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A simple expression for discrete Painlevé equations

By

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Abstract

A simple expression of discrete Painlevé equations and their Lax pair is obtained by using an interpolation problem. We discuss mainly the case of $q$-Painlevé equation of type $E_8^{(1)}$.

§ 1. Structure of discrete Painlevé equations

The second order discrete Painlevé equations were classified by Sakai [13] as follows:

- **Elliptic**
  \[ E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow A_{2+1}^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)} \]

- **Multiplicative**
  \[ E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow A_{2+1}^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)} \]

- **Additive**
  \[ E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow A_{2+1}^{(1)} \rightarrow A_{1+1}^{(1)} \rightarrow A_1^{(1)} \]

Each discrete Painlevé equation, represented as a rational map on $\mathbb{P}^1 \times \mathbb{P}^1$,

\[
T : (f, g) \mapsto (\bar{f}, \bar{g}) = \left( \frac{\psi_1(f, g)}{\psi_0(f, g)}, \frac{\phi_1(f, g)}{\phi_0(f, g)} \right),
\]

has eight singular points where $\psi_0 = \psi_1 = 0$ or $\phi_0 = \phi_1 = 0$. Conversely, a configuration of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ (or nine points on $\mathbb{P}^2$) characterize the Painlevé equation. Let us look at some examples.
Example 1.1. \( q-D_5^{(1)}[3] \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{a_3 a_4 b_1 b_2}{a_1 a_2 b_3 b_4} \).

\[
\overline{f} = \frac{(g - b_1 t)(g - b_2 t)}{(g - b_3)(g - b_4)} a_3 a_4, \quad \overline{g} = \frac{(f - a_1 t)(f - a_2 t)}{(f - a_3)(f - a_4)} b_3 b_4.
\]

(1.2)

The 8 singular points are on the four lines \( f = 0, f = \infty, g = 0 \) and \( g = \infty \):

Example 1.2. \( q-E_6^{(1)}[12][14][11] \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{b_5 b_6 b_7 b_8}{b_1 b_2 b_3 b_4} \).

\[
\frac{(fg - 1)(\overline{f}g - 1)}{fg} = \frac{qt^2(b_1 g - 1)(b_2 g - 1)(b_3 g - 1)(b_4 g - 1)}{b_5 b_6 (b_7 g t - 1)(b_8 g t - 1)},
\]

(1.3)

\[
\frac{(f\overline{g} - 1)(f\overline{g}t - 1)}{g\overline{g}} = \frac{q^2 t^2(b_1 - \overline{f})(b_2 - \overline{f})(b_3 - \overline{f})(b_4 - \overline{f})}{(b_5 - fqt)(b_6 - fqt)}. \]

The 8 singular points are on the two lines \( f = 0, g = 0 \) and one curve \( fg = 1 \):

Example 1.3. \( q-E_7^{(1)}[1][11] \): \( T(t, f, g) = (qt, \overline{f}, \overline{g}) \), \( q = \frac{b_5 b_6 b_7 b_8}{b_1 b_2 b_3 b_4} \).

\[
\frac{(fg - 1)(\overline{f}g - 1)}{(fgt^2 - 1)(\overline{f}gqt^2 - 1)} = \frac{(b_1 g - 1)(b_2 g - 1)(b_3 g - 1)(b_4 g - 1)}{(b_5 g t - 1)(b_6 g t - 1)(b_7 g t - 1)(b_8 g t - 1)},
\]

(1.4)

\[
\frac{(\overline{f}g - 1)(f\overline{g}qt^2 - 1)}{(\overline{f}gt^2 - 1)(f\overline{g}q^2 t^2 - 1)} = \frac{(b_1 - \overline{f})(b_2 - \overline{f})(b_3 - \overline{f})(b_4 - \overline{f})q}{(b_5 - \overline{f}qt)(b_6 - \overline{f}qt)(b_7 - \overline{f}qt)(b_8 - \overline{f}qt)}. \]

The 8 singular points are on the two curves \( fg = 1 \) and \( fgt^2 = 1 \):
A simple expression for discrete Painlevé equations

Example 1.4. \( q\cdot \text{E}^{(1)}_{8} \) [11]: 
\[
T(k, \ell, f, g) = (\frac{k}{q}, q\ell, \overline{f}, \overline{g}), \quad q = \frac{k^{2}\ell^{2}}{u_{1}\cdots u_{8}}.
\]

\[
\frac{(\overline{f} - g)(f - g) - (\frac{k}{q} - \ell)(k - \ell)}{(\frac{q}{k} - \frac{1}{\ell})(\frac{k}{q} - \frac{1}{\ell})\ell} = \frac{k^{2}}{q} A(\ell, g) B(\ell, g),
\]

\[
\frac{(\overline{f} - g)(\overline{f}) - (\frac{k}{q} - \ell q)(\frac{k}{q} - \ell)}{(\frac{q}{k} - \frac{1}{\ell q})(\frac{q}{k} - \frac{1}{\ell q})\ell} = q\ell^{2} A(\frac{k}{q}, \overline{f}) B(\frac{k}{q}, \overline{f}),
\]

where \( A(h, x) \), \( B(h, x) \) are polynomials in \( x \) of degree 4 given by

\[
A(h, x) = \left( \frac{m_{0}}{h^{2}} - \frac{m_{2}}{h} + m_{4} - hm_{6} + h^{2}m_{8} \right) + \left( \frac{m_{1}}{h^{2}} - m_{5} + 2hm_{7} \right)x + \left( \frac{m_{0}}{h^{3}} - m_{6} - 3hm_{8} \right)x^{2} - m_{7}x^{3} + m_{8}x^{4},
\]

\[
B(h, x) = \left( \frac{m_{0}}{h^{2}} - \frac{m_{2}}{h} + m_{4} - hm_{6} + h^{2}m_{8} \right) + \left( \frac{2m_{1}}{h^{2}} - \frac{m_{3}}{h} + hm_{7} \right)x + \left( \frac{3m_{0}}{h^{3}} + \frac{m_{2}}{h^{2}} - hm_{8} \right)x^{2} - \frac{m_{1}}{h^{3}}x^{3} + \frac{m_{0}}{h^{4}}x^{4},
\]

\[
U(z) = \frac{1}{z^{4}} \prod_{i=1}^{8} (z - u_{i}) = \frac{1}{z^{4}} \sum_{i=0}^{8} (-1)^{i}m_{i}z^{i}.
\]

Though it is not so obvious in this form, the singularity of this equation are given by

\[
(f, g) = (F(u_{i}), G(u_{i})) \quad (i = 1, \cdots, 8)
\]

where

\[
(1.8) \quad F(u) = u + \frac{k}{u}, \quad G(u) = u + \frac{\ell}{u}.
\]

These points are on the curve of bi-degree (2, 2)

\[
(1.9) \quad (f - g)(\frac{f}{k} - \frac{g}{\ell}) - (k - \ell)(\frac{1}{k} - \frac{1}{\ell}) = 0,
\]

which has a node at \((\infty, \infty)\).
In the next section, we will rewrite the $q$-$E_{8}^{(1)}$ equation in simpler form where the singularity structure is manifest (see Theorem.2.2).

**Example 1.5.** The most generic equation, the elliptic-$E_{8}^{(1)}$ [13][11], is more complicated than $q$-$E_{8}^{(1)}$ case. The 8 points are on a smooth bi-degree $(2, 2)$ curve (i.e. an elliptic curve):

![Elliptic Curve](image)

There have been many challenges to obtain an explicit expression of the elliptic $E_{8}^{(1)}$ equation (e.g.[4][7][8]). We will give one simple expression (Theorem.3.1) which was obtained in [10] by a similar method as the $q$-case discussed below.

§2. **An approach from the Padé interpolation**

There exists a simple method to derive a Lax pair for Painlevé equations[15]. Using a discrete version of it, we will derive a simple form of $q$-Painlevé equation of type $E_{8}^{(1)}$. Here, we use parameters $a_{1}, a_{2}, a_{3}, b_{1}, b_{2}, b_{3}, k, \ell \in \mathbb{C}$ and $m, n \in \mathbb{Z}_{\geq 0}$ with constraints

\[
q^{-1}k\ell = q^{-n}a_{1}a_{2}a_{3} = q^{-m}b_{1}b_{2}b_{3}.
\]

The time evolution is $\overline{k} = \frac{k}{q}, \overline{\ell} = q\ell$ (and $\overline{x} = x$ for $x = a_{i}, b_{i}, m, n$). The parameters $u_{i}$ in Example.1.4 are related to the parameters $a_{i}, b_{i}, k, \ell, m, n$ by $(u_{1}, \cdots, u_{8}) = (a_{1}, a_{2}, a_{3}, q^{-m-n}, b_{1}, b_{2}, b_{3}, q)$. 
Our starting point is the following Padé interpolation

\[
Y_s = \frac{q^{ns}(\frac{k}{a_1}, \frac{k}{a_2}, \frac{k}{a_3}, b_1, b_2, b_3)_s}{q^{ms}(a_1, a_2, a_3, \frac{k}{b_1}, \frac{k}{b_2}, \frac{k}{b_3})_s} = \frac{P_m}{Q_n}, \quad x = x_s, \quad (s = 0, 1, \ldots, m + n),
\]

(2.2)

\[\prod_{j=0}^{i-1}(1-aj), \quad (a, \ldots, b)_s = (a)_s \cdots (b)_s,\]

where \(P_m, Q_n\) are polynomials of degree \(m, n\) in variable \(x\). In the followings, we use variable \(z\) such that \(x = z + \frac{k}{qz}\), hence \(P_m, Q_n\) are Laurent polynomials of the form

\[
P_m(z) = \sum_{i=0}^{m} u_i(z + \frac{k}{qz})^i, \quad Q_n(z) = \sum_{i=0}^{n} v_i(z + \frac{k}{qz})^i.
\]

(2.3)

The interpolating points are \(x_s = q^{-s} + kq^{s-1}\) \((q\text{-quadratic grid})\) in variable \(x\), and hence \(z_s = q^{-s}\) \((q\text{-grid})\) in variable \(z\).

The main ingredients are the contiguous relations satisfied by \(u(z) = P_m(z)\) and \(v(z) = Y(z)Q_n(z)\) where \(Y(z)\) is a function such as \(Y(q^{-s}) = Y_s\). For instance, the relation between \(y(z), y(\frac{z}{q}), \overline{y}(\frac{z}{q})\) is obtained by evaluating the Casorati determinant

\[
\begin{vmatrix}
  y(z) & y(\frac{z}{q}) & \overline{y}(\frac{z}{q}) \\
  u(z) & u(\frac{z}{q}) & \overline{u}(\frac{z}{q}) \\
  v(z) & v(\frac{z}{q}) & \overline{v}(\frac{z}{q})
\end{vmatrix} = 0.
\]

(2.4)

This determinant divided by \(Y(z)\) is a Laurent polynomial in \(z\) and has many known zeros due to the interpolating condition \(u(z_s) = v(z_s)\). Hence one can determine the structure of the contiguous relations without knowing the explicit form of \(P_m\) and \(Q_n\).

**Proposition 2.1.** The following relations hold for \(y(z) = P_m(z), Y(z)Q_n(z)\):

\[
L_2(z) : \quad B_2(z) \left\{ g - G(\frac{k}{z}) \right\} y(z) - B_2(\frac{k}{qz}) \left\{ g - G(z) \right\} y(\frac{z}{q}) + c \left\{ f - F(z) \right\} (z - \frac{k}{z}) \overline{y}(\frac{z}{q}) = 0,
\]

(2.5)

\[
L_3(z) : \quad B_1(z) \left\{ g - G(\frac{k}{qz}) \right\} y(\frac{z}{q}) - B_1(\frac{k}{qz}) \left\{ g - G(z) \right\} \overline{y}(z) + \frac{w}{c} \left\{ \overline{f} - \overline{F}(z) \right\} (z - \frac{k}{qz}) y(z) = 0,
\]

(2.6)

where

\[
B_1(z) = \frac{1}{z^2} \prod_{i=1}^{4}(z - u_i), \quad B_2(z) = \frac{1}{z^2} \prod_{i=5}^{8}(z - u_i),
\]

\[
F(z) = z + \frac{k}{z}, \quad G(z) = z + \frac{\ell}{z},
\]

(2.7)
and \( f, g, c, w \) are some constants (independent of \( z \)).

Combining \( L_2 \) and \( L_3 \), one can obtain the three term relation \( L_1 \) between \( y(qz), y(z), y(z/q) \).

Though the explicit form of the \( L_1 \) equation is complicated, it can be characterized by the following properties [16]: (1) As a polynomial in \( (f, g) \), it is of bi-degree \((3, 2)\).

(2) It vanishes when \( f = F(u) \), \( g = G(u) \) with \( u = u_1, \cdots, u_8, qz, \frac{k}{z} \), and \( f = F(u) \), \( g = G(\frac{k}{u}) \) with \( u = z, qz \). Hence, the \( L_1 \) equation is equivalent with the \( L_1 \) equation in [17] up to some gauge transformations, and the equations \( L_2, L_3 \) (or \( L_1 \)) can be considered as a Lax pair for \( qE_{8}^{(1)} \).

Theorem 2.2. The compatibility of the equations \( L_2, L_3 \) (2.5)(2.6) is equivalent to the relations

\[
\frac{(f - F(z))\{\overline{f} - \overline{F}(z)\}}{(f - F(\frac{k}{z}))\{\overline{f} - \overline{F}(\frac{k}{z})\}} = \frac{U(z)}{U(\frac{k}{z})}, \quad \text{for} \quad g = G(z),
\]

(2.8)

\[
\frac{(g - G(z))\{\overline{g} - \overline{G}(z)\}}{(g - G(\frac{k}{qz}))\{\overline{g} - \overline{G}(\frac{k}{qz})\}} = \frac{U(z)}{U(\frac{k}{qz})}, \quad \text{for} \quad \overline{f} = \overline{F}(z),
\]

(2.9)

along with an additional relation

\[
w = \frac{(k - \ell)(k - q\ell)U(z)}{k^2\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}, \quad \text{for} \quad g = G(z).
\]

(2.10)

where \( U(z) = B_1(z)B_2(z) = \frac{1}{z^4} \prod_{i=1}^{8} (z - u_i) \).

Proof. Putting \( g = G(z) \) in equations \( L_2(z) \) and \( L_3(z) \), we have the relation (2.10). Since \( G(z) = G(\frac{k}{z}) \), the relation (2.10) holds also when \( z \) is replaced by \( \frac{k}{z} \). Taking the ratio of these two relations, we obtain the equation (2.8). Putting \( \overline{f} = \overline{F}(z) \) in equations \( L_2(z) \) and \( L_3(z) \), we get the equation (2.9). Sufficiency of the equations (2.8) (2.9) (2.10) for the compatibility can be checked by a direct computation. \( \square \)
The equations (2.8)(2.9) are the desired simple expression for the $q$-$E_{8}^{(1)}$. In fact, by eliminating the variable $z$, they correctly reproduce the equations (1.5)(1.6), where the polynomials $A, B$ are given by

\begin{equation}
A(h, z + \frac{h}{z}) = \frac{zU(z) - \frac{h}{z}U(\frac{h}{z})}{z - \frac{h}{z}}, \quad B(h, z + \frac{h}{z}) = \frac{zU(\frac{h}{z}) - \frac{h}{z}U(z)}{z - \frac{h}{z}}.
\end{equation}

Remark. Up to now, we used only the defining relation $Y_s = \frac{P_m(x_s)}{Q_n(x_s)}$ for $P_m(x)$ and $Q_n(x)$. If we know the explicit forms of $P_m(x), Q_n(x)$, then we can determine the Painlevé variables $f, g$ explicitly. For the interpolation problem with general $Y_s$ and $x_s$, the following formula has been classically known by Cauchy and Jacobi

\begin{equation}
P_m(x) = f(x) \det \left( W_{i,j}^{-} \right)_{i,j=0}^{n}, \quad Q_n(x) = \det \left( W_{i,j}^{+} \right)_{i,j=0}^{n-1},
\end{equation}

where

\begin{equation}
W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{Y_s}{f'(x_s) x_s^{i+j} (x-x_s)^{\pm 1}}, \quad f(x) = \prod_{s=0}^{m+n} (x-x_s).
\end{equation}

Applying this for the $q$-quadratic grid: $x = z + \frac{k}{q^2 z}, \quad x_s = q^{-s} + kq^{s-1}$, we have

\begin{equation}
W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n} \frac{(1-kq^{2s-1})}{(1-kq^{-1})} \frac{(k/q, q^{-m-n})_s}{(q,kq^{m+n})_s} Y_s q^{(n-m)s} x_s^{i+j} (x-x_s)^{\pm 1}.
\end{equation}

These expressions give the special solutions for $q$-$E_{8}^{(1)}$ Painlevé equation in terms of the $10 \, W_9$ hypergeometric functions and their determinants (c.f.[5][6]).

Remark. The Padé approach to the degenerate cases were studied in [2][9]. The corresponding Padé problems are

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$q$-$P$ & $q$-$E_{7}^{(1)}$ & $q$-$E_{6}^{(1)}$ & $q$-$D_{5}^{(1)}$ & $q$-$A_{4}^{(1)}$ & $q$-$A_{2}^{(1)}$ \\
\hline
$Y_s$ & $\frac{(b_1, b_2, b_3)_s}{(a_1, a_2, a_3)_s}$ & $\frac{(b_1, b_2)_s}{(a_1, a_2)_s}$ & $cs(s)_s$ & $cs(s)_s$ & $cs(s)_s$ \\
\hline
\end{tabular}
\end{table}

with the grid $x_s = q^s$. There is a constraint $a_1 a_2 q^m = b_1 b_2 q^n$ for $q$-$E_{7}^{(1)}$ case.

§ 3. Elliptic case

As before, we use multiplicative parameters $k, \ell, u_1, \cdots, u_8, q (k^2 \ell^2 = qu_1 \cdots u_8)$ and $p$, where $q$ is the base for the $q$-difference and $p$ is the period of the elliptic functions. Let $[z]$ be a theta function such that $[pz] = [z^{-1}] = -z^{-1}[z]$, and define

\begin{equation}
a(z) = \frac{\alpha}{z} [\frac{k}{\alpha z}], \quad b(z) = \frac{\beta}{z} [\frac{k}{\beta z}],
\end{equation}

\begin{equation}
c(z) = \frac{\alpha}{z} [\frac{\ell}{\alpha z}], \quad d(z) = \frac{\beta}{z} [\frac{\ell}{\beta z}],
\end{equation}

where $\alpha, \beta, \ell, k$ are constants.
(α ≠ β). Then the functions

\[(3.2) \quad F(z) = \frac{b(z)}{a(z)}, \quad G(z) = \frac{d(z)}{c(z)},\]

are elliptic functions such that

\[(3.3) \quad F(pz) = F(z) = F\left(\frac{k}{z}\right), \quad G(pz) = G(z) = G\left(\frac{\ell}{z}\right),\]

which gives the parametrization \(f = F(z), \quad g = G(z)\) of the elliptic curve in Example 1.5. By the same method as the \(q\)-case in previous section, we have [10]

**Theorem 3.1.** The elliptic difference Painlevé equation of type \(E_{8}^{(1)}\) can be written in the form \((k, \ell, f, g) \mapsto (k/q, q\ell, \overline{f}, \overline{g})\), where \(\overline{f}, \overline{g}\) are given by

\[(3.4) \quad \frac{(a(\bar{z})f - b(\bar{z})) (a(z)f - b(z))}{(a(z)f - b(z))(a(\bar{z})f - b(\bar{z}))} = \frac{z^2}{z^2} \prod_{i=1}^{8} \left[\frac{u_i}{z}\right], \quad \text{for} \quad g = G(z), \quad \bar{z} = \frac{\ell}{z},\]

\[(3.5) \quad \frac{(c(\bar{z})g - d(\bar{z})) (c(z)g - d(z))}{(c(z)g - d(z))(c(\bar{z})g - d(\bar{z}))} = \frac{z^2}{z^2} \prod_{i=1}^{8} \left[\frac{u_i}{z}\right], \quad \text{for} \quad \bar{f} = \overline{F}(z), \quad \bar{z} = \frac{k}{qz}.\]

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**References**


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