, \quad x = x_s, \quad (s = 0, 1, \cdots, m+n), \quad (2.2) \]

\[ (a)_i = \prod_{j=0}^{i-1} (1 - aq^j), \quad (a, \cdots, b)_s = (a)_s \cdots (b)_s, \]

where \( P_m, Q_n \) are polynomials of degree \( m, n \) in variable \( x \). In the followings, we use variable \( z \) such that \( x = z + \frac{k}{qz} \), hence \( P_m, Q_n \) are Laurent polynomials of the form

\[ P_m(z) = \sum_{i=0}^{m} u_i (z + \frac{k}{qz})^i, \quad Q_n(z) = \sum_{i=0}^{n} v_i (z + \frac{k}{qz})^i. \quad (2.3) \]

The interpolating points are \( x_s = q^{-s} + kq^{s-1} \) (q-quadratic grid) in variable \( x \), and hence \( z_s = q^{-s} \) (q-grid) in variable \( z \).

The main ingredients are the contiguous relations satisfied by \( u(z) = P_m(z) \) and \( v(z) = Y(z)Q_n(z) \) where \( Y(z) \) is a function such as \( Y(q^{-s}) = Y_s \). For instance, the relation between \( y(z), y(\frac{z}{q}), \overline{y}(\frac{z}{q}) \) is obtained by evaluating the Casorati determinant

\[ \begin{vmatrix} y(z) & y(\frac{z}{q}) \overline{y}(\frac{z}{q}) \\ u(z) & u(\frac{z}{q}) \overline{u}(\frac{z}{q}) \\ v(z) & v(\frac{z}{q}) \overline{v}(\frac{z}{q}) \end{vmatrix} = 0. \quad (2.4) \]

This determinant divided by \( Y(z) \) is a Laurent polynomial in \( z \) and has many known zeros due to the interpolating condition \( u(z_s) = v(z_s) \). Hence one can determine the structure of the contiguous relations without knowing the explicit form of \( P_m \) and \( Q_n \).

Proposition 2.1. The following relations hold for \( y(z) = P_m(z), Y(z)Q_n(z) \):

\[ L_2(z) : \quad B_2(z) \left\{ g - G(\frac{k}{z}) \right\} y(z) - B_2(\frac{k}{z}) \left\{ g - G(z) \right\} y(\frac{z}{q}) = 0, \quad (2.5) \]

\[ + c \left\{ f - F(z) \right\} (z - \frac{k}{z}) \overline{y}(\frac{z}{q}) = 0, \]

\[ L_3(z) : \quad B_1(z) \left\{ g - G(\frac{k}{qz}) \right\} y(\frac{z}{q}) - B_1(\frac{k}{qz}) \left\{ g - G(z) \right\} \overline{y}(z) = 0, \quad (2.6) \]

\[ + \frac{w}{c} \left\{ f - F(z) \right\} (z - \frac{k}{qz}) y(z) = 0, \]

where

\[ B_1(z) = \frac{1}{z^2} \prod_{i=1}^{4} (z - u_i), \quad B_2(z) = \frac{1}{z^2} \prod_{i=5}^{8} (z - u_i), \quad (2.7) \]

\[ F(z) = z + \frac{k}{z}, \quad G(z) = z + \frac{\ell}{z}, \]
and $f, g, c, w$ are some constants (independent of $z$).

Combining $L_2$ and $L_3$, one can obtain the three term relation $L_1$ between $y(qz), y(z), y(z/q)$.

Though the explicit form of the $L_1$ equation is complicated, it can be characterized by the following properties [16]: (1) As a polynomial in $(f, g)$, it is of bi-degree $(3, 2)$. (2) It vanishes when $f = F(u), g = G(u)$ with $u = u_1, \cdots, u_8, qz, \frac{k}{z}$, and $f = F(u), \frac{g - G(\frac{k}{u})}{g - G(u)} = \frac{B_2(\frac{k}{u})}{B_2(u)}$ with $u = z, qz$. Hence, the $L_1$ equation is equivalent with the $L_1$ equation in [17] up to some gauge transformations, and the equations $L_2, L_3$ (or $L_1$) can be considered as a Lax pair for $q\text{-}E_8^{(1)}$.

**Theorem 2.2.** The compatibility of the equations $L_2, L_3$ (2.5)(2.6) is equivalent to the relations

\[
\frac{(f - F(z))\{\bar{f} - \bar{F}(z)\}}{(f - F(\frac{k}{z}))\{\bar{f} - \bar{F}(\frac{k}{z})\}} = \frac{U(z)}{U(\frac{k}{z})}, \quad \text{for } g = G(z),
\]

\[
\frac{(g - G(z))\{\bar{g} - \bar{G}(z)\}}{(g - G(\frac{k}{qz}))\{\bar{g} - \bar{G}(\frac{k}{qz})\}} = \frac{U(z)}{U(\frac{k}{qz})}, \quad \text{for } \bar{f} = \bar{F}(z),
\]

along with an additional relation

\[
w = \frac{(k - \ell)(k - q\ell)U(z)}{k^2(f - F(z))\{\bar{f} - \bar{F}(z)\}}, \quad \text{for } g = G(z),
\]

where $U(z) = B_1(z)B_2(z) = \frac{1}{z^4}\prod_{i=1}^{8}(z - u_i)$.

**Proof.** Putting $g = G(z)$ in equations $L_2(z)$ and $L_3(z)$, we have the relation (2.10). Since $G(z) = G(\frac{k}{z})$, the relation (2.10) holds also when $z$ is replaced by $\frac{k}{z}$. Taking the ratio of these two relations, we obtain the equation (2.8). Putting $\bar{f} = \bar{F}(z)$ in equations $L_2(z)$ and $L_3(z)$, we get the equation (2.9). Sufficiency of the equations (2.8) (2.9) (2.10) for the compatibility can be checked by a direct computation. \qed
The equations (2.8)(2.9) are the desired simple expression for the $q$-$E_8^{(1)}$. In fact, by eliminating the variable $z$, they correctly reproduce the equations (1.5)(1.6), where the polynomials $A, B$ are given by

\begin{equation}
A(h, z + \frac{h}{z}) = \frac{zU(z) - \frac{h}{z}U(h)}{z - \frac{h}{z}}, \quad B(h, z + \frac{h}{z}) = \frac{zU(h) - \frac{h}{z}U(z)}{z - \frac{h}{z}}.
\end{equation}

**Remark.** Up to now, we used only the defining relation $Y_s = \frac{P_m(x_s)}{Q_n(x_s)}$ for $P_m(x)$ and $Q_n(x)$. If we know the explicit forms of $P_m(x), Q_n(x)$, then we can determine the Painlevé variables $f, g$ explicitly. For the interpolation problem with general $Y_s$ and $x_s$, the following formula has been classically known by Cauchy and Jacobi

\begin{equation}
P_m(x) = f(x) \det \left( W_{i,j}^{(-)} \right)_{i,j=0}^{n}, \quad Q_n(x) = \det \left( W_{i,j}^{(+)\pm} \right)_{i,j=0}^{n-1}, \quad f(x) = \prod_{s=0}^{m+n} (x-x_s)\end{equation}

These expressions give the special solutions for $q$-$E_8^{(1)}$ Painlevé equation in terms of the 10W9 hypergeometric functions and their determinants (c.f.[5][6]).

**Remark.** The Padé approach to the degenerate cases were studied in [2][9]. The corresponding Padé problems are

<table>
<thead>
<tr>
<th>$q$-$P$</th>
<th>$q$-$E_7^{(1)}$</th>
<th>$q$-$E_6^{(1)}$</th>
<th>$q$-$D_5^{(1)}$</th>
<th>$q$-$A_4^{(1)}$</th>
<th>$q$-$A_{2+1}^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_s$</td>
<td>$(b_1, b_2, b_3)_s$</td>
<td>$(b_1, b_2)_s$</td>
<td>$(a_1, a_2)_s$</td>
<td>$(a_s)_s$</td>
<td>$c^s q^{\frac{s(s-1)}{2}}$</td>
</tr>
</tbody>
</table>

with the grid $x_s = q^s$. There is a constraint $a_1 a_2 a_3 q^m = b_1 b_2 b_3 q^n$ for $q$-$E_7^{(1)}$ case.

§ 3. Elliptic case

As before, we use multiplicative parameters $k, \ell, u_1, \cdots, u_8, q \ (k^2 \ell^2 = qu_1 \cdots u_8)$ and $p$, where $q$ is the base for the $q$-difference and $p$ is the period of the elliptic functions. Let $[z]$ be a theta function such that $[pz] = [z^{-1}] = -z^{-1}[z]$, and define

\begin{equation}
a(z) = \left[ \frac{\alpha}{z} \right] [\sqrt[k]{\alpha z}], \quad b(z) = \left[ \frac{\beta}{z} \right] [\sqrt[k]{\beta z}], \quad c(z) = \left[ \frac{\ell}{z} \right] [\sqrt[\ell]{\ell z}], \quad d(z) = \left[ \frac{\alpha}{z} \right] [\sqrt[\alpha]{\alpha z}],
\end{equation}

\[1\]
Then the functions

\begin{equation}
F(z) = \frac{b(z)}{a(z)}, \quad G(z) = \frac{d(z)}{c(z)},
\end{equation}

are elliptic functions such that

\begin{equation}
F(pz) = F(z) = F\left(\frac{k}{z}\right), \quad G(pz) = G(z) = G\left(\frac{\ell}{z}\right),
\end{equation}

which gives the parametrization $f = F(z), g = G(z)$ of the elliptic curve in Example 1.5.

By the same method as the $q$-case in previous section, we have [10]

**Theorem 3.1.** The elliptic difference Painlevé equation of type $E_8^{(1)}$ can be written in the form $(k, \ell, f, g) \mapsto (k/q, q\ell, \overline{f}, \overline{g})$, where $\overline{f}, \overline{g}$ are given by

\begin{align}
\frac{\{a(\bar{z})f - b(\bar{z})\}\{\bar{a}(\bar{z})\bar{f} - \bar{b}(\bar{z})\}}{\{a(z)f - b(z)\}\{\bar{a}(z)\bar{f} - \bar{b}(z)\}} &= \frac{z^2}{\tilde{z}^2} \prod_{i=1}^{8} \left[ \frac{u_i}{\tilde{z}} \right] \left[ \frac{\tilde{u}_i}{z} \right], \quad \text{for} \quad g = G(z), \quad \tilde{z} = \frac{\ell}{z},
\end{align}

\begin{align}
\frac{\{c(\bar{z})g - d(\bar{z})\}\{\bar{c}(\bar{z})\bar{g} - \bar{d}(\bar{z})\}}{\{c(z)g - d(z)\}\{\bar{c}(z)\bar{g} - \bar{d}(z)\}} &= \frac{\tilde{z}^2}{z^2} \prod_{i=1}^{8} \left[ \frac{u_i}{\tilde{z}} \right] \left[ \frac{\tilde{u}_i}{z} \right], \quad \text{for} \quad \bar{f} = \overline{F}(z), \quad \tilde{z} = \frac{k}{qz}.
\end{align}

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**References**


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