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On the Hardy type inequality in critical Sobolev-Lorentz spaces

By

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Abstract

In this paper, we establish the Hardy inequality of the logarithmic type in the critical Sobolev-Lorentz spaces. More precisely, we generalize the Hardy type inequality obtained in Edmunds-Triebel [8]. The generalized inequality allows us to take the exponents appearing in the inequality more flexibly and their optimality is discussed in detail. O'Neil's inequality and its reverse play an essential role for the proof.

§ 1. Introduction and main theorem

In this paper, we shall give a systematic treatment concerning the Hardy type inequalities on the critical Sobolev-Lorentz spaces $H^s_{p,q}(\mathbb{R}^n)$ with $n \in \mathbb{N}$, $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le q \le \infty$, where the space $H^s_{p,q}(\mathbb{R}^n)$ can be characterized in terms of the Bessel potential such as $H^s_{p,q}(\mathbb{R}^n) := (1-\Delta)^{-\frac{s}{2}} L_{p,q}(\mathbb{R}^n)$ with the Lorentz space $L_{p,q}(\mathbb{R}^n)$. We collect precise definitions of those function spaces and related properties in Section 2.

We recall the Sobolev embedding theorem on $H^{\frac{n}{p_1}}_{p_1,p_2}(\mathbb{R}^n)$ which states that the continuous inclusions $H^{\frac{n}{p_1}}_{p_1,p_2}(\mathbb{R}^n) \hookrightarrow L_{q_1,q_2}(\mathbb{R}^n)$ hold for all $q_1 \in [p_1, \infty)$ and $q_2 \in [p_2, \infty]$. However, the limiting case $q_1 = \infty$ in this embedding fails provided that $(p_2, q_2) \neq (1, \infty)$. This implies that functions in the space $H^{\frac{n}{p_1}}_{p_1,p_2}(\mathbb{R}^n)$ can have a local singularity at some point in $\mathbb{R}^n$. In fact, the critical Sobolev space $H^{\frac{n}{p}}_{p}(\mathbb{R}^n)$, which is

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identical with the critical Sobolev-Lorentz space $H^{\frac{n}{p_1}}_{p_1,p_2}(\mathbb{R}^n)$ with $p_1 = p_2 =: p$, admits a singularity of the logarithmic order, see Adams-Fournier [1] and Maz'ya [20]. As a characterization of $H^{\frac{n}{p}}_p(\mathbb{R}^n)$, Edmunds-Triebel [8] proved the corresponding Hardy type inequality with a logarithmic correction as follows:

Theorem A (Edmunds-Triebel [8, Theorem 2.8]). *Let $n \in \mathbb{N}$ and $1 < p < \infty$. Then there exists a positive constant C such that the inequality*

$$\left(\int_{\{|x|<\frac{1}{2}\}} \left(\frac{|u(x)|}{|\log |x||} \right)^p \frac{dx}{|x|^n} \right)^{\frac{1}{p}} \leq C \|u\|_{H^{\frac{n}{p}}_p} \tag{1.1}$$

holds for all $u \in H^{\frac{n}{p}}_p(\mathbb{R}^n)$.

The main purpose in this paper is to generalize (1.1) into two directions. First, we shall prove the corresponding logarithmic Hardy type inequality in the critical Sobolev Lorentz space $H^{\frac{n}{p_1}}_{p_1,p_2}(\mathbb{R}^n)$, which coincides with (1.1) when $p_1 = p_2 =: p$. Furthermore, we investigate the possibility whether the exponents appearing in the inequalities can be taken more flexibly including the consideration on its optimality. Indeed, our main result now reads:

Theorem 1.1. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \leq \infty$ and $1 < \alpha, \beta < \infty$. Then the inequality*

$$\left(\int_{\{|x|<\frac{1}{2}\}} \frac{|u(x)|^\alpha}{|\log |x||^\beta} \frac{dx}{|x|^n} \right)^{\frac{1}{\alpha}} \leq C \|u\|_{H^{\frac{n}{p}}_{p,q}} \tag{1.2}$$

holds for all $u \in H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ if and only if one of the following conditions (i), (ii) and (iii) is fulfilled:

$$\begin{cases} (i) \quad 1 + \alpha - \beta < 0\,; \\ (ii) \quad 1 + \alpha - \beta \geq 0 \ \text{ and } \ q < \dfrac{\alpha}{1 + \alpha - \beta}\,; \\ (iii) \quad 1 + \alpha - \beta > 0, \quad q = \dfrac{\alpha}{1 + \alpha - \beta} \ \text{ and } \ \alpha \geq \beta. \end{cases} \tag{1.3}$$

Remark. The condition (ii) in (1.3) allows us to take $1+\alpha-\beta = 0$, which implies $\frac{\alpha}{1+\alpha-\beta} = \infty$. In the special case of $p = q = \alpha = \beta$, the inequality (1.2) is precisely the inequality (1.1) by Edmunds-Triebel [8]. Also note that the value $q = \dfrac{\alpha}{1+\alpha-\beta}$ is the critical exponent in the sense that the inequality (1.2) holds or not. Moreover, Theorem 1.1 states that when $q = \dfrac{\alpha}{1+\alpha-\beta}$, the inequality (1.2) holds if $\alpha \geq \beta$ and fails if $\alpha < \beta$. In particular, the inequality (1.2) fails for the marginal case $q = \infty$ and $1 + \alpha - \beta = 0$. Indeed, the function u_0 defined by $u_0(x) := \eta(x)\,|\log|x||$ belongs to $H^{\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$, where η is a cut-off function supported near the origin, while

$$\int_{\{|x|<\frac{1}{2}\}} \frac{|u_0(x)|^\alpha}{|\log|x||^{1+\alpha}} \frac{dx}{|x|^n} = +\infty.$$

There is a number of both mathematical and physical applications of Hardy type inequalities. Among others, we refer the reader to Adimurthi-Chaudhuri-Ramaswamy [2], Beckner [4], Bradley [6], Brézis-Marcus [7], Edmunds-Triebel [8], García-Peral [9], Gurka-Opic [10], Herbst [11], Kalf-Walter [12], Kerman-Pick [13, 14, 15], Ladyzhenskaya [17], Machihara-Ozawa-Wadade [18], Matsumura-Yamagata [19], Nagayasu-Wadade [22], Ozawa-Sasaki [24], Pick [25], Reed-Simon [26], Triebel [28] and Zhang [29]. Especially, in Bradley [6] and Edmunds-Triebel [8], the inequalities of the type similar to (1.2) were considered in terms of Besov type spaces.

This paper is organized as follows. Section 2 is devoted to the definition of the Sobolev-Lorentz space as well as several lemmas needed for the proof of Theorem 1.1. We shall prove Theorem 1.1 in Section 3 and Section 4.

§ 2. Preliminaries

In this section, we first recall the definition of the Lorentz spaces. To this end, we define the rearrangement of measurable functions. For a measurable function f on $\mathbb{R}^n$ with $n \in \mathbb{N}$, $f_* : [0,\infty) \to [0,\infty]$ denotes the distribution function of f given by

$$f_*(\lambda) := |\{x \in \mathbb{R}^n\,;\, |f(x)| > \lambda\}| \quad \text{for } \lambda \geq 0,$$

and then the rearrangement $f^* : [0,\infty) \to [0,\infty]$ of f is defined by

$$f^*(t) := \inf\{\lambda > 0\,;\, f_*(\lambda) \leq t\} \quad \text{for } t \geq 0.$$

Moreover, $f^{**} : (0,\infty) \to [0,\infty]$ denotes the average function of f^* defined by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(\tau)d\tau \quad \text{for } t > 0.$$

In what follows, we assume $f^*(t) < +\infty$ for all $t > 0$. Then f^* is right-continuous and non-increasing on $(0,\infty)$, and hence, f^{**} is continuous and non-increasing on $(0,\infty)$ with $f^*(t) \leq f^{**}(t)$ for all $t > 0$. We now introduce the Lorentz space by using the rearrangement. Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then the Lorentz space $L_{p,q}(\mathbb{R}^n)$ is defined as a function space equipped with the following norm,

$$\|f\|_{L_{p,q}} := \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty\,; \\ \displaystyle\sup_{t>0}\left(t^{\frac{1}{p}} f^*(t)\right) & \text{if } q = \infty. \end{cases} \tag{2.1}$$

We can take f^{**} replaced by f^* in the definition (2.1) as another equivalent norm on $L_{p,q}(\mathbb{R}^n)$ if $p \neq 1$. Indeed, the following Hardy inequality guarantees its equivalence,

$$\left(\int_0^\infty \left(\frac{t^{\frac{1}{p}}}{t}\int_0^t f(\tau)d\tau\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \le p'\left(\int_0^\infty \left(t^{\frac{1}{p}}f(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{2.2}$$

for non-negative measurable functions f for which the integral on the right-hand side in (2.2) is finite. Remark that the inequality (2.2) is still valid for the case $q = \infty$ by replacing the integral by the supremum. For the proof of (2.2), see O'Neil [23, Lemma 2.3] and references therein. Furthermore, since f^* and f^{**} are both monotonically non-increasing functions in $(0,\infty)$, we easily get the following decay estimates. For any $t > 0$, we have

$$f^*(t) \le \left(\frac{q}{p}\right)^{\frac{1}{q}} t^{-\frac{1}{p}} \|f\|_{L_{p,q}} \tag{2.3}$$

and if $p > 1$, together with the inequality (2.2), we also have for any $t > 0$,

$$f^{**}(t) \le p'\left(\frac{q}{p}\right)^{\frac{1}{q}} t^{-\frac{1}{p}} \|f\|_{L_{p,q}}. \tag{2.4}$$

Note that the inequalities (2.3) and (2.4) are also valid for the marginal case $q = \infty$ and we will utilize them frequently for the proof of the main theorem in Section 3.

We also make use of the celebrated Hardy-Littlewood inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le \int_0^\infty f^*(t)g^*(t)dt \tag{2.5}$$

for all measurable functions f and g. The proof of (2.5) can be found in Bennett-Sharpley [5, Theorem 2.2].

Next, we recall the pointwise rearrangement inequality for the convolution of functions proved by O'Neil [23, Theorem 1.7]. In fact, for measurable functions f and g on $\mathbb{R}^n$, we have

$$(f * g)^{**}(t) \le t\, f^{**}(t)g^{**}(t) + \int_t^\infty f^*(\tau)g^*(\tau)d\tau \quad \text{for } t > 0. \tag{2.6}$$

Moreover, we make use of the reverse O'Neil inequality established in Kozono-Sato-Wadade [16, Lemma 2.2]. Indeed, there exists a positive constant C such that the inequality

$$(f * g)^{**}(t) \ge C\left(t\, f^{**}(t)g^{**}(t) + \int_t^\infty f^*(\tau)g^*(\tau)d\tau\right) \tag{2.7}$$

holds for all $t > 0$ and for all measurable functions f and g on $\mathbb{R}^n$ which are both non-negative, radially symmetric and non-increasing in the radial direction.

In this paper, we frequently use the Bessel potential $G_s* := (1-\Delta)^{-\frac{s}{2}}$ and the Riesz potential $I_s* := (-\Delta)^{-\frac{s}{2}}$ for $0 < s < n$. More precisely, the kernel functions I_s and G_s are defined respectively by

$$\begin{cases} I_s(x) := \dfrac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} |x|^{-(n-s)}\,; \\ G_s(x) := \dfrac{1}{(4\pi)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)} \displaystyle\int_0^\infty e^{-\pi\frac{|x|^2}{t} - \frac{t}{4\pi}}\, t^{-\frac{n-s}{2}} \frac{dt}{t} \end{cases}$$

for $x \in \mathbb{R}^n \setminus \{0\}$, where Γ denotes the Gamma function. Based on the Lorentz space, we define the Sobolev-Lorentz space $H^s_{p,q}(\mathbb{R}^n)$ by $H^s_{p,q}(\mathbb{R}^n) := (I-\Delta)^{-\frac{s}{2}} L_{p,q}(\mathbb{R}^n) = G_s * L_{p,q}(\mathbb{R}^n)$ equipped with the norm $\|u\|_{H^s_{p,q}} := \|(I-\Delta)^{\frac{s}{2}} u\|_{L_{p,q}}$. The space $H^s_{p,q}(\mathbb{R}^n)$ is a generalization of the usual Sobolev space $H^s_p(\mathbb{R}^n)$ since we have $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ due to the norm-invariance of $\|u\|_{L_{p,p}} = \|u\|_{L_p}$. We now collect the elementary properties of I_s and G_s in the following lemma.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $0 < s < n$.*

(i) I_s and G_s are non-negative, radially symmetric and non-increasing in the radial direction, so that $I^*_s(t) = I_s(x)$ and $G^*_s(t) = G_s(x)$ if $|x| = \left(\frac{t}{\omega_n}\right)^{\frac{1}{n}} > 0$, where $\omega_n := \frac{2\pi^{\frac{\pi}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$ denotes the volume of the unit ball in $\mathbb{R}^n$.

(ii) $G_s(x) \le I_s(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$, which implies $G^*_s(t) \le I^*_s(t)$, $G^{**}_s(t) \le I^{**}_s(t)$ for all $t > 0$, and $\displaystyle\lim_{|x|\downarrow 0} \frac{G_s(x)}{I_s(x)} = \lim_{t \downarrow 0} \frac{G^*_s(t)}{I^*_s(t)} = 1$.

(iii) $\|G_s\|_{L_1} = 1$ and there exists a positive constant C such that the following inequalities hold,

$$G_s(x) \le \begin{cases} C\,|x|^{-(n-s)} & \text{for } x \in \mathbb{R}^n \setminus \{0\}\,; \\ C\,e^{-|x|} & \text{for } x \in \mathbb{R}^n \quad \text{with } |x| \ge 1. \end{cases}$$

Since the facts in Lemma 2.1 are well-known, we omit the detailed proof here, see Stein [27] for instance. Furthermore, we refer to Almgren-Lieb [3] and Bennett-Sharpley [5] for further information about the rearrangement theory.

At the end of this section, we shall show the following one-dimensional Hardy inequality of logarithmic type:

Lemma 2.2. *Let $1 < \alpha, \beta < \infty$. Then there exists a positive constant C such that the inequality*

$$(2.8) \qquad \left(\int_0^{\frac{1}{2}} \left(\int_t^{\frac{1}{2}} |\phi(s)| ds\right)^\alpha |\log t|^{-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \le C \left(\int_0^{\frac{1}{2}} |\phi(t)|^\alpha |\log t|^{-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}}$$

holds for all measurable functions ϕ *such that the integral on the right-hand side of* (2.8) *is finite.*

Furthermore, we can show the following dual variant of the inequality (2.8).

Lemma 2.3. *Let* $1 < \beta \le \alpha < \infty$ *and* $q := \frac{\alpha}{1+\alpha-\beta}$. *Then there exists a positive constant* C *such that the inequality*

$$\left(\int_0^{\frac{1}{2}} \left(\int_t^{\frac{1}{2}} |\phi(s)| ds\right)^{\alpha} |\log t|^{-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \le C \left(\int_0^{\frac{1}{2}} (t|\phi(t)|)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{2.9}$$

holds for all measurable functions ϕ *such that the integral on the right-hand side of* (2.9) *is finite.*

We shall apply Lemma 2.3 for the proof of the sufficiency part of Theorem 1.1 in Section 3, and Lemma 2.2 will be used for the proof of the necessity part of Theorem 1.1 in Section 4. Lemma 2.2 and Lemma 2.3 can be obtained as corollaries of the following weighted inequalities obtained in Bradley [6] and Muckenhoupt [21] :

Theorem B (Bradley [6], Muckenhoupt [21]). *Let* $1 < \rho \le \sigma < \infty$ *and let* U *and* V *be measurable weights.*

(i) There exists a positive constant C *such that the inequality*

$$\left(\int_0^{\infty} \left|U(t) \int_0^{t} |\psi(s)| ds\right|^{\sigma} dt\right)^{\frac{1}{\sigma}} \le C \left(\int_0^{\infty} |V(t)\psi(t)|^{\rho} dt\right)^{\frac{1}{\rho}} \tag{2.10}$$

holds for all measurable functions ψ *such that the integral on the right-hand side of* (2.10) *is finite if and only if*

$$\sup_{r>0} \left(\int_r^{\infty} |U(t)|^{\sigma} dt\right)^{\frac{1}{\sigma}} \left(\int_0^{r} |V(t)|^{-\rho'} dt\right)^{\frac{1}{\rho'}} < +\infty.$$

(ii) There exists a positive constant C *such that the inequality*

$$\left(\int_0^{\infty} \left|U(t) \int_t^{\infty} |\psi(s)| ds\right|^{\sigma} dt\right)^{\frac{1}{\sigma}} \le C \left(\int_0^{\infty} |V(t)\psi(t)|^{\rho} dt\right)^{\frac{1}{\rho}} \tag{2.11}$$

holds for all measurable functions ψ *such that the integral on the right-hand side of* (2.11) *is finite if and only if*

$$\sup_{r>0} \left(\int_0^{r} |U(t)|^{\sigma} dt\right)^{\frac{1}{\sigma}} \left(\int_r^{\infty} |V(t)|^{-\rho'} dt\right)^{\frac{1}{\rho'}} < +\infty.$$

Now we shall show Lemma 2.2 and Lemma 2.3 by applying Theorem B (i) and Theorem B (ii), respectively.

Proof of Lemma 2.2. Define the weights U_1 and V_1 by

$$U_1(t) := \begin{cases} |\log t|^{-\frac{\beta}{\alpha}} t^{-\frac{1+\alpha}{\alpha}} & \text{for } 0 < t < \frac{1}{2}; \\ 0 \quad \text{for } t \geq \frac{1}{2} \end{cases}$$

and

$$V_1(t) := \begin{cases} |\log t|^{-\frac{\beta}{\alpha}} t^{-\frac{1}{\alpha}} & \text{for } 0 < t < \frac{1}{2}; \\ 1 \quad \text{for } t \geq \frac{1}{2}. \end{cases}$$

Then the direct calculation shows

$$\sup_{r>0} \left(\int_r^\infty |U_1(t)|^\alpha dt \right)^{\frac{1}{\alpha}} \left(\int_0^r |V_1(t)|^{-\alpha'} dt \right)^{\frac{1}{\alpha'}} < +\infty.$$

Thus Theorem B (i) implies

$$\left(\int_0^\infty \left| U_1(t) \int_0^t |\psi(s)| ds \right|^\alpha dt \right)^{\frac{1}{\alpha}} \leq C \left(\int_0^\infty |V_1(t)\psi(t)|^\alpha dt \right)^{\frac{1}{\alpha}},$$

namely,

$$\begin{aligned} &\left(\int_0^{\frac{1}{2}} \left(\frac{1}{t} \int_0^t |\psi(s)| ds \right)^\alpha |\log t|^{-\beta} \frac{dt}{t} \right)^{\frac{1}{\alpha}} \\ &\leq C \left(\int_0^{\frac{1}{2}} |\psi(t)|^\alpha |\log t|^{-\beta} \frac{dt}{t} + \int_{\frac{1}{2}}^\infty |\psi(t)|^\alpha dt \right)^{\frac{1}{\alpha}} \end{aligned}$$

for all measurable functions ψ. Taking $\phi = \chi_{(0,\frac{1}{2})}\psi$ yields the desired inequality (2.8). □

Proof of Lemma 2.3. Define the weights U_2 and V_2 by

$$U_2(t) := \begin{cases} |\log t|^{-\frac{\beta}{\alpha}} t^{-\frac{1}{\alpha}} & \text{for } 0 < t < \frac{1}{2}; \\ 0 \quad \text{for } t \geq \frac{1}{2} \end{cases}$$

and

$$V_2(t) := \begin{cases} t^{\frac{q-1}{q}} & \text{for } 0 < t < \frac{1}{2}; \\ t^{\frac{2q-1}{q}} & \text{for } t \geq \frac{1}{2}. \end{cases}$$

Then the direct calculation shows

$$\sup_{r>0} \left(\int_0^r |U_2(t)|^\alpha dt \right)^{\frac{1}{\alpha}} \left(\int_r^\infty |V_2(t)|^{-q'} dt \right)^{\frac{1}{q'}} < +\infty.$$

Since $\alpha \geq \beta$ implies $\alpha \geq q$, by applying Theorem B (ii), we obtain

$$\begin{aligned}&\left(\int_0^{\frac{1}{2}} \left(\int_t^\infty |\psi(s)| ds\right)^\alpha |\log t|^{-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \\ &\leq C \left(\int_0^{\frac{1}{2}} (t|\psi(t)|)^q \frac{dt}{t} + \int_{\frac{1}{2}}^\infty t^{2q-1} |\psi(t)|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

for all measurable functions ψ. Taking $\phi = \chi_{(0,\frac{1}{2})}\psi$ yields the desired inequality (2.9). □

§ 3. Proof of the sufficiency part of Theorem 1.1

In this section, we consider the sufficiency part of Theorem 1.1. To this end, it suffices to show the following key lemmas.

Lemma 3.1. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \leq \infty$ and $1 < \alpha, \beta < \infty$. Assume one of the conditions (i), (ii) and (iii) in* (1.3) *holds. Then there exists a positive constant C such that the inequality*

$$\left(\int_0^{\frac{1}{2}} \frac{u^*(t)^\alpha}{|\log t|^\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \leq C \|u\|_{H^{\frac{n}{p}}_{p,q}} \tag{3.1}$$

holds for all $u \in H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$.

Lemma 3.2. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \leq \infty$ and $1 < \alpha, \beta < \infty$. Assume one of the conditions (i), (ii) and (iii) in* (1.3) *holds. Then there exists a positive constant C such that the inequality*

$$\left(\int_{\mathbb{R}^n} |w(x)u(x)|^\alpha dx\right)^{\frac{1}{\alpha}} \leq C \left(\sup_{0<t<\frac{1}{2}} t^{\frac{1}{\alpha}} |\log t|^{\frac{\beta}{\alpha}} w^*(t)\right) \|u\|_{H^{\frac{n}{p}}_{p,q}} \tag{3.2}$$

holds for all $u \in H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ and for all measurable function w satisfying

$$|\operatorname{supp} w| < \frac{1}{2} \quad \text{and} \quad \sup_{0<t<\frac{1}{2}} t^{\frac{1}{\alpha}} |\log t|^{\frac{\beta}{\alpha}} w^*(t) < \infty.$$

Remark. By taking $w(x) := |\log|x||^{-\frac{\beta}{\alpha}} |x|^{-\frac{n}{\alpha}} \chi_{\{|x|<\varepsilon\}}(x)$ with small $\varepsilon > 0$ in Lemma 3.2, we can prove the sufficiency part of Theorem 1.1, where $\chi_{\{|x|<\varepsilon\}}$ is a characteristic function on $\{|x| < \varepsilon\}$.

First, we shall prove Lemma 3.2 by applying Lemma 3.1.

Proof of Lemma 3.2. By using the inequality (2.5) with $|\operatorname{supp} w| < \frac{1}{2}$ and applying Lemma 3.1, we see

$$\begin{aligned}
&\int_{\mathbb{R}^n} |w(x)u(x)|^\alpha dx = \int_0^{\frac{1}{2}} (wu)^*(t)^\alpha dt \le \int_0^{\frac{1}{2}} w^*(t)^\alpha u^*(t)^\alpha dt \\
&= \int_0^{\frac{1}{2}} \left(t^{\frac{1}{\alpha}} |\log t|^{\frac{\beta}{\alpha}} w^*(t) \right)^\alpha \frac{u^*(t)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\
&\le \left(\sup_{0<t<\frac{1}{2}} t\,|\log t|^\beta\, w^*(t)^\alpha \right) \int_0^{\frac{1}{2}} \frac{u^*(t)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\
&\le C \left(\sup_{0<t<\frac{1}{2}} t\,|\log t|^\beta\, w^*(t)^\alpha \right) \|u\|^\alpha_{H^{\frac{n}{p}}_{p,q}},
\end{aligned}$$

which is exactly the inequality (3.2). $\square$

We are now in a position to prove Lemma 3.1.

Proof of Lemma 3.1. First, by letting $(1-\Delta)^{\frac{n}{2p}} u = f \in L_{p,q}(\mathbb{R}^n)$, Lemma 3.1 can be rewritten as the following equivalent form,

$$\left(\int_0^{\frac{1}{2}} \frac{\left(G_{\frac{n}{p}} * f\right)^*(t)^\alpha}{|\log t|^\beta} \frac{dt}{t} \right)^{\frac{1}{\alpha}} \le C\,\|f\|_{L_{p,q}} \tag{3.3}$$

for $f \in L_{p,q}(\mathbb{R}^n)$. Hence, we concentrate our attention on the proof of (3.3) below. By the O'Neil inequality (2.6) and decay estimates (2.3) and (2.4), we have for $0 < t < \frac{1}{2}$,

(3.4)

$$\begin{aligned}
&\left(G_{\frac{n}{p}} * f\right)^*(t) \le \left(G_{\frac{n}{p}} * f\right)^{**}(t) \\
&\le t\, G^{**}_{\frac{n}{p}}(t) f^{**}(t) + \int_t^\infty G^*_{\frac{n}{p}}(s) f^*(s) ds \\
&= t\, G^{**}_{\frac{n}{p}}(t) f^{**}(t) + \int_{\frac{1}{2}}^\infty G^*_{\frac{n}{p}}(s) f^*(s) ds + \int_t^{\frac{1}{2}} G^*_{\frac{n}{p}}(s) f^*(s) ds \\
&\le C \left(\|G_{\frac{n}{p}}\|_{L_{p',\infty}} \|f\|_{L_{p,q}} + \|G_{\frac{n}{p}}\|_{L_1} \|f\|_{L_{p,q}} \int_{\frac{1}{2}}^\infty s^{-(1+\frac{1}{p})} ds \right) + \int_t^{\frac{1}{2}} G^*_{\frac{n}{p}}(s) f^*(s) ds \\
&= C\,\|f\|_{L_{p,q}} + \int_t^{\frac{1}{2}} G^*_{\frac{n}{p}}(s) f^*(s) ds.
\end{aligned}$$

Thus from (3.4), we obtain

$$\left(\int_0^{\frac{1}{2}} \frac{\left(G_{\frac{n}{p}} * f\right)^*(t)^\alpha}{|\log t|^\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \leq C\left(\int_0^{\frac{1}{2}} |\log t|^{-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \|f\|_{L_{p,q}} + \left(\int_0^{\frac{1}{2}} \left(\int_t^{\frac{1}{2}} G^*_{\frac{n}{p}}(s) f^*(s) ds\right)^\alpha |\log t|^{-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}}, \tag{3.5}$$

where the integral of the first term on the right-hand side of (3.5) is finite since $\beta > 1$. We further estimate the integral of the second term below.

Note that the conditions (i), (ii) and (iii) in (1.3) can be rewritten equivalently as follows:

$$\text{(i)} \quad \beta > 1 + \frac{\alpha}{q'} \quad \text{or} \quad \text{(ii)} \quad 1 + \frac{\alpha}{q'} = \beta \ \text{ and } \ \alpha \geq \beta. \tag{3.6}$$

Case 1. Assume (i) in (3.6). For $0 < t < \frac{1}{2}$, by Lemma 2.1 (ii) and Hölder's inequality, we see

$$\begin{aligned}
&\int_t^{\frac{1}{2}} G^*_{\frac{n}{p}}(s) f^*(s) ds \leq C \int_t^{\frac{1}{2}} s^{\frac{1}{p}} f^*(s) \frac{ds}{s} \\
&\leq C\left(\int_t^{\frac{1}{2}} \frac{ds}{s}\right)^{\frac{1}{q'}} \left(\int_t^{\frac{1}{2}} \left(s^{\frac{1}{p}} f^*(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} \leq C\,|\log t|^{\frac{1}{q'}} \|f\|_{L_{p,q}}.
\end{aligned}$$

Note that the above calculation is also valid for the case $q = \infty$. Thus we have

$$\begin{aligned}
&\left(\int_0^{\frac{1}{2}} \left(\int_t^{\frac{1}{2}} G^*_{\frac{n}{p}}(s) f^*(s) ds\right)^\alpha |\log t|^{-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \\
&\leq C\left(\int_0^{\frac{1}{2}} |\log t|^{\frac{\alpha}{q'}-\beta} \frac{dt}{t}\right)^{\frac{1}{\alpha}} \|f\|_{L_{p,q}} \leq C\|f\|_{L_{p,q}},
\end{aligned} \tag{3.7}$$

where we have used the condition $\beta > 1 + \frac{\alpha}{q'}$, which ensures that the integral on the middle-hand side of (3.7) is finite. Thus combining (3.5) with (3.7), we obtain the desired estimate.

Case 2. Assume (ii) in (3.6). By Lemma 2.1 (ii) and Lemma 2.3, we have

$$\begin{aligned}(3.8)\qquad &\left(\int_0^{\frac{1}{2}}\left(\int_t^{\frac{1}{2}} G^*_{\frac{n}{p}}(s)f^*(s)ds\right)^{\alpha}|\log t|^{-\beta}\frac{dt}{t}\right)^{\frac{1}{\alpha}}\\ &\le C\left(\int_0^{\frac{1}{2}}\left(t\,G^*_{\frac{n}{p}}(t)f^*(t)\right)^{q}\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\le C\left(\int_0^{\frac{1}{2}}\left(t^{\frac{1}{p}}f^*(t)\right)^{q}\frac{dt}{t}\right)^{\frac{1}{q}}\le C\,\|f\|_{L_{p,q}}.\end{aligned}$$

Thus combining (3.5) with (3.8), we obtain the desired estimate. □

§ 4. Proof of the necessity part of Theorem 1.1

In this final section, we shall prove the necessity part of Theorem 1.1. To this end, we shall construct a concrete function in the critical Sobolev-Lorentz space $H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$.

Proof of the necessity part of Theorem 1.1. First, by putting $(1-\Delta)^{\frac{n}{2p}}u = f$, the inequality (1.2) can be rewritten as

$$(4.1)\qquad \left(\int_{\{|x|<\frac{1}{2}\}}\frac{|G_{\frac{n}{p}}*f(x)|^{\alpha}}{|\log|x||^{\beta}}\frac{dx}{|x|^n}\right)^{\frac{1}{\alpha}}\le C\,\|f\|_{L_{p,q}}.$$

Therefore, it is enough to show the breakdown of the inequality (4.1) under the following conditions which are the negations of (1.3) or (3.6).

$$(4.2)\qquad \begin{cases}\text{(i)}\quad \beta<1+\frac{\alpha}{q'}\ \text{ and }\ q<\infty\,;\\ \text{(ii)}\quad \beta\le 1+\frac{\alpha}{q'}(=1+\alpha)\ \text{ and }\ q=\infty\,;\\ \text{(iii)}\quad \beta=1+\frac{\alpha}{q'},\quad q<\infty\ \text{ and }\ \alpha<\beta.\end{cases}$$

Case 1. Assume (i) in (4.2). In this case, we define the function f_ε by

$$(4.3)\qquad f_\varepsilon(x) := |\log|x||^{-\frac{1+\varepsilon}{q}}\,|x|^{-\frac{n}{p}}\chi_{\{|x|<\varepsilon\}}(x)$$

for small $\varepsilon>0$. Then we see that for sufficiently small $\varepsilon>0$, f_ε becomes non-negative and non-increasing with repect to the radial direction $|x|$. Thus we have for small $t>0$,

$$(4.4)\qquad f^*_\varepsilon(t)=\tilde{f}_\varepsilon\left(\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}\right)\simeq|\log t|^{-\frac{1+\varepsilon}{q}}\,t^{-\frac{1}{p}}=:g_\varepsilon(t),$$

where $\tilde{f}_\varepsilon(|x|):=f_\varepsilon(x)$. More precisely, (4.4) implies that there exist positive constants δ small enough, C and $\tilde{C}$ such that the inequalities

$$(4.5)\qquad C\,g_\varepsilon(t)\le f^*_\varepsilon(t)\le\tilde{C}\,g_\varepsilon(t)$$

hold for all $0 < t < \delta$. By using (4.5), it is easy to see $f_\varepsilon \in L_{p,q}(\mathbb{R}^n)$. Indeed, from (4.5), we obtain

$$\int_0^\delta \left(t^{\frac{1}{p}} f_\varepsilon^*(t)\right)^q \frac{dt}{t} \le \tilde{C} \int_0^\delta \left(t^{\frac{1}{p}} g_\varepsilon(t)\right)^q \frac{dt}{t} = \tilde{C} \int_0^\delta |\log t|^{-(1+\varepsilon)} \frac{dt}{t} < \infty.$$

On the other hand, since f_ε is non-negative and non-increasing with respect to the radial direction, so is $G_{\frac{n}{p}} * f_\varepsilon$. Thus noting $G_{\frac{n}{p}} * f_\varepsilon(x) = \left(G_{\frac{n}{p}} * f_\varepsilon\right)^* (\omega_n r^n)$ if $|x| = r > 0$, we see by changing a variable $\omega_n r^n = t$,

$$\begin{aligned}(4.6) \qquad & \int_{\{|x|<\frac{1}{2}\}} \frac{\left|G_{\frac{n}{p}} * f_\varepsilon(x)\right|^\alpha}{|\log |x||^\beta} \frac{dx}{|x|^n} \\ &= n\omega_n \int_0^{\frac{1}{2}} \frac{\left(G_{\frac{n}{p}} * f_\varepsilon\right)^* (\omega_n r^n)^\alpha}{|\log r|^\beta} \frac{dr}{r} \ge C \int_0^\delta \frac{\left(G_{\frac{n}{p}} * f_\varepsilon\right)^* (t)^\alpha}{|\log t|^\beta} \frac{dt}{t}\end{aligned}$$

for small $\delta > 0$. Furthermore, by using Lemma 2.2 and the reverse O'Neil inequality (2.7), we have

$$\begin{aligned}(4.7) \qquad & \int_0^\delta \frac{\left(G_{\frac{n}{p}} * f_\varepsilon\right)^* (t)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\ &\ge C \int_0^\delta \frac{\left(G_{\frac{n}{p}} * f_\varepsilon\right)^{**} (t)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\ &\ge C \int_0^\delta \frac{\left(tG_{\frac{n}{p}}^{**}(t) f_\varepsilon^{**}(t) + \int_t^\infty G_{\frac{n}{p}}^*(\tau) f_\varepsilon^*(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\ &\ge C \int_0^\delta \frac{\left(\int_t^\delta G_{\frac{n}{p}}^*(\tau) f_\varepsilon^*(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t}.\end{aligned}$$

Thus by Lemma 2.1 (ii) and (4.5), we have for small $\delta > 0$,

$$\begin{aligned}(4.8) \qquad & \int_0^\delta \frac{\left(\int_t^\delta G_{\frac{n}{p}}^*(\tau) f_\varepsilon^*(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t} \ge C \int_0^\delta \frac{\left(\int_t^\delta I_{\frac{n}{p}}^*(\tau) f_\varepsilon^*(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\ &\ge C \int_0^\delta \frac{\left(\int_t^\delta \tau^{-\frac{1}{p'}} g_\varepsilon(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t}.\end{aligned}$$

Take $\varepsilon > 0$ small enough so that $1 - \frac{1+\varepsilon}{q} > 0$, which is possible since $q > 1$. Thus we have for any $0 < t < \frac{\delta}{2}$ with small $\delta > 0$,

$$(4.9) \qquad \int_t^\delta \tau^{-\frac{1}{p'}} g_\varepsilon(\tau) d\tau = \frac{q}{q-(1+\varepsilon)} \left(|\log t|^{1-\frac{1+\varepsilon}{q}} - |\log \delta|^{1-\frac{1+\varepsilon}{q}}\right) \ge C \, |\log t|^{1-\frac{1+\varepsilon}{q}}.$$

Summing up all estimates (4.6), (4.7), (4.8), and (4.9), we obtain

$$\int_{\{|x|<\frac{1}{2}\}} \frac{\left|G_{\frac{n}{p}} * f_\varepsilon(x)\right|^\alpha}{|\log|x||^\beta} \frac{dx}{|x|^n} \geq C \int_0^{\frac{\delta}{2}} |\log t|^{\left(1-\frac{1+\varepsilon}{q}\right)\alpha-\beta} \frac{dt}{t}. \tag{4.10}$$

However, the integral on the right-hand side of (4.10) diverges provided that $\varepsilon > 0$ is taken small enough so that $\left(1 - \frac{1+\varepsilon}{q}\right)\alpha - \beta + 1 \geq 0$, which is possible since $\frac{\alpha}{q'} - \beta + 1 > 0$ by the assumption. Thus the inequality (4.1) fails under the condition (i) in (4.2).

Case 2. Assume (ii) in (4.2). In this case, we utilize $f_0(x) := |x|^{-\frac{n}{p}}$ instead of $f_\varepsilon(x)$ used in Case 1. Then it is easily seen $f_0 \in L_{p,\infty}(\mathbb{R}^n)$. On the other hand, in a quite similar way carried out in Case 1, we see

$$\begin{aligned}
&\int_{\{|x|<\frac{1}{2}\}} \frac{\left|G_{\frac{n}{p}} * f_0(x)\right|^\alpha}{|\log|x||^\beta} \frac{dx}{|x|^n} \geq C \int_0^\delta \frac{\left(G_{\frac{n}{p}} * f_0\right)^*(t)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\
&\geq C \int_0^\delta \frac{\left(G_{\frac{n}{p}} * f_0\right)^{**}(t)^\alpha}{|\log t|^\beta} \frac{dt}{t} \geq C \int_0^\delta \frac{\left(\int_t^\delta G^*_{\frac{n}{p}}(\tau) f_0^*(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\
&\geq C \int_0^\delta \frac{\left(\int_t^\delta I^*_{\frac{n}{p}}(\tau) f_0^*(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t} \geq C \int_0^{\frac{\delta}{2}} |\log t|^{\alpha-\beta} \frac{dt}{t}
\end{aligned}$$

for small δ, where the last integral diverges if $\alpha - \beta + 1 \geq 0$, that is, $\beta \leq 1 + \alpha$. Thus the inequality (4.1) fails under the condition (ii) in (4.2).

Case 3. Assume (iii) in (4.2), which implies $\frac{\alpha}{q} = 1 + \alpha - \beta < 1$, namely, $q > \alpha$. In this case, we make use of the function f_ε with small $\varepsilon > 0$ defined by

$$f_\varepsilon(x) := |\log|x||^{-\frac{1}{q}} |\log|\log|x|||^{-\frac{1+\varepsilon}{q}} |x|^{-\frac{n}{p}} \chi_{\{|x|<\varepsilon\}}(x).$$

Since f_ε is non-negative and non-increasing in the radial direction $|x|$ with small $\varepsilon > 0$, we see

$$f_\varepsilon^*(t) \simeq |\log t|^{-\frac{1}{q}} |\log|\log t||^{-\frac{1+\varepsilon}{q}} t^{-\frac{1}{p}} =: g_\varepsilon(t)$$

for small $t > 0$, namely, there exist positive constants δ small enough, C and $\tilde{C}$ such that the inequalities

$$C g_\varepsilon(t) \leq f_\varepsilon^*(t) \leq \tilde{C} g_\varepsilon(t) \tag{4.11}$$

hold for all $0 < t < \delta$. By using (4.11), it is easy to see $f_\varepsilon \in L_{p,q}(\mathbb{R}^n)$. Indeed,

$$\begin{aligned}
&\int_0^\delta \left(t^{\frac{1}{p}} f_\varepsilon^*(t)\right)^q \frac{dt}{t} \leq \tilde{C} \int_0^\delta \left(t^{\frac{1}{p}} g_\varepsilon(t)\right)^q \frac{dt}{t} \\
&= \tilde{C} \int_0^\delta |\log t|^{-1} |\log|\log t||^{-(1+\varepsilon)} \frac{dt}{t} < \infty.
\end{aligned}$$

On the other hand, in the quite same estimates from below as in (4.6), (4.7) and (4.8) in Case 1, we obtain

$$\int_{\{|x|<\frac{1}{2}\}} \frac{\left|G_{\frac{n}{p}} * f_\varepsilon(x)\right|^\alpha}{|\log|x||^\beta} \frac{dx}{|x|^n} \geq C \int_0^\delta \frac{\left(\int_t^\delta \tau^{-\frac{1}{p'}} g_\varepsilon(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t}. \tag{4.12}$$

Furthermore, we can easily see

$$\int_t^\delta \tau^{-\frac{1}{p'}} g_\varepsilon(\tau) d\tau = \int_t^\delta |\log \tau|^{-\frac{1}{q}} |\log|\log \tau||^{-\frac{1+\varepsilon}{q}} \frac{d\tau}{\tau} \simeq |\log t|^{1-\frac{1}{q}} |\log|\log t||^{-\frac{1+\varepsilon}{q}}$$

for small $t > 0$. In particular, for any $0 < t < \frac{\delta}{2}$ with small $\delta > 0$, we have

$$\int_t^\delta \tau^{-\frac{1}{p'}} g_\varepsilon(\tau) d\tau \geq C |\log t|^{1-\frac{1}{q}} |\log|\log t||^{-\frac{1+\varepsilon}{q}}. \tag{4.13}$$

Thus combining (4.12) with (4.13), we see

$$\begin{aligned} \int_{\{|x|<\frac{1}{2}\}} \frac{\left|G_{\frac{n}{p}} * f_\varepsilon(x)\right|^\alpha}{|\log|x||^\beta} \frac{dx}{|x|^n} &\geq C \int_0^{\frac{\delta}{2}} \frac{\left(\int_t^\delta \tau^{-\frac{1}{p'}} g_\varepsilon(\tau) d\tau\right)^\alpha}{|\log t|^\beta} \frac{dt}{t} \\ &\geq C \int_0^{\frac{\delta}{2}} |\log t|^{\frac{\alpha}{q'}-\beta} |\log|\log t||^{-\frac{1+\varepsilon}{q}\alpha} \frac{dt}{t} \\ &= C \int_0^{\frac{\delta}{2}} |\log t|^{-1} |\log|\log t||^{-\frac{1+\varepsilon}{q}\alpha} \frac{dt}{t}. \end{aligned} \tag{4.14}$$

However, the last integral in (4.14) diverges provided that $\varepsilon > 0$ is taken small so that $-\frac{1+\varepsilon}{q}\alpha + 1 \geq 0$, which is possible since $q > \alpha$. Thus the inequality (4.1) fails under the condition (iii) in (4.2). $\square$

Remark. (3.1) in Lemma 3.1 is equivalent to (1.2) in Theorem 1.1. Indeed, we have already seen in Section 3 that Lemma 3.1 implies Theorem 1.1. On the other hand, (1.2) is equivalent to (4.1), and since the weighted norm in the left-hand side of (4.1) is non-decreasing under the rearrangement, (4.1) can be reduced to (3.3), which is equivalent to (3.1).

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