RIMS Kôkyûroku Bessatsu
B49 (2014), 131–137

Topological instability of laminar flows for the two-dimensional Navier-Stokes equation with circular arc no-slip boundary conditions

By

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Abstract

In the non-stationary two-dimensional Navier-Stokes equation with circular arc no-slip boundary conditions, topologically changing flow, namely, some kind of instability is observed.

§ 1. Introduction and main result

Ohya and Karasudani [12] developed a new wind turbine system that consists of a diffuser shroud with a broad-ring at the exit periphery and a wind turbine inside it. Their experiments show that a diffuser-shaped (not nozzle-shaped) structure can accelerate the wind at the entrance of the body (we say "wind-lends phenomena"). A strong vortex formation with a low-pressure region is created behind the broad brim. Accordingly, the wind flows into a low-pressure region, the wind velocity is accelerated further near the entrance of the diffuser. In general, creation of a vortex needs separation phenomena near a boundary (namely, topologically changing phenomena), and before separating from the boundary, the flow moves toward reverse direction near the boundary against the laminar flow direction.

In "boundary layer theory" (BLT) point of view, such phenomena itself is well studied. Our main purpose is just propose "local pressure analysis method" through (well-known) separation phenomena. In the beginning of 20th century, Prandtl proposed BLT, and it has been developing extensively (see Rosenhead [13] and Bakker [1] for example). Basically, BLT equations can be deduced from the Navier-Stokes equations. Van Dommelen and Shen [3] made a key observation of shock singularities, which helps us to

Received September 26, 2013.
2010 Mathematics Subject Classification(s): 35Q30, 76D05, 76D10, 53A04
Key Words: Navier-Stokes equation, laminar flow, no-slip boundary condition
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analyze separation phenomena deeply. Ma and Wang [9] provided a characterization of the boundary layer separation of 2-D incompressible viscous fluids. They considered a separation equation linking a separation location and a time with the Reynolds number, the external forcing and the initial velocity field. Due to the limitation of space and the vast literature in the BLT, we do not try to do a complete survey here.
However, we need to mention the results related to the BLT (in other words, wake region) in pure mathematics. Using the Oseen system is one of the mathematical approach to analyze the wake region. For the detailed discussion of the Oseen system, we refer the reader to [6]. In a convex obstacle case, the character of the system is elliptic in front of the obstacle. To the contrary, its character changes into parabolic type (wake region) behind the obstacle (see [8] for example). Maekawa [10] considered the two-dimensional Navier-Stokes equations in a half plane under the no-slip boundary condition. He established a solution formula for the vorticity equations and got a sufficient condition on the initial data for the vorticity to blow up to the inviscid limit.
In this paper we show that a diffuser-shaped boundary induces the reverse flow even near the entrance of the diffuser (by using "local pressure analysis method"). Let us be more precise. We consider the two-dimensional Navier-Stokes equation in $\Omega \subset \mathbb{R}^2$ (define Ω later) with no-slip and inflow-outflow conditions on $\partial\Omega$. We need to handle a shape of the boundary $\partial\Omega$ precisely, thus we set a parametrized smooth boundary $\varphi : [0, S] \to \mathbb{R}^2$ as $|\partial_s\varphi(s)| = 1$, $|\partial_s^2\varphi(s)| = \kappa$ (curvature), $\varphi(0) = (0, 0)$, $\partial_s\varphi(0) = (1, 0)$, $\partial_s^2\varphi(0) = (0, -\kappa)$. We choose S later (should be sufficiently small). We define $n = n(s) := (\partial_s\varphi(s))^\perp$ as a unit normal vector and $\tau = \tau(s) = \partial_s\varphi(s)$ as a unit tangent vector, where $\perp$ represents upward direction. In order to define the domain Ω, we need the following coordinate.

Definition 1.1. (Normal coordinate.) For $s \in [0, S]$ and $r \in [0, R]$, let

$$\Phi(s, r) = \Phi_\varphi(s, r) := n(s)r + \varphi(s).$$

Remark 1.2. Since $\partial_s n(s) = \kappa\tau(s)$ (Frenet-Serret formulas), we see that

$$(\partial_r\Phi)(s, r) = n(s) \quad \text{and} \quad (\partial_s\Phi)(s, r) = (r\kappa + 1)\tau(s).$$

Now we define the domain Ω as follows:

$$\Omega = \Omega_{S,R} := \{\Phi(s, r) \in \mathbb{R}^2 : s \in (0, S),\ r \in (0, R)\}.$$

Note that we will take S and R to be sufficiently small depending on the initial data and the inflow condition (see Remark 1.6). The non-stationary two-dimensional Navier-Stokes equation is expressed as

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u = -\nabla p, \quad \nabla\cdot u = 0 \quad in \quad \Omega \subset \mathbb{R}^2, \\ u|_{\cup_{s=0}^S \varphi(s)} = 0, \end{cases} \tag{1.1}$$

where $u = u(x) = u(x,t) = (u^1(x_1,x_2,t), u^2(x_1,x_2,t))$. In this paper we sometimes abbreviate the time t not x.

Definition 1.3. (Inflow condition) Let $u_{in}(r) := (u\cdot\tau)(\Phi(0,r))$ be a (smooth) inflow with rightward direction, namely, $\partial_r u_{in}(r) > 0$. Also assume $(u\cdot n)(\Phi(0,r)) = 0$ (this condition is just for a technical reason, expressing parallel profile to the boundary). Let $\alpha_1, \alpha_2, \alpha_3 \in C^\infty([0,\infty))$ be coefficients of the time dependent inflow condition, namely,

$$u_{in}(r) = (u\cdot\tau)(\Phi(0,r)) = \alpha_1(t)r - \frac{\alpha_2(t)}{2!}r^2 + \frac{\alpha_3(t)}{3!}r^3 + O(r^4).$$

Since $\partial_r u_{in}(r) > 0$, we see that $\alpha_1(t) > 0$. Assume also the inflow does not grow polynomially for r direction (this is due to the observation of "boundary layer", since the inflow profile should be a uniform one away from the boundary), thus, it is reasonable to focus on the following two cases:

- (Poiseuille type profile) $\alpha_1(t) > 0$, $\alpha_2(t) > 0$ and $\alpha_3(t)$ is small compare with $\alpha_1(t)$ and $\alpha_2(t)$,
- (Before separation profile) $\alpha_2(t) < 0$, $\alpha_3(t) < 0$ and $\alpha_1(t)$ is small compare with $\alpha_2(t)$ and $\alpha_3(t)$.

In this point of view, the following assumption is acceptable:

$$\kappa^2\alpha_1(t) + 2\kappa\alpha_2(t) - \alpha_3(t) > C > 0 \quad \text{(see Theorem 1.10)}.$$

Remark 1.4. The case $\alpha_1(t) \searrow 0$ $(t \to t_0)$ expresses nothing more than ∂-singular (see [9]) which represents separation at the origin and a time t_0. Namely, if separation occurs, $\alpha_1(t_0)$ must be zero.

We assume that there exists a smooth solution except for the origin, namely, assume that there exists a pair of solution (u,p) to (1.1) in

$$u,p \in C^\infty([0,T]\times D) \cap C^\infty((0,T]\times(\bar{\Omega}\setminus B_\epsilon)) \quad \text{for any} \quad D \Subset \Omega \quad \text{and} \quad \epsilon > 0,$$

where $B_\epsilon = \{x \in \mathbb{R}^2 : |x| < \epsilon\}$. In the physical point of view, finite energy should be required. More precisely,

$$u \in L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \nabla u \in L^2(0,T;L^2(\Omega))$$

should be required. Since we only consider the flow near the origin, "local finite energy near the origin" must be the condition we need to check. It is out of the main topic in this paper, thus we do not mention more about it.

Remark 1.5. Combining a result of Navier-Stokes initial value problem in Lipschitz domain [11], a boundary regularity result [7] (We believe we can generalize their result to various smooth domains) and an inhomogeneous boundary result [5] (see also [4]), the above existence and smoothness assumptions should become true. However, regularity at the origin should be more delicate. If the origin is smooth, the following ODE (which comes from the inflow condition) must have a solution (see Remark 1.11 also):

$$\partial_t \alpha_1(t) = -\nu(4\alpha_1(t)\beta(t) + \kappa^2\alpha_1(t) + 2\kappa\alpha_2(t) - \alpha_3(t)),$$

where $\beta(t)$ is a quantified geometrical behavior of the laminar flow (see Definition 1.8). The point it that the coefficient $\alpha_1(t)$ is determined by $\alpha_2(t)$, $\alpha_3(t)$, $\beta(t)$ and κ. This means that we cannot set arbitrary smooth inflow in order to have the smoothness at the origin.

Remark 1.6. We can avoid interior blow-up by taking sufficiently small R. Thus we only need to care boundary regularity not interior regularity. Moreover we can also avoid boundary blow-up except for the origin by taking sufficiently small S. Thus it is reasonable to assume T to be sufficiently large (for sufficiently small S and R).

Definition 1.7. (Laminar flow.) u is "laminar flow" (near the origin) iff u is smooth (including the origin) in $\bar{\Omega}$, $|u(x)| \neq 0$ for $x \in \Omega$ and the flow u is to the rightward direction (laminar flow direction), namely,

$$(u \cdot \tau)(x) > 0$$

for $x \in \Omega$.

We mainly consider a geometrical shape of the laminar flow near the origin. In this case, one of the five situations only occur (for fixed time t): (geometrically) diffusing, almost parallel, concentrating laminar flows, topologically changing flow (inducing the reverse flow) or non-smoothness (singularity) at the origin. Sometimes we write $u \cdot \tau = (u \cdot \tau)(s,r) = (u \cdot \tau)(s,r,t) = (u \cdot \tau)(x,t)$ with $x = \Phi(s,r)$ unless confusion occurs.

Definition 1.8. (Classification of Navier-Stokes flow for fixed time.) Let

$$\mathcal{L}_t(s,r) = \mathcal{L}(s,r) := (r\kappa + 1)\frac{u \cdot n}{u \cdot \tau} \quad \text{(slope of the velocity with Riemannian metric)}$$

and let β be a quantified geometrical behavior of the laminar flow (near the origin):

$$\beta = \beta(t) := \lim_{s,r \to 0} \partial_s \partial_r \mathcal{L}(s,r).$$

- Diffusing laminar flow: We call (geometrically) diffusing laminar flow iff $u, p \in C^\infty(\Omega)$ and
$$\beta(t) > 0.$$
- Almost parallel laminar flow: We call (geometrically) almost parallel laminar flow iff $u, p \in C^\infty(\Omega)$ and
$$\beta(t) = 0.$$
- Concentrating laminar flow: We call (geometrically) concentrating laminar flow iff $u, p \in C^\infty(\Omega)$ and
$$\beta(t) < 0.$$
- Topologically changing flow (not laminar flow case): We say topologically changing flow iff $u, p \in C^\infty(\Omega)$ and there is $x \in \Omega$ such that $|u(x)| = 0$ or $(u \cdot \tau)(x) \leq 0$.
- Non-smoothness at the origin: We say non-smoothness at the origin (for fixed t) iff
$$u(\cdot, t) \notin C^\infty(\Omega \cap B_\epsilon) \quad \text{or} \quad p(\cdot, t) \notin C^\infty(\Omega \cap B_\epsilon) \quad \text{for} \quad \epsilon > 0.$$

In order to give the main theorem, we need to define "trajectory".

Definition 1.9. (Trajectory.) Let $\tilde{\gamma}_X : [0, T) \to \Omega$ be such that
$$\partial_t \tilde{\gamma}_X(t) = u(\tilde{\gamma}_X(t), t), \quad \gamma_X(0) = X \in \Omega.$$

Note that the equation (1.1) can be rewritten to $\partial_t(u(\tilde{\gamma}(t), t)) = (\Delta u - \nabla p)(\tilde{\gamma}(t), t)$.

The following is the main theorem.

Theorem 1.10. *(Horizontally stopping particles phenomena.) Let the initial datum u_0 satisfies the diffusing laminar flow condition, namely, $\beta(0) > 0$. For any given smooth inflow $u_{in}(r)$ with $\kappa^2\alpha_1 + 2\kappa\alpha_2 - \alpha_3 > C > 0$ (see Definition 1.3) then the topologically changing flow (or non-smoothness at the origin) must occur in finite time. In other words, particles near the boundary slow down and finally stop horizontally in finite time. More precisely, there is $\bar{R} < R$ such that if $\bar{r} < \bar{R}$, then*
$$\lim_{t \to \tilde{T}} (u \cdot \tau)(\tilde{\gamma}_{\Phi(0,\bar{r})}(t), t) = 0,$$
where $\tilde{T}(< T)$ is depending on $\bar{r}$, ν, κ, α_1, α_2 and α_3.

Remark 1.11. In order to keep the smoothness at the origin, $\alpha_1(t)$ must satisfy
$$\partial_t \alpha_1(t) = -\nu(4\alpha_1(t)\beta(t) + \kappa^2\alpha_1(t) + 2\kappa\alpha_2(t) - \alpha_3(t)).$$
Otherwise, non-smoothness immediately occurs. This is due to the "breaking effect" (see [15]). In this case, we have $\alpha_1(t) \searrow 0$ as $t \to \tilde{T}$ (this expresses ∂-singular, see [9]).

Remark 1.12. There are direct and indirect evidences for the validity of the "Kutta condition" in restricted regions (see [2]). The method used in the above theorem may give another support for the validity of the Kutta condition in pure mathematical sense. Moreover, we may be able to apply the method to "Taylor vortices" (see Chapter II, Section 4 in [14]) which is closely related to the bifurcation theory.

Now we give outline of the proof briefly. Basically, we need to estimate trajectory of a particle near the boundary. In order to do so, we need to estimate each Δu and ∇p near the boundary. First we construct " streamline coordinate " and then we can estimate Δu directly. Next we construct "pressure coordinate " based on level set of the pressure and no-slip boundary condition. In this case, $\Delta u = \nabla p$ on the boundary is the crucial point. Third we calculate some kind of Riemannian metric of the "pressure coordinate" at the origin (the pressure is nonlocal operator, nevertheless we can estimate it by using orders of approximation). For the detailed proof, see [15].

Acknowledgments. The author would like to thank Professors Takashi Sakajo, Hideo Kozono and Hisashi Okamoto for letting me know the articles [2], [14] and [1, 13] respectively. The author is grateful for Professor Chi-hin Chan for his interest and expository comments. The author is partially supported by JSPS KAKENHI Grant Number 25870004.

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