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Boundedness of Littlewood-Paley operators

By

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Abstract

We survey some results related to $L^p$ boundedness of Littlewood-Paley operators on homogeneous groups. Also, we give proofs of some results in the survey.

§1. Introduction

Let $f \in L^p(\mathbb{T})$ $(1 < p < \infty)$, where $\mathbb{T}$ is the one-dimensional torus, which is identified with $\mathbb{R}/\mathbb{Z}$ ($\mathbb{Z}$ denotes the integer group), and let

$$
\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k \theta}
$$

be the Fourier series of $f$, where

$$c_k = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} \, dx
$$

is the Fourier coefficient.

The Littlewood-Paley function $\gamma(f)$ is defined as

$$
\gamma(f)(\theta) = \left( \sum_{m=0}^{\infty} |\Delta_m(\theta)|^2 \right)^{1/2},
$$

where

$$
\Delta_m(\theta) = \sum_{2^{m-1} \leq |k| < 2^m} c_k e^{2\pi i k \theta}
$$
if $m$ is a positive integer and $\Delta_0 = c_0$. Then Littlewood and Paley proved

$$A_p \|f\|_{L^p} \leq \|\gamma(f)\|_{L^p} \leq B_p \|f\|_{L^p}$$

for some positive constants $A_p$, $B_p$. This can be applied in proving the multiplier theorems of Marcinkiewicz type and in studying the lacunary convergence of the Fourier series.

A result analogous to (1.1) for the $g$ function on $\mathbb{T}$ defined by

$$g(f)(\theta) = \left( \int_0^1 (1-t)|\partial f/\partial t| P_t * f(\theta)|^2 dt \right)^{1/2}$$
was also shown by Littlewood and Paley, where

$$P_t(\theta) = \frac{1-t^2}{1-2t\cos(2\pi \theta) + t^2}$$
is the Poisson kernel for the unit disk. (See Littlewood and Paley [22, 23, 24] and also Zygmund [43, Chap. XV] for the results above).

In this note we consider analogues on the Euclid spaces $\mathbb{R}^n$ and on the homogeneous groups of the Littlewood-Paley function $g(f)$ in (1.2). We survey a paper [10] and some back ground results in Sections 2–4. (See [37, 39, 43] for relevant results.) Also, in Sections 5–7, we shall give proofs of three results stated in Sections 2 and 3. Finally, in Section 8, we shall see some results related to Littlewood-Paley operators arising from the Bochner-Riesz means and the spherical means.

§2. Littlewood-Paley functions on $\mathbb{R}^n$

Let $\psi$ be a function in $L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \psi(x) dx = 0.$$  

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$S_\psi(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2} ,$$
where $\psi_t(x) = t^{-n} \psi(t^{-1}x)$.

Let $Q(x) = [(\partial/\partial t)P_t(x)]_{t=1}$, where

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}$$
is the Poisson kernel on the upper half space $\mathbb{R}^n \times (0, \infty)$. Then $S_Q(f)$ is a version on $\mathbb{R}^n$ of the Littlewood-Paley function $g(f)$.
If $H(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x)$ is the Haar function on $\mathbb{R}$, then $S_H(f)$ coincides with the Marcinkiewicz integral

$$
\mu(f)(x) = \left( \int_{0}^{\infty} |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},
$$

where $F(x) = \int_{0}^{x} f(y) dy$. Here $\chi_E$ denotes the characteristic function of a set $E$. We can easily see that $S_Q$ and $S_H$ are $L^p$ $(1 < p < \infty)$ bounded on $\mathbb{R}^n$ and $\mathbb{R}$, respectively, from the following well-known result of Benedek, Calderón and Panzone [2].

**Theorem A.** Suppose that $\psi$ satisfies (2.1) and

\begin{align*}
(2.2) & \quad |\psi(x)| \leq C(1 + |x|)^{-n-\epsilon}, \\
(2.3) & \quad \int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)| \, dx \leq C |y|^\epsilon
\end{align*}

for some positive constant $\epsilon$. Then

1. $S_\psi$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$;
2. $S_\psi$ is of weak type $(1,1)$ on $\mathbb{R}^n$.

It is known that for the $L^p$ boundedness, the condition (2.3) is superfluous, which can be seen from the following result when $p = 2$.

**Theorem B.** $S_\psi$ is bounded on $L^2(\mathbb{R}^n)$ if $\psi$ satisfies (2.1) and (2.2) with $\epsilon = 1$.

We refer to Coifman and Meyer [8, p. 148] for this. A proof can be found in Journé [20]; see [20, pp. 81-82].

Let $H_\psi(x) = \sup_{|y| \geq |x|} |\psi(y)|$ be the least non-increasing radial majorant of $\psi$. Also, define

$$
B_\epsilon(\psi) = \int_{|x| > 1} |\psi(x)| |x|^\epsilon \, dx \quad \text{for} \quad \epsilon > 0,
$$

$$
D_u(\psi) = \left( \int_{|x| < 1} |\psi(x)|^u \, dx \right)^{1/u} \quad \text{for} \quad u > 1.
$$

In [28], part (1) of Theorem A and Theorem B are improved as follows.

**Theorem C.** Let $\psi \in L^1(\mathbb{R}^n)$. Suppose that $\psi$ satisfies (2.1) and the conditions

1. $B_\epsilon(\psi) < \infty$ for some $\epsilon > 0$;
(2) $D_u(\psi) < \infty$ for some $u > 1$;

(3) $H_\psi \in L^1(\mathbb{R}^n)$.

Then

$$\|S_\psi(f)\|_{L^p_w} \leq C_{p,w}\|f\|_{L^p_w}$$

for all $p \in (1, \infty)$ and $w \in A_p$.

As usual $L^p_w(\mathbb{R}^n)$ denotes the weighted $L^p$ space of those functions $f$ which satisfy $\|f\|_{L^p_w} = \|fw^{1/p}\|_p < \infty$. Also, here we recall the weight class $A_p$ of Muckenhoupt. We say that $w \in A_p$ ($1 < p < \infty$) if

$$\sup_B \left( |B|^{-1} \int_B w(x) \, dx \right) \left( |B|^{-1} \int_B w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure. Let $M$ be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls $B$ containing $x$. We then say that $w \in A_1$ if there exists a constant $C$ such that $M(w)(x) \leq C w(x)$ for almost every $x$.

We now see some applications of Theorem C from [28].

**Corollary 1.** Suppose that $\psi \in L^1$ satisfies (2.1) and (2.2). Let $b \in BMO$ and $w \in A_2$. We define the measure $\nu$ on the upper half space $\mathbb{R}^n \times (0, \infty)$ by

$$d\nu(x,t) = |b \ast \psi_t(x)|^2 \frac{dt}{t} w(x) \, dx.$$ 

Then, the measure $\nu$ is a Carleson measure with respect to the measure $w(x) \, dx$, that is,

$$\nu(S(Q)) \leq C_w \|b\|^2_{BMO} \int_Q w(x) \, dx$$

for all cubes $Q$ in $\mathbb{R}^n$, where

$$S(Q) = \{(x,t) \in \mathbb{R}^n \times (0, \infty) : x \in Q, 0 < t \leq \ell(Q)\}$$

with $\ell(Q)$ denoting sidelength of $Q$.

This follows from the $L^2_w$-boundedness of the operator $S_\psi$. See [20, pp. 85–87]. From Corollary 1 we get the following (see [20, p. 87]).
Corollary 2. Let $b \in \text{BMO}$. Suppose that $\varphi$ satisfies (2.2) and that $\psi$ satisfies (2.1), (2.2). Then
$$
\|T_b(f)\|_{L_p^w} \leq C_{p,w} \|b\|_{\text{BMO}} \|f\|_{L_p^w}
$$
for all $p \in (1, \infty)$ and $w \in A_p$, where
$$
T_b(f)(x) = \left(\int_0^{\infty} |b \ast \psi_t(x)|^2 |f \ast \varphi_t(x)|^2 \frac{dt}{t}\right)^{1/2}.
$$

We note that the conditions (2.1), (2.2) only are required for $\psi$ in Corollaries 1, 2 (no additional regularity condition for $\psi$ is needed).

By Corollary 2 and Theorem C we have the following.

Corollary 3. We assume that $\psi$ satisfies (2.1), (2.2) and that $\varphi$ satisfies (2.2). Let $b \in \text{BMO}$. Furthermore, let $\eta$ be a function in $L^1(\mathbb{R}^n)$ satisfying all the conditions of Theorem C imposed on $\psi$. Define a paraproduct $\pi_b$ by the equation
$$
\pi_b(f)(x) = \int_0^{\infty} \eta_t \ast ((b \ast \psi_t)(f \ast \varphi_t))(x) \frac{dt}{t}.
$$
Then
$$
\|\pi_b(f)\|_{L_p^w} \leq C_{p,w} \|b\|_{\text{BMO}} \|f\|_{L_p^w}
$$
for all $p \in (1, \infty)$ and $w \in A_p$.

The class $L(\log L)^{\alpha}(\mathbb{R}^n)$, $\alpha > 0$, is defined to be the collection of the functions $f$ on $\mathbb{R}^n$ such that
$$
\int_{\mathbb{R}^n} |f(x)||\log(2 + |f(x)|)|^\alpha \, dx < \infty.
$$
Similarly, let $L(\log L)^{\alpha}(S^{n-1})$ be the class of the functions $\Omega$ on $S^{n-1}$ satisfying
$$
\int_{S^{n-1}} |\Omega(\theta)||\log(2 + |\Omega(\theta)|)|^\alpha \, d\sigma(\theta) < \infty,
$$
where $d\sigma$ denotes the Lebesgue surface measure on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

For the rest of this section we consider the cases where $\psi$ is compactly supported. In [31] the following result was proved.

Theorem D. The operator $S_{\psi}$ is bounded on $L_p(\mathbb{R}^n)$ for all $2 \leq p < \infty$ if $\psi$ is a function in $L(\log L)^{1/2}(\mathbb{R}^n)$ with compact support and satisfies (2.1).

This improves on a previous result of [17] which guarantees $L_p$ boundedness of $S_{\psi}$ for $p \in [2, \infty)$ under a more restrictive condition that $\psi \in L^q(\mathbb{R}^n)$ with some $q > 1$.

For $p < 2$, Duoandikoetxea [12] proved the following result.
Theorem E. We assume that $\psi$ has compact support.

(1) Suppose that $1 < q \leq 2$ and $0 < 1/p < 1/2 + 1/q'$. Then $S_\psi$ is bounded on $L^p(\mathbb{R}^n)$ if $\psi$ is in $L^q(\mathbb{R}^n)$ and satisfies (2.1).

(2) Let $1 < q < 2$ and $1/p > 1/2 + 1/q'$. Then there exists $\psi \in L^q(\mathbb{R}^n)$ such that $S_\psi$ is not bounded on $L^p(\mathbb{R}^n)$.

Here $q'$ denotes the exponent conjugate to $q$. See also [6] for a previous result for $p < 2$. Theorem E (1) was shown by arguments involving a theory of weights (see also [14]).

Let $\psi^{(\alpha)}$ be a function on $\mathbb{R}$ defined by

$$\psi^{(\alpha)}(x) = \begin{cases} \alpha(1 - |x|)^{\alpha-1} \text{sgn}(x), & x \in (-1, 1), \\ 0, & \text{otherwise}. \end{cases}$$

Suppose that $1 < p < 2$, $1 < q < 2$ and $1/q' < \alpha < 1/p - 1/2$. Then $\psi^{(\alpha)} \in L^q(\mathbb{R})$; also, Remark 2 of [17] implies that $S_{\psi^{(\alpha)}}$ is not bounded on $L^p$ and $S_{\psi^{(\alpha)}}$ is of weak type $(p, p)$ if $\alpha = 1/p - 1/2$.

The following result is a particular case of part (1) of Theorem E.

**Proposition 1.** If $\psi$ is compactly supported and belongs to $L^2(\mathbb{R}^n)$, then $S_\psi$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

This can be proved by combining results of [28] and the weight theory of [12]. We shall give the proof in Section 5.

The Marcinkiewicz integral $\mu_\Omega(f)$ of Stein [36] (see also Hörmander [19]) is defined by $\mu_\Omega(f) = S_\psi(f)$ with

$$\psi(x) = |x|^{-n+1} \Omega(x') \chi_{(0, 1)}(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\},$$

where $x' = x/|x|$, $\Omega \in L^1(S^{n-1})$, $\int_{S^{n-1}} \Omega \, d\sigma = 0$.

Al-Salman, Al-Qassem, Cheng and Pan [1] proved the following.

**Theorem F.** $\mu_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ if $\Omega \in L(\log L)^{1/2}(S^{n-1})$.

See Walsh [42] for the case $p = 2$. In Section 3, we shall consider an analogue of Theorem F on homogeneous groups.

§ 3. Littlewood-Paley functions on homogeneous groups

We consider Littlewood-Paley functions on homogeneous groups. We also regard $\mathbb{R}^n$, $n \geq 2$, as a homogeneous group with multiplication given by a polynomial mapping.

So, we have a dilation family $\{A_t\}_{t > 0}$ on $\mathbb{R}^n$ such that

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \ldots, t^{a_n} x_n), \quad x = (x_1, \ldots, x_n),$$
with some real numbers $a_1, \ldots, a_n$ satisfying $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and such that each $A_t$ is an automorphism of the group structure (see [18], [41] and [25, Section 2 of Chapter 1]). We also write $\mathbb{H} = \mathbb{R}^n$. $\mathbb{H}$ is equipped with a homogeneous nilpotent Lie group structure; the underlying manifold is $\mathbb{R}^n$ itself. We recall that Lebesgue measure is a bi-invariant Haar measure, the identity is the origin $0$ and $x^{-1} = -x$. Multiplication $xy$, $x, y \in \mathbb{H}$, satisfies the following.

(1) $A_t(xy) = A_txA_ty$, $x, y \in \mathbb{H}$, $t > 0$;
(2) $(ux)(vx) = ux + vx$, $x \in \mathbb{H}$, $u, v \in \mathbb{R}$;
(3) if $z = xy$, $z = (z_1, \ldots, z_n)$, $z_k = P_k(x, y)$, then

$$P_1(x, y) = x_1 + y_1,$$
$$P_k(x, y) = x_k + y_k + R_k(x, y) \text{ for } k \geq 2,$$

where $R_k(x, y)$ is a polynomial depending only on $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$.

We have a norm function $r(x)$ satisfying the following.

(1) $r(A_{t}x) = tr(x)$, for all $t > 0$ and $x \in \mathbb{R}^n$;
(2) $r$ is continuous on $\mathbb{R}^n$ and smooth in $\mathbb{R}^n \setminus \{0\}$;
(3) $r(x + y) \leq N_1(r(x) + r(y))$, $r(xy) \leq N_2(r(x) + r(y))$ for some positive constants $N_1, N_2$;
(4) $r(x^{-1}) = r(x)$;
(5) if $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$, $\Sigma$ coincides with $S^{n-1}$;
(6) there exist positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$c_1|x|^\alpha \leq r(x) \leq c_2|x|^\alpha \text{ if } r(x) \geq 1,$$
$$c_3|x|^\beta \leq r(x) \leq c_4|x|^\beta \text{ if } r(x) \leq 1.$$

Let $\gamma = a_1 + \cdots + a_n$ (the homogeneous dimension of $\mathbb{H}$). Then $dx = t^{\gamma-1} dS dt$, that is,

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{\Sigma} f(A_t\theta)t^{\gamma-1} dS(\theta) dt$$

with $dS = \omega d\sigma$, where $\omega$ is a strictly positive $C^\infty$ function on $\Sigma$ and $d\sigma$ is the Lebesgue surface measure on $\Sigma$ as above.

The Heisenberg group $\mathbb{H}_1$ is an example of the homogeneous groups. Let

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2)$$
for \((x, y, u), (x', y', u') \in \mathbb{R}^3\). Then, with this group law, \(\mathbb{R}^3\) is the Heisenberg group \(\mathbb{H}_1\).

A dilation is defined by

\[ A_t(x, y, u) = (tx, ty, t^2u) \quad (2\text{-step}). \]

Also, we can adopt

\[ A'_t(x, y, u) = (tx, t^2y, t^3u) \quad (3\text{-step}) \]

as an automorphism dilation.

For a function \(f\) on \(\mathbb{H}\), let

\[ f_t(x) = \delta_t f(x) = t^{-\gamma} f(A_t^{-1}x). \]

Convolution on \(\mathbb{H}\) is defined as

\[ f * g(x) = \int_{\mathbb{H}} f(y)g(y^{-1}x) \, dy. \]

Then \((f * g) * h = f * (g * h), (f * g)^\sim = \tilde{g} * \tilde{f}\) if \(\tilde{f}(x) = f(x^{-1})\).

We consider the Littlewood-Paley function on \(\mathbb{H}\) defined by

\[ S_\psi(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}, \]

where \(\psi\) is in \(L^1(\mathbb{H})\) and satisfies (2.1). Let \(\Omega\) be locally integrable in \(\mathbb{H} \setminus \{0\}\). We assume that \(\Omega\) is homogeneous of degree 0 with respect to the dilation group \(\{A_t\}\), which means that \(\Omega(A_t x) = \Omega(x)\) for \(x \neq 0, t > 0\). Also, we assume that

\[ \int_{\Sigma} \Omega(\theta) \, dS(\theta) = 0. \tag{3.1} \]

Let \(\mu_\Omega = S_\Psi\) with

\[ \Psi(x) = r(x)^{-\gamma + a} \Omega(x') \chi_{(0,1]}(r(x)), \quad a > 0, \tag{3.2} \]

where \(x' = A_{r(x)^{-1}} x\) for \(x \neq 0\). The spaces \(L^p(\Sigma), L(\log L)^a(\Sigma)\) are defined with respect to the measure \(dS\).

We recall a result of Ding and Wu [11].

**Theorem G.** We assume in (3.2) that \(a = 1\) and that \(\Omega\) is a function in \(L \log L(\Sigma)\) satisfying (3.1). Then \(\mu_\Omega\) is bounded on \(L^p(\mathbb{H})\) for \(p \in (1, 2]\) and is of weak type \((1, 1)\).

The result on the \(L^p\) boundedness of Theorem G was improved by [10] as follows.

**Theorem 1.** \(\mu_\Omega\) is bounded on \(L^p(\mathbb{H})\) for all \(p \in (1, \infty)\) if \(\Omega\) is in \(L(\log L)^{1/2}(\Sigma)\) and satisfies (3.1).
To prove Theorem 1 we decompose $\Psi(x) = \sum_{k<0} 2^{ka} \Psi^{(k)}(x)$, where
\[
\Psi^{(k)}(x) = 2^{-ka} r(x)^{a-\gamma} \Omega(x') \chi_{[1,2]}(2^{-k} r(x)).
\]
A change of variables and the property $\delta_s \delta_t = \delta_{st}$ of operators $\delta_t$ imply
\[
S_{\Psi^{(k)}} f(x) = S_{r^{-k} \Psi^{(k)}} f(x) = S_{\Psi^{(0)}} f(x).
\]
Thus, by the sublinearity we have
\[
S_\Psi f(x) \leq \sum_{k<0} 2^{ka} S_{\Psi^{(k)}} f(x) = c_a S_{\Psi^{(0)}} f(x).
\]
(See [16] for this observation.) So, we consider a function of the form
\[
(3.3) \quad \Psi(x) = \ell(r(x)) \frac{\Omega(x')}{r(x)^{\gamma}},
\]
where $\ell$ is in $\Lambda^\eta_\infty$ (see [33]) for some $\eta > 0$ and supported in the interval $[1, 2]$.

Now we recall the definition of $\Lambda^\eta_q$ (the definition of $\Lambda^\eta_q$, $1 \leq q \leq \infty$, can be found in [33]). Let $h$ be a locally integrable function on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$. For $t \in (0, 1]$, define
\[
\omega(h, t) = \sup_{|s|<tR/2} \int_{R}^{2R} |h(r-s) - h(r)| \frac{dr}{r},
\]
where the supremum is taken over all $s$ and $R$ such that $|s| < tR/2$ (see [34]). Define $\Lambda^\eta$, $\eta > 0$, to be the family of the functions $h$ such that
\[
\|h\|_{\Lambda^\eta} = \sup_{t \in (0, 1]} t^{-\eta} \omega(h, t) < \infty.
\]
Let $\Lambda^\eta_\infty = L^\infty(\mathbb{R}_+) \cap \Lambda^\eta$ with $\|h\|_{\Lambda^\eta_\infty} = \|h\|_{\infty} + \|h\|_{\Lambda^\eta}$ for $h \in \Lambda^\eta_\infty$. Then $\Lambda^\eta_\infty \subset \Lambda^\eta_\infty$ if $\eta_2 \leq \eta_1$.

Theorem 1 is a consequence of the following.

**Theorem 2.** Let $\Psi$ be as in (3.3). Then $S_\Psi$ is bounded on $L^p(\mathbb{H})$ for all $p \in (1, \infty)$ if $\Omega$ is in $L(\log L)^{1/2}(\Sigma)$ and satisfies (3.1).

Extrapolation arguments using the following estimates can prove Theorem 2 (see [32]).

**Theorem 3.** Suppose that $\Psi$ is as in (3.3) with $\Omega$ belonging to $L^s(\Sigma)$ for some $s \in (1, 2]$ and satisfying (3.1). Let $1 < p < \infty$. Then
\[
\|S_\Psi f\|_p \leq C_p (s - 1)^{-1/2} \|\Omega\|_s \|f\|_p,
\]
where the constant $C_p$ is independent of $s$ and $\Omega$. 
For $F \in L(\log L)^a(\Sigma)$, $a > 0$, recall that

$$\|F\|_{L(\log L)^a} = \inf \left\{ \lambda > 0 : \int_{\Sigma} \frac{|F|}{\lambda} \left[ \log \left( 2 + \frac{|F|}{\lambda} \right) \right]^a dS \leq 1 \right\}.$$

Then, under the assumptions of Theorem 2, we can in fact prove that

$$\|S \Psi f\|_p \leq C_p \|\Omega\|_{L(\log L)^{1/2}} \|f\|_p$$

for a constant $C_p$ independent of $\Omega$, which is not stated explicitly in Theorem 2. We shall give a proof of (3.4) in Section 6 by applying Theorem 3.

To prove Theorem 3 we apply certain vector valued inequalities, which will be controlled by a maximal function of the form

$$M_\psi(f)(x) = \sup_{t > 0} |f * |\psi|_t(x)|.$$

**Lemma 1.** Let $\Psi$ be as in (3.3) and $p > 1$. Suppose that $\Omega$ is in $L^1(\Sigma)$. Then

$$\|M_\Psi f\|_p \leq C_p \|\Omega\|_{1} \|f\|_p.$$

For $\theta \in \Sigma$, let

$$M_\theta f(x) = \sup_{s > 0} \frac{1}{s} \int_{0}^{s} |f(x(A_{st}\theta)^{-1})| dt$$

be the maximal function on $\mathbb{H}$ along a curve homogeneous with respect to the dilation $A_t$. To prove Lemma 1, we apply a result of M. Christ [7].

**Lemma 2.** Let $p > 1$. Then, there exists a constant $C_p$ independent of $\theta$ such that

$$\|M_\theta f\|_p \leq C_p \|f\|_p.$$

We can easily prove Lemma 1 by applying Lemma 2.

**Proof of Lemma 1.** By a change of variables, we have

$$f * |\psi|_t(x) = \int f(xy^{-1})|\psi|_t(y) dy$$

$$= \int_1^2 \int_{\Sigma} f(x(A_{st}\theta)^{-1})|\Omega(\theta)\ell(s)|s^{-1} dS(\theta) ds.$$

It follows that

$$M_\psi f(x) \leq C\|\ell\|_\infty \int_{\Sigma} M_\theta f(x)|\Omega(\theta)| dS(\theta).$$

Thus, Minkowski’s inequality and Lemma 2 imply the conclusion. $\square$
As indicated in [7], if we consider the Heisenberg group with 2-step dilation, then Lemma 2 can be proved by the boundedness of a maximal function along a curve in $\mathbb{R}^2$ (see (7.5)), which was studied by [40]. In Section 7, we shall give a straightforward proof of this fact.

Let $\mathcal{H} = L^2((0, \infty), dt/t)$. For each $k \in \mathbb{Z}$ and $\rho \geq 2$ we consider an operator $T_k$ defined by

$$(T_k(f)(x))(t) = T_k(f)(x, t) = f * \Psi_t(x) \chi_{(1, \rho)}(\rho^{-k}t),$$

where $\Psi$ is as in (3.3). The operator $T_k$ maps functions on $\mathbb{H}$ to $\mathcal{H}$-valued functions on $\mathbb{R}$ and we see that

$$|T_k(f)(x)|_{\mathcal{H}} = \left( \int_{\rho^k}^{\rho^{k+1}} |f * \Psi_t(x)|^2 \frac{dt}{t} \right)^{1/2} = \left( \int_{1}^{\rho} |f * \Psi_{\rho^k t}(x)|^2 \frac{dt}{t} \right)^{1/2}.$$ 

By Lemma 1, we have the following vector valued inequality, which will be useful in proving Theorem 3.

**Lemma 3.** Let $1 < s < \infty$. Then

$$\left\| \left( \sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s \leq C(\log \rho)^{1/2} \| \Omega \|_1 \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_s.$$ 

We can apply the converse of Hölder’s inequality and Lemma 1 to prove this (see [13]).

### § 4. Outline of the proof of Theorem 3

Let $\phi$ be a $C^\infty$ function supported in $\{1/2 < r(x) < 1\}$ such that $\int \phi = 1$, $\phi(x) = \tilde{\phi}(x)$, $\phi(x) \geq 0$ for all $x \in \mathbb{H}$. For $\rho \geq 2$, we define

$$\Delta_k = \delta_{\rho^{k-1}} \phi - \delta_{\rho^k} \phi, \quad k \in \mathbb{Z}.$$ 

Then, supp$(\Delta_k) \subset \{\rho^{k-1}/2 < r(x) < \rho^k\}$, $\Delta_k = \tilde{\Delta}_k$ and

$$\sum_k \Delta_k = \delta,$$

where $\delta$ is the delta function.

We decompose

$$f * \Psi_t(x) = \sum_{j \in \mathbb{Z}} F_j(x, t),$$

where

$$F_j(x, t) = \sum_{k \in \mathbb{Z}} f * \Delta_{j+k} * \Psi_t(x) \chi_{(\rho^k, \rho^{k+1})}(t).$$
Define
\[ U_j f(x) = \left( \int_0^\infty |F_j(x,t)|^2 \frac{dt}{t} \right)^{1/2} = \left( \sum_{k \in \mathbb{Z}} \int_1^\rho |f \ast \Delta_{j+k} \ast \Psi_{\rho^k t}|^2 \frac{dt}{t} \right)^{1/2} \]
\[ = \left( \sum_k |T_k(f \ast \Delta_{j+k})|^2_{\mathcal{H}} \right)^{1/2}. \]

**Lemma 4.** Let \( 1 < s \leq 2 \) and \( \rho = 2^s \). Then, there exist positive constants \( C, \epsilon \) independent of \( s \) and \( \Omega \in L^s(\Sigma) \) such that
\[ \|U_j f\|_2 \leq C(s-1)^{-1/2} 2^{-\epsilon |j|} \|\Omega\|_s \|f\|_2. \]

We choose \( \psi_j \in C_0^\infty(\mathbb{R}) \), \( j \in \mathbb{Z} \), such that
\[
\operatorname{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0, \\
\log 2 \sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t > 0, \\
|\frac{d}{dt}^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \ldots ,
\]
where \( c_m \) is a constant independent of \( \rho \geq 2 \). Decompose
\[ \frac{\Omega(x')}{r(x)^\gamma} = \sum_{j \in \mathbb{Z}} S_j(x), \]
where
\[ S_j(x) = \int_0^\infty \psi_j(t) \delta_t K_0(x) \frac{dt}{t} = \frac{\Omega(x')}{r(x)^\gamma} \int_{1/2}^1 \psi_j(tr(x)) \frac{dt}{t} \]
with
\[ K_0(x) = \frac{\Omega(x')}{r(x)^\gamma} \chi_{[1,2]}(r(x)). \]

We observe that \( S_j \) is supported in \( \{\rho^j \leq r(x) \leq 2\rho^{j+2}\} \). Let
\[ L_{m}^{(t)}(x) = \ell(t^{-1}r(x))S_m(x). \]
Then by the restraint of the support of \( \ell \) we have
\[ \Psi_t(x)\chi_{[\rho^k, \rho^{k+1}]}(t) = \sum_{m=k-3}^{k+3} L_{m}^{(t)}(x)\chi_{[\rho^k, \rho^{k+1}]}(t). \]
Consequently,
\[ F_j(x, t) = \sum_{k \in \mathbb{Z}} \sum_{m=k-3}^{k+3} f \ast \Delta_{j+k} \ast L_{m}^{(t)}(x)\chi_{[\rho^k, \rho^{k+1}]}(t). \]
Using this expression of $F_j$ and an analogue of the estimates in Lemma 1 of [33] (see also [9] for related results on product homogeneous groups), which can be proved by methods based on Tao [41], we can prove Lemma 4.

Now we are able to prove Theorem 3. First we recall the Littlewood-Paley inequality
\[
\left\| \left( \sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_r \leq C_r \|f\|_r, \quad 1 < r < \infty,
\]
where $C_r$ is independent of $\rho$. Let $1 < p < \infty$, $\rho = 2^{s'}, 1 < s \leq 2$. By Lemma 3 and the Littlewood-Paley inequality we have
\begin{align*}
\| U_j(f) \|_r & = \left\| \left( \sum_k |T_k(f * \Delta_{j+k})|^2 \right)^{1/2} \right\|_r \\
& \leq C (\log \rho)^{1/2} \| \Omega \|_1 \left\| \left( \sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_r \\
& \leq C (\log \rho)^{1/2} \| \Omega \|_1 \| f \|_r
\end{align*}
for all $r \in (1, \infty)$. Also, by Lemma 4
\begin{align*}
\| U_j f \|_2 & \leq C (\log \rho)^{1/2} 2^{-\epsilon |j|} \| \Omega \|_s \| f \|_2.
\end{align*}
Thus, interpolating between (4.1) and (4.2), we have
\[
\| U_j f \|_p \leq C (\log \rho)^{1/2} 2^{-\epsilon |j|} \| \Omega \|_s \| f \|_p
\]
with some $\epsilon > 0$, which implies
\[
\| S_{\Psi} f \|_p \leq \sum_j \| U_j f \|_p \leq C_p (s-1)^{-1/2} \| \Omega \|_s \| f \|_p.
\]
This completes the proof of Theorem 3.

§ 5. A proof of Proposition 1

Let
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \langle x, \xi \rangle} \, dx
\]
be the Fourier transform of $f$, where
\[
\langle x, \xi \rangle = \sum_{j=1}^n x_j \xi_j, \quad x = (x_1, \ldots, x_n), \quad \xi = (\xi_1, \ldots, \xi_n).
\]
To prove Proposition 1 we apply the following Fourier transform estimates.
Lemma 5. Let $\psi \in L^2(\mathbb{R}^n)$. Suppose that $\psi$ is compactly supported and satisfies (2.1). Then

$$\int_1^2 |\hat{\psi}(t\xi)|^2 dt \leq C \min \left( |\xi|^{\epsilon}, |\xi|^{-\epsilon} \right) \quad \text{for all} \quad \xi \in \mathbb{R}^n$$

with some $\epsilon \in (0, 1)$.

Also, we need the following.

Lemma 6. Suppose that $\psi$ is a function in $L^2(\mathbb{R}^n)$ with compact support. Let $w \in A_1$. If $v = w$ or $w^{-1}$, then we have

$$\sup_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_1^2 |f \ast \psi_{t2^k}(x)|^2 dt \, v(x) \, dx \leq C \|f\|_{L^2_v}^2.$$

For a proof of Lemma 5 see [28].

Proof of Lemma 6. When $v = w$, Lemma 6 was proved in [28] (the author has learned from [12] that Lemma 6 is also valid for $v = w^{-1}$ and that it is useful for application). Now we recall the proof. We may assume that $\text{supp}(\psi) \subset \{|x| \leq 1\}$. Then, by Schwarz’s inequality we see that

$$|f \ast \psi_t(x)|^2 \leq t^{-n} \|\psi\|_2^2 \int_{|y|<t} |f(x-y)|^2 dy.$$

Since $w \in A_1$, integration with respect to the measure $w(x) \, dx$ gives

$$\int |f \ast \psi_t(x)|^2 w(x) \, dx \leq \|\psi\|_2^2 \int |f(y)|^2 t^{-n} \int_{|x-y|<t} w(x) \, dx \, dy \leq C_w \|\psi\|_2^2 \int |f(y)|^2 w(y) \, dy$$

uniformly in $t$. Also, by duality we can prove the uniform estimate

$$\int |f \ast \psi_t(x)|^2 w^{-1}(x) \, dx \leq C_w \|\psi\|_2^2 \int |f(y)|^2 w^{-1}(y) \, dy.$$

The conclusion easily follows from the estimates (5.1) and (5.2).

We choose $\Psi \in C^\infty$ that is supported in $\{1/2 \leq |\xi| \leq 2\}$ and satisfies

$$\sum_{j \in \mathbb{Z}} \Psi(2^j \xi) = 1 \quad \text{for} \quad \xi \neq 0.$$

Define

$$\hat{D_j(f)}(\xi) = \Psi(2^j \xi) \hat{f}(\xi) \quad \text{for} \quad j \in \mathbb{Z},$$
and decompose
\[ f * \psi_t(x) = \sum_{j \in \mathbb{Z}} F_j(x, t), \]
where
\[ F_j(x, t) = \sum_{k \in \mathbb{Z}} D_{j+k}(f * \psi_t)(x) \chi_{[2^k, 2^{k+1})}(t). \]
Let
\[ T_j(f)(x) = \left( \int_{0}^\infty |F_j(x, t)|^2 \frac{dt}{t} \right)^{1/2}. \]

We write \( A_j = \{2^{-1-j} \leq |\xi| \leq 2^{1-j}\} \). Then, by the Plancherel theorem and Lemma 5 we see that

\[ \Vert T_j(f) \Vert_2^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |D_{j+k}(f * \psi_t)(x)|^2 \frac{dt}{t} dx \]
\[ \leq \sum_{k \in \mathbb{Z}} C \int_{A_{j+k}} \left( \int_{2^k}^{2^{k+1}} \left| \hat{\psi}(t \xi) \right|^2 \frac{dt}{t} \right) \left| \hat{f}(\xi) \right|^2 d\xi \]
\[ \leq \sum_{k \in \mathbb{Z}} C \int_{A_{j+k}} \min(|2^k \xi|^\epsilon, |2^k \xi|^{-\epsilon}) \left| \hat{f}(\xi) \right|^2 d\xi \]
\[ \leq C 2^{\epsilon+j} \sum_{k \in \mathbb{Z}} \int_{A_{j+k}} \left| \hat{f}(\xi) \right|^2 d\xi. \]

Since the sets \( A_j \) are finitely overlapping, (5.3) implies that

\[ \Vert T_j(f) \Vert_2^2 \leq C 2^{\epsilon+j} \left| \hat{f} \right|_2^2 = C 2^{\epsilon+j} \left| f \right|_2^2. \]

Let \( w \in A_1 \). If \( v = w \) or \( w^{-1} \), by Lemma 6 and the Littlewood-Paley inequality for \( L_v^2 \) (note that \( v \in A_2 \)) we see that

\[ \Vert T_j(f) \Vert_{L_v^2}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |D_{j+k}(f) * \psi_t(x)|^2 \frac{dt}{t} v(x) dx \]
\[ \leq \sum_{k \in \mathbb{Z}} C \int_{\mathbb{R}^n} |D_{j+k}(f)(x)|^2 v(x) dx \]
\[ \leq C \left| f \right|_{L_v^2}^2. \]

Thus, by interpolation with change of measures between (5.4) and (5.5)

\[ \Vert T_j(f) \Vert_{L_{v^a}^2} \leq C 2^{-\epsilon(1-a)\left| j \right|/2} \left| f \right|_{L_{v^a}^2}, \]

for \( a \in (0, 1) \). Choosing \( a \) so that \( w^{1/a} \in A_1 \), by (5.6) we have

\[ \Vert T_j(f) \Vert_{L_v^2} \leq C 2^{-\epsilon(1-a)\left| j \right|/2} \left| f \right|_{L_v^2}. \]
From this it follows that

\[(5.7) \quad \|S_{\psi}(f)\|_{L^{p}_{v}} \leq \sum_{j \in \mathbb{Z}} \|T_{j}(f)\|_{L^{p}_{v}} \leq C \|f\|_{L^{p}_{v}}.\]

Let \(M\) be the Hardy-Littlewood maximal operator (see Section 2) and \(M_{s}(f) = (M(|f|^{s})(x))^{1/s}\). To prove Proposition 1, by Theorem D we may assume that \(p < 2\). Now we apply the idea of [12]. If \(1 < s < p/(2 - p)\), then \(M_{s}(|f|^{2-p})\) is in \(A_{1}\) (we may assume that \(0 < M_{s}(|f|^{2-p}) < \infty\)) and \(M_{s}\) is bounded on \(L^{p/(2-p)}\). Thus by Hölder’s inequality and (5.7) with \(v = M_{s}(|f|^{2-p})^{-1}\), we have

\[
\int S_{\psi}(f)(x)^{p} dx = \int S_{\psi}(f)(x)^{p} M_{s}(|f|^{2-p})(x)^{p/2} M_{s}(|f|^{2-p})(x)^{-p/2} dx \leq \left( \int S_{\psi}(f)(x)^{2} M_{s}(|f|^{2-p})(x)^{-1} dx \right)^{p/2} \left( \int M_{s}(|f|^{2-p})(x)^{p/(2-p)} dx \right)^{1-p/2} \leq C \left( \int |f(x)|^{2} M_{s}(|f|^{2-p})(x)^{-1} dx \right)^{p/2} \|f\|^{p(1-p/2)} = C \|f\|^{p}.
\]

This completes the proof of Proposition 1.

§ 6. Proof of (3.4)

We can prove Theorem 2 by extrapolation arguments using Theorem 3. More specifically, we can prove the estimate (3.4).

Let \(a > 0\). We define the space \(N_{a}(\Sigma)\) to be the class of the functions \(F \in L^{1}(\Sigma)\) for which we can find a sequence \(\{F_{m}\}_{m=1}^{\infty}\) of functions on \(\Sigma\) and a sequence \(\{b_{m}\}_{m=1}^{\infty}\) of non-negative real numbers such that

1. \(F = \sum_{m=1}^{\infty} b_{m} F_{m},\)
2. \(\sup_{m \geq 1} \|F_{m}\|_{1+1/m} \leq 1,\)
3. \(\int_{\Sigma} F_{m} dS = 0,\)
4. \(\sum_{m=1}^{\infty} m^{a} b_{m} < \infty.\)

For \(F \in N_{a}(\Sigma)\), let

\[\|F\|_{N_{a}} = \inf_{\{b_{m}\}} \sum_{m=1}^{\infty} m^{a} b_{m},\]

where the infimum is taken over all such non-negative sequences \( \{b_m\} \). We note that \( \int_{\Sigma} F \, dS = 0 \) if \( F \in \mathcal{N}_a(\Sigma) \).

By well-known arguments we have the following (see [43, Chap. XII, pp. 119–120] for relevant results).

**Proposition 2.** Suppose that \( F \in L^1(\Sigma) \) and \( a > 0 \). Then, the following two statements (1), (2) are equivalent:

1. \( F \in L(\log L)^a(\Sigma) \) and \( \int_{\Sigma} F \, dS = 0 \);
2. \( F \in \mathcal{N}_a(\Sigma) \).

Moreover,

3. there exist positive constants \( A, B \) such that
   \[
   \|F\|_{L(\log L)^a} \leq A\|F\|_{\mathcal{N}_a}, \quad \|F\|_{\mathcal{N}_a} \leq B\|F\|_{L(\log L)^a}
   \]
   for \( F \in \mathcal{N}_a(\Sigma) \).

To prove Proposition 2 we use the following two elementary results.

**Lemma 7.** Let \( 1 < p < \infty, a > 0, x \geq 2 \). Then, there exists a positive constant \( C_a \) depending only on \( a \) such that
   \[
   x(\log x)^a \leq C_a(p-1)^{-a}x^p.
   \]
   This was also used in [32].

**Lemma 8.** Let \( f \) be a continuous, non-negative, convex function on \([0, \infty)\) such that \( f(0) = 0 \). Suppose that a series \( \sum_{k=1}^{\infty} c_k a_k \) converges, where \( c_k \geq 0, \sum_{k=1}^{\infty} c_k \leq 1, a_k \in \mathbb{C} \). Then
   \[
   f\left(\sum_{k=1}^{\infty} c_k a_k\right) \leq \sum_{k=1}^{\infty} c_k f(|a_k|).
   \]

**Proof of Proposition 2.** We first see that part (1) follows from part (2). Let \( F \in \mathcal{N}_a(\Sigma) \). We have already noted that \( \int_{\Sigma} F \, dS = 0 \). For any \( \epsilon > 0 \) there exist a sequence \( \{b_m\} \) of non-negative real numbers and a sequence \( \{F_m\} \) of functions on \( \Sigma \) with the properties required in the definition of \( \mathcal{N}_a(\Sigma) \) such that
   \[
   \|F\|_{\mathcal{N}_a} \leq \sum_{m=1}^{\infty} m^a b_m < \|F\|_{\mathcal{N}_a} + \epsilon.
   \]

Let \( \lambda = \|F\|_{\mathcal{N}_a} + \epsilon \). By Lemma 8 with \( f(x) = x[\log(2 + x)]^a \) and \( c_k = b_k/\lambda \), we have
   \[
   \int_{\Sigma} \frac{|F|}{\lambda} \left[\log \left(2 + \frac{|F|}{\lambda}\right)\right]^a \, dS \leq \sum_{m=1}^{\infty} \lambda^{-1} b_m \int_{\Sigma} |F_m| \left[\log (2 + |F_m|)\right]^a \, dS.
   \]
It follows from Lemma 7 with \( p = 1 + 1/m \) that
\[
|F_m| \left( \log (2 + |F_m|) \right)^a \leq C_a m^a (2 + |F_m|)^{1+1/m} \\
\leq C_a m^a 2^{1/m} (2^{1+1/m} + |F_m|^{1+1/m}) \\
\leq 2C_a m^a (4 + |F_m|^{1+1/m}).
\]

Thus
\[
\int_{\Sigma} \frac{|F|}{\lambda} \left[ \log \left( 2 + \frac{|F|}{\lambda} \right) \right]^a dS \leq \sum_{m=1}^{\infty} \lambda^{-1} b_m 2C_a m^a \int_{\Sigma} (4 + |F_m|^{1+1/m}) dS \\
= \sum_{m=1}^{\infty} \lambda^{-1} b_m 2C_a m^a (4S(\Sigma) + \|F_m\|_{1+1/m}^{1+1/m}) \\
\leq \sum_{m=1}^{\infty} \lambda^{-1} b_m 2C_a m^a (4S(\Sigma) + 1) \\
\leq 2C_a (4S(\Sigma) + 1).
\]

This implies that \( F \) belongs to \( L(\log L)^a(\Sigma) \) and
\[
\|F\|_{L(\log L)^a} \leq A\lambda = A(\|F\|_N + \epsilon)
\]
for some \( A > 0 \). Letting \( \epsilon \) tend to 0, we see that the first inequality of part (3) holds.

Next we prove that part (1) implies part (2). We take \( \lambda > 0 \) such that
\[
\int_{\Sigma} \frac{|F|}{\lambda} \left[ \log \left( 2 + \frac{|F|}{\lambda} \right) \right]^a dS \leq 1.
\]

Let \( F_\lambda = F/\lambda \). We define
\[
U_m = \{ \theta \in \Sigma : 2^{m-1} < |F_\lambda(\theta)| \leq 2^m \} \quad \text{for} \quad m \geq 2, \\
U_1 = \{ \theta \in \Sigma : |F_\lambda(\theta)| \leq 2 \}
\]
and decompose \( F_\lambda = \sum_{m=1}^{\infty} \tilde{F}_{\lambda,m} \), where
\[ \tilde{F}_{\lambda,m} = F_\lambda \chi_{U_m} - S(\Sigma)^{-1} \int_{U_m} F_\lambda dS. \]

Note that \( \int \tilde{F}_{\lambda,m} dS = 0 \). If we put \( e_m = S(U_m), \) \( m \geq 1 \), then
\[
(6.1) \quad \|\tilde{F}_{\lambda,m}\|_{1+1/m} \leq 22^m e_m^{m/(m+1)} \quad \text{for} \quad m \geq 1.
\]

Define
\[
F_{\lambda,m} = \begin{cases} 
2^{-m-1} e_m^{-m/(m+1)} \tilde{F}_{\lambda,m}, & \text{if} \ e_m \neq 0, \\
0, & \text{if} \ e_m = 0.
\end{cases}
\]
Let \( b_m = 2^{m+1}e_m^{m/(m+1)} \) for \( m \geq 1 \). Then
\[
F_\lambda = \sum_{m=1}^{\infty} b_m F_{\lambda,m}, \quad \int_{\Sigma} F_{\lambda,m} \, dS = 0.
\]

Also, by (6.1) we see that \( \sup_{m \geq 1} \|F_{\lambda,m}\|_{1+1/m} \leq 1 \). Furthermore, applying Young’s inequality, we have

\[
\begin{align*}
\sum_{m=1}^{\infty} m^a b_m &= \sum_{m=1}^{\infty} m^a 2^{m+1} e_m^{m/(m+1)} \\
&\leq 2 \sum_{m=1}^{\infty} \frac{m}{m+1} m^a 2^{(m+1)(1+1/m)} e_m + 2 \sum_{m=1}^{\infty} m^a 2^{-m-1/(m+1)} \\
&\leq C \sum_{m=1}^{\infty} m^a 2^m e_m + C \\
&\leq C \int_{\Sigma} |F_\lambda| (\log(2 + |F_\lambda|))^a \, dS + C \\
&\leq C.
\end{align*}
\]

Collecting results, we see that \( F \in \mathcal{N}_a \) and, since \( F = \sum_{m=1}^{\infty} \lambda b_m F_{\lambda,m} \),
\[
\sum_{m=1}^{\infty} m^a b_m \geq \lambda^{-1} \|F\|_{\mathcal{N}_a},
\]
which combined with (6.2) implies that \( \|F\|_{\mathcal{N}_a} \leq B\lambda \) for some \( B > 0 \). So, taking the infimum over \( \lambda \), we get the second inequality of part (3). \( \square \)

Let \( \Omega \) and \( \Psi \) be as in Theorem 2. By Proposition 2 we can decompose \( \Omega \) as
\[
\Omega = \sum_{m=1}^{\infty} b_m \Omega_m,
\]
where \( \sup_{m \geq 1} \|\Omega_m\|_{1+1/m} \leq 1 \) and each \( \Omega_m \) satisfies (3.1), while \( \{b_m\} \) is a sequence of non-negative real numbers such that \( \sum_{m=1}^{\infty} m^{1/2} b_m < \infty \). Accordingly,
\[
\Psi = \sum_{m=1}^{\infty} \Psi_m, \quad \Psi_m(x) = b_m \ell(r(x)) \frac{\Omega_m(x')}{r(x)^{\gamma}}.
\]

Let \( 1 < p < \infty \). By Theorem 3 with \( s = 1 + 1/m \) we have
\[
\|S_{\Psi_m} f\|_p \leq C_p m^{1/2} b_m \|\Omega_m\|_{1+1/m} \|f\|_p \leq C_p m^{1/2} b_m \|f\|_p,
\]
which implies
\[
\|S\Psi f\|_p \leq \sum_{m=1}^{\infty} \|S_{\Psi_m} f\|_p \leq C_p (\sum_{m=1}^{\infty} m^{1/2} b_m) \|f\|_p.
\]
Taking the infimum over \( \{b_m\} \) and applying Proposition 2, we get
\[
\|S_{\Psi}f\|_p \leq C_p\|\Omega\|_{N_{1/2}}\|f\|_p \leq C_p B\|\Omega\|_{L(\log L)^{1/2}}\|f\|_p.
\]
This completes the proof of (3.4).

§ 7. Maximal functions on the Heisenberg group with two-step dilation

We give a proof of Lemma 2 for the maximal function \( M_\theta \) on the Heisenberg group \( \mathbb{H}_1 \) with 2-step dilation by applying the boundedness of the maximal function \( \mathfrak{M}_g \) on \( \mathbb{R}^2 \) (see (7.5)).

Let \( \theta = (\theta_1, \theta_2, \theta_3) \in S^2 \) and \( d_\theta = |\theta_1 \theta_2 \theta_3| \). We may assume that \( d_\theta \neq 0 \). Let
\[
T_\theta x = (\theta_1^{-1}x_1, \theta_2^{-1}x_2, \theta_3^{-1}x_3).
\]
It is convenient to define a group law \( u \circ_{\theta} v \) on \( \mathbb{R}^3 \) so that
\[
T_\theta x \circ_{\theta} T_\theta y = T_\theta (xy).
\]
If \( u = T_\theta x, \ v = T_\theta y \), this requires that
\[
u \circ_{\theta} v = T_\theta x \circ_{\theta} T_\theta y = T_\theta (xy)
\]
\[
= T_\theta (x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1y_2 - y_1x_2)/2)
\]
\[
= (\theta_1^{-1}(x_1 + y_1), \theta_2^{-1}(x_2 + y_2), \theta_3^{-1}(x_3 + y_3) + \theta_3^{-1}(x_1y_2 - y_1x_2)/2)
\]
\[
= (u_1 + v_1, u_2 + v_2, u_3 + v_3 + (2\theta_3)^{-1}\theta_1 \theta_2(u_1v_2 - v_1u_2)).
\]

Since \( A_t x = (tx_1, tx_2, t^2x_3) \), if \( a(t) = (t, t, t^2) \),
\[
f(x(A_t \theta)^{-1}) = f(T_\theta^{-1}((T_\theta x) \circ_{\theta} a(t)^{-1})) = f_\theta((T_\theta x) \circ_{\theta} a(t)^{-1}),
\]
where \( f_\theta(x) = f(T_\theta^{-1}x) \) and \( a(t)^{-1} = (-t, -t, -t^2) \). Thus, by a change of variables, we have
\[
(7.1) \quad \int_{\mathbb{H}_1} \left( \sup_{r>0} \frac{1}{r} \int_0^r |f(x(A_t \theta)^{-1})| \, dt \right)^p \, dx
\]
\[
= d_\theta \int_{\mathbb{H}_1} \left( \sup_{r>0} \frac{1}{r} \int_0^r |f_\theta(y \circ_{\theta} a(t)^{-1})| \, dt \right)^p \, dy.
\]
Let \( c_\theta = (2\theta_3)^{-1}\theta_1 \theta_2 \). Then we note that
\[
y = (y_1, y_2, y_3) = (0, y_2 - y_1, 0) \circ_{\theta} (y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1)).
\]
The thus
\[
y \circ_{\theta} a(t)^{-1} = ((0, y_2 - y_1, 0) \circ_{\theta} (y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1))) \circ_{\theta} a(t)^{-1}
\]
\[
= (0, y_2 - y_1, 0) \circ_{\theta} ((y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1)) \circ_{\theta} a(t)^{-1}).
\]
By (7.1) and (7.2), applying a change of variables, we have

\[(7.3)\]
\[
\int_{\mathbb{H}_{1}} \left( \sup_{r>0} \frac{1}{r} \int_{0}^{r} |f(x(A_t \theta)^{-1})| \, dt \right)^p \, dx
\]
\[= d_\theta \int_{\mathbb{H}_{1}} \left( \sup_{r>0} \frac{1}{r} \int_{0}^{r} |f_\theta((0, y_2 - y_1, 0) \circ \theta ((y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1)) \circ \theta a(t)^{-1}))| \, dt \right)^p \, dy
\]
\[= d_\theta \int_{\mathbb{H}_{1}} \left( \sup_{r>0} \frac{1}{r} \int_{0}^{r} |f_\theta((0, y_2, 0) \circ \theta ((y_1, y_1, y_3) \circ \theta a(t)^{-1}))| \, dt \right)^p \, dy.
\]

We observe that

\[
(y_1, y_1, y_3) \circ \theta a(t)^{-1} = (y_1 - t, y_1 - t, y_3 - t^2).
\]

Thus (7.3) implies that

\[(7.4)\]
\[
\int_{\mathbb{H}_{1}} \left( \sup_{r>0} \frac{1}{r} \int_{0}^{r} |f(x(A_t \theta)^{-1})| \, dt \right)^p \, dx
\]
\[= d_\theta \int_{\mathbb{H}_{1}} \left( \sup_{r>0} \frac{1}{r} \int_{0}^{r} |f_\theta((0, y_2, 0) \circ \theta (y_1 - t, y_1 - t, y_3 - t^2))| \, dt \right)^p \, dy
\]
\[= d_\theta \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{2}} (\mathfrak{M} f_{\theta, y_2}(y_1, y_3))^p \, dy_1 \, dy_3 \right) \, dy_2,
\]

where \( f_{\theta, y_2}(y_1, y_3) = f_\theta((0, y_2, 0) \circ \theta (y_1, y_1, y_3)) \) and

\[(7.5)\]
\[
\mathfrak{M} g(y_1, y_3) = \sup_{r>0} \frac{1}{r} \int_{0}^{r} |g(y_1 - t, y_3 - t^2)| \, dt.
\]

It is known that

\[
\|\mathfrak{M} g\|_{L^p(\mathbb{R}^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}, \quad p > 1
\]

(see [40]). Applying this and a change of variables, we see that

\[(7.6)\]
\[
d_\theta \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{2}} (\mathfrak{M} f_{\theta, y_2}(y_1, y_3))^p \, dy_1 \, dy_3 \right) \, dy_2
\]
\[\leq C_p^p d_\theta \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{2}} |f_{\theta, y_2}(y_1, y_3)|^p \, dy_1 \, dy_3 \right) \, dy_2
\]
\[= C_p^p d_\theta \int_{\mathbb{H}_{1}} |f_{\theta}(y_1, y_1 + y_2, y_3 - c_\theta y_1 y_2)|^p \, dy_1 \, dy_2 \, dy_3
\]
\[= C_p^p d_\theta \int_{\mathbb{H}_{1}} |f_{\theta}(y)|^p \, dy
\]
\[= C_p^p \int_{\mathbb{H}_{1}} |f(y)|^p \, dy.
\]
Combining (7.4) and (7.6), we get the conclusion.

§ 8. Littlewood-Paley operators related to Bochner-Riesz means and spherical means

Let

\[ S_R^\delta(f)(x) = \int_{|\xi|<R} \hat{f}(\xi)(1 - R^{-2}|\xi|^2)^\delta e^{2\pi i\langle x, \xi \rangle} d\xi = H_{R-1}^\delta * f(x) \]

be the Bochner-Riesz mean of order \( \delta \) on \( \mathbb{R}^n \), \( \delta > -1 \), where

\[ H^\delta(x) = \pi^{-\delta} \Gamma(\delta + 1)|x|^{-(n/2+\delta)} J_{n/2+\delta}(2\pi|x|) \]

with \( J_\nu \) denoting the Bessel function of the first kind of order \( \nu \).

For \( \beta > 0 \), let

\[ M_t^\beta(f)(x) = c_\beta t^{-n} \int_{|y|<t} (1 - t^{-2}|y|^2)^{\beta-1} f(x-y) dy, \]

where

\[ c_\beta = \frac{\Gamma(\beta + \frac{n}{2})}{\pi^{\frac{n}{2}} \Gamma(\beta)}. \]

By taking the Fourier transform, we can embed these operators in an analytic family of operators in \( \beta \) so that

\[ M_t^0(f)(x) = c \int_{S^{n-1}} f(x-ty) d\sigma(y). \]

Now we define a Littlewood-Paley operator \( \sigma_\delta, \delta > 0 \), from the Bochner-Riesz means as

\[ \sigma_\delta(f)(x) = \left( \int_0^\infty |(\partial/\partial R)S_R^\delta(f)(x)|^2 R dR \right)^{1/2} \]

\[ = \left( \int_0^\infty |(-2\delta (S_R^\delta(f)(x) - S_R^{\delta-1}(f)(x))|^2 \frac{dR}{R} \right)^{1/2}, \]

and also another Littlewood-Paley operator \( \nu_\beta, \beta + n/2 - 1 > 0 \), from the spherical means as

\[ \nu_\beta(f)(x) = \left( \int_0^\infty |(\partial/\partial t) M_t^\beta(f)(x)|^2 t dt \right)^{1/2} \]

\[ = \left( \int_0^\infty |(-2(\beta + n/2 - 1) (M_t^\beta(f)(x) - M_t^{\beta-1}(f)(x))|^2 \frac{dt}{t} \right)^{1/2}. \]

These Littlewood-Paley functions are related as follows.
**Theorem H.** Suppose that $\delta = \beta + n/2 - 1 > 0$. Then, there exist positive constants $A, B$ such that for all $x \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space) we have
\[
\sigma_\delta(f)(x) \leq A \nu_\beta(f)(x), \quad \nu_\beta(f)(x) \leq B \sigma_\delta(f)(x).
\]
This was proved by Kaneko and Sunouchi [21].

Also, we recall a result of Carbery, Rubio de Francia and Vega [5].

**Theorem I.** If $\delta > 1/2$ and $-1 < \alpha \leq 0$, then
\[
\int_{\mathbb{R}^n} |\sigma_\delta(f)(x)|^2 |x|^{\alpha} \, dx \leq C_{\delta, \alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{\alpha} \, dx.
\]
See Rubio de Francia [27] for a different proof. Theorems H and I imply the following.

**Proposition 3.** Suppose that $\beta > 3/2 - n/2$ and $-1 < \alpha \leq 0$. Then
\[
\int_{\mathbb{R}^n} |\nu_\beta(f)(x)|^2 |x|^{\alpha} \, dx \leq C_{\beta, \alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{\alpha} \, dx.
\]

Let
\[
M_\alpha^\beta(f)(x) = \sup_{t>0} |M_t^\alpha(f)(x)|.
\]
The following weighted $L^2$ estimate can be deduced from Proposition 3.

**Proposition 4.** Suppose that $\text{Re}(\beta) > 3/2 - n/2$ and $-1 < \alpha \leq 0$. Then
\[
\int_{\mathbb{R}^n} \left|M_\alpha^{\beta-1/2}(f)(x)\right|^2 |x|^{\alpha} \, dx \leq C_{\beta, \alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{\alpha} \, dx.
\]
This is due to [38] when $\alpha = 0$.

To prove Proposition 4 we use the following relation.

**Lemma 9.** If $\text{Re}(\alpha) > \text{Re}(\alpha') > -n/2$,
\[
M_t^\alpha(f)(x) = \frac{2\Gamma(\alpha + n/2)}{\Gamma(\alpha - \alpha') \Gamma(\alpha' + n/2)} \int_0^1 M_t^{\alpha'}(f)(x)(1 - s^2)^{\alpha - \alpha'-1} s^{\alpha' + 2\alpha' - 1} \, ds.
\]
See [38] and [40, p. 1270].

**Proof of Proposition 4.** Let $k$ be the smallest non-negative integer such that $1 < \text{Re}(\beta) + k$. Let $3/2 - n/2 < \eta < \text{Re}(\beta)$. Then, by Lemma 9 and the Schwarz inequality we have
\[
M_\alpha^{\beta-1/2}(f)(x) \leq CM^{\eta - 1}(f)(x),
\]
where

\[ M^{\eta-1}(f)(x) = \sup_{t>0} \left( \frac{1}{t} \int_{0}^{t} |M^{\eta-1}_s(f)(x)|^2 \, ds \right)^{1/2}. \]

Also, we easily see that

\[ M^{\eta-1}(f)(x) \leq C\nu_{\eta}(f)(x) + C\nu_{\eta+1}(f)(x) + \cdots + C\nu_{\eta+k}(f)(x) + CM^{\eta+k}(f)(x). \]

Note that \( M^{\eta+k}(f) \) is bounded by the Hardy-Littlewood maximal function if \( \eta \) is sufficiently close to \( \text{Re}(\beta) \). Thus, applying Proposition 3, we get the weighted inequality as claimed.

Define the spherical maximal operator \( \mathcal{M} \) by

\[ \mathcal{M}(f)(x) = \sup_{t>0} \left| \int_{S^{n-1}} f(x-ty) \, d\sigma(y) \right|. \]

We note that \( \mathcal{M}(f)(x) = cM^{0}_*(f)(x) \). The following weighted norm inequality for \( \mathcal{M} \) is due to Duoandikoetxea and Vega [15].

**Theorem J.** Suppose that \( n \geq 2 \) and \( n/(n-1) < p \). Then the inequality

\[ \int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p |x|^\alpha \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha \, dx \]

holds for \( 1-n < \alpha < p(n-1)-n \).

This was partly proved by Rubio de Francia [26].

When \( \alpha = 0 \), Theorem J was proved by Stein [38] for \( n \geq 3 \) and by Bourgain [3] for \( n = 2 \). We can find in Sogge [35] a proof of the result of Bourgain which has some features in common with a proof, also given in [35], of Carbery’s result [4] for the maximal Bochner-Riesz operator on \( \mathbb{R}^2 \).

We can give a different proof of Theorem J when \( n \geq 3, \ 1-n < \alpha \leq 0 \) and \( p > n/(n-1) \) by applying Proposition 4. To see this, first we note that

(8.1) \[ \int_{\mathbb{R}^n} |M^{\beta}_*(f)(x)|^p |x|^\alpha \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha \, dx \]

when \( 1 < p < \infty, \ -n < \alpha < n(p-1) \) and \( \text{Re}(\beta) \geq 1 \), since \( M^{\beta}_*(f) \) is pointwise bounded by the Hardy-Littlewood maximal function. On the other hand, by Proposition 4 we have

(8.2) \[ \int_{\mathbb{R}^n} |M^{\beta}_*(f)(x)|^2 |x|^\alpha \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha \, dx, \]
if $\text{Re}(\beta) > (2-n)/2$ and $-1 < \alpha \leq 0$. By an interpolation argument involving (8.1) and (8.2), we see that for any $p > n/(n-1)$ and $\alpha \in (1-n, 0)$, there exist $r \in (n/(n-1), p)$ and $\tau \in (1-n, \alpha)$ such that

$$
\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^r |x|^\tau \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^r |x|^\tau \, dx.
$$

Interpolating between this estimate and the unweighted $L^r$ estimate for $\mathcal{M}$, since $\tau < \alpha < 0$, we have

$$
\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^r |x|^\alpha \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^r |x|^\alpha \, dx.
$$

Since $r < p < \infty$, interpolating between this and the obvious $L^\infty(|x|^\alpha)$ estimate for $\mathcal{M}$, we get the $L^p(|x|^\alpha)$ boundedness of $\mathcal{M}$ as claimed. (A similar argument can be found in [29]; see also [30].)

Finally, we prove Theorem J when $n \geq 2$, $0 \leq \alpha < p(n-1)-n$ and $p > n/(n-1)$ by the methods of [15]. We write $w_\alpha(x) = |x|^\alpha$. It is known that the pointwise inequality $\mathcal{M}(w_\alpha) \leq Cw_\alpha$ holds if and only if $\alpha \in (1-n, 0]$ (see [15]). Let

$$
T_\alpha(g) = w_\alpha^{-1} \mathcal{M}(w_\alpha g)
$$

for $\alpha \in (1-n, 0]$. Then, $T_\alpha$ is bounded on $L^\infty$, as we see that

(8.3) \[ \|T_\alpha(g)\|_\infty \leq \|g\|_\infty \|w_\alpha^{-1} \mathcal{M}(w_\alpha)\|_\infty \leq C\|g\|_\infty. \]

Let $r \in (n/(n-1), p)$. Since $\mathcal{M}$ is bounded on $L^r$, we have

(8.4) \[ \int_{\mathbb{R}^n} |T_\alpha(g)(x)|^r w_\alpha^r(x) \, dx = \int_{\mathbb{R}^n} |\mathcal{M}(w_\alpha g)(x)|^r \, dx \leq C \int_{\mathbb{R}^n} |g(x)|^r w_\alpha^r(x) \, dx. \]

Interpolation between (8.3) and (8.4) will imply that

$$
\int_{\mathbb{R}^n} |T_\alpha(g)(x)|^p w_\alpha^r(x) \, dx \leq C \int_{\mathbb{R}^n} |g(x)|^p w_\alpha^r(x) \, dx.
$$

This can be expressed as

$$
\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p w_\alpha^{r-p}(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w_\alpha^{r-p}(x) \, dx
$$

for any $\alpha \in (1-n, 0]$ and $r \in (n/(n-1), p)$, which implies the result as claimed.

References


LITTLEwood-Paley operators


