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The growth series for pure Artin monoids of dihedral type

By

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Abstract

We study a positive monoid for the pure Artin group of dihedral type $I_2(k)$, where the pure Artin group of dihedral type $P_{I_2(k)}$ is the kernel of the natural projection from the Artin group of dihedral type to the corresponding Coxeter group. We call this monoid the pure Artin monoid of dihedral type and denote it by $P^+_{I_2(k)}$. We show that $P^+_{I_2(k)}$ is naturally embedded in $P_{I_2(k)}$. Moreover, we give a normal form for an element of $P^+_{I_2(k)}$, and present an exact rational function form for the spherical growth series of $P^+_{I_2(k)}$ with respect to its natural generating set.

§ 1. Introduction

Let G be a finitely generated group with a finite generating set S. Set $S^{-1} := \{s^{-1} \mid s \in S\}$. The word length $||g||$ of an element $g \in G$ is the smallest integer $n \geq 0$ for which there exist $s_1, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$. The spherical growth series of G relative to S is the formal power series

$$\mathcal{S}_{G,S}(t) := \sum_{q \geq 0} \alpha_q(G, S)\, t^q \in \mathbf{Z}[[t]],$$

where $\alpha_q(G,S) := \sharp\{g \in G \mid ||g|| = q\}$ for each $q \geq 0$.

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The spherical growth series directly describes the distribution of elements in the group according to length. To determine the spherical growth series of a group relative to a generating set leads to a deep understanding of a combinatorial structure of the group with respect to the generating set. It is obvious by the definition that the spherical growth series strongly depends on a generating set. For many cases, including the Coxeter groups with the standard generators, the spherical growth series are known to be rational (see [4], [9], [10], [7], [8], [11], [13], [14], [15] and [28]). In general, however, even if we can easily compute the spherical growth series for a certain generating set, it might be quite difficult to determine it for another generating set. For instance, even though Charney [10] determined the spherical growth series of the Artin group of finite type for some generating set, it is still an open problem to determine that of it for the standard Artin generators.

In order to compute the spherical growth series of a group, we have to choose a unique geodesic representative for each element of the group, and count all of them. Mairesse and Mathéus [23] have succeeded to do it for the Artin group $G_{I_2(k)}$ of dihedral type with respect to the standard Artin generators. More precisely, they constructed finite-state automata which recognize a unique geodesic representative of each element of $G_{I_2(k)}$ over the standard Artin generators, and obtained a concrete rational function expression of its growth series. We remark that before Mairesse and Mathéus, Sabalka [25] obtained an exact rational function expression for the case $k = 3$. For general Artin groups, however, there are few computations for their growth series with respect to the standard Artin generators.

In this paper, we consider the kernel of the projection from the Artin group $G_{I_2(k)}$ of dihedral type to its Coxeter group $\overline{G}_{I_2(k)}$. We call it the pure Artin group of dihedral type, and denote it by $P_{I_2(k)}$. For $k = 3$, $P_{I_2(3)}$ is isomorphic to the pure braid group with three strands. To begin with, by using the Reidemeister-Schreier method, we give the following finite presentation of $P_{I_2(k)}$ for $k \geq 3$:

$$P_{I_2(k)} := \langle\, a_1, \dots, a_k \mid a_1 \cdots a_k = a_2 \cdots a_k a_1 = \cdots = a_k a_1 \cdots a_{k-1} \,\rangle,$$

(see Proposition 2.1). Since all the words in the relations above consist of positive letters, we can consider a positive monoid

$$P^+_{I_2(k)} := \langle\, a_1, \dots, a_k \mid a_1 \cdots a_k = a_2 \cdots a_k a_1 = \cdots = a_k a_1 \cdots a_{k-1} \,\rangle^+,$$

associated to the group $P_{I_2(k)}$. (The definition of the right-hand side is given in Section 2.) We call $P^+_{I_2(k)}$ the pure Artin monoid of dihedral type. One of the main theorems of this paper is the following:

Theorem 1.1 (= Theorem 2.9). *The natural monoid homomorphism $P^+_{I_2(k)} \to P_{I_2(k)}$ is injective.*

We prove this theorem by using the structure of the quotient group of $P_{I_2(k)}$ by its center, which can be identified with the free group generated by $a_1, \ldots, a_{k-1}$. In a subsequent paper [17], Fujii gives another proof of Theorem 1 based on a work of Garside [19]. Theorem 1 implies that all the words containing only the positive letters, $a_1, \ldots, a_k$, are geodesic in the Cayley graph of $P_{I_2(k)}$ with respect to the generators $a_1, \ldots, a_k$. Hence, the coefficient of the spherical growth series of $P^+_{I_2(k)}$ is smaller than or equal to that of $P_{I_2(k)}$ for each degree.

In this paper, we consider an element $\nabla := a_1 a_2 \cdots a_k$ in the free monoid over the set $\{a_1, \ldots, a_k\}$. The element ∇ is a pure Artin group analogue of the fundamental element given for the Artin group (cf. [6]). We also introduce fundamental blocks of $P^+_{I_2(k)}$. (The definition of them is given in Section 2.) Let FB^+ be the set of all of the fundamental blocks of $P^+_{I_2(k)}$. Then, from Theorem 1.1, we provide a normal form of an element of $P^+_{I_2(k)}$ (see Proposition 2.10). In Section 3, by using the normal form, we construct deterministic finite-state automata over subsets of $\mathrm{FB}^+ \cup \{\nabla\}$ that recognize a unique geodesic representative for each element of $P^+_{I_2(k)}$. These automata lead us to determine the spherical growth series $\mathcal{S}(t)$ of $P^+_{I_2(k)}$ in principle. In fact, in Section 4, by considering the structure of the automata, we determine the spherical growth series $\mathcal{S}(t)$ exactly as follows.

Theorem 1.2 (= Theorem 4.3). *For $k \geq 3$,*

$$\mathcal{S}(t) = \frac{1}{1 - kt + (k-1)t^k}.$$

We remark that in a subsequent paper [17], Fujii obtained a rational function form of the spherical growth series of $P_{I_2(k)}$ with respect to the generating set $\{a_1, \ldots, a_k\}$.

Contents

§ 2. Pure Artin groups and monoids of dihedral type

In this section, for each $k \in \mathbf{Z}_{\geq 3}$, we define the pure Artin group $P_{I_2(k)}$ of dihedral type, and give its finite presentation. Moreover, we introduce the positive monoid $P^+_{I_2(k)}$

for $P_{I_2(k)}$ in order to investigate the combinatorial group structure of $P_{I_2(k)}$ with respect to the finite presentation. In particular, we show that $P^+_{I_2(k)}$ is naturally embedded in $P_{I_2(k)}$, and give a normal form of each element of $P^+_{I_2(k)}$.

Let k be an integer greater than two, and $G_{I_2(k)}$ the *Artin group of dihedral type* $I_2(k)$ defined by

$$G_{I_2(k)} \;:=\; \langle\ \sigma_1,\ \sigma_2 \mid \langle\sigma_1\sigma_2\rangle^k = \langle\sigma_2\sigma_1\rangle^k\ \ \rangle,$$

where we define

$$\langle\sigma_i\sigma_j\rangle^k \quad:=\quad \underbrace{\sigma_i\sigma_j\sigma_i\sigma_j\sigma_i\cdots}_{k \text{ letters}}.$$

The *Coxeter group of dihedral type* is the group presented by

$$\overline{G}_{I_2(k)} \;:=\; \langle\ \sigma_1,\ \sigma_2 \mid \langle\sigma_1\sigma_2\rangle^k = \langle\sigma_2\sigma_1\rangle^k,\ \ \sigma_1^2 = \sigma_2^2 = 1\ \rangle.$$

The group $\overline{G}_{I_2(k)}$ is isomorphic to the dihedral group of order $2k$. If we set $\sigma := \sigma_1\sigma_2$ and $\tau := \sigma_2$, then we have the following usual presentation:

$$\overline{G}_{I_2(k)} = \langle\ \sigma,\ \tau \mid \sigma^k = \tau^2 = (\sigma\tau)^2 = 1\ \rangle.$$

There is a natural homomorphism

$$p : G_{I_2(k)} \to \overline{G}_{I_2(k)}.$$

We call its kernel the *pure Artin group of dihedral type*, and denote it by $P_{I_2(k)}$. First, we give a finite presentation of $P_{I_2(k)}$.

Proposition 2.1. *For any $k \geq 3$, the group $P_{I_2(k)}$ has the following finite presentation:*

$$P_{I_2(k)} \;=\; \langle\ a_1,\ldots,a_k \mid a_1\cdots a_k = a_2\cdots a_k a_1 = a_3\cdots a_k a_1 a_2 = \cdots = a_k a_1\cdots a_{k-1}\ \rangle. \tag{2.1}$$

Proof. In order to obtain the required presentation, we use the Reidemeister-Schreier method. (For details, see Chapter II. 4 in [22] for example.)

Case 1. $k = 2l$ and $l \geq 2$.
Here, we have

$$G_{I_2(k)} \;=\; \langle\ \sigma_1,\ \sigma_2 \mid (\sigma_1\sigma_2)^l = (\sigma_2\sigma_1)^l\ \ \rangle.$$

If we set $\sigma := \sigma_1\sigma_2$ and $\tau := \sigma_2$, we have

$$G_{I_2(k)} \;=\; \langle\ \sigma,\ \tau \mid \sigma^l = \tau\sigma^l\tau^{-1}\ \ \rangle,$$

by the Tietze transformation. Then, by applying the Reidemeister-Schreier method to a generating set $X := \{\sigma, \tau\}$ for $G_{I_2(k)}$ and a Schreier transversal

$$T := \{1, \sigma, \ldots, \sigma^{k-1}, \tau, \sigma\tau, \ldots, \sigma^{k-1}\tau\} \subset G_{I_2(k)},$$

we obtain finitely many generators and relations of $P_{I_2(k)}$ as follows:

- A generating set of $P_{I_2(k)}$ is given by

$$\Gamma := \{(t,x) := tx(\overline{tx})^{-1} \,|\, x \in X,\ t \in T,\ (t,x) \neq 1\},$$

where for any $y \in G_{I_2(k)}$, an element $\overline{y} \in T$ is defined by $p(y) = p(\overline{y})$. In fact, Γ consists of the following finitely many elements:

$$\begin{aligned} &a := \sigma^k, \\ &b_0 := \tau\sigma\tau^{-1}\sigma^{-(k-1)}, \quad b_i := \sigma^i\tau\sigma\tau^{-1}\sigma^{-(i-1)} \text{ for } 1 \le i \le k-1, \\ &c_i := \sigma^i\tau^2\sigma^{-i} \text{ for } 0 \le i \le k-1. \end{aligned} \tag{2.2}$$

- A set of finitely many relations is given by

$$\{\varphi(t \cdot \sigma^l\tau\sigma^{-l}\tau^{-1} \cdot t^{-1}) = 1 \,|\, t \in T\}, \tag{2.3}$$

where $\varphi(w)$ is an element in the free group generated by Γ that is obtained from w by rewriting w as a product of a, b_i and c_i. In fact, this set consists of the following finitely many relations:

$$\begin{aligned} (\mathbf{R1}):\ & b_0b_{2l-1}b_{2l-2}\cdots b_{l+1} = 1,\ b_1b_0b_{2l-1}\cdots b_{l+2} = 1,\ \cdots\cdots,\ b_{l-1}b_{l-2}\cdots b_1b_0 = 1, \\ (\mathbf{R2}):\ & a = b_lb_{l-1}\cdots b_1,\ a = b_{l+1}b_l\cdots b_2,\ \cdots\cdots,\ a = b_{2l-1}b_{2l-2}\cdots b_{l+1}b_l, \\ (\mathbf{R3}):\ & b_0b_{2l-1}b_{2l-2}\cdots b_{l+1}c_lc_0^{-1} = 1,\ b_1b_0b_{2l-1}\cdots b_{l+2}c_{l+1}c_1^{-1} = 1,\ \cdots\cdots, \\ & b_{l-1}b_{l-2}\cdots b_1b_0c_{2l-1}c_{l-1}^{-1} = 1, \\ (\mathbf{R4}):\ & b_lb_{l-1}\cdots b_1c_0a^{-1}c_l^{-1} = 1,\ b_{l+1}b_l\cdots b_2c_1a^{-1}c_{l+1}^{-1} = 1,\ \cdots\cdots, \\ & b_{2l-1}b_{2l-2}\cdots b_{l+1}b_lc_{l-1}a^{-1}c_{2l-1}^{-1} = 1. \end{aligned}$$

Below, we demonstrate how to derive the above relations. First, we consider the relations

$$\varphi(\sigma^i \cdot \sigma^l\tau\sigma^{-l}\tau^{-1} \cdot \sigma^{-i}) = 1,$$

where $0 \le i \le k-1$. If $i = 0$, we have

$$\begin{aligned} \sigma^l\tau\sigma^{-1}\tau^{-1}\sigma^{-(l+1)} \cdot \sigma^{l+1}\tau\sigma^{-1}\tau^{-1}\sigma^{-(l+2)} \cdots \sigma^{2l-1}\tau\sigma^{-1}\tau^{-1} = 1 \\ \iff b_{l+1}^{-1}b_{l+2}^{-1}\cdots b_{2l-1}^{-1}b_0^{-1} = 1 \\ \iff b_0b_{2l-1}b_{2l-2}\cdots b_{l+1} = 1. \end{aligned}$$

The relation appearing on the last line is equal to the first relation of **(R1)**. Similarly, for the cases where $i = 1, 2, \ldots, k-1$, we obtain the rest of **(R1)** and **(R2)**. Next, consider the relations

$$\varphi(\sigma^i\tau \cdot \sigma^l\tau\sigma^{-l}\tau^{-1} \cdot \tau^{-1}\sigma^{-i}) = 1,$$

where $0 \leq i \leq k-1$. If $i = 0$, we have

$$\tau\sigma\tau^{-1}\sigma^{-(2l-1)} \cdot \sigma^{2l-1}\tau\sigma\tau^{-1}\sigma^{-(2l-2)} \cdots \sigma^{l+1}\tau\sigma\tau^{-1}\sigma^{-l} \cdot \sigma^{l}\tau^{2}\sigma^{-l} \cdot \tau^{-2} = 1$$
$$\iff b_0 b_{2l-1} b_{2l-2} \cdots b_{l+1} c_l c_0^{-1} = 1.$$

The relation appearing on the last line is equal to the first relation of **(R3)**. Similarly, for the cases where $i = 1, 2, \ldots, k-1$, we obtain the rest of **(R3)** and **(R4)**.

Using **(R1)** and **(R2)**, we can transform the relations **(R3)** and **(R4)** into

$$(\mathbf{R3})' : \; c_l = c_0, \;\; c_{l+1} = c_1, \;\; \cdots\cdots, \;\; c_{2l-1} = c_{l-1},$$
$$(\mathbf{R4})' : \; ac_0 = c_l a, \;\; ac_1 = c_{l+1}a, \;\; \cdots\cdots, \;\; ac_{l-1} = c_{2l-1}a,$$

respectively. Hence, it is seen that $P_{I_2(k)}$ has the generators,

$$a, b_1, \ldots, b_{l-1}, c_0, \ldots, c_{l-1},$$

with the relations,

$$(\mathbf{R4})'' : \; ac_0 = c_0 a, \;\; ac_1 = c_1 a, \;\; \cdots\cdots, \;\; ac_{l-1} = c_{l-1}a,$$
$$(\mathbf{R5}) \;\; : \; ab_1 = b_1 a, \;\; ab_2 = b_2 a, \;\; \cdots\cdots, \;\; ab_{l-1} = b_{l-1}a,$$

because we can remove the generators, $b_l, b_{l+1}, b_{l+2}, \ldots, b_{2l-1}, b_0$ and $c_l, c_{l+1}, \ldots, c_{2l-1}$, by using the relations,

the first relation in $(\mathbf{R2})$, and all relations in $(\mathbf{R1})$ and $(\mathbf{R3})'$.

Finally, if we set

$$a_1 := b_1, \; a_2 := b_2, \; \ldots, \; a_{l-1} := b_{l-1}, \; a_l := c_0, \; a_{l+1} := c_1, \; \ldots, \; a_{2l-1} := c_{l-1},$$

and $a_{2l} := a_{2l-1}^{-1} a_{2l-2}^{-1} \cdots a_1^{-1} a$, then we obtain the desired presentation.

Case 2. $k = 2l+1$ and $l \geq 1$.
If we set $\sigma := \sigma_1\sigma_2$ and $\tau := \sigma_2$, we have

$$G_{I_2(k)} \;=\; \langle\; \sigma, \; \tau \mid \sigma^{l+1} = \tau\sigma^{l}\tau^{-1} \;\;\rangle.$$

By an argument similar to Case 1, it is seen that generators of $P_{I_2(k)}$ are given as in

(2.2) and relations of $P_{I_2(k)}$ are written as

$$\begin{aligned}
(\mathbf{S1}):\ & c_{l+1}^{-1}b_{l+2}^{-1}b_{l+3}^{-1}\cdots b_{2l-1}^{-1}b_0^{-1}=1,\ \ c_{l+2}^{-1}b_{l+3}^{-1}b_{l+4}^{-1}\cdots b_0^{-1}b_1^{-1}=1,\ \ \cdots\cdots,\\
& c_{2l}^{-1}b_0^{-1}b_1^{-1}\cdots b_{l-2}^{-1}b_{l-1}^{-1}=1,\\
(\mathbf{S2}):\ & ac_0^{-1}b_1^{-1}b_2^{-1}\cdots b_l^{-1}=1,\ \ ac_1^{-1}b_2^{-1}b_3^{-1}\cdots b_{l+1}^{-1}=1,\ \ \cdots\cdots,\\
& ac_l^{-1}b_{l+1}^{-1}b_{l+2}^{-1}\cdots b_{2l}^{-1}=1,\\
(\mathbf{S3}):\ & b_0b_{2l}b_{2l-1}\cdots b_{l+1}c_0^{-1}=1,\ \ b_1b_0b_{2l}\cdots b_{l+2}c_1^{-1}=1,\ \ \cdots\cdots,\\
& b_lb_{l-1}\cdots b_1b_0c_l^{-1}=1,\\
(\mathbf{S4}):\ & b_{l+1}b_l\cdots b_1a^{-1}c_{l+1}^{-1}=1,\ \ b_{l+2}b_{l+1}\cdots b_2a^{-1}c_{l+2}^{-1}=1,\ \ \cdots\cdots,\\
& b_{2l}b_{2l-1}\cdots b_la^{-1}c_{2l}^{-1}=1.
\end{aligned}$$

The relations in **(S1)** and **(S3)** are transformed into

$$\begin{aligned}
(\mathbf{S1})':\ & c_{l+1}=b_{l+2}^{-1}b_{l+3}^{-1}\cdots b_{2l-1}^{-1}b_0^{-1},\ \ c_{l+2}=b_{l+3}^{-1}b_{l+4}^{-1}\cdots b_0^{-1}b_1^{-1},\cdots\cdots,\\
& c_{2l}=b_0^{-1}b_1^{-1}\cdots b_{l-2}^{-1}b_{l-1}^{-1},\\
(\mathbf{S3})':\ & c_0=b_0b_{2l}b_{2l-1}\cdots b_{l+1},\ \ c_1=b_1b_0b_{2l}\cdots b_{l+2},\ \ \cdots\cdots,\\
& c_l=b_lb_{l-1}\cdots b_1b_0,
\end{aligned}$$

respectively. Then, by removing the generators, $c_0, c_1, \ldots, c_{2l}$, with using the relations $(\mathbf{S1})'$ and $(\mathbf{S3})'$, we have that $P_{I_2(k)}$ has the generators,

$$a, b_0, \ldots, b_{2l},$$

with the relations,

$$a=b_{2l}b_{2l-1}\cdots b_1b_0=b_{2l-1}b_{2l-2}\cdots b_0b_{2l}=\cdots=b_0b_{2l}\cdots b_2b_1.$$

Finally, if we remove the generator a from the relations above, we obtain the required presentation. This completes the proof of Proposition 2.1. □

Example 2.2. The pure braid group with three strands, P_3, is a geometric realization of the pure Artin group,

$$P_{I_2(3)}\ =\ \langle\ a_1,\ a_2,\ a_3\ |\ a_1a_2a_3=a_2a_3a_1=a_3a_1a_2\ \rangle.$$

The generators a_1, a_2 and a_3 are themselves braids and are given in terms of the standard Artin generators of the braid group with three strands, σ_1 and σ_2, as follows: $a_1=\sigma_1^2\sigma_2^2$, $a_2=\sigma_1\sigma_2^2\sigma_1$ and $a_3=\sigma_1^{-2}$. Now, let $A_{12}:=a_3^{-1}=\sigma_1^2$, $A_{13}:=a_3a_2=\sigma_1^{-1}\sigma_2^2\sigma_1=\sigma_2\sigma_1^2\sigma_2^{-1}$ and $A_{23}:=a_3a_1=\sigma_2^2$. Then, we obtain the same presentation of $P_{I_2(k)}$:

$$P_{I_2(3)}\ =\ \langle\ A_{12},\ A_{13},\ A_{23}\ |\ A_{12}A_{13}A_{23}=A_{13}A_{23}A_{12}=A_{23}A_{12}A_{13}\ \rangle.$$

Here, note that $\{A_{12}, A_{13}, A_{23}\}$ is the standard generating set of the pure braid group with three strands (cf. [3]).

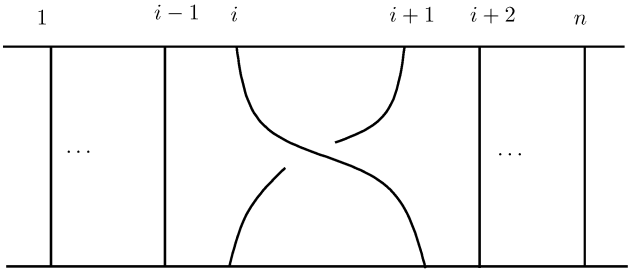

Figure 1. A geometrical braid corresponding to σ_i

Now, define

$$\begin{aligned} A^+ &:= \{a_1, \ldots, a_k\}, \\ A^- &:= \{a_1^{-1}, \ldots, a_k^{-1}\}, \\ A &:= A^+ \cup A^-. \end{aligned}$$

For any subsets $\Sigma \subset A$, $\Sigma^+ \subset A^+$ and $\Sigma^- \subset A^{-1}$, let Σ^*, $(\Sigma^+)^*$ and $(\Sigma^-)^*$ be the free monoids generated by Σ, Σ^+ and Σ^-, respectively. We call an element of Σ a *letter*, and an element of Σ^* a *word*. An element of Σ^+ (resp. Σ^-) is called a *positive letter* (resp. *negative letter*), that of $(\Sigma^+)^*$ (resp. $(\Sigma^-)^*$) a *positive word* (resp. *negative word*). The length of a word w is the number of letters in w, which is denoted by $|w|$. The length of the null word, ε, is zero. The null word is the identity of each monoid.

We write the canonical monoid homomorphism as $\pi : A^* \to P_{I_2(k)}$. If u and v are words in A^*, then $u = v$ means that $\pi(u) = \pi(v)$, and $u \equiv v$ means that u and v are identical letter by letter. A word $w \in \pi^{-1}(g)$ is called a *representative* of g. The length of a group element g is defined by

$$\|g\| := \min\{l \mid g = \pi(s_1 \cdots s_l),\ s_i \in A\}.$$

A word $w \in A^*$ is *geodesic* if $|w| = \|\pi(w)\|$. A word $w_1 \cdots w_m \in A^*$ is called a *reduced* word if $w_i \neq w_{i+1}^{-1}$ for all $i \in \{1, \ldots, m-1\}$. A geodesic representative is a reduced word.

For each $q \in \mathbf{Z}_{\geq 0}$, we define

$$\alpha_q(P_{I_2(k)}, A^+) := \sharp\{g \in P_{I_2(k)} \mid \|g\| = q\}.$$

The *spherical growth series* of $P_{I_2(k)}$ with respect to the generating set A^+ is the following formal power series

$$\mathcal{S}_{P_{I_2(k)}, A^+}(t) := \sum_{q=0}^{\infty} \alpha_q(P_{I_2(k)}, A^+)\ t^q.$$

It is well-known that the radius of convergence of the growth series of any finitely generated group is strictly greater than 0. Thus, the growth series $\mathcal{S}_{P_{I_2(k)},A^+}(t)$ is a holomorphic function near the origin 0. In a subsequent paper [17], Fujii determined a rational function expression of $\mathcal{S}_{P_{I_2(k)},A^+}(t)$ for any $k \geq 3$.

Consider a positive word $\nabla := a_1 a_2 \cdots a_k \in (A^+)^*$. Then, we have

$$\nabla = a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1}$$

in $P_{I_2(k)}$. Then, it is readily seen that the following lemma holds.

Lemma 2.3.

(1) *The element* ∇ *generates the infinite cyclic group* Z *contained in the center of* $P_{I_2(k)}$. *In particular, we have*

$$a\nabla^{\pm 1} = \nabla^{\pm 1} a,$$

for any $a \in A = A^+ \cup A^-$.

(2) *The quotient group* $P_{I_2(k)}/Z$ *is the free group* F_{k-1} *generated by (the coset classes of)* $a_1, \ldots, a_{k-1}$.

From Lemma 2.3, we see that Z coincides with the center of $P_{I_2(k)}$. Hence, we have that $Z \cong \mathbf{Z}$, and see that $P_{I_2(k)}$ is isomorphic to $\mathbf{Z} \times F_{k-1}$ as a group. From this aspect, the group structure of $P_{I_2(k)}$ is quite simple. In this paper, however, we consider the generating set $\{a_1, \ldots, a_k\}$ of $P_{I_2(k)}$ and investigate the growth series of $P_{I_2(k)}$ with respect to this generating set.

The *pure Artin monoid* of dihedral type is the monoid presented by

$$P^+_{I_2(k)} := \langle\, a_1, \ldots, a_k \mid a_1 \cdots a_k = a_2 \cdots a_k a_1 = \cdots = a_k a_1 \cdots a_{k-1} \,\rangle^+,$$

where the right-hand side is the quotient of the free monoid $(A^+)^*$ by an equivalence relation on $(A^+)^*$ defined as follows:

(i) Two positive words $\omega, \omega' \in (A^+)^*$ are *elementarily equivalent* if there are positive words $u, v \in (A^+)^*$ and indices $i, j \in \{1, \ldots, k\}$ such that $\omega \equiv u(a_i \cdots a_k a_1 \cdots a_{i-1})v$ and $\omega' \equiv u(a_j \cdots a_k a_1 \cdots a_{j-1})v$.

(ii) Two positive words $\omega, \omega' \in (A^+)^*$ are equivalent if there is a sequence $\omega_0, \omega_1, \cdots, \omega_l$ for some $l \in \mathbf{Z}_{\geq 0}$ such that ω_s is elementarily equivalent to ω_{s+1} for $s = 0, \cdots, l-1$, and $\omega_0 \equiv \omega$, $\omega_l \equiv \omega'$.

Let π^+ be the canonical monoid homomorphism from $(A^+)^*$ to $P^+_{I_2(k)}$. If u and v are words in $(A^+)^*$, then $u \doteq v$ means that $\pi^+(u) = \pi^+(v)$. There is a natural monoid homomorphism $P^+_{I_2(k)} \to P_{I_2(k)}$. Below, we show that this map is injective. To begin with, we show the following:

Proposition 2.4. *Each element of $P^+_{I_2(k)}$ has a unique representative of the following type:*

$$v_1 a_k^{e_1} v_2 a_k^{e_2} \cdots v_n a_k^{e_n} v_{n+1} (a_1 \cdots a_k)^m \in (A^+)^*, \tag{2.4}$$

where

(i) $m \in \mathbf{Z}_{\geq 0}$,
(ii) $e_1, \ldots, e_n \in \mathbf{N}$,
(iii) $v_1, \ldots, v_{n+1} \in \{a_1, \ldots, a_{k-1}\}^*$ *with* $v_2, \ldots, v_n \neq \varepsilon$,
(iv) $v_1 a_k^{e_1} v_2 a_k^{e_2} \cdots v_n a_k^{e_n} v_{n+1}$ *does not contain a positive word* u *satisfying* $u \doteq \nabla$.

Let q be the natural projection from $P_{I_2(k)}$ to $P_{I_2(k)}/Z$ $(= F_{k-1} = \langle a_1, \ldots, a_{k-1} \rangle)$. For an element $\xi \in \{a_1^{\pm 1}, ..., a_{k-1}^{\pm 1}\}^*$, we denote by $\mathrm{red}(\xi)$ the reduced word expression for ξ within F_{k-1}. Let $\theta : P^+_{I_2(k)} \to F_{k-1}$ be the composition of the natural homomorphisms:

$$\theta : P^+_{I_2(k)} \to P_{I_2(k)} \xrightarrow{q} P_{I_2(k)}/Z = F_{k-1} = \langle a_1, \ldots, a_{k-1} \rangle.$$

We prepare the following three lemmas.

Lemma 2.5. *For each $1 \leq j \leq n$, consider the image of a subword $v_j a_k^{e_j} v_{j+1}$ of (2.4) by the map $\theta \circ \pi^+$:*

$$\theta(\pi^+(v_j a_k^{e_j} v_{j+1})) = \mathrm{red}(v_j (a_{k-1}^{-1} \cdots a_2^{-1} a_1^{-1})^{e_j} v_{j+1}) \in F_{k-1}.$$

Then, there exists a certain letter a_r^{-1} appearing in $(a_{k-1}^{-1} \cdots a_2^{-1} a_1^{-1})^{e_j}$ that cannot disappear when we take the reduced expression of the word $v_j (a_{k-1}^{-1} \cdots a_2^{-1} a_1^{-1})^{e_j} v_{j+1}$ within F_{k-1}.

Proof of Lemma 2.5. If all letters in $(a_{k-1}^{-1} \cdots a_2^{-1} a_1^{-1})^{e_j}$ are canceled for some j, then by the uniqueness of the reduced word expression, we have the following three cases:

(i) $v_j \doteq v'_j a_1 a_2 \cdots a_{k-1}$ for some $v'_j \in (A^+ \setminus \{a_k\})^*$,

(ii) $v_{j+1} \doteq a_1 a_2 \cdots a_{k-1} v'_{j+1}$ for some $v'_{j+1} \in (A^+ \setminus \{a_k\})^*$,

(iii) $e_j = 1$ and $v_j \doteq v'_j a_i a_{i+1} \cdots a_{k-1}$, $v_{j+1} \doteq a_1 a_2 \cdots a_{i-1} v'_{j+1}$ for some $v'_j, v'_{j+1} \in (A^+ \setminus \{a_k\})^*$.

In each case, the condition (iv) given in Proposition 2.4 is not satisfied. Thus, we obtain a contradiction. □

Lemma 2.6. *Consider the image of the word (2.4) by the map $\theta \circ \pi^+$:*

$$\mathrm{red}(v_1 (a_{k-1}^{-1} \cdots a_2^{-1} a_1^{-1})^{e_1} v_2 \cdots v_j (a_{k-1}^{-1} \cdots a_2^{-1} a_1^{-1})^{e_j} v_{j+1} \cdots v_n (a_{k-1}^{-1} \cdots a_2^{-1} a_1^{-1})^{e_n} v_{n+1}). \tag{2.5}$$

Suppose that there exists a letter a_i^{-1} *appearing in* $(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_j}$ *that is canceled when we take the reduced expression of the word (2.5). Then, the letter* a_i^{-1} *is canceled with a letter* a_i *coming from the words* v_j *or* v_{j+1}*.*

Proof of Lemma 2.6. Assume that for some $j \in \{1,\dots,n\}$, there exists a letter a_i^{-1} appearing in $(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_j}$ such that it is canceled with a letter a_i coming from v_l, where $l \neq j, j+1$. There are the following two cases: $l \geq j+2$ and $l \leq j-1$. Now, consider the case $l \geq j+2$. Among such indices l, we choose the smallest one, $\bar{l}$. Then, it is readily seen that all letters in $(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_{j+1}}$ are canceled in the reduce expression of the word (2.5). Since $\bar{l}$ is smallest, all letters in $(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_{j+1}}$ are canceled with letters coming from v_{j+1} or v_{j+2}. This, however, contradicts with Lemma 2.5. Similarly, we obtain a contradiction in the case where $l \leq j-1$. □

Lemma 2.7. *Consider the image of the word (2.4) by the map* $\theta\circ\pi^+$ *as in lemma 2.6. Then, for any* $1 \leq j \leq n$*, there exists a letter* a_r^{-1} *appearing in* $(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_j}$ *that does not disappear when we take the reduced expression of the word (2.5) within* F_{k-1}*.*

Proof of Lemma 2.7. Assume that for some $1 \leq j \leq n$, all letters in $(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_j}$ are canceled. Then, by Lemma 2.6, all letters in $(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_j}$ must be canceled within $v_j(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_j}v_{j+1}$. This contradicts with Lemma 2.5. □

Now, it is ready to prove Proposition 2.4. It is easily seen that for any element $v \in P^+_{I_2(k)}$, v has a representative as in (2.4), since ∇ belongs to the center of $P^+_{I_2(k)}$. Hence, it is sufficient to show the uniqueness of the expression. Take any two elements of $(A^+)^*$,

$$v \equiv v_1 a_k^{e_1} v_2 a_k^{e_2}\cdots v_n a_k^{e_n} v_{n+1}(a_1\cdots a_k)^m$$

and

$$w \equiv w_1 a_k^{e'_1} w_2 a_k^{e'_2}\cdots w_{n'} a_k^{e'_n} w_{n'+1}(a_1\cdots a_k)^{m'}$$

of type (2.4), and assume that $v \doteq w$. First, consider their images by $\theta\circ\pi^+$. Then we have

$$\begin{aligned} &v_1(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_1}v_2\cdots v_n(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_n}v_{n+1} \\ &\quad = w_1(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e'_1}w_2\cdots w_{n'}(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e'_{n'}}w_{n'+1} \end{aligned}$$

in F_{k-1}. Then, from Lemma 2.7, we see that $n = n'$ by taking the reduced expressions of both sides. Hence,

$$\begin{aligned} &v_1(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_1}v_2\cdots v_n(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e_n}v_{n+1} \\ &\quad = w_1(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e'_1}w_2\cdots w_n(a_{k-1}^{-1}\cdots a_2^{-1}a_1^{-1})^{e'_n}w_{n+1}. \end{aligned} \tag{2.6}$$

Step 1. Consider the subwords v_1 and w_1. Then, we will show that $v_1 \equiv w_1$. Since v_1 does not contain a positive word u satisfying $u \doteq \nabla$, v_1 takes one of the following types:

(i) The last letter of v_1 is not a_{k-1},

(ii) $v_1 \equiv v_1' a_{j_1} a_{j_1+1} \cdots a_{k-1}$ for some $2 \le j_1 \le k-1$ such that the last letter of v_1' is not a_{j_1-1}.

Assume that v_1 is of type (ii). Then, we have

$$\begin{aligned} v_1(a_{k-1}^{-1} &\cdots a_2^{-1}a_1^{-1})^{e_1} v_2 \cdots v_n (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_n} v_{n+1} \\ &= v_1' a_{j_1-1}^{-1} \cdots a_2^{-1}a_1^{-1}(a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_1-1} v_2 \cdots v_n (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_n} v_{n+1}. \end{aligned}$$

We claim that the letter $a_{j_1-1}^{-1}$ standing on the right neighbor of v_1' is not canceled in the reduced expression of the word. In fact, if this $a_{j_1-1}^{-1}$ is canceled in the reduced expression, then it is canceled with a_{j_1-1} coming from v_2 by Lemma 2.6. This implies that $v_2 \equiv (a_1a_2 \cdots a_{k-1})^{e_1-1} a_1 a_2 \cdots a_{j_1-1} v_2'$ for some positive word v_2'. This, however, contradicts with the condition (iv) given in Proposition 2.4. Therefore, the negative letter, $a_{j_1-1}^{-1}$, is the leftmost negative letter in the left-hand side of (2.6). Similarly, by observing the right-hand side of (2.6), we have that if w_1 is also of the form (ii), that is, $w_1 \equiv w_1' a_{j_1} a_{j_1+1} \cdots a_{k-1}$ for some w_1' such that the last word of w_1' is not a_{j_1-1}, then the leftmost negative letter in the right-hand side of (2.6) is also $a_{j_1-1}^{-1}$. Hence, we obtain $v_1' \equiv w_1'$. Therefore, we have $v_1 \equiv w_1$.

If v_1 is of type (i), so is w_1. Then the leftmost negative letters of both sides of (2.6) are a_{k-1}^{-1}. Hence, we also obtain $v_1 \equiv w_1$.

Step 2. We will show that $e_1 = e_1'$. By Step 1, we have

$$\begin{aligned} (a_{k-1}^{-1} &\cdots a_2^{-1}a_1^{-1})^{e_1} v_2 \cdots v_n (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_n} v_{n+1} \\ &= (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_1'} w_2 \cdots w_n (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_n'} w_{n+1} \end{aligned}$$

in F_{k-1}. First, assume $e_1 > e_1'$. Then, we have

$$\begin{aligned} (a_{k-1}^{-1} &\cdots a_2^{-1}a_1^{-1})^{e_1-e_1'} v_2 \cdots v_n (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_n} v_{n+1} \\ &= w_2 (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_2'} \cdots w_n (a_{k-1}^{-1} \cdots a_2^{-1}a_1^{-1})^{e_n'} w_{n+1} \end{aligned} \tag{2.7}$$

in F_{k-1}. Then, from Lemma 2.7, by observing the negative letters in both sides of (2.7), we have $n = n-1$. This is a contradiction. Hence, we have $e_1 \le e_1'$. Similarly, we obtain $e_1 \ge e_1'$. Therefore, we have $e_1 = e_1'$.

By repeating Step 1 and Step 2, we can show that $v_i \equiv w_i$ and $e_i = e_i'$ for any $1 \le i \le n$. Since $v \doteq w$, we have

$$(a_1 \cdots a_k)^m = (a_1 \cdots a_k)^{m'} \in P_{I_2(k)}.$$

Since ∇ generates the infinite cyclic subgroup of $P_{I_2(k)}$, we obtain $m = m'$. Hence, we conclude that $v \equiv w$. This completes the proof of Proposition 2.4. □

Lemma 2.8. *The monoid homomorphism* $\theta : P^+_{I_2(k)} \to P_{I_2(k)}/Z = F_{k-1}$ *is injective.*

Proof of Lemma 2.8. For elements $v, w \in P^+_{I_2(k)}$, take representatives v' and w' as in (2.4), respectively. Assume that $\theta \circ \pi^+(v') = \theta \circ \pi^+(w')$. Then, by the argument given in the proof of Proposition 2.4, we obtain $v \equiv w$. This shows that θ is injective. □

As a corollary, we have the following:

Theorem 2.9. *The natural monoid homomorphism* $P^+_{I_2(k)} \to P_{I_2(k)}$ *is injective.*

In the following, we consider $P^+_{I_2(k)}$ to be a submonoid of $P_{I_2(k)}$ through this embedding, identifying the null word ε with the identity of $P_{I_2(k)}$. Then, we have the following commutative diagram:

$$\begin{array}{ccc} (A^+)^* & \subset & A^* \\ {\scriptstyle \pi^+}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ P^+_{I_2(k)} & \subset & P_{I_2(k)}. \end{array}$$

Now, in order to give a standard representative for each element of $P^+_{I_2(k)}$, let us introduce the concept of fundamental blocks. A *fundamental block* is a positive word in $(A^+)^*$ with length smaller than k that appears as a subword in representatives of $\pi(\nabla)$. There are $k(k-1)$ fundamental blocks. We list all of them below:

$$\begin{array}{ll} \text{length } k-1: & a_1 \cdots a_{k-1},\ a_2 \cdots a_k,\ a_3 \cdots a_k a_1,\ \ldots,\ a_k a_1 \cdots a_{k-2} \\ \text{length } k-2: & a_1 \cdots a_{k-2},\ a_2 \cdots a_{k-1},\ a_3 \cdots a_k,\ a_4 \cdots a_k a_1,\ \ldots,\ a_k a_1 \cdots a_{k-3} \\ \quad\cdots & \cdots \\ \text{length } 2 & : a_1 a_2,\ a_2 a_3,\ \ldots,\ a_{k-1} a_k,\ a_k a_1 \\ \text{length } 1 & : a_1,\ a_2,\ \ldots,\ a_k \end{array}$$

Next, we give several definitions concerning the fundamental blocks. Set

$$\mathrm{FB}^+ := \{\mu \in (A^+)^* \mid \mu \text{ is a fundamental block }\}.$$

For any $I \in \{0, \ldots, k-1\}$, set

$$\begin{aligned} \mathrm{FB}^+_I &:= \{\mu \in \mathrm{FB}^+ \mid |\mu| = I\}, \\ \mathrm{FB}^+_{\leq I} &:= \{\mu \in \mathrm{FB}^+ \mid |\mu| \ \leq \ I\}. \end{aligned}$$

For any $\mu = a_i \cdots a_k a_1 \cdots a_j \in \mathrm{FB}^+$, define

$$\mathcal{L}(\mu) := a_i, \quad \mathcal{R}(\mu) := a_j.$$

For any $\mu = a_i \cdots a_k a_1 \cdots a_j \in \mathrm{FB}^+$, we call a_{j+1} the *letter subsequent to* μ. When $\mu = a_i \cdots a_k$, we call a_1 the letter subsequent to μ. The letter subsequent to μ is denoted by $\mathcal{N}(\mu)$.

Let g be an element of $P^+_{I_2(k)}$. We can choose the following representative $\xi \in (A^+)^*$ of g as in Proposition 2.4:

$$\xi \equiv v_1 a_k^{e_1} v_2 a_k^{e_2} \cdots v_n a_k^{e_n} v_{n+1} \cdot \nabla^d.$$

Moreover, we can represent $v_1 a_k^{e_1} v_2 a_k^{e_2} \cdots v_n a_k^{e_n} v_{n+1}$ as a product of elements of FB^+ uniquely as follows.

$$v_1 a_k^{e_1} v_2 a_k^{e_2} \cdots v_n a_k^{e_n} v_{n+1} \equiv \mu_1 \cdots \mu_m \in (\mathrm{FB}^+)^*,$$

where $\mathcal{N}(\mu_j) \neq \mathcal{L}(\mu_{j+1})$ for $1 \leq j \leq m-1$. Hence, combining this observation and Proposition 2.4, we obtain

Proposition 2.10 (Normal form). *For any $g \in P^+_{I_2(k)}$, there exist unique $\mu_1 \cdots \mu_m \in (\mathrm{FB}^+)^*$ and $d \in \mathbf{Z}_{\geq 0}$ such that $\xi := \mu_1 \cdots \mu_m \cdot \nabla^d$ is a representative of g, and $\mathcal{N}(\mu_j) \neq \mathcal{L}(\mu_{j+1})$ for $1 \leq j \leq m-1$.*

We call $\xi \equiv \mu_1 \cdots \mu_m \cdot \nabla^d$ the *normal form* of g, and $\mu_1 \cdots \mu_m$ the *non-∇ part* of the normal form. From Theorem 2.9 and Lemma 2.3, we can show that every positive word is geodesic (see [17]). Hence for each element $g \in P^+_{I_2(k)}$, the normal form of g is a geodesic representative of g.

§ 3. Automata for geodesic representatives of $P^+_{I_2(k)}$

In this section, we construct deterministic finite-state automata over subsets of $\mathrm{FB}^+ \cup \{\nabla\}$ that recognize the normal forms of elements of $P^+_{I_2(k)}$. (Refer to [20] for a general reference on automata.)

Let $\overline{\Sigma}$ be a subset of $\mathrm{FB}^+ \cup \{\nabla\}$, and $\overline{\Sigma}^*$ the free monoid generated by $\overline{\Sigma}$. Here we naturally consider $\overline{\Sigma}^*$ as a subset of A^*. For any $g \in P^+_{I_2(k)}$, take the normal form ξ of g as in Proposition 2.10. Set

$$\mathrm{Pos}(\xi) := \begin{cases} \max\{|\mu_j| \mid 1 \leq j \leq m\} & \text{if } d = 0, \\ k & \text{if } d > 0. \end{cases}$$

Then, we have $0 \leq \mathrm{Pos}(\xi) \leq k$.

Define $\mathbf{\Gamma}_k$ to be the set of all the normal forms ξ of elements $g \in P^+_{I_2(k)}$ such that $\mathrm{Pos}(\xi) = k$. Namely,

$$\mathbf{\Gamma}_k := \left\{ \xi \in (\mathrm{FB}^+ \cup \{\nabla\})^* \;\middle|\; \begin{array}{l} \xi \equiv \mu_1 \cdots \mu_m \cdot \nabla^d, \;\; \mu_1, \ldots, \mu_m \in \mathrm{FB}^+ \cup \{\varepsilon\}, \\ d \geq 1, \text{ and } \mathcal{N}(\mu_j) \neq \mathcal{L}(\mu_{j+1}) \text{ for } 1 \leq {}^\forall j \leq m-1 \end{array} \right\}.$$

Similarly, for any $0 \leq P \leq k-1$, define

$$\mathbf{\Gamma}_P := \left\{ \xi \in (\mathrm{FB}^+_{\leq P})^* \;\middle|\; \begin{array}{l} \xi \equiv \mu_1 \cdots \mu_m,\ \ \mu_1, \ldots, \mu_m \in \mathrm{FB}^+ \cup \{\varepsilon\}, \\ \mathcal{N}(\mu_j) \neq \mathcal{L}(\mu_{j+1}) \text{ for } 1 \leq {}^\forall j \leq m-1, \text{ and } \mathrm{Pos}(\xi) = P \end{array} \right\},$$

and

$$\mathbf{\Gamma} := \mathbf{\Gamma}_k \cup \bigcup_{0 \leq P \leq k-1} \mathbf{\Gamma}_P \quad \text{(disjoint union).} \tag{3.1}$$

Since every element of $P^+_{I_2(k)}$ has a unique geodesic representative in $\mathbf{\Gamma}$, the restriction of π^+ to $\mathbf{\Gamma}$ is a bijective map to $P^+_{I_2(k)}$.

We now proceed to construct automata that recognize all words in $\mathbf{\Gamma}$.

[**Case 1.** ($P = k$.)] It is clear that every word of the set $\mathbf{\Gamma}_k$ is recognized by the deterministic finite-state automaton $\mathbf{A}_k$ over $\mathrm{FB}^+ \cup \{\nabla\}$ defined by

(i) **States**: $\{\varepsilon\} \cup \mathrm{FB}^+ \cup \{\nabla\}$;
Initial state: $\{\varepsilon\}$; Accept state: $\{\nabla\}$;
(ii) **Transitions**:
(ii-1) ${}^\forall v \in \mathrm{FB}^+ \cup \{\nabla\}$, $\varepsilon \xrightarrow{v} v$,
(ii-2) ${}^\forall u, {}^\forall v \in \mathrm{FB}^+$, $u \xrightarrow{v} v$
if $\mathcal{N}(u) \neq \mathcal{L}(v)$,
(ii-3) ${}^\forall u \in \mathrm{FB}^+ \cup \{\nabla\}$, $u \xrightarrow{\nabla} \nabla$.

[**Case 2.** ($0 \leq P \leq k-1$.)] It is also readily seen that every word of the set $\bigcup_{0 \leq p \leq P} \mathbf{\Gamma}_p$ is recognized by the deterministic finite-state automaton $\mathbf{A}_{\leq P}$ over $\mathrm{FB}^+_{\leq P}$ defined by

(i) **States**: $\{\varepsilon\} \cup \mathrm{FB}^+_{\leq P}$;
Initial state: $\{\varepsilon\}$; Accept states: $\{\varepsilon\} \cup \mathrm{FB}^+_{\leq P}$;
(ii) **Transitions**:
(ii-1) ${}^\forall v \in \mathrm{FB}^+_{\leq P}$, $\varepsilon \xrightarrow{v} v$,
(ii-2) ${}^\forall u, {}^\forall v \in \mathrm{FB}^+_{\leq P}$, $u \xrightarrow{v} v$
if $\mathcal{N}(u) \neq \mathcal{L}(v)$.

Example 3.1. (Case $k = 3$) Set $a := a_1$, $b := a_2$ and $c := a_3$. Then, the fundamental blocks are the followings:

$$a,\ b,\ c,\ ab,\ bc,\ ca.$$

The automaton $\mathbf{A}_{\leq 2}$ is depicted as in Figure 2.

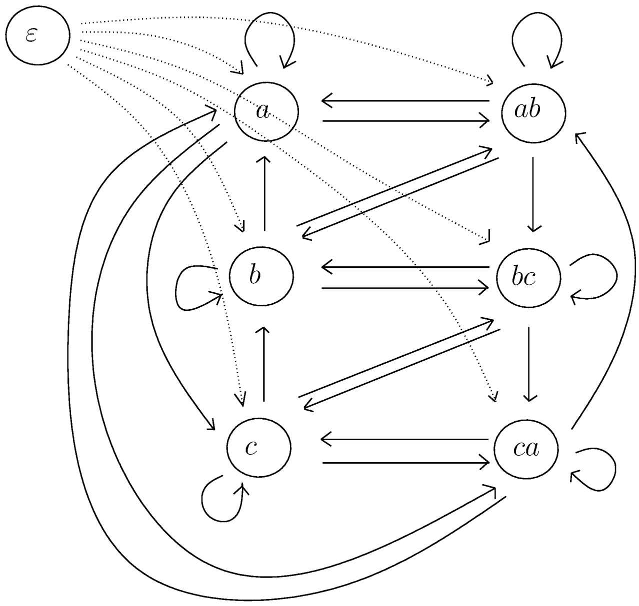

Figure 2. The automaton $\mathbf{A}_{\leq 2}$ for the case $k = 3$.

§ 4. The spherical growth series of $P^+_{I_2(k)}$

In this section, under the identification of $P^+_{I_2(k)}$ with $\bigcup_{0 \leq P \leq k} \mathbf{\Gamma}_P$, by considering the structure of the automata constructed in Section 3, we give a rational function expression of the spherical growth series $\mathcal{S}(t)$ of $P^+_{I_2(k)}$ with respect to the standard generators $a_1, \ldots, a_k$.

For each $0 \leq P \leq k$, let

$$\mathcal{S}_P(t) := \sum_{q=0}^{\infty} \sharp\{\xi \in \mathbf{\Gamma}_P \mid |\xi| = q\}\ t^q$$

be the spherical growth series for $\mathbf{\Gamma}_P$. Then, from (3.1), we have

$$\mathcal{S}(t) = \mathcal{S}_k(t) + \sum_{0 \leq P \leq k-1} \mathcal{S}_P(t). \tag{4.1}$$

In order to simplify the presentation of the growth series, for each $n \in \mathbf{Z}_{\geq 0}$, we use the following notations:

$$\begin{cases} T_n := t + t^2 + \cdots + t^n \ \text{ for } \ n \geq 1, \\ T_0 := 0. \end{cases}$$

First, we consider the case where $0 \leq P \leq k - 1$.

Proposition 4.1. *For each $0 \leq P \leq k-1$, we have*

$$\sum_{0 \leq p \leq P} \mathcal{S}_p(t) = \frac{1+T_P}{1-(k-1)T_P}.$$

Proof. Take $P \in \{0, \dots, k-1\}$. For any $q \in \mathbf{Z}_{\geq 0}$, set

$$B_q(P) := \Big\{ \xi \equiv \mu_1 \cdots \mu_m \in \bigcup_{0 \leq p \leq P} \mathbf{\Gamma}_p \;\Big|\; \xi : \text{normal form},\ \mu_i \in \mathrm{FB}^+,\ |\xi| = q \Big\},$$

and set

$$\beta_q(P) := \sharp B_q(P).$$

Then,

$$\sum_{0 \leq p \leq P} \mathcal{S}_p(t) = \sum_{q=0}^{\infty} \beta_q(P)\ t^q.$$

Next, we calculate $\beta_q(P)$. Clearly,

$$\beta_0(P) = 1. \tag{4.2}$$

Lemma 4.2. *We have the following recursive formula:*

$$\beta_q(P) = (k-1)\{\beta_{q-1}(P) + \cdots + \beta_{q-P}(P)\}, \tag{4.3}$$

for $q \geq P+1$.

Proof of Lemma 4.2. By following the automaton $\mathbf{A}_{\leq P}$, we see that for each $I \in \{1, \dots, P\}$ and each $v_1 \cdots v_{m-1} \in B_{q-I}(P)$, there are $k-1$ choices of v_m from FB_I^+ such that $v_1 \cdots v_{m-1} \cdot v_m \in B_q(P)$ by (ii-2) in $\mathbf{A}_{\leq P}$. Thus, we obtain the recursive formula (4.3). This completes the proof of Lemma 4.2. □

On the other hand, for $1 \leq q \leq P$, we have

$$\beta_q(P) = k^q. \tag{4.4}$$

Thus, by the recursive formula (4.3) with (4.2) and (4.4), we can see

$$\Big(\sum_{0 \leq p \leq P} \mathcal{S}_p(t) \Big) \times \{1 - (k-1)(t + t^2 + \cdots + t^P)\} = 1 + t + t^2 + \cdots + t^P.$$

Hence,

$$\sum_{0 \leq p \leq P} \mathcal{S}_p(t) = \frac{1 + t + t^2 + \cdots + t^P}{1 - (k-1)(t + t^2 + \cdots + t^P)} = \frac{1 + T_P}{1-(k-1)T_P},$$

for each t in a sufficiently small neighborhood of the origin 0. This completes the proof of Proposition 4.1. □

Finally, we consider the case where $P = k$ and the spherical growth series $\mathcal{S}(t)$ of $P^+_{I_2(k)}$.

Theorem 4.3. *Let $D_k(t)$ be the polynomial defined by*

$$D_k(t) := 1 - kt + (k-1)t^k = (1-t)\{1-(k-1)T_{k-1}\}.$$

Then, we have

$$\mathcal{S}_k(t) = \frac{t^k}{D_k(t)},$$

and

$$\mathcal{S}(t) = \sum_{0 \le P \le k} \mathcal{S}_P(t) = \frac{1}{D_k(t)}.$$

Proof. Every words in the set $P^+_{I_2(k)} = \bigcup_{0 \le P \le k} \mathbf{\Gamma}_P$ is recognized by the automata $\mathbf{A}_{\le k-1}$ and $\mathbf{A}_k$. Thus, its spherical growth series $\mathcal{S}(t)$ has a rational function expression. Then, put

$$\mathcal{S}(t) := \frac{G(t)}{F(t)},$$

where $F(t), G(t)$ are polynomials in t.

By the case $P = k-1$ in Proposition 4.1, we have

$$\sum_{0 \le p \le k-1} \mathcal{S}_p(t) = \frac{1 + T_{k-1}}{1-(k-1)T_{k-1}}. \tag{4.5}$$

This is the growth series for the maximal subset of $P^+_{I_2(k)}$ whose elements contain no word u satisfying $u = \nabla$.

On the other hand, by observing the normal form of an element of $P^+_{I_2(k)}$, it is seen that the series

$$\sum_{q=0}^{\infty} \sharp\{\xi \in P^+_{I_2(k)} \mid |\xi| = q\}\ t^{q+k}$$

is the spherical growth series for $\mathbf{\Gamma}_k$. Thus,

$$\mathcal{S}_k(t) = \mathcal{S}(t) \times t^k = \frac{G(t)}{F(t)} t^k.$$

Hence, with (4.5), we have

$$\frac{G(t)}{F(t)} - \frac{G(t)}{F(t)} t^k = \sum_{0 \le p \le k-1} \mathcal{S}_p(t) = \frac{1 + T_{k-1}}{1-(k-1)T_{k-1}}.$$

By solving this equality for $G(t)/F(t)$, we obtain

$$\mathcal{S}(t) = \frac{G(t)}{F(t)} = \frac{1}{(1-t)\{1-(k-1)(t+t^2+\cdots+t^{k-1})\}} = \frac{1}{1-kt+(k-1)t^k}.$$

Also, we have

$$\mathcal{S}_k(t) = \frac{G(t)}{F(t)}\, t^k = \frac{t^k}{1-kt+(k-1)t^k}.$$

This completes the proof of Theorem 4.3. □

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