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# Trace identity for parabolic elements of $SL(2,\mathbb{C})$, II

*Dedicated to Professor Hiroshige Shiga on the occassion of his sixtieth birthday*

By

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**Abstract**

Let $\mathcal{P}$ be the set of all parabolic elements in $SL(2,\mathbb{C})$ with trace $-2$. If $P_1$ and $P_2$ in $\mathcal{P}$ do not commute, then the complex lambda length between $P_1$ and $P_2$ is the trace of a matrix $Q \in SL(2,\mathbb{C})$ satisfying $Q^2 = -P_1P_2$, which is determined uniquely up to sign. For each $n$-gon $(P_1, P_2, ..., P_n)$ in $\mathcal{P}$ consider the tuples $(Q_1, Q_2, ..., Q_n)$ with $Q_i^2 = -P_iP_{i+1}$ with $P_{n+1} = P_1$. The tuples are classified into tuples of $(-)$-system and tuples of $(+)$-system. Suppose that $(P_1, ..., P_n)$ is divided into subpolygons $(P_1, P_2, ..., P_m)$ and $(P_1, P_m, P_{m+1}, ..., P_n)$, and $R_m$ and $S_m \in SL(2,\mathbb{C})$ with $R_m^2 = -P_mP_1$, $S_m^2 = -P_1P_m$ and $\mathrm{tr}R_m = \mathrm{tr}S_m$ are given. We show that if $(Q_1, ..., Q_{m-1}, R_m)$ and $(S_m, Q_m, ..., Q_n)$ are $(-)$-systems, then $(Q_1, Q_2, ..., Q_n)$ is also a $(-)$-system.

## § 1. Introduction and the main result

This paper is a continuation of [4] which established the "ideal Ptolemy identity" for complex $\lambda$-lengths introduced in [2] and [3] following Penner's paper [5]. We define

$$\mathcal{P} = \{P \in SL(2,\mathbb{C}) : P \text{ is parabolic with } \mathrm{tr}P = -2\}.$$

Note that $\mathcal{P}$ is the conjugacy class of

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \tag{1.1}$$

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and hence two matrices in $\mathcal{P}$ are conjugate to each other in $SL(2,\mathbb{C})$. If two elements $P_1$ and $P_2 \in \mathcal{P}$ do not commute, then there exists a square root $Q$ of $-P_1P_2$, that is, a matrix in $SL(2,\mathbb{C})$ such that

$$Q^2 = -P_1P_2. \tag{1.2}$$

$Q$ is determined up to sign, satisfies $\mathrm{tr}(P_1P_2) = 2 - (\mathrm{tr}Q)^2$ and also

$$P_2 = Q^{-1}P_1Q, \text{ and } Q^{-1}P_1 \text{ and } Q^{-1}P_2 \text{ are elliptic of order 2.} \tag{1.3}$$

(Here the order of an elliptic $A$ in $SL(2,\mathbb{C})$ means the order of the Möbius transformation $A(z)$.) In order to see this, it suffices to consider the normalized pair

$$P_1 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \; P_2 = \begin{pmatrix} -1 & 0 \\ \lambda & -1 \end{pmatrix}$$

with $\lambda \neq 0$. Then $Q$ must be of the form

$$Q = \pm \begin{pmatrix} \sqrt{\lambda} & -1/\sqrt{\lambda} \\ \sqrt{\lambda} & 0 \end{pmatrix}.$$

With this we can verify (1.3) and also

$$\mathrm{tr}Q \neq 0. \tag{1.4}$$

In what follows the diagram

$$P_1 \xrightarrow{Q} P_2 \tag{1.5}$$

means that $P_1$ and $P_2 \in \mathcal{P}$ do not commute and $Q^2 = -P_1P_2$.

*Definition* 1.1. A cycle $(P_1, P_2, ..., P_n)$, $P_{n+1} = P_1$, of elements in $\mathcal{P}$ is called an *$n$-gon* if $P_i$ and $P_j$ do not commute for $i \neq j$. If, in particular, $n = 3$ or 4, then it is called a *triangle* or *quadrangle*, respectively. Two $n$-gons $(P_1, P_2, ..., P_n)$ and $(R_1, R_2, ..., R_n)$ are *congruent* if there exists $T \in SL(2,\mathbb{C})$ such that $R_j = T^{-1}P_jT$ for $j = 1, ..., n$.

Let $(P_1, ..., P_n)$ be an $n$-gon in $\mathcal{P}$. Then there exists a square root $Q_i$ of $-P_iP_{i+1}$ for $i = 1, 2, ..., n$. Since from (1.3)

$$P_2 = Q_1^{-1}P_1Q_1, \; P_3 = Q_2^{-1}P_2Q_2, ..., \; P_1 = Q_n^{-1}P_nQ_n,$$

$Q_1Q_2 \cdots Q_n$ commutes with $P_1$ and hence $\mathrm{tr}Q_1Q_2 \cdots Q_n$ is either $-2$ or $+2$.

*Definition* 1.2. $(Q_1, Q_2, ..., Q_n)$ is called a *$(-)$-system* if $\mathrm{tr}Q_1Q_2 \cdots Q_n = -2$ and a *$(+)$-system* if $\mathrm{tr}Q_1Q_2 \cdots Q_n = +2$.

Let $(P_1, P_2, ..., P_n)$ be an $n$-gon and $Q_j$ be such that $P_j \xrightarrow{Q_j} P_{j+1}$ for $j = 1, ..., n$. If $2 < m < n$, then the "diagonal" $P_1P_m$ divides the $n$-gon into an $m$-gon $(P_1, P_2, ..., P_m)$ and an $(n - m + 1)$-gon $(P_1, P_m, P_{m+1}, ..., P_n)$. Choose $R_m$ and $S_m \in SL(2, \mathbb{C})$ such that

$$P_m \xrightarrow{R_m} P_1, \qquad P_1 \xrightarrow{S_m} P_m,$$

and that $\mathrm{tr}R_m = \mathrm{tr}S_m$. So $S_m = P_1R_mP_1^{-1}$. The main objective of this paper is to prove

*Theorem* 1.1. If two among $(Q_1, Q_2, ..., Q_{m-1}, R_m)$, $(S_m, Q_m, Q_{m+1}, ..., Q_n)$ and $(Q_1, Q_2, ..., Q_n)$ are $(-)$-systems, then so is the rest.

In [4] we showed this theorem for $n = 4$ and $m = 3$. In this case, if both of $(Q_1, Q_2, R_3)$ and $(S_3, Q_3, Q_4)$ are $(-)$-systems, then $(Q_1, Q_2, Q_3, Q_4)$ is also a $(-)$-system. We choose $R_2$ and $S_2$ so that

$$P_2 \xrightarrow{R_2} P_4, \qquad P_4 \xrightarrow{S_2} P_2,$$

and that $\mathrm{tr}R_2 = \mathrm{tr}S_2$. See Figure 1. If $(Q_1, R_2, Q_4)$ is a $(-)$-system, then from Theorem 1.1, $(Q_2, Q_3, S_2)$ is also a $(-)$-system. In this situation the following "ideal Ptolemy identity" holds ([4, Theorem 0.1])

$$\mathrm{tr}R_2\mathrm{tr}R_3 = \mathrm{tr}Q_1\mathrm{tr}Q_3 + \mathrm{tr}Q_2\mathrm{tr}Q_4. \tag{1.6}$$

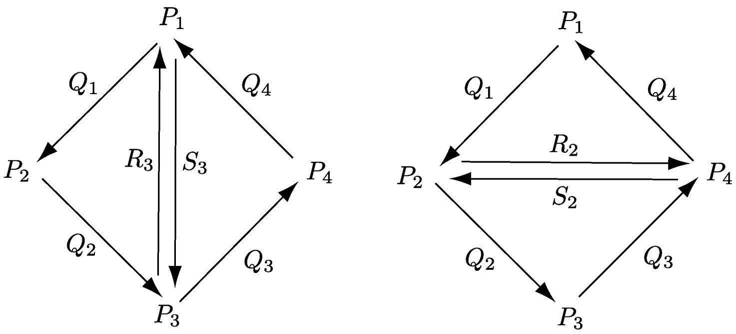


Figure 1. A decomposition of a quadrangle into triangles

Theorem 1.1 follows immediately from

*Lemma* 1.1. With the notation as above the following identity holds:

$$(\mathrm{tr}Q_1Q_2 \cdots Q_{m-1}R_m)(\mathrm{tr}S_mQ_m \cdots Q_n) = -2\mathrm{tr}Q_1Q_2 \cdots Q_n. \tag{1.7}$$

We prove (1.7) in Section 3.

*Remark* 1.1. Let $\bar{S}$ be an oriented closed surface of genus $g$ and $P = \{x_1, ..., x_n\}$ a non-empty set of distinct points on $\bar{S}$. Let $S = \bar{S} - P$. We assume that $2g-2+n > 0$. Let $\mathcal{R}(S)$ denote the space of all conjugacy classes of faithful representations $\rho : \pi_1(S) \to SL(2, \mathbb{C})$ such that if $\delta$ is the homotopy class of a loop which goes around a puncture $x_j$ once, then $\rho(\delta) \in \mathcal{P}$. Let $\Delta = \{c_1, c_2, ..., c_d\}$, where $d = 6g - 6 + 3n$, be an arbitrary ideal triangulation of $S$ (see [5]). Let $c = c_i \in \Delta$ and suppose that $x_j$ and $x_k$ are the end points of $c$. Choose a point $y$ of $c$. We define $\delta_1$ to be the loop which goes from $y$ to $x_j$ along $c$ and turns around $x_j$ in the positive direction and goes back to $y$ along $c$. We define $\delta_2$ in the same way for $x_k$. Choose an arc $\delta_0$ from the base point of $\pi_1(S)$ to $y$. Let $[\rho] \in \mathcal{R}(S)$. Then homotopy classes of $\delta_0\delta_1\delta_0^{-1}$ and $\delta_0\delta_2\delta_0^{-1}$ determine two elements $P_1 = \rho(\delta_0\delta_1\delta_0^{-1})$ and $P_2 = \rho(\delta_0\delta_2\delta_0^{-1})$ in $\mathcal{P}$. Since $\rho$ is faithful, $P_1$ and $P_2$ do not commute. Choose $Q_i$ so that $P_1 \xrightarrow{Q_i} P_2$. The value

$$\lambda_i = \lambda(c_i, \rho) = \mathrm{tr}Q_i.$$

depends only on the class $[\rho]$ and the homotopy class of $c_i$. This value $\lambda_i$ is called in [2] and [3] the *complex $\lambda$-length* of $c_i$ associated to $[\rho]$ . The positive branch of $\lambda_i$ restricted to the Fuchsian representation space of $\pi_1(S)$ coincides with the $\lambda$-length (for a special choice of horocycles around punctures) introduced by Penner [5].

Since $\lambda_i$ is determined up to sign, the tuple $(\lambda_1, ..., \lambda_d)$ defines a map $\underline{\Lambda}_\Delta : \mathcal{R}(S) \to (\mathbb{C}/\{\pm 1\})^d$. If it is restricted to, for example, the subspace $\mathcal{QF}$ of quasifuchsian representations, which is simply connected, the map $\underline{\Lambda}_\Delta$ can be lifted to a holomorphic injection $\Lambda_\Delta$ of $\mathcal{QF}$ into $\mathbb{C}^d$, and it is possible to choose a lift $\Lambda_\Delta$ so that $\lambda_1$,..., $\lambda_d$ satisfy the condition that $(Q_i, Q_j, Q_k)$ are $(-)$-systems for all triangles $(c_i, c_j, c_k)$ in $\Delta$, see [3] for details. By using (1.6) we can show just as in [5] that, for two ideal triangulations $\Delta_1$ and $\Delta_2$, the coordinate change between $\Lambda_{\Delta_1}(\mathcal{QF})$ and $\Lambda_{\Delta_2}(\mathcal{QF})$ is a rational transformation. Thus the faithful representation of the mapping class group of $S$ by a group of rational transformations for its action on the decorated Teichmüller space ([5]) is naturally extended to its action on $\mathcal{QF}$.

## § 2. Trace identities

We shall use repeatedly the following basic trace identities which hold for matrices in $SL(2, \mathbb{C})$ (see [1, 3.4]):

(2.1) $$\mathrm{tr}Y^{-1}XY = \mathrm{tr}X,$$

(2.2) $$\mathrm{tr}XY + \mathrm{tr}XY^{-1} = \mathrm{tr}X\mathrm{tr}Y,$$

From (2.1), $\mathrm{tr}X_1X_2\cdots X_n = \mathrm{tr}X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(n)}$ for any cyclic permutation $\sigma$ on $\{1, 2, ..., n\}$. So (2.2) yields

(2.3) $$\mathrm{tr}XYZ = \mathrm{tr}Y\mathrm{tr}XZ - \mathrm{tr}XY^{-1}Z$$

for $X$, $Y$ and $Z \in SL(2,\mathbb{C})$. The following trace identities are proved in [2, Proposition 1.1] and [4, Lemma 1.3], respectively.

*Lemma* 2.1. If $A$,$B$,$C$ and $D \in SL(2,\mathbb{C})$ are such that $\mathrm{tr}ABCD = -2$, then

$$\begin{aligned}&(\mathrm{tr}AB + \mathrm{tr}CD)(\mathrm{tr}BC + \mathrm{tr}AD)\\ &= (\mathrm{tr}A + \mathrm{tr}BCD)(\mathrm{tr}C + \mathrm{tr}ABD) + (\mathrm{tr}B + \mathrm{tr}ACD)(\mathrm{tr}D + \mathrm{tr}ABC).\end{aligned} \tag{2.4}$$

*Lemma* 2.2. Let $X$, $Y_1$,..., $Y_{n+1} \in SL(2,\mathbb{C})$, where $n \geq 1$. If $\mathrm{tr}Y_1 = \cdots = \mathrm{tr}Y_{n+1}$, then

$$\begin{aligned}&\sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_n}\mathrm{tr}XY_1^{\epsilon_1}Y_2^{\epsilon_1+\epsilon_2}\cdots Y_n^{\epsilon_{n-1}+\epsilon_n}Y_{n+1}^{\epsilon_n+1}\\ &= \sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_n}\mathrm{tr}XY_1^{\epsilon_1+1}Y_2^{\epsilon_1+\epsilon_2}\cdots Y_n^{\epsilon_{n-1}+\epsilon_n}Y_{n+1}^{\epsilon_n}.\end{aligned} \tag{2.5}$$

*Lemma* 2.3. Let $X \in SL(2,\mathbb{C})$ and $P_1$,..., $P_n \in \mathcal{P}$ with $n \geq 2$. Then

$$\begin{aligned}&\sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} \mathrm{tr}XP_1^{\epsilon_1}P_2^{\epsilon_2}\cdots P_n^{\epsilon_n}\\ &= \sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}+1}\mathrm{tr}XP_1^{1+\epsilon_1}P_2^{\epsilon_1+\epsilon_2}\cdots P_n^{\epsilon_{n-1}+\epsilon_n}.\end{aligned} \tag{2.6}$$

*Proof.* If $n = 2$, then by using (2.3) and $\mathrm{tr}P_1 = \mathrm{tr}P_2 = -2$, we can deform the right had side of (2.6) to the left hand side as follows:

$$\begin{aligned}&\sum_{\epsilon_1,\epsilon_2\in\{0,1\}} (-1)^{\epsilon_1+1}\mathrm{tr}XP_1^{1+\epsilon_1}P_2^{\epsilon_1+\epsilon_2} = -\mathrm{tr}XP_1 + \mathrm{tr}XP_1^2P_2 - \mathrm{tr}XP_1P_2 + \mathrm{tr}XP_1^2P_2^2\\ &= -\mathrm{tr}XP_1 + (-2\mathrm{tr}XP_1P_2 - \mathrm{tr}XP_2) - \mathrm{tr}XP_1P_2 + (-2\mathrm{tr}XP_1^2P_2 - \mathrm{tr}XP_1^2)\\ &= -\mathrm{tr}XP_1 - \mathrm{tr}XP_2 - 3\mathrm{tr}XP_1P_2\\ &\qquad +(-2(-2\mathrm{tr}XP_1P_2 - \mathrm{tr}XP_2) + 2\mathrm{tr}XP_1 + \mathrm{tr}X)\\ &= \mathrm{tr}X + \mathrm{tr}XP_1 + \mathrm{tr}XP_2 + \mathrm{tr}XP_1P_2.\end{aligned}$$

We prove (2.6) for $n > 2$ by induction. We divide the sum in the right hand side into the sum for $\epsilon_1 = 0$ and that for $\epsilon_1 = 1$. Then it equals

$$\begin{aligned}&\sum_{\epsilon_2,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_2+\cdots+\epsilon_{n-1}+1}\mathrm{tr}XP_1P_2^{-1}P_2^{1+\epsilon_2}P_3^{\epsilon_2+\epsilon_3}\cdots P_n^{\epsilon_{n-1}+\epsilon_n}\\ &- \sum_{\epsilon_2,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_2+\cdots+\epsilon_{n-1}+1}\mathrm{tr}XP_1^2P_2^{1+\epsilon_2}P_3^{\epsilon_2+\epsilon_3}\cdots P_n^{\epsilon_{n-1}+\epsilon_n}.\end{aligned}$$

We assume that (2.6) holds for $n-1$ and we apply it to $P_2$,..., $P_n$ and $X$ replaced by $XP_1P_2^{-1}$ and $XP_1^2$. Then the last term equals

$$\sum_{\epsilon_2,\dots,\epsilon_n\in\{0,1\}} \mathrm{tr}XP_1P_2^{-1}P_2^{\epsilon_2}\cdots P_n^{\epsilon_n} - \sum_{\epsilon_2,\dots,\epsilon_n\in\{0,1\}} \mathrm{tr}XP_1^2P_2^{\epsilon_2}\cdots P_n^{\epsilon_n}. \tag{2.7}$$

Let $Y = P_2^{\epsilon_2}P_3^{\epsilon_3}\cdots P_n^{\epsilon_n}$. From (2.3) $\mathrm{tr}XP_1P_2^{-1}Y = -\mathrm{tr}XP_1P_2Y - 2\mathrm{tr}XP_1Y$ and $\mathrm{tr}XP_1^2Y = -2\mathrm{tr}XP_1Y - \mathrm{tr}XY$. Then we have with $Z = P_3^{\epsilon_3}\cdots P_n^{\epsilon}$

$$\begin{aligned}
&\sum_{\epsilon_2\in\{0,1\}} \mathrm{tr}XP_1P_2^{-1}(P_2^{\epsilon_2}Z) - \sum_{\epsilon_2\in\{0,1\}} \mathrm{tr}XP_1^2(P_2^{\epsilon_2}Z)\\
&= -\sum_{\epsilon_2\in\{0,1\}} \mathrm{tr}XP_1P_2P_2^{\epsilon_2}Z + \sum_{\epsilon_2\in\{0,1\}} \mathrm{tr}XP_2^{\epsilon_2}Z\\
&= -\mathrm{tr}XP_1P_2Z - \mathrm{tr}XP_1P_2^2Z + \mathrm{tr}XZ + \mathrm{tr}XP_2Z\\
&= -\mathrm{tr}XP_1P_2Z - (-2\mathrm{tr}XP_1P_2Z - \mathrm{tr}XP_1Z) + \mathrm{tr}XZ + \mathrm{tr}XP_2Z\\
&= \mathrm{tr}XZ + \mathrm{tr}XP_1Z + \mathrm{tr}XP_2Z + \mathrm{tr}XP_1P_2Z.
\end{aligned}$$

Summing the last term over $\epsilon_3$,..., $\epsilon_n$, we obtain the left hand side of (2.6). Thus (2.6) holds for all $n$. □

*Lemma* 2.4. Let $P_1, P_2 \in \mathcal{P}$ and $X, Y \in SL(2,\mathbb{C})$. Then

$$\sum_{\epsilon_1,\epsilon_2\in\{0,1\}} \mathrm{tr}P_2^{\epsilon_1}P_1^{\epsilon_2}Y\cdot \sum_{\epsilon_3,\epsilon_4\in\{0,1\}} \mathrm{tr}P_1^{\epsilon_3}P_2^{\epsilon_4}X = (\mathrm{tr}P_1P_2-2) \sum_{\epsilon_1,\epsilon_2\in\{0,1\}} \mathrm{tr}P_1^{\epsilon_1}YP_2^{\epsilon_2}X. \tag{2.8}$$

*Proof.* We can substitute $A = P_1$, $B = P_1^{-1}XP_1$, $C = P_1^{-1}X^{-1}Y^{-1}$ and $D = YP_2$ into (2.4), because $\mathrm{tr}ABCD = \mathrm{tr}P_2 = -2$. We have

$$\begin{aligned}
&\mathrm{tr}A + \mathrm{tr}BCD = \mathrm{tr}P_1 + \mathrm{tr}P_1^{-1}P_2\\
&= \mathrm{tr}P_1 + (-2\mathrm{tr}P_1 - \mathrm{tr}P_1P_2) = -\mathrm{tr}P_1 - \mathrm{tr}P_1P_2.
\end{aligned}$$

Likewise we obtain

$$\begin{aligned}
&\mathrm{tr}A + \mathrm{tr}BCD = 2 - \mathrm{tr}P_1P_2, && \mathrm{tr}B + \mathrm{tr}ACD = -\mathrm{tr}X - \mathrm{tr}XP_2,\\
&\mathrm{tr}C + \mathrm{tr}ABD = \mathrm{tr}XP_1Y + \mathrm{tr}XP_1YP_2, && \mathrm{tr}D + \mathrm{tr}ABC = \mathrm{tr}Y + \mathrm{tr}YP_2,\\
&\mathrm{tr}AB + \mathrm{tr}CD = -\mathrm{tr}XP_1 - \mathrm{tr}XP_1P_2, && \mathrm{tr}BC + \mathrm{tr}AD = \mathrm{tr}P_1Y + \mathrm{tr}P_1YP_2.
\end{aligned}$$

Therefore (2.4) in this case equals

$$\begin{aligned}
&(\mathrm{tr}XP_1 + \mathrm{tr}XP_1P_2)(\mathrm{tr}P_1Y + \mathrm{tr}P_1YP_2)\\
&= (\mathrm{tr}P_1P_2 - 2)(\mathrm{tr}XP_1Y + \mathrm{tr}XP_1YP_2) + (\mathrm{tr}X + \mathrm{tr}XP_2)(\mathrm{tr}Y + \mathrm{tr}YP_2).
\end{aligned} \tag{2.9}$$

Substituting $P_1^{-1}Y$ to $Y$ in this equation, we obtain

$$\begin{aligned}
&(\mathrm{tr}XP_1 + \mathrm{tr}XP_1P_2)(\mathrm{tr}Y + \mathrm{tr}YP_2) \\
&= (\mathrm{tr}P_1P_2 - 2)(\mathrm{tr}XY + \mathrm{tr}XYP_2) + (\mathrm{tr}X + \mathrm{tr}XP_2)(\mathrm{tr}P_1^{-1}Y + \mathrm{tr}P_1^{-1}YP_2). \\
&= (\mathrm{tr}P_1P_2 - 2)(\mathrm{tr}XY + \mathrm{tr}XYP_2) \\
&\qquad + (\mathrm{tr}X + \mathrm{tr}XP_2)(-2\mathrm{tr}Y - \mathrm{tr}P_1Y - 2\mathrm{tr}YP_2 - \mathrm{tr}P_1YP_2).
\end{aligned} \tag{2.10}$$

By adding (2.9) and (2.10) we obtain (2.8).

## § 3. Proof of the main theorem

Let $(P_1, P_2, ..., P_n)$ be an $n$-gon in $\mathcal{P}$, where $n \geq 4$, and $Q_i \in SL(2, \mathbb{C})$ be such that $P_i \xrightarrow{Q_i} P_{i+1}$ for $i = 1, 2, ..., n$.

*Lemma* 3.1.

$$\mathrm{tr}Q_1\mathrm{tr}Q_2 \cdots \mathrm{tr}Q_n\mathrm{tr}Q_1 \cdots Q_n = \sum_{\epsilon_1,\dots,\epsilon_n \in \{0,1\}} 2\mathrm{tr}P_1^{\epsilon_1}P_2^{\epsilon_2} \cdots P_n^{\epsilon_n}. \tag{3.1}$$

*Proof.* By (2.2) we have with $X_{n-1} = Q_1 \cdots Q_{n-1}$

$$\mathrm{tr}Q_n\mathrm{tr}Q_1 \cdots Q_n = \mathrm{tr}X_{n-1}Q_n^2 + \mathrm{tr}X_{n-1}Q_nQ_n^{-1} = \mathrm{tr}X_{n-1}Q_n^2 + \mathrm{tr}X_{n-1}$$

and then with $X_{n-2} = Q_1 \cdots Q_{n-2}$

$$\begin{aligned}
&\mathrm{tr}Q_{n-1}\mathrm{tr}Q_n\mathrm{tr}Q_1 \cdots Q_n \\
&= (\mathrm{tr}Q_n^2X_{n-2}Q_{n-1}^2 + \mathrm{tr}Q_n^2X_{n-2}) + (\mathrm{tr}X_{n-2}Q_{n-1}^2 + \mathrm{tr}X_{n-2}) \\
&= \sum_{\epsilon_{n-1},\epsilon_n \in \{0,1\}} \mathrm{tr}X_{n-2}Q_{n-1}^{2\epsilon_{n-1}}Q_n^{2\epsilon_n}..
\end{aligned}$$

By proceeding in this manner we have

$$\mathrm{tr}Q_1\mathrm{tr}Q_2 \cdots \mathrm{tr}Q_n\mathrm{tr}Q_1 \cdots Q_n = \sum_{\epsilon_1,\dots,\epsilon_n \in \{0,1\}} \mathrm{tr}Q_1^{2\epsilon_1}Q_2^{2\epsilon_2} \cdots Q_n^{2\epsilon_n}.$$

Thus

$$\begin{aligned}
&\mathrm{tr}Q_1\mathrm{tr}Q_2 \cdots \mathrm{tr}Q_n\mathrm{tr}Q_1 \cdots Q_n \\
&= \sum_{\epsilon_1,\dots,\epsilon_n \in \{0,1\}} \mathrm{tr}(-P_1P_2)^{\epsilon_1}(-P_2P_3)^{\epsilon_2} \cdots (-P_nP_1)^{\epsilon_n} \\
&= \sum_{\epsilon_1,\dots,\epsilon_n \in \{0,1\}} (-1)^{\epsilon_1+\epsilon_2+\dots+\epsilon_n}\mathrm{tr}P_1^{\epsilon_n+\epsilon_1}P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}+\epsilon_n}
\end{aligned}$$

We divide the last sum into the sum for $\epsilon_n = 0$ and the sum for $\epsilon_n = 1$ and apply (2.5) to the second term by setting $X = P_1$ and $Y_i = P_i$ for $i = 1, ..., n$. Then we obtain

$$\begin{aligned}
&\sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} \mathrm{tr} P_1^{\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}} \\
&\quad + \sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{1+\epsilon_1+\cdots+\epsilon_{n-1}} \mathrm{tr} P_1^{1+\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}+1} \\
&= \sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} \mathrm{tr} P_1^{\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}} \\
&\quad + \sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{1+\epsilon_1+\cdots+\epsilon_{n-1}} \mathrm{tr} P_1^{2+\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}}. \qquad (3.2)
\end{aligned}$$

Let $Y = P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}}$. Then from (2.3)

$$\mathrm{tr} P_1^{\epsilon_1} Y - \mathrm{tr} P_1^{2+\epsilon_1} Y = 2\mathrm{tr} P_1^{1+\epsilon_1} Y + 2\mathrm{tr} P_1^{\epsilon_1} Y.$$

Taking the sum over $\epsilon_1, ..., \epsilon_{n-1}$ we see that (3.2) equals

$$\begin{aligned}
&\sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} 2\mathrm{tr} P_1^{1+\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}} \\
&\quad + \sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} 2\mathrm{tr} P_1^{\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}}.
\end{aligned}$$

We apply (2.5) to the first term in this expression, then it equals

$$\begin{aligned}
&\sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} 2\mathrm{tr} P_1^{\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}+1} \\
&\quad + \sum_{\epsilon_1,\dots,\epsilon_{n-1}\in\{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} 2\mathrm{tr} P_1^{\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}} \\
&= \sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1}} 2\mathrm{tr} P_1^{\epsilon_1} P_2^{\epsilon_1+\epsilon_2} \cdots P_n^{\epsilon_{n-1}+\epsilon_n}
\end{aligned}$$

Let $a_{(1,2,\dots,n)}$ denote the last expression. Then by dividing the sum in it into the sum for $\epsilon_1 = 0$ and the sum for $\epsilon_1 = 1$,

$$\begin{aligned}
a_{(1,2,\dots,n)} &= \sum_{\epsilon_2,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_2+\cdots+\epsilon_{n-1}} 2\mathrm{tr} P_2^{\epsilon_2} P_3^{\epsilon_2+\epsilon_3} \cdots P_n^{\epsilon_{n-1}+\epsilon_n} \\
&\quad + \sum_{\epsilon_2,\dots,\epsilon_n\in\{0,1\}} (-1)^{1+\epsilon_2+\cdots+\epsilon_{n-1}} 2\mathrm{tr} P_1 P_2^{1+\epsilon_2} P_3^{\epsilon_2+\epsilon_3} \cdots P_n^{\epsilon_{n-1}+\epsilon_n}.
\end{aligned}$$

From (2.6) follows

$$a_{(1,2,\dots,n)} = a_{(2,3,..,n)} + \sum_{\epsilon_2,\dots,\epsilon_n\in\{0,1\}} 2\mathrm{tr} P_1 P_2^{\epsilon_2} P_3^{\epsilon_3} \cdots P_n^{\epsilon_n}. \qquad (3.3)$$

We have

$$\begin{aligned}
a_{((n-1)n)} &= \sum_{\epsilon_{n-1},\epsilon_n\in\{0,1\}} (-1)^{\epsilon_{n-1}} 2\mathrm{tr} P_{n-1}^{\epsilon_{n-1}} P_n^{\epsilon_{n-1}+\epsilon_n} \\
&= 2\mathrm{tr} I + 2\mathrm{tr} P_n - 2\mathrm{tr} P_{n-1}P_n - 2\mathrm{tr} P_{n-1}P_n^2 \\
&= 2\mathrm{tr} I + 2\mathrm{tr} P_n - 2\mathrm{tr} P_{n-1}P_n - 2(-2\mathrm{tr} P_{n-1}P_n - \mathrm{tr} P_{n-1}) \\
&= 2\mathrm{tr} I + 2\mathrm{tr} P_{n-1} + 2\mathrm{tr} P_n + 2\mathrm{tr} P_{n-1}P_n,
\end{aligned}$$

where $I$ is the unit matrix. From this and (3.3) we can obtain (3.1) by induction on $n$. □

Now we prove the identity (1.7) in Lemma 1.1 from which Theorem 1.1 is easily obtained. From (3.1) we see that

$$(\mathrm{tr} Q_1 \cdots \mathrm{tr} Q_{m-1} \mathrm{tr} R_m)(\mathrm{tr} Q_1 \cdots Q_{m-1} R_m) \cdot (\mathrm{tr} S_m \mathrm{tr} Q_m \cdots \mathrm{tr} Q_n)(\mathrm{tr} S_m Q_m \cdots Q_n)$$

equals

$$\sum_{\eta_m,\eta_1,\epsilon_2,\dots,\epsilon_{m-1}\in\{0,1\}} 2\mathrm{tr} P_m^{\eta_m} P_1^{\eta_1} P_2^{\epsilon_2} \cdots P_{m-1}^{\epsilon_{m-1}} \cdot \sum_{\epsilon_1,\epsilon_m,\dots,\epsilon_n\in\{0,1\}} 2\mathrm{tr} P_1^{\epsilon_1} P_m^{\epsilon_m} P_{m+1}^{\epsilon_{m+1}} \cdots P_n^{\epsilon_n}$$

By replacing $P_2$, $X$ and $Y$ in (2.8) by $P_m$, $P_{m+1}^{\epsilon_{m+1}} \cdots P_n^{\epsilon_n}$ and $P_2^{\epsilon_2} \cdots P_{m-1}^{\epsilon_{m-1}}$, respectively, we see that the last expression equals

$$\begin{aligned}
&4(\mathrm{tr} P_1 P_m - 2) \sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} P_1^{\epsilon_1} P_2^{\epsilon_2} \cdots P_n^{\epsilon_n} \\
&= -2(\mathrm{tr} R_m)^2 \mathrm{tr} Q_1 \mathrm{tr} Q_2 \cdots \mathrm{tr} Q_n \mathrm{tr} Q_1 Q_2 \cdots Q_n.
\end{aligned}$$

Here we used $-\mathrm{tr} P_1 P_m = \mathrm{tr} R_m^2 = (\mathrm{tr} R_m)^2 - 2$ and (3.1). Since $\mathrm{tr} R_m = \mathrm{tr} S_m$ and none of $\mathrm{tr} R_m$, $\mathrm{tr} Q_1$,..., $\mathrm{tr} Q_n$ are non-zero (see (1.4)), we obtain (1.7).

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