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A variation of McShane's identity for 2-bridge links and its possible generalization

By

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In his doctoral thesis [21], G. McShane proved the following striking identity concerning the lengths of simple closed geodesics on a once-punctured torus T with a complete hyperbolic structure of finite area:

$$\sum_{\gamma} \frac{1}{1+e^{l(\gamma)}} = \frac{1}{2}.$$

Here γ runs over all simple closed geodesics on T and $l(\gamma)$ denotes the length of γ.

This identity has been generalized to cusped hyperbolic surfaces by McShane himself [22]. Since then numerous variations of the identity have been obtained (see [2, 4, 7, 8, 9, 15, 20, 23, 28, 29, 30, 32]), and a wonderful application to the Weil-Petersson volume of the moduli spaces of bordered hyperbolic surfaces was established by Mirzakhani [20]. In [3], the first and the third authors, together with H. Miyachi, gave a review of this topic up to 2003, laying emphasis on [4], which gives a variation of McShane's identity for hyperbolic surface bundles.

The purpose of this note is to give a review of the joint work of the second and the third authors [15], which gives a variation of McShane's identity for 2-bridge links, and to report an experimental study by the first and the last authors aiming at a geometric

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proof of the variation, by generalizing it to a certain continuous family of hyperbolic cone manifolds.

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§ 1. Bowditch's generalization of McShane's identity

In this section, we recall a generalization of McShane's identity established by Bowditch [8] and slightly extended by [2] and [29].

Let $\boldsymbol{T} := (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ be the once-punctured torus. Then the curve complex of $\boldsymbol{T}$ is identified with the *Farey tessellation* $\mathcal{D}$, the tessellation of the hyperbolic plane $\mathbb{H}^2$ obtained from the ideal triangle $\langle 0, 1, \infty\rangle$ by successive reflection in its edges. The vertex set of the Farey tessellation is equal to $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\} \subset \partial\mathbb{H}^2$ and is identified with the set of the isotopy classes of essential simple loops in $\boldsymbol{T}$ by the following rule: a representative of the isotopy class corresponding to $r \in \mathcal{D}^{(0)} = \hat{\mathbb{Q}}$ is the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope r. The element $r \in \hat{\mathbb{Q}}$ associated to a loop or an arc is called its *slope*, and an essential simple loop of slope r in $\boldsymbol{T}$ is denoted by β_r. We abuse notation to denote by β_s an element of $\pi_1(\boldsymbol{T})$ represented by the simple loop β_s of slope s.

Let $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2, \mathbb{C})$ be a *type-preserving representation*, namely ρ is irreducible and sends peripheral elements to parabolic transformations. Since $\pi_1(\boldsymbol{T})$ is a free group, ρ lifts to a representation $\tilde{\rho} : \pi_1(\boldsymbol{T}) \to \mathrm{SL}(2, \mathbb{C})$. Let $\phi = \phi_{\tilde{\rho}}$ be the map from $\mathcal{D}^{(0)} = \hat{\mathbb{Q}}$ to $\mathbb{C}$ defined by $\phi(s) = \mathrm{tr}(\tilde{\rho}(\beta_s))$. Then it is a *Markoff map* in the sense of [8]:

(i) For any Farey triangle $\langle s_0, s_1, s_2\rangle$, the triple $(\phi(s_0), \phi(s_1), \phi(s_2))$ is a *Markoff triple*, that is, it is a nontrivial solution of the Markoff equation

$$x^2 + y^2 + z^2 = xyz.$$

(ii) For any pair of Farey triangles $\langle s_0, s_1, s_2\rangle$ and $\langle s_1, s_2, s_3\rangle$ of $\mathcal{D}$ sharing a common edge $\langle s_1, s_2\rangle$, we have

$$\phi(s_0) + \phi(s_3) = \phi(s_1)\phi(s_2).$$

Let $\mathcal{T}$ be a binary tree (a countably infinite simplicial tree all of whose vertices have degree 3) properly embedded in $\mathbb{H}^2$ dual to $\mathcal{D}$. A *directed edge*, $\vec{e}$, of $\mathcal{T}$ can be thought of as an ordered pair of adjacent vertices of $\mathcal{T}$, referred to as the *head* and *tail* of $\vec{e}$. Following [8], we use the notation $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$ to mean that s_0, s_1, s_2 and s_3 are the ideal vertices of $\mathcal{D}$ such that

(i) the Farey edge $\langle s_1, s_2\rangle$ is the dual to $\vec{e}$, and

(ii) the Farey triangle $\langle s_0, s_1, s_2\rangle$ ($\langle s_1, s_2, s_3\rangle$, respectively) is dual to the head (tail, respectively) of $\vec{e}$.

If $\phi(s_1)\phi(s_2) \neq 0$, then we set

$$\psi_\phi(\vec{e}) := \frac{\phi(s_0)}{\phi(s_1)\phi(s_2)}.$$

We regard $\psi = \psi_\phi$ as a map from the set of oriented edges $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$ of $\mathcal{T}$ such that $\phi(s_1)\phi(s_2) \neq 0$, and we call it the *complex probability map* associated with the Markoff map ϕ. We note that this map is determined by the type-preserving representation $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2, \mathbb{C})$, from whose lift $\tilde{\rho} : \pi_1(\boldsymbol{T}) \to \mathrm{SL}(2, \mathbb{C})$ the Marokoff map ϕ is constructed. So we also call ψ the complex probability map associated with ρ.

By a *complementary region* of $\mathcal{T}$, we mean the closure of a connected component of $\mathbb{H}^2 - \mathcal{T}$. Let Ω be the set of complementary regions of $\mathcal{T}$. Then there is a natural bijection from Ω to $\hat{\mathbb{Q}}$. In the following we identify Ω with $\hat{\mathbb{Q}}$. Let $\vec{e} \leftrightarrow (s_1, s_2; s_0, s_3)$ be a directed edge of $\mathcal{T}$. If we remove the interior of e from $\mathcal{T}$, we are left with two disjoint subsets, which we denote by $\mathcal{T}^{\pm}(\vec{e})$, so that $e \cap \mathcal{T}^+(\vec{e})$ is the head of $\vec{e}$ and $e \cap \mathcal{T}^-(\vec{e})$ is its tail. Let $\Omega^{\pm}(\vec{e}) \subset \Omega$ be the set of regions whose boundaries lie in $\mathcal{T}^{\pm}(\vec{e})$, and set $\Omega^0(e) = \{s_1, s_2\}$. We see that Ω can be written as the disjoint union: $\Omega = \Omega^0(e) \cup \Omega^+(\vec{e}) \cup \Omega^-(\vec{e})$. Set $\Omega^{0-}(\vec{e}) = \Omega^0(e) \cup \Omega^-(\vec{e})$ and $\Omega^{0+}(\vec{e}) = \Omega^0(e) \cup \Omega^+(\vec{e})$.

The following result obtained by [2, Proposition 5.2] is a slight extension of a result obtained by Bowditch [8, Proposition 3.13], where the condition $\Omega^{0-}(\vec{e}) \cap \phi^{-1}[-2, 2] = \emptyset$, which is stronger than (ii), was required (see [29] for further extension).

Theorem 1.1. *Let ϕ be a Markoff map and $\vec{e}$ a directed edge of $\mathcal{T}$ which satisfy the "extended BQ-condition" on $\Omega^{0-}(\vec{e})$, namely ϕ satisfies the following conditions.*

(i) *The set $\{s \in \Omega^-(\vec{e}) \mid |\phi(s)| \leq 2\}$ is finite.*

(ii) $\Omega^{0-}(\vec{e}) \cap \phi^{-1}(-2, 2) = \emptyset$.

Then the following identity holds:

$$\psi(\vec{e}) = \sum_{s \in \Omega^0(e)} h(\phi(s)) + 2 \sum_{s \in \Omega^-(\vec{e})} h(\phi(s)).$$

Here, $h : \mathbb{C} - [-2, 2] \to \mathbb{C}$ is defined by $h(x) = \frac{1}{2}\left(1 - \sqrt{1 - 4/x^2}\right)$, where we adopt the convention that the real part of a square root is always non-negative. For each $s \in \Omega = \hat{\mathbb{Q}}$, let $l(\rho(\beta_s)) \in \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ be the *complex translation length* of the isometry

$\rho(\beta_s)$ of $\mathbb{H}^3$, i.e., $\Re(l(\rho(\beta_s))) \geq 0$ is the translation length of $\rho(\beta_s)$ along its axis and $\Im(\rho(\beta_s))$ is the rotation angle of $\rho(\beta_s)$ around its axis. Then the following holds (see [8, p.721]):

$$h(\phi(s)) = \frac{1}{1 + e^{l(\rho(\beta_s))}}.$$

Thus Theorem 1.1 can be regarded as a refined generalization of McShane's identity. In fact, the theorem implies the following theorem (see [8, Theorem 3], [2], [29]).

Theorem 1.2. *Let ϕ be a Markoff map which satisfies the "extended BQ-condition", namely ϕ satisfies the following conditions.*

(i) *The set $\{s \in \hat{\mathbb{Q}} \mid |\phi(s)| \leq 2\}$ is finite.*

(ii) $\phi^{-1}(-2, 2) = \emptyset$.

Then the following identity holds:

$$\sum_{s \in \hat{\mathbb{Q}}} \frac{1}{1 + e^{l(\rho(\beta_s))}} = \frac{1}{2}.$$

Proof. Let $\{\vec{e}_i\}_{i=0,1,2}$ be a set of the oriented edges of $\mathcal{T}$ sharing the same head. Let ϕ be a Markoff map satisfying the extended BQ-condition, and let ψ be the complex probability map associated with ϕ. Then the Markoff identity implies $\psi(\vec{e}_0) + \psi(\vec{e}_1) + \psi(\vec{e}_2) = 1$. By the assumption, ϕ satisfies the extended BQ-condition on $\Omega^{0-}(\vec{e}_i)$ for each $i = 0, 1, 2$. Hence we obtain the desired identity by applying Theorem 1.1 to the left hand side of the above identity. $\square$

If $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2, \mathbb{C})$ is a type-preserving quasifuchsian representation, then the corresponding Markoff map ϕ satisfies the extended BQ-condition and so the identity in Theorem 1.2 holds. Thus Theorem 1.2 gives a generalization of the original McShane's identity.

Remark. Bowditch [8, Conjecture A] proposed a very interesting conjecture that a type-preserving representation $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2, \mathbb{C})$ is quasifuchsian if and only if the corresponding Markoff map ϕ satisfies the BQ-condition, namely ϕ satisfies the following conditions.

(i) The set $\{s \in \hat{\mathbb{Q}} \mid |\phi(s)| \leq 2\}$ is finite.

(ii) $\phi^{-1}[-2, 2] = \emptyset$.

§ 2. A variation of McShane's identity for 2-bridge links

In this section, we explain a variation of McShane's identity for 2-bridge links obtained by the second and the third authors in [15].

Let $\boldsymbol{S}$ and $\boldsymbol{O}$, respectively, be the 4-times punctured sphere and the $S^2(2,2,2,\infty)$-orbifold (i.e., the orbifold with underlying space a once-punctured sphere and with three cone points of cone angle π). Then $\boldsymbol{T}$, $\boldsymbol{S}$ and $\boldsymbol{O}$ have $\mathbb{R}^2 - \mathbb{Z}^2$ as a common covering space. To be precise, let H and $\tilde{H}$, respectively, be the groups of transformations on $\mathbb{R}^2 - \mathbb{Z}^2$ generated by π-rotations about points in $\mathbb{Z}^2$ and $(\frac{1}{2}\mathbb{Z})^2$. Then $\boldsymbol{T} = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$, $\boldsymbol{S} = (\mathbb{R}^2 - \mathbb{Z}^2)/H$ and $\boldsymbol{O} = (\mathbb{R}^2 - \mathbb{Z}^2)/\tilde{H}$. In particular, there are a $\mathbb{Z}_2$-covering $\boldsymbol{T} \to \boldsymbol{O}$ and a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-covering $\boldsymbol{S} \to \boldsymbol{O}$: the pair of these coverings is called the *Fricke diagram* and each of $\boldsymbol{T}$, $\boldsymbol{S}$, and $\boldsymbol{O}$ is called a *Fricke surface*.

As for the punctured torus $\boldsymbol{T}$, the isotopy classes of essential simple loops in a Fricke surface are in one-to-one correspondence with $\mathcal{D}^{(0)} = \hat{\mathbb{Q}}$: a representative of the isotopy class corresponding to $r \in \hat{\mathbb{Q}}$ is the projection of a line in $\mathbb{R}^2$. The element $r \in \hat{\mathbb{Q}}$ associated to a loop or an arc is called its *slope*. An essential simple loop of slope r in $\boldsymbol{T}$ or $\boldsymbol{O}$ is denoted by β_r, while that in $\boldsymbol{S}$ is denoted by α_r. The notation reflects the following fact: after an isotopy, the restriction of the projection $\boldsymbol{T} \to \boldsymbol{O}$ to β_r $(\subset \boldsymbol{T})$ gives a homeomorphism from β_r $(\subset \boldsymbol{T})$ to β_r $(\subset \boldsymbol{O})$, while the restriction of the projection $\boldsymbol{S} \to \boldsymbol{O}$ to α_r gives a two-fold covering from α_r $(\subset \boldsymbol{S})$ to β_r $(\subset \boldsymbol{O})$.

Now we recall the definition of a 2-bridge link. To this end, set $(\boldsymbol{S}^2, \boldsymbol{P}) = (\mathbb{R}^2, \mathbb{Z}^2)/H$ and call it the *Conway sphere*. Then $\boldsymbol{S}^2$ is homeomorphic to the 2-sphere, $\boldsymbol{P}$ consists of four points in $\boldsymbol{S}^2$, and $\boldsymbol{S}^2 - \boldsymbol{P}$ is the 4-punctured sphere $\boldsymbol{S}$. We also call $\boldsymbol{S}$ the Conway sphere. A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs properly embedded in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3. By a *rational tangle*, we mean a trivial tangle (B^3, t) which is endowed with a homeomorphism from $\partial(B^3, t)$ to $(\boldsymbol{S}^2, \boldsymbol{P})$. Through the homeomorphism we identify the boundary of a rational tangle with the Conway sphere. Thus the slope of an essential simple loop in $\partial B^3 - t$ is defined. We define the *slope* of a rational tangle to be the slope of an essential loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. (Such a loop is unique up to isotopy on $\partial B^3 - t$ and so the slope of a rational tangle is well defined.)

For each $r \in \hat{\mathbb{Q}}$, the 2-*bridge link* $K(r)$ *of slope* r is defined to be the sum of the rational tangles of slopes ∞ and r, namely, $(S^3, K(r))$ is obtained from $(B^3, t(\infty))$ and $(B^3, t(r))$ by identifying their boundaries through the identity map on the Conway sphere $(\boldsymbol{S}^2, \boldsymbol{P})$. (Recall that the boundaries of rational tangles are identified with the Conway sphere.) $K(r)$ has one or two components according to whether the denominator of r is odd or even.

For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint r, and let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞. Then the region, R, bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$

on $\mathbb{H}^2$ (see Figure 1). Let $I_1(r)$ and $I_2(r)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R.

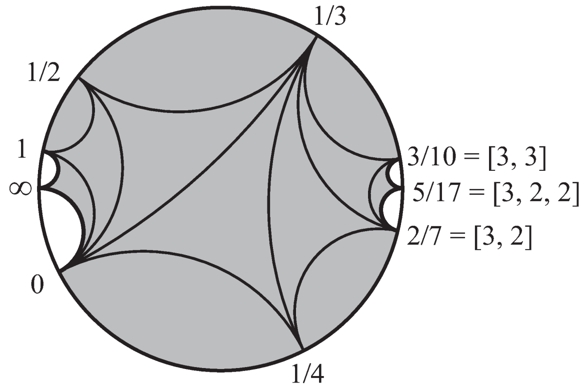

Figure 1. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$. In this case, $I_1(r) = [0, 2/7]$ and $I_2(r) = [3/10, 1]$.

Now assume $r = q/p$, where p and q are relatively prime positive integers such that $q \not\equiv \pm 1 \pmod{p}$. This is equivalent to the condition that $K(r)$ is hyperbolic, namely the link complement $S^3 - K(r)$ admits a complete hyperbolic structure of finite volume. Let ρ_r be the $\mathrm{PSL}(2, \mathbb{C})$-representation of $\pi_1(\boldsymbol{S})$ obtained as the composition

$$\pi_1(\boldsymbol{S}) \to \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(S^3 - K(r)) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C}),$$

where the last homomorphism is the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$. Since $\pi(S^3 - K(r))$ is generated by two meridians, $\rho_r(\pi_1(\boldsymbol{S}))$ is generated by two parabolic transformations. Hence the hyperbolic manifold $S^3 - K(r)$ admits an isometric $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-action, and so the $\mathrm{PSL}(2, \mathbb{C})$-representation ρ_r of $\pi_1(\boldsymbol{S})$ extends to that of $\pi_1(\boldsymbol{O})$. Moreover, this extension is unique (see [5, Proposition 2.2]). So we obtain, in a unique way, a $\mathrm{PSL}(2, \mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$ by restriction. We continue to denote it by ρ_r.

We define the complex number, $\lambda(K(r))$, which expresses the Euclidean structure of the cusp tori of $S^3 - K(r)$. To be precise, each cusp of the hyperbolic manifold $S^3 - K(r)$ carries a Euclidean structure, well-defined up to similarity, and hence it is identified with the quotient of $\mathbb{C}$ (with the natural Euclidean metric) by the lattice $\mathbb{Z} \oplus \mathbb{Z}\lambda$, generated by the translations $[\zeta \mapsto \zeta + 1]$ and $[\zeta \mapsto \zeta + \lambda]$ corresponding to the meridian and a (suitably chosen) longitude, respectively. This λ does not depend on the choice of the cusp, because when $K(r)$ is a two-component 2-bridge link there is an orientation-preserving isometry of $S^3 - K(r)$ interchanging the two cusps. We call λ the *modulus* of the cusp and denote it by $\lambda(K(r))$. Then we have the following formula which describes the modulus $\lambda(K(r))$ in terms of the complex translation lengths of the images by ρ_r of essential simple loops on $\boldsymbol{T}$ (see [15, Theorem 2.2]). This proves a conjecture anticipated by [24].

Theorem 2.1. *For a hyperbolic 2-bridge link $K(r)$, the following identity holds*:

$$2\sum_{s\in\hat{\mathbb{Q}}\cap\mathrm{int}I_1(r)}\frac{1}{1+e^{l(\rho_r(\beta_s))}}l(\rho_r(\beta_s))+2\sum_{s\in\hat{\mathbb{Q}}\cap\mathrm{int}I_2(r)}\frac{1}{1+e^{l(\rho_r(\beta_s))}} + \sum_{s\in\partial I_1(r)\cup\partial I_2(r)}\frac{1}{1+e^{l(\rho_r(\beta_s))}}=-1,$$

where $l(\rho_r(\beta_s))$ denotes the complex translation length of the hyperbolic isometry $\rho_r(\beta_s)$. Furthermore, the modulus $\lambda(K(r))$ of the cusp torus of the cusped hyperbolic manifold $S^3-K(r)$ with respect to a suitable choice of a longitude is given by the following formula:

$$\begin{aligned}\lambda(K(r))&=\frac{4}{|K(r)|}\left\{2\sum_{s\in\hat{\mathbb{Q}}\cap\mathrm{int}I_1(r)}\frac{1}{1+e^{l(\rho_r(\beta_s))}}+\sum_{s\in\partial I_1(r)}\frac{1}{1+e^{l(\rho_r(\beta_s))}}\right\}\\&=\frac{-4}{|K(r)|}\left\{2\sum_{s\in\hat{\mathbb{Q}}\cap\mathrm{int}I_2(r)}\frac{1}{1+e^{l(\rho_r(\beta_s))}}+\sum_{s\in\partial I_2(r)}\frac{1}{1+e^{l(\rho_r(\beta_s))}}+1\right\},\end{aligned}$$

where $|K(r)|$ denotes the number of components of $K(r)$.

For the choice of the longitude in the above theorem, see [15, Proposition 7.1].

§ 3. Outline of the proof of Theorem 2.1

Let $K(r)$ be a hyperbolic 2-bridge link. Then we may assume $r=q/p$, where p and q are relatively prime integers such that $2\le q<p/2$, and so r has the continued fraction expansion

$$r=\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots+\cfrac{1}{a_n}}}=:[a_1,a_2,\dots,a_n],$$

where $n\ge 1$, $(a_1,\dots,a_n)\in(\mathbb{Z}_+)^n$, and $a_n\ge 2$. Set $c=\sum_{i=1}^n a_i$, and let $\Sigma(r)=(\sigma_1,\sigma_2,\dots,\sigma_c)$ be the chain of Farey triangles which intersect the hyperbolic geodesic joining ∞ with r in this order (see Figure 2).

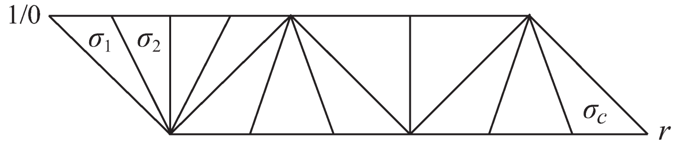

Figure 2. The chain $\Sigma(r)$ of Farey triangles.

Let $\rho_r : \pi_1(\boldsymbol{O}) \to \mathrm{PSL}(2,\mathbb{C})$ denote the type-preserving representation induced by the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$, ϕ_r a Markoff map determined by a lift $\tilde{\rho}_r : \pi_1(\boldsymbol{T}) \to \mathrm{SL}(2,\mathbb{C})$ of the restriction of ρ_r to $\pi_1(\boldsymbol{T})$, and ψ_r the complex probability map determined by ρ_r. We note that the Markoff map ϕ_r satisfies the condition $\phi_r(\infty) = \phi_r(r) = 0$ (cf. [15, Lemma 4.5]).

Let $\mathcal{T}_0(r)$ be the subtree of $\mathcal{T}$ dual to the chain $\Sigma_0(r) := (\sigma_2, \sigma_3, \ldots, \sigma_{c-1})$ obtained from $\Sigma(r)$ by removing σ_1 and σ_c. Let $\vec{E}(r)$ be the set of the oriented edges of $\mathcal{T} - \mathcal{T}_0(r)$ whose head is contained in $\mathcal{T}_0(r)$. For each interval $I_j(r)$ $(j = 1, 2)$, we consider the following set of oriented edges:

$$\vec{E}_j(r) = \{\vec{e} \in \vec{E}(r) \,|\, \Omega^{0-}(\vec{e}) \subset I_j(r)\}.$$

It should be noted that $\vec{E}(r) = \vec{E}_1(r) \sqcup \vec{E}_2(r) \sqcup \{\vec{e}_-, \vec{e}_+\}$, where $\vec{e}_-$ and $\vec{e}_+$ are the elements of $\vec{E}(r)$ with tails dual to σ_1 and σ_c, respectively (see Figure 3).

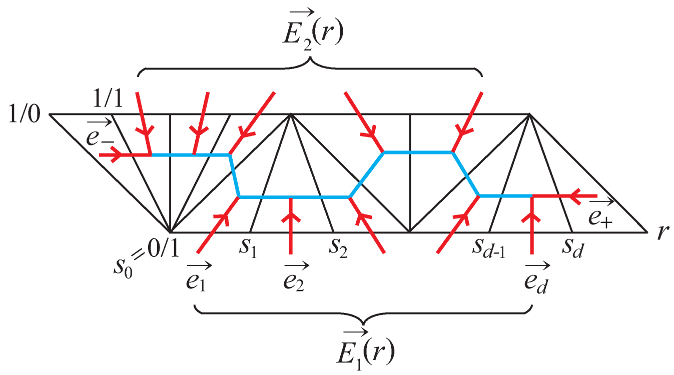

Figure 3. Dual oriented edges.

We now use the fact that the topological ideal triangulation constructed by [25] is isotopic to a geometric triangulation (see [11, 5]). From this fact, we can see that the induced cusp triangulation can be described by using the values of ψ_r on $\vec{E}(r)$, and we can see that the cusp shape $\lambda(K(r))$ with respect to a suitable choice of a longitude is given by the following formula (see Figure 4 and [15, Proposition 5.2]):

$$\frac{|K(r)|}{4}\lambda(K(r)) = \sum_{\vec{e}\in\vec{E}_1(r)} \psi_r(\vec{e}) = -1 - \sum_{\vec{e}\in\vec{E}_2(r)} \psi_r(\vec{e}). \tag{3.1}$$

By the formula (3.1), the proof of Theorem 2.1 is reduced to the following key lemma.

Key Lemma 3.1. *The Markoff map ϕ_r satisfies the extended BQ-condition on $I_1(r) \cup I_2(r)$, namely the following conditions hold.*

(i) *The set $\{s \in I_1(r) \cup I_2(r) \mid |\phi_r(s)| \le 2\}$ is finite.*

(ii) $(I_1(r) \cup I_2(r)) \cap \phi_r^{-1}(-2, 2) = \emptyset$.

Proof of Theorems 2.1 assuming Key Lemma 3.1. Note that Key Lemma 3.1 implies that the Markoff map ϕ_r satisfies the extended BQ condition on $\Omega^{0-}(\vec{e})$ for every member $\vec{e}$ of $\vec{E}_1(r) \cup \vec{E}_2(r)$. Hence Theorem 1.1 implies the following identity for each $j = 1, 2$:

$$\begin{aligned}\sum_{\vec{e}\in\vec{E}_j(r)} \psi_r(\vec{e}) &= \sum_{\vec{e}\in\vec{E}_j(r)} \left\{ \sum_{s\in\Omega^0(e)} h(\phi_r(s)) + 2\sum_{s\in\Omega^-(\vec{e})} h(\phi(s)) \right\} \\ &= 2\sum_{s\in\mathrm{int}I_j(r)} \frac{1}{1+e^{l(\rho_r(\beta_s))}} + \sum_{s\in\partial I_j(r)} \frac{1}{1+e^{l(\rho_r(\beta_s))}}.\end{aligned}$$

By applying this identity to the formula (3.1), we obtain the desired results. □

§ 4. Convergence of the series - Proof of Key Lemma 3.1 -

In the proof of Theorem 2.1, the most difficult part is the proof of Key Lemma 3.1, which in particular implies the absolute convergence of the series, by virtue of Theorem 1.1. Its proof is based on the results obtained in the series of joint work [13, 14] by the second and the third authors (see also the announcement [12]), which gives a complete answer to the following question concerning the simple loops in 2-bridge sphere $\boldsymbol{S}$ of a 2-bridge link $K(r)$.

(1) Which simple loop on $\boldsymbol{S}$ is null-homotopic or peripheral in $S^3 - K(r)$?

(2) For given two simple loops on $\boldsymbol{S}$, when are they homotopic?

In particular, we have the following theorem.

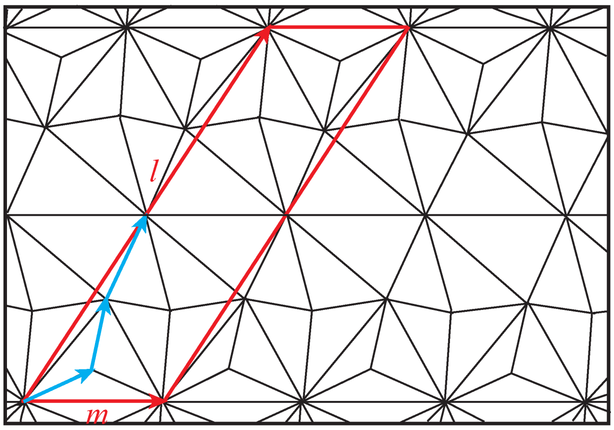

Figure 4. The actual cusp triangulation of $S^3 - K([3,2,2])$. The vectors in the oriented zigzag line segment correspond to the complex numbers $\psi_r(\vec{e})$ in the formula (3.1).

Theorem 4.1. *For a hyperbolic 2-bridge link $K(r)$, the following hold.*

(1) *For any rational number s in $I_1(r) \cup I_2(r)$, α_s is not null-homotopic in $S^3 - K(r)$.*

(2) *There are at most two rational numbers s in $I_1(r) \cup I_2(r)$ such that α_s is peripheral.*

(3) *Except for at most two pairs of rational numbers in $I_1(r) \cup I_2(r)$, the simple loops $\{\alpha_s \,|\, s \in I_1(r) \cup I_2(r)\}$ are not mutually homotopic in $S^3 - K(r)$.*

Proof of Key Lemma 3.1. We first prove the condition (ii). Let s be a rational number contained in $I_1(r) \cup I_2(r)$. Then, by Theorem 4.1(1), α_s determines a non-trivial element of $\pi_1(S^3 - K(r))$. Since ρ_r is induced by the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$, we see that $\rho_r(\alpha_s) = \rho_r(\beta_s^2)$ is neither trivial nor elliptic. Thus $\rho_r(\beta_s)$ is not elliptic, and so $\phi_r(s) = \mathrm{tr}(\tilde{\rho}_r(\beta_s))$ is not contained in $(-2, 2)$. Hence we obtain (ii).

Next we prove the condition (i). Suppose on the contrary that the set $\{s \in I_1(r) \cup I_2(r) \,|\, |\phi_r(s)| \le 2\}$ contains infinitely many elements $\{s_j\}_{j \in \mathbb{Z}}$. By Theorem 4.1(1) and (2), we may assume that $\rho(\alpha_{s_j})$ is neither trivial nor parabolic, and hence, the simple loop α_{s_j} is homotopic to a closed geodesic in the hyperbolic manifold $S^3 - K(r)$. By Theorem 4.1(3), we may also assume that no two of the α_{s_j} are mutually homotopic in $S^3 - K(r)$ and so the corresponding closed geodesics are mutually distinct. On the other hand, the condition $|\phi(s_j)| \le 2$ implies that the real length $L(\rho(\alpha_{s_j})) = 2L(\rho(\beta_{s_j}))$ is bounded from above. This contradicts the discreteness of marked length spectrum of geometrically finite hyperbolic 3-manifolds (see, for example, [1, Theorem 1 in p.73] and [10]). Hence we obtain (i). This completes the proof of Key Lemma 3.1. □

We note that computer experiments show that the convergence of the series in Theorem 2.1 is very slow whereas the convergence of the original McShane's identity is quite fast (see [15, Section 10] and [33]).

§ 5. Extended BQ-conditions for the holonomy representations of certain families of hyperbolic cone manifolds

In the joint work of the first and the third authors with Masaki Wada and Yasushi Yamashita [5], it is announced that, for each hyperbolic 2-bridge link $K(r)$, there is a continuous family of hyperbolic cone manifolds $\{M(r;\theta^-,\theta^+)\}_{0\le\theta^\pm\le 2\pi}$ satisfying the following conditions.

(1) The underlying space of $M(r;\theta^-,\theta^+)$ is $S^3 - K(r)$.

(2) The cone axis of $M(r;\theta^-,\theta^+)$ consists of the core tunnel of $(B^3, t(\infty))$ and that of $(B^3, t(r))$, where the cone angles are θ^- and θ^+, respectively. In particular, $M(r;2\pi,2\pi)$ corresponds to the complete hyperbolic structure on $S^3 - K(r)$.

The first and the third authors have been trying to prove Theorem 2.1, by establishing the following natural generalization of Key Lemma 3.1. (This project actually started a few years before the second and the third authors started the project to prove Theorem 2.1 by using the small cancellation theory.)

Conjecture 5.1. Let $\rho_{(r;\theta^-,\theta^+)} : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2,\mathbb{C})$ be the type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation induced by the holonomy representation of the hyperbolic cone manifold $M(r;\theta^-,\theta^+)$ with $0 \le \theta_\pm \le 2\pi$. Then the Markoff map $\phi_{(r;\theta^-,\theta^+)}$ corresponding to $\rho_{(r;\theta^-,\theta^+)}$ satisfies the extended BQ condition on $I_1(r) \cup I_2(r)$.

Consider the subset, J, of $[0,2\pi]\times[0,2\pi]$ consisting of those points (θ^-,θ^+) for which the conjecture is valid. It is obvious that $(0,0)$ belongs to J and so J is non-empty. Ser Peow Tan pointed out that [27] implies that the set J is open. So what we need to show is that J is closed. Though we have not been able to accomplish this, Conjecture 5.1 for the case when $(\theta^-,\theta^+) = (2\pi, 2\pi/d)$, with $d \ge 2$ a positive integer, are proved in the series of joint work [17, 18, 19] by the second and the third authors (see also the announcement [16]).

Through computer experiments toward the above conjecture, the first and the last authors came to the following conjecture.

Conjecture 5.2. Let $K(r)$ be a hyperbolic 2-bridge link. Then for any rational number $s \in I_1(r) \cup I_2(r)$, the function $(\theta^-, \theta^+) \mapsto |\phi_{(r;\theta^-,\theta^+)}(s)|$ on $[0, 2\pi] \times [0, 2\pi]$ is monotone decreasing, i.e., if $\theta_1^- \le \theta_2^-$ and $\theta_1^+ \le \theta_2^+$, then $|\phi_{(r;\theta_1^-,\theta_1^+)}(s)| \ge |\phi_{(r;\theta_2^-,\theta_2^+)}(s)|$.

Since Conjecture 5.1 for the case $(\theta^-, \theta^+) = (2\pi, 2\pi)$ is already established by Key Lemma 3.1, Conjecture 5.2 implies Conjecture 5.1. This conjecture is based on an intuition that hyperbolic cone manifolds become 'smaller' as the cone angles grow. This intuition is justified for 2-dimensional hyperbolic cone manifolds with a fixed conformal structure by [26, Proposition 2.4]. The following computer experiments made by the first and the third authors also seem to support Conjecture 5.2.

Experiment 5.3. Consider the figure-eight knot $K(2/5)$. Then for each rational number s in one of the following sets

$$\left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{3}{11}, \frac{2}{7}, \frac{3}{10}, \frac{1}{3}\right\} \subset I_1\left(\frac{2}{5}\right) = \left[\frac{0}{1}, \frac{1}{3}\right]$$
$$\left\{\frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\} \subset I_2\left(\frac{2}{5}\right) = \left[\frac{1}{2}, \frac{1}{1}\right],$$

Figures 5, 6 and 7 support Conjecture 5.2 for the slope s. The graphs in Figure 5 show that the functions $\theta \mapsto |\phi_{(r;\theta,\theta)}(s)|$ on $[0, 2\pi]$ are monotone decreasing for all such slopes s. Each figure in Figures 6 and 7 contains the graphs of the 16 functions $|\phi_{(2/5;\theta_j^-,\theta^+)}(s)|$ of the variable $\theta^+ \in [0, 2\pi]$, where $\theta_j^- = 2\pi j/15$ for $j \in \{0, 1, \dots, 15\}$. It shows that each function $|\phi_{(2/5;\theta_j^-,\theta^+)}(s)|$ is monotone decreasing and that the following inequality holds for any $\theta^+ \in [0, 2\pi]$:

$$|\phi_{(2/5;\theta_0^-,\theta^+)}(s)| < |\phi_{(2/5;\theta_1^-,\theta^+)}(s)| < \cdots < |\phi_{(2/5;\theta_{15}^-,\theta^+)}(s)|.$$

Moreover, we are convineced that we can prove Conjecture 5.1 for $K(2/5)$ if we assume the above experimental results. We hope to discuss this on another occasion.

We finally note that Conjecture 5.2 does not hold if we drop the condition that $s \in I_1(r) \cup I_2(r)$. In fact, for $r = 2/5$ and $s = -4/1,\ 9/22 \notin I_1(r) \cup I_2(r)$ the function $\theta \mapsto |\phi_{(r;\theta,\theta)}(s)|$ is not monotone decreasing, as shown in Figure 8.

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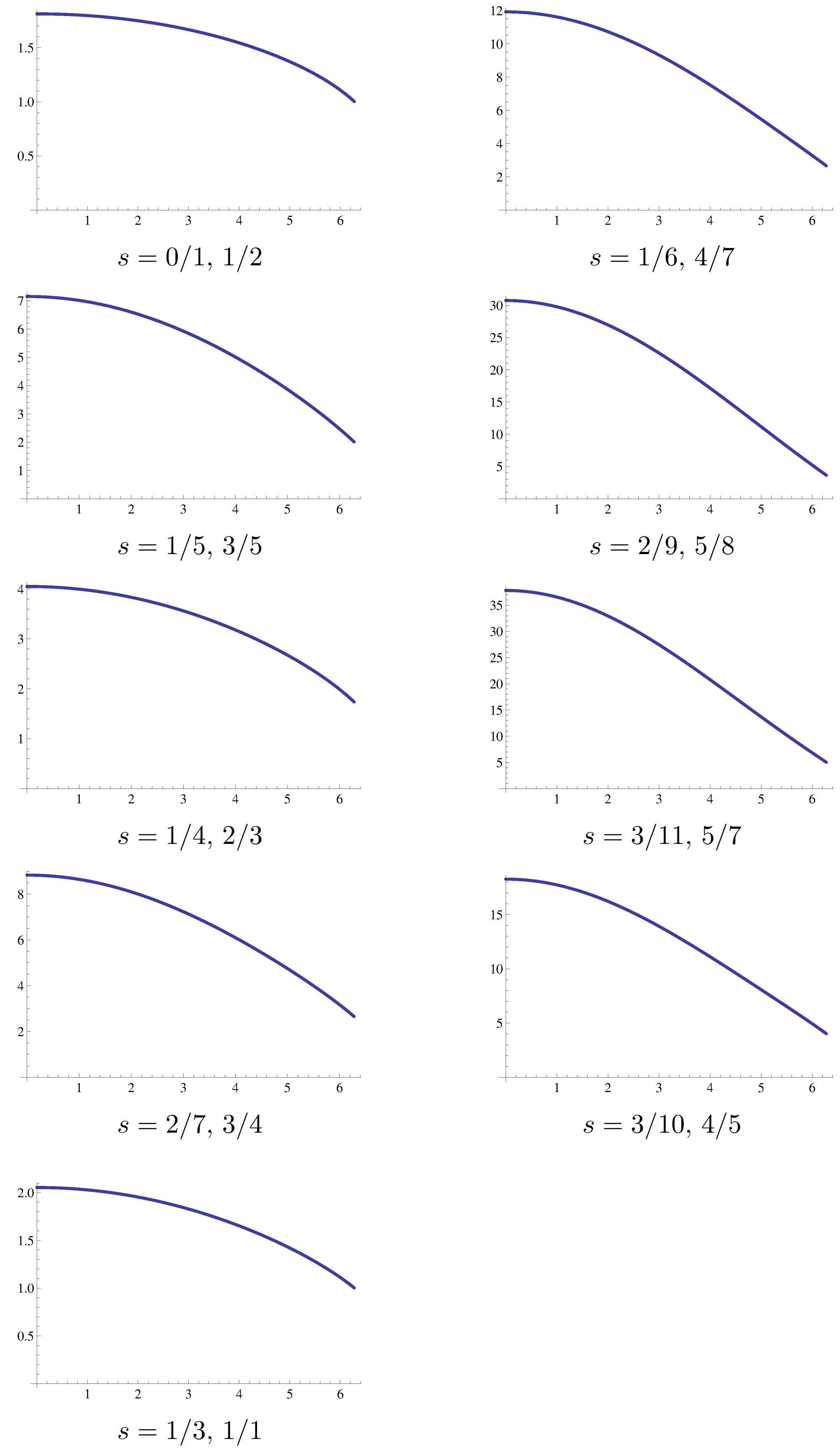

Figure 5. Graphs of $|\phi_{(2/5;\theta,\theta)}(s)|$: The horizontal axis corresponds to the cone angle θ with range $[0, 2\pi]$.

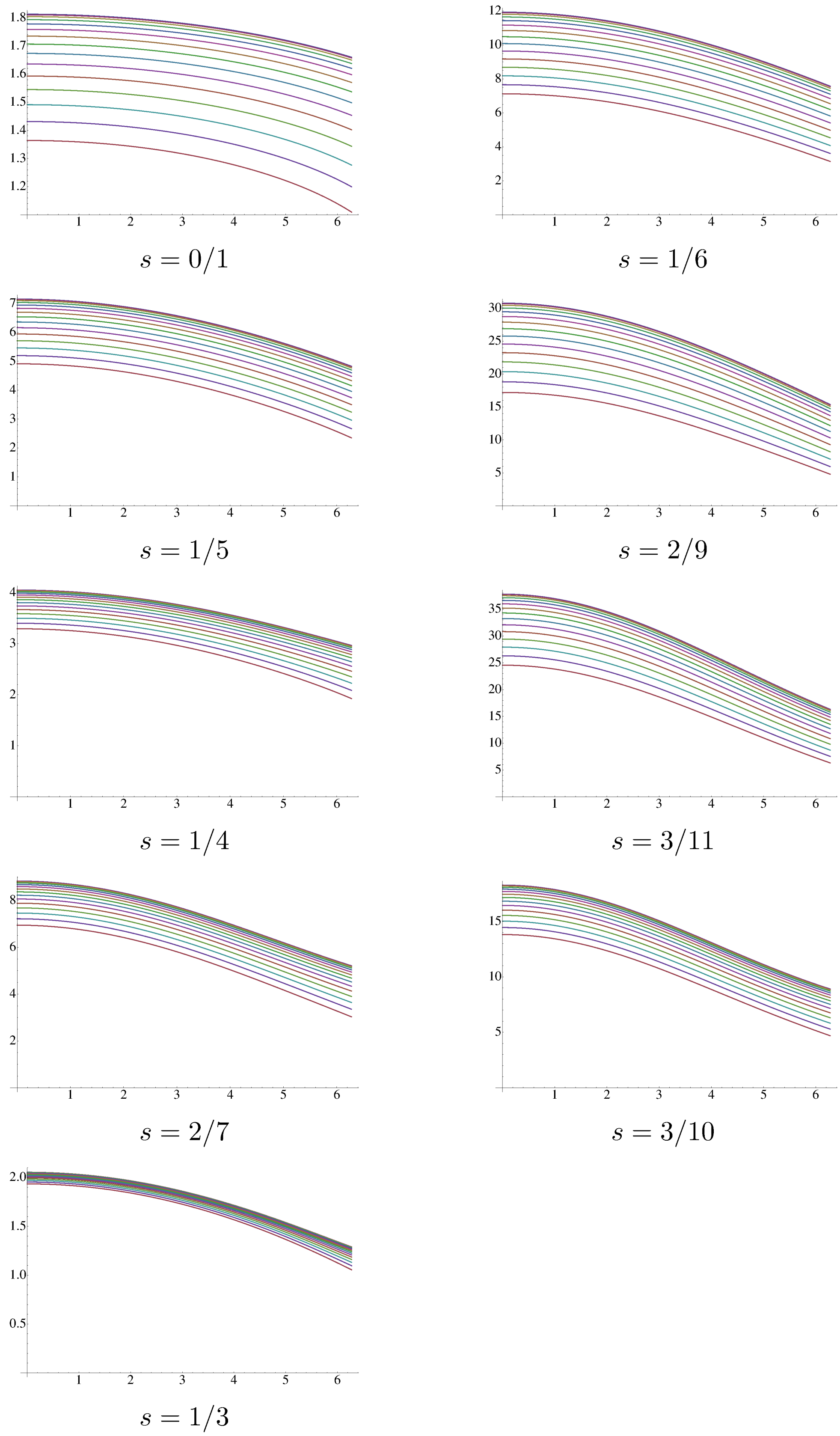

Figure 6. Graphs of $|\phi_{(2/5;\theta_j^-,\theta^+)}(s)|$: The horizontal axis corresponds to the cone angle θ^+ with range $[0, 2\pi]$, and each picture contains 16 graphs corresponding to $\theta_j^- = 2\pi j/15$ with $j \in \{0, 1, \ldots, 15\}$.

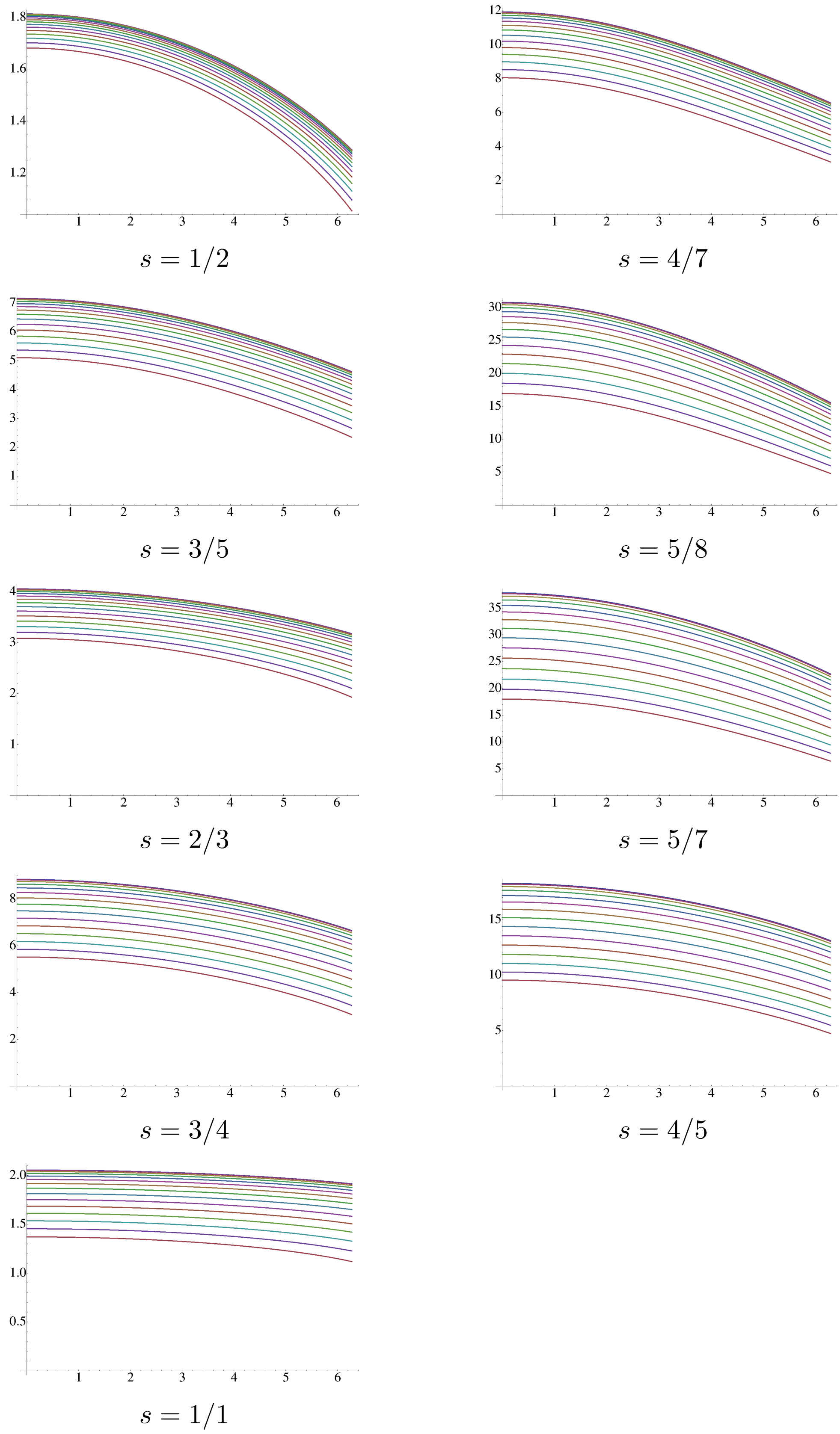

Figure 7. Graphs of $|\phi_{(2/5;\theta_j^-,\theta^+)}(s)|$: The horizontal axis corresponds to the cone angle θ^+ with range $[0, 2\pi]$, and each picture contains 16 graphs corresponding to $\theta_j^- = 2\pi j/15$ with $j \in \{0, 1, \ldots, 15\}$.

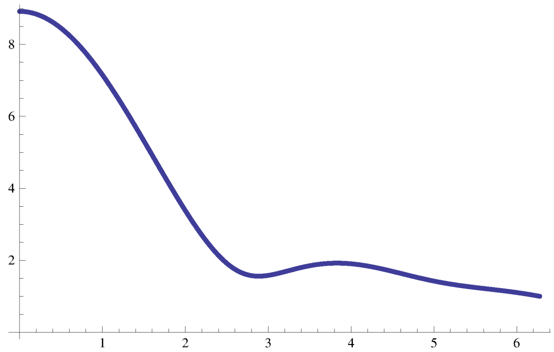

Figure 8. Graph of $|\phi_{(2/5;\theta,\theta)}(s)|$ with $s = -4/1, 9/22$: the horizontal axis corresponds to the cone angle θ with range $[0, 2\pi]$.

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