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L^2-stability of solitary waves for the KdV equation via Pego and Weinstein's method

By

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Abstract

In this article, we will prove $L^2(\mathbb{R})$-stability of 1-solitons for the KdV equation by using exponential stability property of the semigroup generated by the linearized operator. The proof follows the lines of recent stability argument of Mizumachi ([25]) and Mizumachi, Pego and Quintero ([29]) which show stability in the energy class by using strong linear stability of solitary waves in exponentially weighted spaces.

This gives an alternative proof of Merle and Vega ([23]) which shows $L^2(\mathbb{R})$-stability of 1-solitons for the KdV equation by using the Miura transformation. Our argument is a refinement of Pego and Weinstein ([34]) that proves asymptotic stability of solitary waves in exponentially weighted spaces. We slightly improve the H^1-stability of the modified KdV equation as well.

§ 1. Introduction

In this article, we discuss stability of solitary waves for the generalized KdV equations

$$\partial_t u + \partial_x^3 u + 3\partial_x(u^p) = 0 \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{1.1}$$

The case $p = 2$ corresponds to the KdV equation and describes a motion of shallow water waves. The case $p = 3$ corresponds to the modified KdV equation. The generalized KdV

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equations have a family of solitary wave solutions $\{\varphi_c(x-ct+\gamma) \mid c>0\,,\ \gamma\in\mathbb{R}\}$, where

$$\varphi_c(x)=\alpha_c\operatorname{sech}^{2/(p-1)}(\beta_c x)\,,\quad \alpha_c=\left(\frac{(p+1)c}{6}\right)^{1/(p-1)}\,,\quad \beta_c=\frac{p-1}{2}\sqrt{c}\,, \tag{1.2}$$

and φ_c is a solution of

$$\varphi_c''-c\varphi_c+3\varphi_c^p=0\quad\text{for } x\in\mathbb{R}. \tag{1.3}$$

Solitary waves play an important role among the solutions of (1.1). Indeed, solutions of the KdV equation resolve into a train of solitary waves and an oscillating tail if the initial data are rapidly decreasing functions (see [37]).

Stability of solitary waves has been studied for many years since Benjamin ([2]) and Bona ([3]). Let us briefly introduce their result by Weinstein's argument ([5, 38]). Eq. (1.1) has conserved quantities

$$\int_{\mathbb{R}} u^2(x)\,dx\quad\text{(the momentum)},$$

$$E(u)=\int_{\mathbb{R}}\left(\frac{1}{2}(\partial_x u)^2(x)-\frac{3}{p+1}u^{p+1}(x)\right)dx\quad\text{(the Hamiltonian)}.$$

Let $\mathcal{M}_c:=\{u\in H^1(\mathbb{R})\mid \|u\|_{L^2}=\|\varphi_c\|_{L^2}\}$. The set $\mathcal{M}_c$ is invariant under the flow generated by (1.1) and the fact that φ_c minimizes $E|_{\mathcal{M}_c}$ for $p=2$, 3 and 4 implies orbital stability of φ_c, that is, for any $\varepsilon>0$, there exists a $\delta>0$ such that if $u(0,x)=\varphi_c(x)+v_0(x)$ and $\|v_0\|_{H^1}<\delta$, then

$$\sup_{t\in\mathbb{R}}\inf_{\gamma\in\mathbb{R}}\|u(t,\cdot+\gamma)-\varphi_c\|_{H^1}<\varepsilon\,.$$

In order to study blow up problems of (1.1) with $p=5$, Martel and Merle ([18]) established a Liouville theorem for solutions around solitary wave solutions of (1.1). Using the Liouville theorem, they prove that solitary wave solutions are asymptotic stabile in $H^1_{loc}(\mathbb{R})$ if $p=2$, 3 and 4. Later, they gave a more direct proof by using a time global virial estimate around solitary wave solutions ([20]). We refer [21] for recent developments such as inelastic collision of solitary waves for (1.1) with $p=4$.

The $L^2(\mathbb{R})$-stability of solitary wave solutions was first studied by Merle and Vega ([23]) for the KdV equation by using the Miura transformation and the fact that kink solutions of the defocusing mKdV equation is stable to perturbations in $H^1(\mathbb{R})$. Indeed, a combination of the Miura transformation and the Galilean transformation

$$u(t,x)=M^+(v)(t,x-3ct)+\frac{c}{2}\,,\quad M^+(v)=\partial_x v-v^2\,,$$

is isomorphic between an L^2-neighborhood of a 1-soliton $\varphi_c(x-ct)$ and an $H^1(\mathbb{R})\times\mathbb{R}$-neighborhood of $(\psi_c(x+ct),c)$, where $\psi_c=\sqrt{c/2}\tanh(\sqrt{c/2}x)$ and $\psi_c(x+ct)$ is a kink solution of the defocusing mKdV $\partial_t v+\partial_x^3 v-2\partial_x(v^3)=0$.

Their result has been extended to prove $L^2(\mathbb{R}_x \times \mathbb{T}_y)$-stability of line solitons for the KP-II equation ([31]), $L^2(\mathbb{R})$-stability of 1-solitons for the $1d$-cubic NLS ([30]) and $L^2(\mathbb{R})$-stability of N-solitons for KdV ([1]) and the structural stability of 1-solitons for KdV in $H^{-1}(\mathbb{R})$ ([6]). These results rely on the Bäcklund transformations which are peculiar to the integrable systems. In this article, we will show $L^2(\mathbb{R})$-stability of KdV 1-solitons without using the integrability of the KdV equation.

It is common for the long wave models that the main solitary wave moves faster than the other parts of the solution, which leads to strong linear stability of solitary waves in exponentially weighted spaces (see e.g. [24, 32, 34, 35]). This property was first used by Pego and Weinstein ([34]) to prove asymptotic stability of solitary waves of the generalized KdV equations to exponentially localized perturbations. Their argument turns out to be useful especially when solitary waves cannot be characterized as (constrained) minimizers of conserved quantities. Applying the idea of [34], Friesecke and Pego ([9, 10, 11, 12]) proved that solitary waves to the FPU lattices are stable for exponentially localized perturbations (see also [28]). Mizumachi ([25, 26]) extended [10] and prove stability of N-soliton like solutions in the energy class by suitability decomposing the remainder part of the solution into a sum of small waves which moves much slower than the main waves and exponentially localized parts. The argument has been applied to the Benney-Luke equation which is one of bidirectional models of the water waves whose solitary waves in the weak surface tension regime are infinitely indefinite saddle point of the energy-momentum functional ([29]). Recently, Mizumachi ([27]) has proved transversal stability of line solitons for the KP-II equation in exponentially weighted space. We expect the argument used in [25, 29] is useful to prove stability of line solitons for the KP-II equation in unweighted spaces. In this article, we will apply the argument used in [25, 29] to the KdV equation and give an alternative proof of the following result by Merle and Vega ([23]).

Theorem 1.1 ([23]). *Let $p = 2$ and $c_0 > \sigma > 0$. Then there exist positive constants C and δ satisfying the following. Suppose that $u(t,x)$ is a solution of* (1.1) *satisfying $u(0,x) = \varphi_{c_0}(x) + v_0(x)$ and $\|v_0\|_{L^2} < \delta$. Then there exist $c_+ > 0$ and a C^1-function $x(t)$ such that*

$$\sup_{t\ge 0} \|u(t,\cdot) - \varphi_{c_0}(\cdot - x(t))\|_{L^2} \le C\|v_0\|_{L^2}^{1/2}, \tag{1.4}$$

$$c_+ = \lim_{t\to\infty} \dot{x}(t), \tag{1.5}$$

$$|c_+ - c_0| + \sup_{t\ge 0} |\dot{x}(t) - c_0| \le C\|v_0\|_{L^2}, \tag{1.6}$$

$$\lim_{t\to\infty} \left\|u(t,\cdot) - \varphi_{c_+}(\cdot - x(t))\right\|_{L^2(x\ge\sigma t)} = 0. \tag{1.7}$$

Remark. The $L^2(\mathbb{R})$ well-posedness of the KdV equation was proved by Bourgain

([4]).

Remark. We expect that stability argument of N-solitary wave solutions to the FPU lattices ([26]) is applicable to N-solitary wave solutions of the long wave models as well.

For the mKdV equation, we slightly improve orbital stability of 1-solitons in $H^1(\mathbb{R})$. Note that the mKdV equation is well-posed in $H^s(\mathbb{R})$ with $s \ge 1/4$. See [16, 17].

Theorem 1.2. *Let $p = 3$ and $c_0 > \sigma > 0$. Then there exist positive constants C and δ satisfying the following. Suppose that $u(t,x)$ is a solution of* (1.1) *satisfying $u(0,x) = \varphi_{c_0}(x) + v_0(x)$ and $\|v_0\|_{L^2}^{3/4}\|v_0\|_{H^1}^{1/4} < \delta$. Then there exist $c_+ > 0$ and a C^1-function $x(t)$ such that*

$$\sup_{t\ge 0}\|u(t,\cdot) - \varphi_{c_0}(\cdot - x(t))\|_{L^2} \le C\|v_0\|_{L^2}^{1/2}\,, \tag{1.8}$$

$$c_+ = \lim_{t\to\infty}\dot{x}(t)\,, \tag{1.9}$$

$$|c_+ - c_0| + \sup_{t\ge 0}|\dot{x}(t) - c_0| \le C\|v_0\|_{L^2}\,, \tag{1.10}$$

$$\lim_{t\to\infty}\left\|u(t,\cdot) - \varphi_{c_+}(\cdot - x(t))\right\|_{L^2(x\ge\sigma t)} = 0\,. \tag{1.11}$$

Finally, let us introduce several notations. Let L^p_a $(1 \le p \le \infty)$ and H^k_a $(k \in \mathbb{N})$ be exponentially weighted spaces, writing

$$L^p_a = \{g \mid e^{ax}g \in L^p(\mathbb{R})\}\,, \quad H^k_a = \{g \mid \partial_x^j g \in L^2_a \text{ for } 0 \le j \le k\}\,,$$

with norms

$$\|g\|_{L^p_a} = \|e^{a\cdot}g\|_{L^p(\mathbb{R})}\,, \quad \|g\|_{H^k_a} = \left(\sum_{0\le j\le k}\|\partial_x^j g\|_{L^2_a}^2\right)^{1/2}\,.$$

We define $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{t,x}$ as

$$\langle u_1, u_2\rangle = \int_{\mathbb{R}} u_1(x)u_2(x)\,dx\,, \quad \langle v_1, v_2\rangle_{t,x} = \int_{\mathbb{R}}\int_{\mathbb{R}} v_1(t,x)v_2(t,x)\,dxdt\,.$$

For any Banach spaces X, Y, we denote by $B(X,Y)$ the space of bounded linear operators from X to Y. We abbreviate $B(X,X)$ as $B(X)$. We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \le Cb$. Various constants will be simply denoted by C and C_i $(i \in \mathbb{N})$ in the course of the calculations.

§ 2. Linear stability of 1-solitons

In this section, we recall strong linear stability of 1-soliton solutions in the exponentially weighted space L^2_a for the sake of self-containedness. Let

$$u(t,x) = \varphi_c(y) + v(t,y)\,, \quad y = x - ct$$

and linearize nonlinear terms of (1.1) around $v = 0$. Then we have

$$\partial_t v + \mathcal{L}_c v = 0\,, \tag{2.1}$$

where $\mathcal{L}_c = \partial_y(\partial_y^2 - c + f'(\varphi_c))$ with $f(u) = 3u^p$. The linearized operator $\mathcal{L}_c$ has a generalized kernel associated with the infinitesimal changes in the location and the speed of the solitary waves. Let $\xi_c^1(y) = \partial_y\varphi_c(y)$, $\xi_c^2(y) = \partial_c\varphi_c(y)$ and

$$\zeta_c^1(y) = -\theta_1(c)\int_{-\infty}^{y} \partial_c\varphi_c(y_1)\,dy_1 + \theta_2(c)\varphi_c(y)\,, \quad \zeta_c^2(y) = \theta_1(c)\varphi_c(y)\,, \tag{2.2}$$

where $\theta_1(c) = 1/\int_{\mathbb{R}}\varphi_c(y)\partial_c\varphi_c(y)\,dy$ and $\theta_2(c) = \theta_1(c)^2\left(\int_{\mathbb{R}}\partial_c\varphi_c(y)\,dy\right)^2/2$. Differentiating (1.3) twice with respect to y and c, we have

$$\mathcal{L}_c\xi_c^1 = 0\,, \quad \mathcal{L}_c\xi_c^2 = \xi_c^1\,. \tag{2.3}$$

In view of (2.3) and the fact that (formally) $\partial_y\mathcal{L}_c^* = -\mathcal{L}_c\partial_y$,

$$\mathcal{L}_c^*\zeta_c^1 = \zeta_c^2\,, \quad \mathcal{L}_c^*\zeta_c^2 = 0\,. \tag{2.4}$$

For i, j=1, 2,

$$\int_{\mathbb{R}} \xi_c^i(y)\zeta_c^j(y)\,dy = \delta_{ij}\,.$$

Since $d\|\varphi_c\|_{L^2}^2/dc \neq 0$ for $p \neq 5$, the algebraic multiplicity of the eigenvalue 0 is two if $p \neq 5$. Let $P_c : L^2_a \to L^2_a$ $(0 < a < \sqrt{c})$ be the spectral projection associated with the generalized kernel of $\mathcal{L}_c$ and let $Q_c = I - P_c$. Then

$$P_c v = \langle v\,,\,\zeta_c^1\rangle\xi_c^1 + \langle v\,,\,\zeta_c^2\rangle\xi_c^2\,,$$
$$\mathrm{Range}(\mathcal{L}_cP_c) \subset \mathrm{Range}(P_c)\,, \quad \mathrm{Range}(\mathcal{L}_cQ_c) \subset \mathrm{Range}(Q_c)\,.$$

Next we recall the spectrum of the linearized operator $\mathcal{L}_c$. The spectrum of $\mathcal{L}_c$ in $L^2(\mathbb{R})$ consists of $i\mathbb{R}$ (see [33]). However if $0 < a < \sqrt{c}$, the essential spectrum of $\mathcal{L}_c$ in L^2_a locates in the stable half plane. Indeed, the spectrum of $\mathcal{L}_c$ in L^2_a is equivalent to the spectrum of $e^{a\cdot}\mathcal{L}_ce^{-a\cdot}$ in $L^2(\mathbb{R})$ and by Weyl's essential spectrum theorem,

$$\sigma_{ess}(\mathcal{L}_c) = \{ip(\xi + ia) \mid \xi \in \mathbb{R}\}\,, \quad p(\xi) = -(\xi^3 + c\xi)\,.$$

Since

$$p(\xi+ia) = -\{\xi^3+(c-3a^2)\xi\} - i\{a(c-a^2)+3a\xi^2\}\,, \tag{2.5}$$

we have $\sigma_{ess}(\mathcal{L}_c) \subset \{\lambda\in\mathbb{C} \mid \Re\lambda \ge a(c-a^2)>0\}$ if $a\in(0,\sqrt{c})$.

The complement of $\sigma_{ess}(\mathcal{L}_c)$ consists of two connected components. We denote by $\Omega(a)$ one of these components which contains the unstable half plane. Pego and Weinstein [34, Proposition 2.6, Theorem 3.1] prove spectral stability of $\mathcal{L}_c$ in the exponentially weighted space L^2_a.

Proposition 2.1 (Spectral stability of 1-solitons ([34]))**.** *Let $p=2$ or 3. Suppose $c>0$ and $a\in(0,\sqrt{c})$. Then the operator $\mathcal{L}_c$ in L^2_a has no eigenvalue in $\Omega(a)$ other than 0 whose algebraic multiplicity is two and $\sigma_{ess}(\mathcal{L}_c)\subset\{\lambda\in\mathbb{C}\mid \Re\lambda\ge a(c-a^2)>0\}$.*

Let $R(\lambda)=(i\lambda+\mathcal{L}_c)^{-1}$ and $\Sigma_b:=\{\lambda\in\mathbb{C}\mid \Im\lambda<b\}$. The spectral stability of $\mathcal{L}_c$ implies that

$$\sup_{\lambda\in\Sigma_b}\|R(\lambda)Q_c\|_{B(L^2_a)}<\infty \quad \text{for } b \text{ satisfying } 0<b<a(c-a^2). \tag{2.6}$$

Let $\mathcal{L}_0=\partial_y^3-c\partial_y$, $R_0(\lambda)=(i\lambda+\mathcal{L}_0)^{-1}$ and $V=\partial_y(f'(\varphi_c)\cdot)$. By Plancherel's theorem and (2.5),

$$\begin{aligned}\|R_0(\lambda)\|_{B(L^2_a,H^k_a)} \lesssim & \sup_{\xi\in\mathbb{R}}\frac{1+|\xi+ia|^k}{|\lambda+p(\xi+ia)|}\\ \lesssim & (1+|\lambda|)^{-(2-k)/3} \quad \text{for } \lambda\in\Sigma_b \text{ and } 0\le k\le 2.\end{aligned} \tag{2.7}$$

In view of (2.7) and the fact that $f'(\varphi_c)$ is exponentially localized, we see that $VR_0(\lambda)$ is compact on L^2_a and that $I+VR_0(\lambda)$ has a bounded inverse unless λ is an eigenvalue of $\mathcal{L}_c$. Hence it follows from Proposition 2.1 that $\|(I+VR_0(\lambda))^{-1}Q_c\|_{B(L^2_a)}$ is bounded on any compact subset of Σ_b. Moreover, Eq. (2.7) with $k=1$ implies that $\|VR_0(\lambda)\|_{B(L^2_a)}\le\frac12$ for large λ. Thus we have

$$\sup_{\lambda\in\Sigma_b}\|R(\lambda)Q_c\|_{B(L^2_a;H^2_a)} \lesssim \sup_{\lambda\in\Sigma_b}\|R_0(\lambda)\|_{B(L^2_a;H^2_a)}<\infty \tag{2.8}$$

since $R(\lambda)=R_0(\lambda)(I+VR_0(\lambda))^{-1}$.

Once (2.6) is established, the Gearhart-Prüss theorem ([13, 36]) on C_0-semigroups on Hilbert spaces implies exponential linear stability of $e^{-t\mathcal{L}_c}Q_c$.

Proposition 2.2 (Linear stability of 1-solitons ([34]))**.** *Let $p=2$ or 3, $c>0$ and $a\in(0,\sqrt{c})$. Then there exist positive constants K and b such that for every $f\in L^2_a$ and $t\ge0$,*

$$\|e^{-t\mathcal{L}_c}Q_cf\|_{L^2_a}\le Ke^{-bt}\|f\|_{L^2_a}\,. \tag{2.9}$$

Exponential stability of $e^{-t\mathcal{L}_c}Q_c$ reflects that the largest solitary wave moves faster to the right than any other waves.

Kato [15] tells us that $e^{-t\partial_x^3}$ has a strong smoothing effect on L^2_a. We will use the property to deal with nonlinear terms.

Corollary 2.3. *Let $p=2$ or 3, $c>0$ and $0<a<\sqrt{c}$. Then there exist positive constants K' and b' such that*

$$\|e^{-t\mathcal{L}_c}Q_c\partial_y^j f\|_{L^2_a} \le K'e^{-b't}t^{-(2j+1)/4}\|f\|_{L^1_a} \quad \text{for } j=0,\,1,\ f\in L^1_a \text{ and } t>0, \tag{2.10}$$

$$\|e^{-t\mathcal{L}_c}Q_c\partial_y f\|_{L^2_a} \le K'e^{-b't}t^{-1/2}\|f\|_{L^2_a} \quad \text{for } f\in L^2_a \text{ and } t>0. \tag{2.11}$$

Proof. By the definition of the Fourier transform and Plancherel's theorem,

$$\|f\|_{L^2_a} = \|\hat f(\cdot+ia)\|_{L^2} \quad \text{for } f\in L^2_a. \tag{2.12}$$

By (2.12) and (2.5), we have for $j\in\mathbb{Z}_{\ge 0}$,

$$\begin{aligned}\|\partial_y^j e^{-t\mathcal{L}_0}f\|_{L^2_a} =& \left(\int_{\mathbb{R}}\left|(\xi+ia)^j e^{-itp(\xi+ia)}\hat f(\xi+ia)\right|^2 d\xi\right)^{1/2}\\ \le& e^{-a(c-a^2)t}\left(\int_{\mathbb{R}}\left|(\xi+ia)^j e^{-3at\xi^2}\hat f(\xi+ia)\right|^2 d\xi\right)^{1/2}.\end{aligned}$$

Since $\|\xi^k e^{-3at\xi^2}\|_{L^2}\lesssim t^{-(2k+1)/4}$ for $k\ge 0$ and $\|\hat f(\cdot+ia)\|_{L^\infty}\lesssim \|f\|_{L^1_a}$,

$$\|\partial_y^j e^{-t\mathcal{L}_0}f\|_{L^2_a} \lesssim e^{-a(c-a^2)t}(1+t^{-(2j+1)/4})\|f\|_{L^1_a} \quad \text{for } j\in\mathbb{Z}_{\ge 0}. \tag{2.13}$$

Similarly, we have

$$\|\partial_y^j e^{-\mathcal{L}_0}f\|_{L^2_a} \lesssim e^{-a(c-a^2)t}(1+t^{-j/2})\|f\|_{L^2_a} \quad \text{for } j\in\mathbb{Z}_{\ge 0}\,. \tag{2.14}$$

Now we will show (2.10) with $j=0$. Let $v(t)$ be a mild solution of (2.1) with $v(0)=Q_cf$ and $f\in L^1_a$, that is,

$$v(t) = e^{-t\mathcal{L}_0}Q_cf - \int_0^t e^{-(t-s)\mathcal{L}_0}\partial_y\left(f'(\varphi_c)v(s)\right)\,ds =: Tv(t)\,. \tag{2.15}$$

Since $Q_c\in B(L^1_a)$ and $f'(\varphi_c)$ is bounded, it follows from (2.13) and (2.14) that

$$\|Tv\|_{L^2_a} \lesssim e^{-a(c-a^2)t}t^{-1/4}\|f\|_{L^1_a} + \int_0^t e^{-a(c-a^2)(t-s)}(t-s)^{-1/2}\|v(s)\|_{L^2_a}\,ds\,.$$

By the contraction mapping theorem, there exists a unique solution of (2.15) satisfying

$$\sup_{0<t<t_1} t^{1/4}\|v(t)\|_{L^2_a} < \infty \quad \text{for a } t_1>0. \tag{2.16}$$

For $t \geq t_1$, Proposition 2.2 and (2.16) imply

$$\|e^{-t\mathcal{L}_c}Q_c f\|_{L^2_a} \leq \|e^{-(t-t_1)\mathcal{L}_c}Q_c\|_{B(L^2_a)}\|v(t_1)\|_{L^2_a} \lesssim e^{-b(t-t_1)}\|f\|_{L^1_a}\,. \tag{2.17}$$

Combining (2.16) and (2.17), we obtain (2.10) with $j = 0$. We can prove (2.10) with $j = 1$ and (2.11) in exactly the same way. Thus we complete the proof. $\square$

Remark. Suppose $g(t) \in C([0,T]; L^1_a)$ and that $v(t)$ is a solution of

$$\partial_t v + \mathcal{L}_c v = Q_c \partial_y f\,, \quad v(0) = 0\,, \tag{2.18}$$

in the class $C([0,T]; L^2_a)$. Then $v(t)$ can be represented as

$$v(t) = \int_0^t e^{-(t-s)\mathcal{L}_c} Q_c g(s)\, ds \quad \text{for } t \in [0,T]. \tag{2.19}$$

Indeed, Corollary 2.3 ensures that the right hand side of (2.19) is a solution of (2.18) in the class $C([0,T]; L^2_a)$. Note that a solution to (2.18) is unique in the class $C([0,T]; L^2_a)$. Applying Corollary 2.3 to (2.19) we have

$$\|v(t)\|_{L^2_a} \leq K' \int_0^t (t-s)^{-3/4} e^{-b'(t-s)} \|g(s)\|_{L^1_a}\, ds\,. \tag{2.20}$$

In Section 7, we will use (2.20) to estimate quadratic nonlinearities.

To deal with cubic terms of mKdV, we will use the following local smoothing effect of

$$Ag(t) := \int_0^t e^{-(t-s)\mathcal{L}_c} Q_c g(s)\, ds$$

in the exponentially weighted space.

Proposition 2.4. *Let $p = 2$ or 3, $c > 0$ and $0 < a < \sqrt{c}$. Then there exists a positive constant K_1 such that for every $g \in L^2(\mathbb{R}_+; L^2_a)$,*

$$\|Ag\|_{L^2(\mathbb{R}_+; H^2_a)} \leq K_1 \|g\|_{L^2(\mathbb{R}_+; L^2_a)}\,. \tag{2.21}$$

Because of parabolic nature of $e^{-t\partial_x^3}$ on exponential weighted spaces, Proposition 2.4 follows from the argument of [8]. However, we here follow the lines of the proof of [7, Proposition 2.7].

Proof of Proposition 2.4. Fix $T > 0$ and define $g_T(t)$ to be equal to $g(t)$ for $t \in [0,T]$ and 0 elsewhere. Define

$$u_T(t) = \int_0^t e^{-(t-s)\mathcal{L}_c} Q_c g_T(s)\, ds.$$

Then $u_T(t) = Ag(t)$ for $t \in [0, T]$, $u(t) = 0$ for $t \le 0$ and $u_T(t)$ is a constant for $t > T$. We have for $t \in \mathbb{R}$,

$$(\partial_t + \mathcal{L}_c)u_T(t) = Q_c g_T(t)\,.$$

Thanks to the properties of g_T and u_T, we can take the Fourier transform in time (in the lower complex half plane) to get the relation

$$(i\tau + \mathcal{L}_c)\widehat{u_T}(\tau) = Q_c\widehat{g_T}(\tau), \quad \mathrm{Im}(\tau) < 0.$$

Take $\tau = \lambda - i\varepsilon$, $\varepsilon > 0$ and $\lambda \in \mathbb{R}$. Using the resolvent estimate (2.8), letting ε to zero and integrating over λ, we get

$$\|\widehat{u_T}(\lambda)\|_{L^2(\mathbb{R};H^2_a)} \lesssim \|\widehat{g_T}(\lambda)\|_{L^2(\mathbb{R};L^2_a)}\,.$$

Since the Fourier transform of a function from $\mathbb{R}$ to a Hilbert space H is an isometry on $L^2(\mathbb{R}; H)$, we obtain

$$\|u_T\|_{L^2(\mathbb{R};H^2_a)} \lesssim \|g_T\|_{L^2(\mathbb{R};L^2_a)}\,.$$

This is turn implies

$$\|Ag\|_{L^2([0,T];H^2_a)} \lesssim \|g\|_{L^2(\mathbb{R};L^2_a)}\,.$$

Observe that the implicit constant in the last inequality is independent of T. Letting T tends to infinity, we complete the proof of Proposition 2.4. □

§ 3. Decomposition of solutions around 1-solitons

In this section, we will decompose a solution around 1-solitons into a sum of a modulating solitary wave and the remainder part. Let

$$u(t, x) = \varphi_{c(t)}(y) + v(t, y)\,, \quad y = x - x(t)\,. \tag{3.1}$$

Here $c(t)$ and $x(t)$ denote the modulating speed and the modulating phase shift of the main solitary wave at time t and $v(t, y)$ denotes the remainder part of the solution. Substituting (3.1) into (1.1), we obtain

$$\partial_t v + \mathcal{L}_{c(t)}v - (\dot{x}(t) - c(t))\partial_y v + \ell(t) + \partial_y \mathcal{N} = 0\,, \tag{3.2}$$

where

$$\mathcal{N} = f(\varphi_{c(t)} + v) - f(\varphi_{c(t)}) - f'(\varphi_{c(t)})v\,,$$
$$\ell(t) = \dot{c}(t)\partial_c\varphi_{c(t)}(y) - (\dot{x}(t) - c(t))\partial_y\varphi_{c(t)}(y)\,.$$

Suppose that $u(t, x)$ satisfies the initial condition

$$u(0, x) = \varphi_{c_0}(x) + v_0(x)\,.$$

To apply the semigroup estimate directly to v as [34], the perturbation $v(t)$ should belong to an exponentially weighted space. In order to extend Pego-Weinstein's approach for $v_0 \in L^2(\mathbb{R})$ or $v_0 \in H^1(\mathbb{R})$, we further decompose $v(t,y)$ into a sum of a small L^2-solution of the KdV equation and an exponentially localized part. More precisely, let $\tilde{v}_1$ be a solution of (1.1) satisfying $\tilde{v}_1(0,\cdot) = v_0$ and let

$$v_1(t,y) = \tilde{v}_1(t,x)\,, \quad v(t,y) = v_1(t,y) + v_2(t,y)\,.$$

Then

$$\begin{cases} \partial_t v_1 - \dot{x}(t)\partial_y v_1 + \partial_y^3 v_1 + \partial_y f(v_1) = 0\,, \\ v_1(0,y) = v_0(y + x(0))\,, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \partial_t v_2 + \mathcal{L}_{c(t)} v_2 - (\dot{x}(t) - c(t))\,\partial_y v_2 + \ell(t) + \partial_y N(t) = 0\,, \\ v_2(0,x) = 0\,, \end{cases} \tag{3.4}$$

where $N(t) = N_1(t) + N_2(t)$,

$$\begin{aligned} N_1(t) &= f(\varphi_{c(t)} + v_1) - f(\varphi_{c(t)}) - f(v_1)\,, \\ N_2(t) &= f(\varphi_{c(t)} + v) - f(\varphi_{c(t)} + v_1) - f'(\varphi_{c(t)})v_2\,. \end{aligned}$$

The solutions of (3.3) will be evaluated by using a virial estimate first used in the fundamental article by Kato ([15]). The solutions of (3.4) will be estimated by using the linear estimates due to Pego-Weinstein in Section 2.

To begin with, we will show that $v_2(t)$ remains in exponentially weighted spaces as long as the decomposition (3.1) exists and $c(t) - c_0$ remains small.

Lemma 3.1. *Let $p = 2$ and $v_0 \in L^2(\mathbb{R})$ or $p = 3$ and $v_0 \in H^1(\mathbb{R})$. Suppose that $u(t)$ is a solution to (1.1) satisfying $u(0) = \varphi_{c_0} + v_0$ and that and $\tilde{v}_1(t)$ is a solution to (1.1) satisfying $\tilde{v}_1(0) = v_0$. Then for $u(t) - \tilde{v}_1(t) \in C([0,\infty); L^2_a)$ for any $a \in [0, \sqrt{c_0})$.*

Proof. Suppose $p = 2$ and $\bar{u}(t,x) = u(t,x) - \tilde{v}_1(t,x)$. Then

$$\begin{cases} \partial_t \bar{u} + \partial_x^3 \bar{u} + 3\partial_x \{(u + \tilde{v}_1)\bar{u}\} = 0 \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ \bar{u}(0,x) = \varphi_{c_0}(x) \quad \text{for } x \in \mathbb{R}. \end{cases} \tag{3.5}$$

Thanks to the well-posedness of the KdV equation in $L^2(\mathbb{R})$, we have $\bar{u}(t) = u(t) - \tilde{v}_1(t) \in C(\mathbb{R}; L^2(\mathbb{R}))$.

Next we will show that $\bar{u}(t) \in L^\infty(0,T; L^2_a)$ for any $a \in (0, \sqrt{c_0})$ and $T > 0$. Suppose in addition that $v_0 \in H^3(\mathbb{R}) \cap L^2_a$ so that $\bar{u} \in C(\mathbb{R}; H^3(\mathbb{R}))$ and $\tilde{v}_1(t)$, $u(t) \in$

$C([0,\infty);L^2_a)$. At least formally, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} e^{2ax}\bar{u}^2(t,x)\,dx+6a\int_{\mathbb{R}} e^{2ax}(\partial_x\bar{u})^2(t,x)\,dx\\&=\int e^{2ax}\{(2a)^3\bar{u}^2+8a\bar{u}^3\}(t,x)\,dx+I\,,\end{aligned}$$

where $I=12\int_{\mathbb{R}} e^{2ax}(2a\bar{u}^2+\bar{u}\partial_x\bar{u})\tilde{v}_1\,dx$. Since $\|\tilde{v}_1(t)\|_{L^2}=\|v_0\|_{L^2}$ and

$$\|\bar{u}(t)\|_{L^2}\le\|u(t)\|_{L^2}+\|\tilde{v}_1(t)\|_{L^2}\le\|\varphi_{c_0}\|_{L^2}+2\|v_0\|_{L^2}\quad\text{for every } t\in\mathbb{R},$$

there exists a $C>0$ depending only on c_0 and $\|v_0\|_{L^2}$ such that

$$\frac{d}{dt}\|\bar{u}(t)\|^2_{L^2_a}\le C\|\bar{u}(t)\|^2_{L^2_a}\,.$$

Here we use a weighted Sobolev inequality $\|w\|_{L^q_a}\lesssim\|w\|^{1/2+1/q}_{L^2_a}\|w\|^{1/2-1/q}_{H^1_a}$ for $2\le q\le\infty$ (see (9.2) in Section 9). Thus we have

$$\|\bar{u}(t)\|^2_{L^2_a}\le e^{Ct}\|\bar{u}(0)\|^2_{L^2_a}=e^{Ct}\|\varphi_{c_0}\|^2_{L^2_a}\quad\text{for } t\ge 0. \tag{3.6}$$

More precisely, let $\tilde{\chi}_n(x)=e^{2an}(1+\tanh a(x-n))/2$. We have $\tilde{\chi}_n(x)\uparrow e^{2ax}$ and $\tilde{\chi}_n'(x)\uparrow 2ae^{2ax}$ as $n\to\infty$ and $0<\tilde{\chi}_n'(x)\le a\tilde{\chi}_n(x)$ and $|\tilde{\chi}_n'''(x)|\le 4a^2\tilde{\chi}_n'(x)$ for any $x\in\mathbb{R}$. Using the above properties of $\tilde{\chi}_n$ and Lemma 9.1, we can easily justify (3.6) for $v_0\in H^3(\mathbb{R})\cap L^2_a$.

For any $v_0\in L^2(\mathbb{R})$, there exists a sequence $v_{0n}\in H^3(\mathbb{R})\cap L^2_a$ such that $v_{0n}\to v_0$ in $L^2(\mathbb{R})$ as $n\to\infty$. Let $u_n(t)$ and $\tilde{v}_n(t)$ be a solution to (1.1) satisfying $u_n(0)=\varphi_{c_0}+v_{0n}$ and $\tilde{v}_n(0)=v_{0n}$. Then for any $t\in\mathbb{R}$, we have $\lim_{n\to\infty}\|u_n(t)-\tilde{v}_n(t)-\bar{u}(t)\|_{L^2(\mathbb{R})}=0$ and there exists a subsequence of $\{u_n(t)-\tilde{v}_n(t)\}$ that converges to $\bar{u}(t)$ weakly in L^2_a. Thus we have

$$\|\bar{u}(t)\|_{L^2_a}\le\liminf_{n\to\infty}\|u_n(t)-\tilde{v}_n(t)\|_{L^2_a}\le e^{Ct/2}\|\varphi_{c_0}\|_{L^2_a}\,.$$

By the variation of the constant formula,

$$\bar{u}(t)=e^{-t\partial_x^3}\varphi_{c_0}-3\int_0^t e^{-(t-s)\partial_x^3}\partial_x(u(s)+\tilde{v}_1(s))\bar{u}(s)\,ds\,.$$

Since $e^{-t\partial_x^3}$ is a C_0-semigroup on L^2_a and $\|\partial_x e^{-t\partial_x^3}\|_{B(L^1_a;L^2_a)}\lesssim t^{-3/4}$, we easily see that $\bar{u}(t)\in C([0,\infty);L^2_a)$.

The case $p=3$ can be shown in the same way. Thus we complete the proof. ☐

Now we impose the symplectic orthogonality condition on v_2.

$$\int_{\mathbb{R}} v_2(t,y)\zeta^1_{c(t)}(y)\,dy=0\,, \tag{3.7}$$

$$\int_{\mathbb{R}} v_2(t,y)\zeta^2_{c(t)}(y)\,dy=0\,. \tag{3.8}$$

Note that $\zeta^1_{c(t)}$, $\zeta^2_{c(t)} \in L^2_{-a}$ and $v_2(t) \in L^2_a$ for $a \in (0, \sqrt{c_0}/2)$ by Lemma 3.1 as long as $|c(t) - c_0|$ remains small. In an L^2_a-neighborhood of φ_{c_0}, the speed and the phase satisfying the orthogonality conditions can be uniquely chosen.

Lemma 3.2. *Let $c_0 > 0$, $a \in (0, \sqrt{c_0})$. Then there exist positive constants δ_0 and δ_1 such that for each $w \in U_0(\delta_0) := \{w \in L^2_a \mid \|w - \varphi_{c_0}\|_{L^2_a} < \delta_0\}$, there exists a unique $(\gamma\,, c) \in U_1(\delta_1) := \{(\gamma\,, c) \in \mathbb{R}^2 \mid |\gamma| + |c - c_0| < \delta_1\}$ such that*

$$\langle w(\cdot + \gamma) - \varphi_c\,, \zeta^1_c\rangle = \langle w(\cdot + \gamma) - \varphi_c\,, \zeta^2_c\rangle = 0\,. \tag{3.9}$$

Proof. Let $G : L^2_a \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2$ be a mapping defined by

$$G(w, \gamma, c) = \begin{pmatrix} \langle w - \varphi_c(\cdot - \gamma)\,, \zeta^1_c(\cdot - \gamma)\rangle \\ \langle w - \varphi_c(\cdot - \gamma)\,, \zeta^2_c(\cdot - \gamma)\rangle \end{pmatrix}\,. \tag{3.10}$$

Since $G(\varphi_{c_0}, 0, c_0) = {}^t(0, 0)$ and $\nabla_{(\gamma,c)} G(\varphi_{c_0}, 0, c_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is invertible, Lemma 3.2 follows immediately from the implicit function theorem. $\square$

Lemma 3.2 implies that the decomposition

$$u(t, x) - \tilde{v}_1(t, x) = \varphi_{c(t)}(y) + v_2(t, y)\,, \quad y = x - x(t) \tag{3.11}$$

satisfying the orthogonality conditions (3.7) and (3.8) persists as long as $\varphi_{c(t)}(y) + v_2(t, y)$ stays in $U_0(\delta_0)$ and $c(t) - c_0$ remains small.

Lemma 3.3. *Let c_0, δ_0 and δ_1 be as in Lemma 3.2. Suppose that u and $\tilde{v}_1$ be solutions of* (1.1) *satisfying $u(0) = \varphi_{c_0} + v_0$ and $\tilde{v}_1(0) = v_0 \in L^2(\mathbb{R})$. Then there exist $T > 0$ and $c(t)$, $x(t) \in C([0, T]) \cap C^1((0, T))$ such that*

$$c(0) = c_0\,, \quad x(0) = 0\,, \quad \sup_{t \in [0,T]} |c(t) - c_0| < \delta_1\,,$$

and that v_2 defined by (3.11) *satisfies the orthogonality conditions* (3.7) *and* (3.8) *for $t \in [0, T]$. Moreover, if T is finite and*

$$\sup_{t \in [0,T)} \|\varphi_{c(t)} + v_2(t) - \varphi_{c_0}\|_{L^2_a} < \delta_0\,,$$

then T is not maximal.

Proof. Let X be a Banach space with the norm $\|u\|_X = \|(c_0 - \partial_x^2)^{-2} u\|_{L^2_a}$. Then G defined by (3.10) is a smooth mapping from $X \times \mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}^2$. Thus by the implicit function theorem there exist an X-neighborhood $\widetilde{U}_0$ of 0 and an $\mathbb{R}^2$-neighborhood $\widetilde{U}_1$

of $(0, c_0)$ such that for any $w \in \widetilde{U}_0$, there exists a unique $(\gamma, c) \in \widetilde{U}_1$ satisfying (3.9). Moreover, the mapping $\widetilde{U}_0 \ni w \mapsto \Phi(w) = (\gamma, c) \in \widetilde{U}_1$ is smooth. Since

$$\bar{u}(t) = u(t) - \tilde{v}_1(t) \in C([0,\infty); L^2_a) \cap C^1((0,\infty); X)$$

by Lemma 3.1 and (1.1), we have $(x(t)\,,\, c(t)) = \Phi(\bar{u}(t))$ is C^1 on $(0,T)$. The other part of the proof is exactly the same as the proof of [29, Proposition 9.3]. $\square$

In order to prove stability of 1-solitons, we will estimate the following quantities in the subsequent sections. Let

$$\|w\|_W := \left(\int_{\mathbb{R}} e^{-2a|x|} w^2(x)\,dx\right)^{1/2}, \quad \|w\|_{W_1} := \left(\|w\|_W^2 + \|\partial_x w\|_W^2\right)^{1/2},$$

and let

$$\begin{aligned}
&\mathbb{M}_1(T) = \sup_{t\in[0,T]} \|v_1(t)\|_{L^2} + \|v_1(t)\|_{L^2(0,T;W_1)}\,,\\
&\mathbb{M}_2(T) = \begin{cases} \sup_{t\in[0,T]} \|v_2(t)\|_{L^2_a} + \|v_2\|_{L^2(0,T;L^2_a)} \text{ if } p=2,\\ \sup_{t\in[0,T]} \|v_2(t)\|_{L^2_a} + \|v_2\|_{L^2(0,T;H^1_a)} \text{ if } p=3,\end{cases}\\
&\mathbb{M}_v(T) = \sup_{0\le t\le T} \|v(t)\|_{L^2}^2\,, \quad \mathbb{M}_c(T) = \sup_{t\in[0,T]} |c(t)-c_0|\,,\\
&\mathbb{M}_x(T) = \sup_{t\in[0,T]} |\dot{x}(t)-c(t)|\,, \quad \mathbb{M}_\gamma(T) = \sup_{t\in[0,T]} |\dot{\gamma}(t)-c(t)|\,,\\
&\mathbb{M}_{tot}(T) = \begin{cases} \mathbb{M}_1(T)+\mathbb{M}_2(T)+\mathbb{M}_v(T)+\mathbb{M}_c(T)+\mathbb{M}_x(T) \text{ if } p=2,\\ \mathbb{M}_1(T)+\mathbb{M}_2(T)+\mathbb{M}_v(T)+\mathbb{M}_c(T)+\mathbb{M}_\gamma(T) \text{ if } p=3,\end{cases}
\end{aligned}$$

where $\gamma(t)$ is a function which shall be introduced in Lemma 4.2. We remark that $\|v_1(t)\|_{W_1}$ measures the interaction between the solitary wave and v_1.

§ 4. Modulation equations on the speed and the phase shift

In this section, we will derive modulation equations on the speed parameter $c(t)$ and the phase shift parameter $x(t)$.

Differentiating (3.7) and (3.8) with respect to t and substituting (3.4) into the resulting equation, we have for $i = 1$ and 2,

$$\begin{aligned}
0 &= \frac{d}{dt}\langle v_2(t)\,,\, \zeta^i_{c(t)}\rangle\\
&= -\langle v_2(t)\,,\, \mathcal{L}^*_{c(t)}\zeta^i_{c(t)}\rangle - \langle \ell, \zeta^i_{c(t)}\rangle + \dot{c}\langle v_2\,,\, \partial_c\zeta^i_{c(t)}\rangle - (\dot{x}-c)\langle v_2\,,\, \partial_y\zeta^i_{c(t)}\rangle + \langle N\,,\, \partial_y\zeta^i_{c(t)}\rangle\,.
\end{aligned}$$

By (2.4) and (3.8), we have $\langle v_2(t)\,,\,\mathcal{L}^*_{c(t)}\zeta^i_{c(t)}\rangle = 0$ for $i = 1, 2$. Hence it follows that

$$\mathcal{A}(t)\begin{pmatrix} c(t)-\dot{x}(t) \\ \dot{c}(t)\end{pmatrix} = \begin{pmatrix}\langle N\,,\,\partial_y\zeta^1_{c(t)}\rangle \\ \langle N\,,\,\partial_y\zeta^2_{c(t)}\rangle\end{pmatrix}\,, \tag{4.1}$$

where

$$\mathcal{A}(t) = I - \begin{pmatrix}\langle v_2\,,\,\partial_y\zeta^1_c\rangle & \langle v_2\,,\,\partial_c\zeta^1_c\rangle \\ \langle v_2\,,\,\partial_y\zeta^2_c\rangle & \langle v_2\,,\,\partial_c\zeta^2_c\rangle\end{pmatrix}\,.$$

The following lemma provides estimates for $c(t)$ and $x(t)$ in terms of the weighted L^2-norms of v_1 and v_2.

Lemma 4.1. *Let $p = 2$ and $c_0 > 0$ and $0 < a < \sqrt{c_0}/2$. Then there exists a positive constant δ_2 such that if the decomposition* (3.11) *satisfying* (3.7) *and* (3.8) *exists on $[0,T]$ and $\mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_c(T) < \delta_2$, then for $t \in [0,T]$,*

$$\begin{aligned}&|\dot{c}(t)| + |\dot{x}(t) - c(t)| \\ &\lesssim \|v_1(t)\|_W + \|v_2\|_{L^2_a}(\|v_1(t)\|_W + \|v_2(t)\|_{L^2_a})\,.\end{aligned} \tag{4.2}$$

Furthermore,

$$\frac{d}{dt}\left\{c(t) + \theta_1(c(t))\left\langle v_1(t), \varphi_{c(t)}\right\rangle\right\} = O\left(\|v_1(t)\|^2_W + \|v_2(t)\|^2_{L^2_a}\right)\,, \tag{4.3}$$

$$\mathbb{M}_c(T) + \mathbb{M}_x(T) \lesssim \mathbb{M}_1(T) + \mathbb{M}_2(T)^2\,. \tag{4.4}$$

Lemma 4.2. *Let $p = 3$ and $c_0 > 0$ and $0 < a < \sqrt{c_0}/2$. Suppose $v_0 \in H^1(\mathbb{R})$. Then there exists a positive constant δ_2 such that if the decomposition* (3.11) *satisfying* (3.7) *and* (3.8) *exists on $[0,T]$ and $\mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_v(T) + \mathbb{M}_c(T) < \delta_2$, then for $t \in [0,T]$,*

$$|\dot{c}(t)| + |\dot{x}(t) - c(t)| \lesssim \|v_1(t)\|_W + \left(\|v_1(t)\|^2_W + \|v_2\|^2_{L^2_a}\right)(1 + \|v_1(t)\|_{L^\infty} + \|v(t)\|_{L^\infty})\,. \tag{4.5}$$

Furthermore,

$$\frac{d}{dt}\left\{c(t) + \theta_1(c(t))\left\langle v_1(t), \varphi_{c(t)}\right\rangle\right\} = O\left(\|v_1(t)\|_{W_1} + \|v_2(t)\|_{H^1_a}\right)^2\,, \tag{4.6}$$

and there exists a C^1-function $\gamma(t)$ such that $\gamma(0) = 0$,

$$\dot{\gamma}(t) - \dot{x}(t) = O\left((\|v_1(t)\|_{W_1} + \|v_2(t)\|_{H^1_a})^2(\|v_1(t)\|_{L^2} + \|v(t)\|_{L^2})\right)\,, \tag{4.7}$$

$$\dot{\gamma}(t) - c(t) = O\left(\|v_1\|_W + \|v_2(t)\|^2_{L^2_a}\right)\,, \tag{4.8}$$

$$\mathbb{M}_c(T) + \mathbb{M}_\gamma(T) \lesssim \mathbb{M}_1(T) + \mathbb{M}_2(T)^2\,. \tag{4.9}$$

As we will see in Section 8, Lemmas 4.1 and 4.2 imply that the modulating speed $c(t)$ converges to a fixed speed as $t \to \infty$. We will use (4.3) and (4.6) to show that $c(t)$ tends to a fixed speed speed c_+ as $t \to \infty$.

Proof of Lemma 4.1. If δ_2 is small enough, then $2a < \sqrt{c(t)}$ for all $t \in [0, T]$, and it follows from (2.2) that for $i = 1$ and 2, $\|\partial_c \zeta^i_{c(t)}\|_{L^2_{-a}}$ and $\sup_{y\in\mathbb{R}} e^{2a|y|}\partial_y \zeta^i_{c(t)}(y)$ are uniformly bounded on $[0, T]$. Thus we have for $i = 1$ and 2,

$$|\langle v_2, \partial_y \zeta^i_{c(t)}\rangle| + |\langle v_2, \partial_c \zeta^i_{c(t)}\rangle| \lesssim \|v_2\|_{L^2_a} \lesssim \delta_2\,.$$

Moreover,

$$|\langle N_1, \partial_y \zeta^i_{c(t)}\rangle| \lesssim \|v_1(t)\|_W\,, \quad |\langle N_2, \partial_y \zeta^i_{c(t)}\rangle| \lesssim \|v_1(t)\|_W \|v_2\|_{L^2_a}\,, + \|v_2\|^2_{L^2_a}\,.$$

because $N_1 = 6\varphi_c v_1$ and $N_2 = 6v_1v_2 + 3v_2^2$. Combining the above with (4.1), we obtain (4.2). Moreover,

$$\begin{aligned}\dot{c}(t) =& \langle N, \partial_y \zeta^2_{c(t)}\rangle \left(1 + O(\|v_2(t)\|_{L^2_a})\right) + O\left(\|v_2(t)\|_{L^2_a} |\langle N, \partial_y \zeta^1_{c(t)}\rangle|\right) \\ =& \langle N_1, \partial_y \zeta^2_{c(t)}\rangle + O(\|v_1(t)\|^2_W + \|v_2(t)\|^2_{L^2_a})\,.\end{aligned} \tag{4.10}$$

Next we will rewrite $\langle N_1, \partial_y \zeta^2_c\rangle$ as a sum of time derivative $\theta_1(c(t))\langle v_1(t, \cdot), \varphi_{c(t)}\rangle$ and a remainder part which is integrable in time. Substituting (3.3) and integrating the resulting equation by parts, we have

$$\begin{aligned}&\frac{d}{dt}\langle v_1\,,\, \varphi_c\rangle - \dot{c}\langle v_1\,,\, \partial_c\varphi_c\rangle + (\dot{x} - c)\langle v_1\,,\, \varphi_c'\rangle \\ =& \langle v_1\,,\, \varphi_c''' - c\varphi_c'\rangle + 3\langle v_1^2\,,\, \varphi_c'\rangle\,.\end{aligned} \tag{4.11}$$

By (1.3) and (4.11),

$$\begin{aligned}\langle N_1, \partial_y \zeta^2_c\rangle =& 3\theta_1(c)\left\langle v_1\,,\, \partial_y\left(\varphi_c^2\right)\right\rangle \\ =& \theta_1(c)\langle v_1\,,\, c\varphi_c' - \varphi_c'''\rangle \\ =& -\frac{d}{dt}\left(\theta_1(c)\langle v_1, \varphi_c\rangle\right) + \frac{d\theta_1(c)}{dt}\langle v_1, \varphi_c\rangle \\ &+ \theta_1(c)\{\dot{c}\langle v_1\,,\, \partial_c\varphi_c\rangle - (\dot{x} - c)\langle v_1\,,\, \varphi_c'\rangle + 3\langle v_1^2\,,\, \partial_y\varphi_c\rangle\}\,.\end{aligned}$$

Substituting (4.2) into the above, we have

$$\langle N_1, \partial_y \zeta^2_c\rangle + \frac{d}{dt}\left(\theta_1(c)\langle v_1, \varphi_c\rangle\right) = O(\|v_1\|^2_W + \|v_2\|^2_{L^2_a})\,.$$

Thus (4.3) follows from (4.10) and the above. Eq. (4.4) follows immediately from (4.2) and (4.3). $\square$

Proof of Lemma 4.2. By the definition, we have

$$N_1 = N_{11} + N_{12}\,,\quad N_{11} = 9\varphi_c^2 v_1\,,\quad N_{12} = 9\varphi_c v_1^2\,, \tag{4.12}$$

$$N_2 = N_{21} + N_{22}\,,\quad N_{21} = 9\varphi_c v_2(2v_1 + v_2)\,,\quad N_{22} = 3v_2(3v_1^2 + 3v_1v_2 + v_2^2)\,, \tag{4.13}$$

and for $i = 1,\, 2$,

$$|\langle N_{11}, \partial_y \zeta^i_{c(t)}\rangle| \lesssim \|v_1(t)\|_W\,,\quad |\langle N_{12}, \partial_y \zeta^i_{c(t)}\rangle| \lesssim \|v_1(t)\|_W^2\,, \tag{4.14}$$

$$\begin{aligned} &|\langle N_{21}, \partial_y \zeta^i_{c(t)}\rangle| \lesssim (\|v_1(t)\|_W + \|v_2(t)\|_{L^2_a})^2\,,\\ &|\langle N_{22}, \partial_y \zeta^i_{c(t)}\rangle| \lesssim (\|v_1(t)\|_W + \|v_2(t)\|_{L^2_a})^2 \|v_2(t)\|_{L^\infty}\,, \end{aligned} \tag{4.15}$$

in the same way as in the proof of Lemma 4.1. Combining the above with (4.1), we have (4.5).

Next we will show (4.6). Since $|\partial_y \zeta^i_{c(t)}(y)| \lesssim e^{-2a|y|}$, Lemma 9.2 implies that

$$\int_{\mathbb{R}} |\partial_y \zeta^i_{c(t)}(y)||v^3(t,y)|\,dy \lesssim \|v(t)\|_{L^2}\|v(t)\|_{W_1}^2 \quad \text{for } i = 1,\, 2. \tag{4.16}$$

By (4.16) and the Hölder inequality,

$$|\langle N_{22}, \partial_y \zeta^i_{c(t)}\rangle| \lesssim (\|v_1(t)\|_{L^2} + \|v(t)\|_{L^2})(\|v_1(t)\|_{W_1} + \|v_2(t)\|_{H^1_a})^2\,, \tag{4.17}$$

whence

$$\begin{aligned} &|\dot c(t)| + |\dot x(t) - c(t)|\\ \lesssim &\|v_1(t)\|_W + (1 + \|v_1(t)\|_{L^2} + \|v(t)\|_{L^2})(\|v_1(t)\|_{W_1} + \|v_2(t)\|_{H^1_a})^2 \end{aligned} \tag{4.18}$$

follows from (4.1), (4.14), (4.15) and (4.17).

As in the proof of Lemma 4.1, we have

$$\begin{aligned} &\langle N_{11}, \partial_y \zeta^2_{c(t)}\rangle + \frac{d}{dt}\left(\theta_1(c)\langle v_1, \varphi_c\rangle\right)\\ =&\langle v_1, \varphi_c\rangle \frac{d}{dt}\theta_1(c) + \theta_1(c)\left\{\dot c\langle v_1, \partial_c\varphi_c\rangle - (\dot x - c)\langle v_1, \varphi_c'\rangle + 3\langle v_1^3, \varphi_c'\rangle\right\}\\ =&O\left((\|v_1(t)\|_{W_1} + \|v_2(t)\|_{H^1_a})^2\right)\,. \end{aligned} \tag{4.19}$$

In the last line, we use (4.18) and the fact that

$$|\langle v_1^3, \varphi'_{c(t)}\rangle| \lesssim \|v_1(t)\|_{L^2}\|v_1(t)\|_{W_1}^2 = \|v_0\|_{L^2}\|v_1(t)\|_{W_1}^2\,.$$

Combining (4.1), (4.14), (4.15), (4.17) and (4.19), we obtain (4.6).

Finally, we will show (4.7) and (4.8). Let $\gamma(t)$ be a C^1-function satisfying

$$\gamma(0) = 0\,,\quad c(t) - \dot\gamma(t) = (1, 0)\mathcal{A}(t)^{-1}\begin{pmatrix}\langle N_1 + N_{21}, \partial_y\zeta^1_{c(t)}\rangle\\ \langle N_1 + N_{21}, \partial_y\zeta^2_{c(t)}\rangle\end{pmatrix}\,.$$

The (4.8) follows from (4.14) and (4.15). In view of the definition of γ and (4.1),

$$\begin{aligned}|\dot\gamma(t)-\dot x(t)| \lesssim& (1+\|v\|_{L^2_a})\sum_{i=1,2}|\langle N_{22},\partial_y\zeta^i_{c(t)}\rangle| \\ \lesssim& (1+\|v\|_{L^2_a})(\|v_1\|_{L^2}+\|v_2\|_{L^2})(\|v_1\|_W+\|v_2\|_{H^1_a})^2 .\end{aligned}$$

Eq. (4.9) immediately follows from (4.6), (4.7) and (4.8). Thus we complete the proof. □

§ 5. The L^2-estimate of v

In this section, we will estimate v by using the L^2-conservation law of the gKdV equation.

Lemma 5.1. *Suppose $v_0 \in L^2(\mathbb{R})$ if $p=2$ and $v_0\in H^1(\mathbb{R})$ if $p=3$. Then there exist positive constants δ_3 and C such that if* (3.11) *satisfying* (3.8) *exists and $\|v_0\|_{L^2}+\mathbb{M}_2(T)+\mathbb{M}_c(T)<\delta_3$ for a $T\in(0,\infty]$, then*

$$\mathbb{M}_v(T)\le C(\mathbb{M}_c(T)+\|v_0\|_{L^2}) .$$

Proof. Since $v_1(t,x-x(t))$ is a solution of (1.1) satisfying $v_1(0,x)=v_0(x)$,

$$\|v_1(t)\|_{L^2}=\|v_0\|_{L^2} , \tag{5.1}$$

as long as the decomposition (3.11) exists.

Let $u(t)$ be a solution of (1.1) satisfying $u(0)=\varphi_{c_0}+v_0$. By the L^2-conservation law,

$$\|u(t)\|^2_{L^2}=\|\varphi_{c_0}+v_0\|^2_{L^2}=\|\varphi_{c_0}\|^2_{L^2}+O(\|v_0\|_{L^2}) . \tag{5.2}$$

Substituting (3.1) into the left hand side, we have

$$\|u(t)\|^2_{L^2}=\|\varphi_{c(t)}\|^2_{L^2}+2\int_{\mathbb{R}}\varphi_{c(t)}(y)v(t,y)\,dy+\|v(t)\|^2_{L^2} . \tag{5.3}$$

By the orthogonality condition (3.8),

$$\int_{\mathbb{R}}\varphi_{c(t)}(y)v(t,y)\,dy=\int_{\mathbb{R}}\varphi_{c(t)}(y)v_1(t,y)\,dy . \tag{5.4}$$

Combining (5.1)–(5.4), we obtain

$$\begin{aligned}\|v(t)\|^2_{L^2}\le&\left|\|\varphi_{c(t)}\|^2_{L^2}-\|\varphi_{c_0}\|^2_{L^2(\mathbb{R})}\right|+O(\|v_0\|_{L^2(\mathbb{R})}) \\ \lesssim& |c(t)-c_0|+\|v_0\|_{L^2} .\end{aligned}$$

Thus we complete the proof. □

§ 6. The virial estimate of v_1

In this section, we will show that $\|v_1(t)\|_{W_1}$ is square integrable in time by using the virial identity.

Lemma 6.1. *Suppose $p = 2$ and $v_0 \in L^2(\mathbb{R})$. There exist positive constants C and δ_4 such that if $\mathbb{M}_2(T) + \mathbb{M}_c(T) + \mathbb{M}_x(T) + \|v_0\|_{L^2} < \delta_4$, then $\mathbb{M}_1(T) \le C\|v_0\|_{L^2}$.*

Lemma 6.2. *Suppose $p = 3$ and $v_0 \in H^1(\mathbb{R})$. There exist positive constants C and δ_4 such that if $\mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_c(T) + \mathbb{M}_\gamma(T) + \|v_0\|_{L^2} < \delta_4$, then $\mathbb{M}_1(T) \le C\|v_0\|_{L^2}$.*

Let us recall the virial identity for the KdV equation which ensures that $v_1(t) \in L^2(\mathbb{R}_+; W_1)$. Let $\chi_\varepsilon(x) = 1 + \tanh \varepsilon x$, $\tilde{x}(t)$ be a C^1 function and

$$I_{x_0}(t) = \int_{\mathbb{R}} \chi_\varepsilon(x - \tilde{x}(t) - x_0)\tilde{v}_1(t,x)^2\,dx\,.$$

Then we have the following.

Lemma 6.3. *Suppose $v_0 \in L^2(\mathbb{R})$ if $p = 2$ and $v_0 \in H^1(\mathbb{R})$ if $p = 3$. For any $c_1 > 0$, there exist positive constants ε_0 and δ such that if $\inf_t \tilde{x}'(t) \ge c_1$, $\varepsilon \in (0, \varepsilon_0)$ and $\|v_0\|_{L^2} < \delta$, then for any $x_0 \in \mathbb{R}$,*

$$I_{x_0}(t) + \nu \int_0^t \int_{\mathbb{R}} \chi'_\varepsilon(x - \tilde{x}(s) - x_0)\{(\partial_x \tilde{v}_1)^2 + \tilde{v}_1^2\}(s,x)\,dxds \le I_{x_0}(0)\,,$$

where $\nu = \frac{1}{2}\min\{3, c_1\}$.

Proof of Lemmas 6.1 and 6.2. Lemma 6.1 is an immediate consequence of the L^2-conservation law (5.1) Lemma 6.3 with $\tilde{x}(t) = x(t)$ and $x_0 = 0$.

To prove Lemma 6.2, we apply Lemma 6.3 with $\tilde{x}(t) = \gamma(t)$ and $x_0 = 0$. Then

$$\int_0^t \int_{\mathbb{R}} \chi'_\varepsilon(y + h(s))v_1(t,y)^2\,dyds \lesssim \|v_0\|_{L^2}^2\,,$$

where $h(t) = x(t) - \gamma(t)$. By Lemma 4.2,

$$|h(t)| \le \int_0^t |\dot{x}(t) - \dot{\gamma}(t)|\,dt \lesssim \mathbb{M}_1(T)^2 + \mathbb{M}_2(T)^2\,, \tag{6.1}$$

and there exists a positive constant μ depending only on δ_4 such that $\chi'_\varepsilon(y) \le \mu\chi'_\varepsilon(y + h(t))$ for every $y \in \mathbb{R}$ and $t \in [0, T]$. Thus we complete the proof. □

Proof of Lemma 6.3. Suppose that $\tilde{v}_1(t)$ is a smooth solution of (1.1). Then

$$
\begin{aligned}
(6.2)\qquad & I'_{x_0}(t) + \int_{\mathbb{R}} \chi'_\varepsilon(x-\tilde{x}(t)-x_0)\left\{3(\partial_x\tilde{v}_1)^2 + \tilde{x}'(t)\tilde{v}_1^2 - \frac{6p}{p+1}\tilde{v}_1^{p+1}\right\}(t,x)\,dx \\
&= \int_{\mathbb{R}} \chi'''_\varepsilon(x-\tilde{x}(t)-x_0)\tilde{v}_1(t,x)^2\,dx\,.
\end{aligned}
$$

By the definition of χ_ε,

$$
(6.3)\qquad 0<\chi'_\varepsilon(x)<2\varepsilon\chi_\varepsilon(x)\,,\quad |\chi''_\varepsilon(x)|\le 2\varepsilon\chi'_\varepsilon(x)\,,\quad |\chi'''_\varepsilon(x)|\le 4\varepsilon^2\chi'_\varepsilon(x)\quad \text{for } \forall x\in\mathbb{R}.
$$

Integrating (6.2) over $[0,t]$ and using Lemma 9.1, (5.1) and (6.3) to the resulting equation, we obtain

$$
(6.4)\qquad I_{x_0}(t) + \nu\int_0^t\int_{\mathbb{R}} \chi'_\varepsilon(x-\tilde{x}(s)-x_0)\left((\partial_x\tilde{v}_1)^2+\tilde{v}_1^2\right)(s,x)\,dxds \ \le I_{x_0}(0)
$$

provided ε and δ_4 are sufficiently small. Since (1.1) is well-posed in $L^2(\mathbb{R})$ if $p=2$ and in $H^1(\mathbb{R})$ if $p=3$, we can verify (6.4) for any v_0 satisfying the assumption of Lemma 6.3. □

Corollary 6.4. *Under the conditions of Lemma 6.3, if there exists a positive constant σ such that $\inf_{t\ge 0}\tilde{x}'(t)\ge c_1+\sigma$, then*

$$
\int_{\mathbb{R}} \chi_\varepsilon(x-\tilde{x}(t))\tilde{v}_1(t,x)^2\,dx \le \int_{\mathbb{R}} \chi_\varepsilon(x-\tilde{x}(0)-\sigma t)v_0(x)^2\,dx \to 0 \quad \textit{as } t\to\infty.
$$

Proof. Let $t_1>0$ and $\tilde{x}_1(t)=\tilde{x}(t)-\sigma(t-t_1)$. Then $\tilde{x}_1(t_1)=\tilde{x}(t_1)$ and $\tilde{x}'_1(t)\ge c_1$ for every $t\ge 0$. Using $\tilde{x}_1(t)$ in place of $\tilde{x}(t)$ in Lemma 6.3, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \chi_\varepsilon(x-\tilde{x}_1(t_1))\tilde{v}_1(t,x)^2\,dx &\le \int_{\mathbb{R}} \chi_\varepsilon(x-\tilde{x}_1(0))\tilde{v}_1(0,x)^2\,dx \\
&= \int_{\mathbb{R}} \chi_\varepsilon(x-\tilde{x}(0)-\sigma t_1)v_0(x)^2\,dx\,.
\end{aligned}
$$

Thus we complete the proof. □

§ 7. The weighted estimate of v_2

In this section, we will estimate $\|v_2(t)\|_{L^2_a}$ by using the exponential stability property of the linearized operator as in [25, 29, 34]. Thanks to the parabolic smoothing effect of $e^{t\partial_x^3}$ on L^2_a, we do not need re-centering argument as in [29] which is used to avoid a derivative loss caused by the term $(\dot{x}-c)\partial_y v$.

Lemma 7.1. *Let $p = 2$. There exist positive constants C and δ_5 such that if $\|v_0\|_{L^2} + \mathbb{M}_{tot}(T) \le \delta_5$, then $\mathbb{M}_{tot}(T) \le C\|v_0\|_{L^2}$.*

Proof. To begin with, we deduce a priori bounds on $\mathbb{M}_1$, $\mathbb{M}_c$, $\mathbb{M}_x$ and $\mathbb{M}_v$ in terms of $\|v_0\|_{L^2}$ and $\mathbb{M}_2(T)$. Lemma 6.1 implies

$$\mathbb{M}_1(T) \lesssim \|v_0\|_{L^2} . \tag{7.1}$$

By (4.4) and (7.1),

$$\mathbb{M}_c(T) + \mathbb{M}_x(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2 , \tag{7.2}$$

and

$$\mathbb{M}_v(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_c(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2 \tag{7.3}$$

follows from Lemma 5.1 and (7.2). Hence it suffices to show $\mathbb{M}_2(T) \lesssim \|v_0\|_{L^2}$ to prove Lemma 7.1.

Now we will estimate v_2. Eq. (3.4) can be rewritten as

$$\partial_t v_2 + \mathcal{L}_{c_0} v_2 + \ell(t) + \partial_y(N(t) + \widetilde{N}(t)) = 0 , \tag{7.4}$$

where $\widetilde{N}(t) = (c_0 - \dot{x}(t))\, v_2 + 6\left(\varphi_{c(t)} - \varphi_{c_0}\right) v_2$. Using the variation of constants formula, we have

$$Q_{c_0} v_2(t) = -\int_0^t e^{-(t-s)\mathcal{L}_{c_0}} Q_{c_0}\{\ell(s) + \partial_y(N(s) + \widetilde{N}(s))\}\, ds . \tag{7.5}$$

Applying Proposition 2.2 and Corollary 2.3 to (7.5), we have

$$\begin{aligned} \|Q_{c_0} v_2(t)\|_{L^2_a} \lesssim & \int_0^t e^{-b(t-s)} \|\ell(s)\|_{L^2_a}\, ds \\ & + \int_0^t e^{-b'(t-s)} (t-s)^{-1/2} (\|N_1(s)\|_{L^2_a} + \|\widetilde{N}(s)\|_{L^2_a})\, ds \\ & + \int_0^t e^{-b'(t-s)} (t-s)^{-3/4} \|N_2(s)\|_{L^1_a}\, ds . \end{aligned} \tag{7.6}$$

Since $Q_{c(t)} v_2(t) = v_2(t)$ and $\|Q_{c(t)} - Q_{c_0}\|_{B(L^2_a)} = O(|c(t) - c_0|)$,

$$\|v_2(t) - Q_{c_0} v_2(t)\|_{L^2_a} = O(|c(t) - c_0|) \|v_2(t)\|_{L^2_a} .$$

Hence for small δ_5, there exist positive constants d_1 and d_2 such that

$$d_1 \|v_2(t)\|_{L^2_a} \le \|Q_{c_0} v_2(t)\|_{L^2_a} \le d_2 \|v_2(t)\|_{L^2_a} \quad \text{for } t \in [0, T].$$

Let us prove

$$\|N_1\|_{L^\infty(0,T;L^2_a)} + \|N_1\|_{L^2(0,T;L^2_a)} \lesssim \|v_0\|_{L^2}\,, \tag{7.7}$$

$$\|N_2\|_{L^\infty(0,T;L^1_a)} + \|N_2\|_{L^2(0,T;L^1_a)} \lesssim (\|v_0\|_{L^2}^{1/2} + \mathbb{M}_2(T))\mathbb{M}_2(T)\,, \tag{7.8}$$

$$\|\widetilde{N}\|_{L^\infty(0,T;L^2_a)} + \|\widetilde{N}\|_{L^2(0,T;L^2_a)} \lesssim (\|v_0\|_{L^2} + \mathbb{M}_2(T)^2)\mathbb{M}_2(T)\,, \tag{7.9}$$

$$\|\ell\|_{L^\infty(0,T;L^2_a)} + \|\ell\|_{L^2(0,T;L^2_a)} \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2\,. \tag{7.10}$$

If δ_5 is sufficiently small, we have $2a < \inf_{s\in[0,T]} \sqrt{c(s)}$ and

$$\|N_1(s)\|_{L^2_a} \lesssim \|v_1(s)\|_W \tag{7.11}$$

follows from the definition of N_1. Since $|N_2| \lesssim |v_2|(|v_1| + |v_2|)$ and $v_2 = v - v_1$, we have for $s \in [0,T]$,

$$\begin{aligned}\|N_2(s)\|_{L^1_a} \lesssim& \|v_2(s)\|_{L^2_a}(\|v_1(s)\|_{L^2} + \|v_2(s)\|_{L^2})\\ \lesssim&(\mathbb{M}_1(T) + \mathbb{M}_v(T)^{1/2})\|v_2(s)\|_{L^2_a}\,.\end{aligned} \tag{7.12}$$

Combining (7.1)–(7.3) with (7.11) and (7.12), we obtain (7.7) and (7.8). Moreover,

$$\begin{aligned}\|\widetilde{N}(s)\|_{L^2_a} &\lesssim (|\dot{x}(s) - c(s)| + |c(s) - c_0|)\,\|v_2(s)\|_{L^2_a}\\ &\lesssim \left(\|v_0\|_{L^2} + \mathbb{M}_2(T)^2\right)\|v_2(s)\|_{L^2_a}\,.\end{aligned}$$

By (4.2) and the definition of ℓ,

$$\begin{aligned}\|\ell(s)\|_{L^2_a} \lesssim& |\dot{c}(s)| + |\dot{x}(s) - c(s)|\\ \lesssim& \|v_1(s)\|_W + \|v_2(s)\|_{L^2_a}^2 \lesssim \|v_1(s)\|_W + \mathbb{M}_2(T)\|v_2(s)\|_{L^2_a}\,.\end{aligned}$$

Thus we prove (7.7)–(7.10). Since $e^{-bt}(1+t^{-3/4}) \in L^1((0,\infty))$, it follows from Young's inequality and (7.6)–(7.10) that

$$\begin{aligned}\mathbb{M}_2(T) =& \|v_2\|_{L^\infty(0,T;L^2_a)} + \|v_2\|_{L^2(0,T;L^2_a)}\\ \lesssim& \|v_0\|_{L^2} + (\|v_0\|_{L^2}^{1/2} + \mathbb{M}_2(T))\mathbb{M}_2(T)\,.\end{aligned}$$

Thus we have $\mathbb{M}_2(T) \lesssim \|v_0\|_{L^2}$ provided δ_5 is sufficiently small. This completes the proof of Lemma 7.1. □

Lemma 7.2. *Let $p = 3$. There exists a positive constant δ_5 such that if $\mathbb{M}_{tot}(T) + \|v_0\|_{L^2}^{3/4}\|v_0\|_{H^1}^{1/4} < \delta_5$, then $\mathbb{M}_{tot}(T) \lesssim \|v_0\|_{L^2}$.*

To prove Lemma 7.2, we need the H^1-bound of v_1 and v.

Lemma 7.3. *Let $p = 3$ and $\tilde{v}_1$ be a solution of (1.1) satisfying $\tilde{v}_1(0) = v_0 \in H^1(\mathbb{R})$. Then*

$$\|\partial_x \tilde{v}_1(t)\|_{L^2} \le C\left(\|\partial_x v_0\|_{L^2} + \|v_0\|_{L^2}^3\right), \tag{7.13}$$

where C is a constant independent of t and v_0.

Proof. Since $\|\partial_x v_1\|_{L^2}^2 \le 2E(v_1) + \frac{3}{2}\|v_1\|_{L^4}^4$ and $\|v_1\|_{L^4} \lesssim \|v_1\|_{L^2}^{3/4}\|\partial_x v_1\|_{L^2}^{1/4}$,

$$\|\partial_x v_1(t)\|_{L^2}^2 \le 2E(v_1(t)) + \frac{1}{2}\|\partial_x v_1(t)\|_{L^2}^2 + O(\|v(t)\|_{L^2}^6).$$

Combining the above with the L^2 conservation law and the energy conservation law, we obtain (7.13). □

Lemma 7.4. *There exists a positive constant δ' such that if $\|v_0\|_{L^2} + \mathbb{M}_2(T) + \mathbb{M}_v(T) + \mathbb{M}_c(T) < \delta'$, then*

$$\|v(t)\|_{H^1} \le C(\|v_0\|_{H^1} + \|v(t)\|_{L^2}^3 + |c(t) - c_0|) \quad \text{for } t \in [0, T]. \tag{7.14}$$

Proof. Let $S(u) := E(u) + \frac{c_0}{2}\|u\|_{L^2}^2$. Thanks to the energy and the L^2 conservation laws,

$$\begin{aligned} S(\varphi_{c_0} + v_0) =& S(\varphi_{c(t)} + v) \\ =& S(\varphi_{c(t)}) + \langle S'(\varphi_{c(t)}), v\rangle + \frac{1}{2}\langle S''(\varphi_{c(t)})v, v\rangle - R, \end{aligned} \tag{7.15}$$

where

$$R = \frac{3}{4}\int_{\mathbb{R}} \left(4\varphi_{c(t)}v^3 + v^4\right)\, dy.$$

Since $S'(\varphi_{c_0}) = 0$ by (1.3),

$$S(\varphi_{c(t)}) = S(\varphi_{c_0}) + O(|c(t) - c_0|^2). \tag{7.16}$$

By (3.8), the fact that $S'(\varphi_{c(t)}) = (c_0 - c(t))\varphi_{c(t)}$ and (5.1),

$$\langle S'(\varphi_{c(t)}), v\rangle = (c_0 - c(t))\langle v_1, \varphi_{c(t)}\rangle = O\left(|c(t) - c_0|\|v_0\|_{L^2}\right). \tag{7.17}$$

Next, we will show that $S''(\varphi_c)$ is positive definite for v_2. Let $L = S''(\varphi_c) + (c - c_0)I$ and

$$v_2 = a\varphi_c^2 + b\varphi_c' + p, \quad \langle p, \varphi_c^2\rangle = \langle p, \varphi_c'\rangle = 0.$$

Note that

$$L\varphi_c^2 = -3c\varphi_c^2, \quad L\varphi_c' = 0, \tag{7.18}$$

and that L is positive definite on ${}^{\perp}\operatorname{span}\{\varphi_c^2, \varphi_c'\}$ by the Sturm-Liouville theorem. By (7.18),

$$\langle Lv_2, v_2\rangle = \langle Lp, p\rangle - 3ca^2\langle L\varphi_c^2, \varphi_c^2\rangle .$$

Since $\langle a\varphi_c^2 + p, \varphi_c\rangle = \langle v_2, \varphi_c\rangle = 0$ by (3.8) and $d\|\varphi_c\|_{L^2}^2/dc > 0$,

$$\langle Lv_2, v_2\rangle \gtrsim \|a\varphi_c^2 + p\|_{H^1}^2$$

in exactly the same way as [14, Proof of Theorem 3.3]. Thanks to the orthogonality condition (3.7), we have $|b| \lesssim \|a\varphi_c^2+p\|_{H^1}$. Thus there exists a positive constant ν such that

$$\langle S''(\varphi_c)v_2, v_2\rangle \ge \nu\|v_2\|_{H^1}^2 \tag{7.19}$$

provided $|c - c_0|$ is sufficiently small.

By (7.19) and Lemma 7.3,

$$\langle S''(\varphi_{c(t)})v, v\rangle \ge \frac{\nu}{2}\|v_2(t)\|_{H^1}^2 - O(\|v_1(t)\|_{H^1}^2) \ge \frac{\nu}{2}\|v_2(t)\|_{H^1}^2 - O(\|v_0\|_{H^1}^2) . \tag{7.20}$$

By the Sobolev imbedding theorem,

$$|R| \le \frac{\nu}{8}\|\partial_x v\|_{L^2}^2 + O(\|v(t)\|_{L^2}^6) . \tag{7.21}$$

Combining (7.15)–(7.17), (7.20) and (7.21), we obtain (7.14). □

Now we are in position to prove Lemma 7.2.

Proof of Lemma 7.2. By Lemmas 4.2, 5.1 and 6.2,

$$\mathbb{M}_1(T) \lesssim \|v_0\|_{L^2} , \quad \mathbb{M}_c(T) + \mathbb{M}_\gamma(T) + \mathbb{M}_v(T) \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2 . \tag{7.22}$$

Furthermore, it follows from (4.18) and (7.22) that

$$\begin{aligned} &\|\dot c\|_{L^1(0,T)+L^2(0,T)} + \|\dot x - c\|_{L^1(0,T)+L^2(0,T)} \\ \lesssim&\mathbb{M}_1(T) + \mathbb{M}_2(T)^2 \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2 . \end{aligned} \tag{7.23}$$

Now we will estimate $\mathbb{M}_2(T)$. Instead of v_2, we will estimate its small translation. Let $\tilde v_2(t,y) = v(t, y + h(t))$. By (3.4),

$$\begin{cases} \partial_t\tilde v_2 + \mathcal{L}_{c_0}\tilde v_2 + \tau_{h(t)}\ell(t) + \partial_y(\tau_{h(t)}N(t) + \widetilde N(t)) = 0 , \\ \tilde v_2(0, x) = 0 , \end{cases} \tag{7.24}$$

where τ_h is a shift operator defined by $\tau_h g(x) = g(x + h)$ and

$$\widetilde N(t) = (c_0 - \dot\gamma(t))\,\tilde v_2 + 9\left(\tau_{h(t)}\varphi_{c(t)}^2 - \varphi_{c_0}^2\right)\tilde v_2 .$$

Using the variation of constants formula, we have $Q_{c_0}\tilde v_2(t)=v_{21}(t)+v_{22}(t)+v_{23}(t)$, where

$$v_{21}(t)=-\int_0^t e^{-(t-s)\mathcal{L}_{c_0}}Q_{c_0}\tau_{h(s)}\ell(s)\,ds\,,$$
$$v_{22}(t)=-\int_0^t e^{-(t-s)\mathcal{L}_{c_0}}Q_{c_0}\tau_{h(s)}(N(s)+\widetilde{N}(s)-N_{22}(s))\,ds\,,$$
$$v_{23}(t)=-\int_0^t e^{-(t-s)\mathcal{L}_{c_0}}Q_{c_0}\tau_{h(s)}N_{22}(s)\,ds\,.$$

Note that $\|\tilde v_2(t)\|_{H^1_a}\lesssim\|Q_{c_0}\tilde v_2(t)\|_{H^1_a}$ as in the proof of Lemma 7.1. By Proposition 2.2,

$$\|v_{21}(t)\|_{H^1_a}\lesssim\int_0^t e^{-b(t-s)}\|\tau_{h(s)}\ell(s)\|_{H^1_a}\,ds\,. \tag{7.25}$$

By (7.23), (6.1) and the definition of ℓ,

$$\|\tau_{h(t)}\ell(t)\|_{L^1(0,T;H^1_a)+L^2(0,T;H^1_a)}\lesssim\|v_0\|_{L^2}+\mathbb{M}_2(T)^2\,. \tag{7.26}$$

Combining (7.25) and (7.26), we have

$$\|v_{21}\|_{L^\infty(0,T;H^1_a)}+\|v_{21}\|_{L^2(0,T;H^1_a)}\lesssim\|v_0\|_{L^2}+\mathbb{M}_2(T)^2\,. \tag{7.27}$$

Using Corollary 2.3, we can estimate $\sup_{t\in[0,T]}\|v_{22}(t)\|_{L^2_a}$ in the same way as the proof of Lemma 7.1. Indeed,

$$\begin{aligned}\|v_{22}(t)\|_{L^2_a}\lesssim&\int_0^t e^{-b'(t-s)}(t-s)^{-1/2}\left(\|\tau_{h(s)}N_{11}(s)\|_{L^2_a}+\|\widetilde{N}(s)\|_{L^2_a}\right)\,ds\\ &+\int_0^t e^{-b'(t-s)}(t-s)^{-3/4}\left(\|\tau_{h(s)}N_{12}(s)\|_{L^1_a}+\|\tau_{h(s)}N_{21}(s)\|_{L^1_a}\right)\,ds\,.\end{aligned} \tag{7.28}$$

Now we will estimate each term of the right hand side of (7.28). Since $|\widetilde{N}(s)|\lesssim(|\dot\gamma(s)-c(s)|+|c(s)-c_0|)|v_2|$, it follows from (7.22) and (6.1) that

$$\begin{aligned}&\|\tau_{h(s)}\widetilde{N}\|_{L^\infty(0,T;L^2_a)}+\|\tau_{h(s)}\widetilde{N}\|_{L^2(0,T;L^2_a)}\\ \lesssim&(\mathbb{M}_c(T)+\mathbb{M}_\gamma(T))\mathbb{M}_2(T)\lesssim(\|v_0\|_{L^2}+\mathbb{M}_2(T)^2)\mathbb{M}_2(T)\,.\end{aligned} \tag{7.29}$$

By (4.12), (4.13), (7.22) and (6.1),

$$\|\tau_{h(s)}N_{11}\|_{L^\infty(0,T;L^2_a)}+\|\tau_{h(s)}N_{11}\|_{L^2(0,T;L^2_a)}\lesssim\mathbb{M}_1(T)\lesssim\|v_0\|_{L^2}\,, \tag{7.30}$$
$$\|\tau_{h(s)}N_{12}\|_{L^\infty(0,T;L^1_a)}\lesssim\mathbb{M}_1(T)^2\lesssim\|v_0\|_{L^2}^2\,, \tag{7.31}$$

$$\|\tau_{h(s)}N_{21}\|_{L^\infty(0,T;L^1_a)}\lesssim\mathbb{M}_1(T)(\mathbb{M}_1(T)+\mathbb{M}_2(T))\lesssim\|v_0\|_{L^2}^2+\mathbb{M}_2(T)^2\,. \tag{7.32}$$

Combining (7.28)–(7.32) with Young's inequality, we have

$$\|v_{22}\|_{L^\infty(0,T;L^2_a)} \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2 \,. \tag{7.33}$$

Next we will estimate $\|v_2\|_{L^2(0,T;H^1_a)}$. By (9.2) in Lemma 9.2,

$$\begin{aligned}&\|\tau_{h(s)}N_{12}\|_{L^2(0,T;L^2_a)} + \|\tau_{h(s)}N_{21}\|_{L^2(0,T;L^2_a)}\\ \lesssim&\mathbb{M}_1(T)(\mathbb{M}_1(T)+\mathbb{M}_2(T)) \lesssim \|v_0\|_{L^2}^2 + \mathbb{M}_2(T)^2 \,.\end{aligned} \tag{7.34}$$

Combining Corollary 2.3 with (7.29), (7.30) and (7.34), we have

$$\|v_{22}\|_{L^\infty(0,T;H^1_a)} \lesssim \|v_0\|_{L^2} + \mathbb{M}_2(T)^2 \,. \tag{7.35}$$

Finally, we will estimate $v_{23}(t)$. By Corollary 2.3 and (6.1),

$$\begin{aligned}\|v_{23}(t)\|_{L^2_a} \lesssim& \int_0^t e^{-b'(t-s)}(t-s)^{-1/2}\|\tau_{h(s)}N_{22}(s)\|_{L^2_a}\,ds\\ \lesssim& \int_0^t e^{-b'(t-s)}(t-s)^{-1/2}\left(\|v_1^2v_2(s)\|_{L^2_a} + \|v_2^3(s)\|_{L^2_a}\right)\,ds\,.\end{aligned} \tag{7.36}$$

Since $\|f\|_{L^4} \lesssim \|f\|_{L^2}^{3/4}\|\partial_x f\|_{L^2}^{1/4}$ and $\|v_1(t)\|_{L^2} = \|v_0\|_{L^2}$ is small, Lemma 7.3 implies $\|v_1(t)\|_{L^4} \lesssim \|v_0\|_{L^2}^{3/4}\|v_0\|_{H^1}^{1/4}$. Hence by (9.2),

$$\|v_1^2v_2\|_{L^2_a} \lesssim \|v_1\|_{L^4}^2\|v_2\|_{L^\infty_a} \lesssim \|v_0\|_{L^2}^{3/2}\|v_0\|_{H^1}^{1/2}\|v_2\|_{L^2_a}^{1/2}\|v_2\|_{H^1_a}^{1/2}\,.$$

By the definition of $\mathbb{M}_2(T)$ with $p=3$,

$$\|v_1^2v_2\|_{L^2(0,T;L^2_a)} + \|v_1^2v_2\|_{L^4(0,T;L^2_a)} \lesssim \|v_0\|_{L^2}^{3/2}\|v_0\|_{H^1}^{1/2}\mathbb{M}_2(T)\,. \tag{7.37}$$

Lemmas 7.3 and 7.4, (5.1) and (7.22) imply

$$\begin{aligned}\|v_2(t)\|_{L^2} \le& \|v(t)\|_{L^2} + \|v_1(t)\|_{L^2} \lesssim \|v_0\|_{L^2}^{1/2} + \mathbb{M}(T)\,,\\ \|v_2(t)\|_{H^1} \le& \|v(t)\|_{H^1} + \|v_1(t)\|_{H^1} \lesssim \|v_0\|_{H^1} + \mathbb{M}_2(T)^2\,.\end{aligned} \tag{7.38}$$

Combining (7.22), (7.38) and (9.1) in Section 9 with $\theta = 5/7$, we have

$$\begin{aligned}\|v_2^3\|_{L^2_a} \le&\|v_2\|_{L^2}\|v_2^2\|_{L^\infty_a}\\ \lesssim&\|v_2\|_{H^1_a}^{5/7}\|v_2\|_{L^2_a}^{2/7}\|v_2\|_{L^2}^{12/7}\|v_2\|_{H^1}^{2/7}\\ \lesssim&\|v_2\|_{H^1_a}^{5/7}\|v_2\|_{L^2_a}^{2/7}(\|v_0\|_{L^2}^{6/7}+\mathbb{M}_2(T)^{12/7})(\|v_0\|_{H^1}^{2/7}+\mathbb{M}_2(T)^{4/7})\,.\end{aligned}$$

Since $\left\|\|v_2\|_{H^1_a}^{5/7}\|v_2\|_{L^2_a}^{2/7}\right\|_{L^2(0,T)\cap L^{14/5}(0,T)} \lesssim \mathbb{M}_2(T)$,

$$\begin{aligned}&\|v_2^3\|_{L^2(0,T;L^2_a)} + \|v_2^3\|_{L^{14/5}(0,T;L^2_a)}\\ \lesssim&(\|v_0\|_{L^2}^{6/7}+\mathbb{M}_2(T)^{12/7})(\|v_0\|_{H^1}^{2/7}+\mathbb{M}_2(T)^{4/7})\mathbb{M}_2(T)\\ \lesssim&\|v_0\|_{L^2}^{6/7}\|v_0\|_{H^1}^{2/7}\left(1+\frac{\mathbb{M}_2(T)^2}{\|v_0\|_{L^2}}\right)^{6/7}\mathbb{M}_2(T)\\ &+\|v_0\|_{L^2}^{6/7}\mathbb{M}_2(T)^{11/7}+\mathbb{M}_2(T)^{23/7}\,.\end{aligned} \tag{7.39}$$

Substituting (7.37) and (7.39) into (7.36) and using Young's inequality, we have

$$\sup_{t\in[0,T]}\|v_{23}(t)\|_{L^2_a} \lesssim \left\{\eta + \|v_0\|_{H^1}^{2/7}\|v_0\|_{L^2}^{6/7}\left(\frac{\mathbb{M}_2(T)^2}{\|v_0\|_{L^2}}\right)^{6/7}\right\}\mathbb{M}_2(T)\,, \tag{7.40}$$

where $\eta = \|v_0\|_{H^1}^{1/2}\|v_0\|_{L^2}^{3/2} + \|v_0\|_{H^1}^{2/7}\|v_0\|_{L^2}^{6/7} + \|v_0\|_{L^2}^{6/7}\mathbb{M}_2(T)^{4/7} + \mathbb{M}_2(T)^{16/7}$.

On the other hand, applying Proposition 2.4 to $v_{23}(t)$, we have

$$\|v_{23}(t)\|_{L^2(0,T;H^1_a)} \lesssim \left\{\eta + \|v_0\|_{H^1}^{2/7}\|v_0\|_{L^2}^{6/7}\left(\frac{\mathbb{M}_2(T)^2}{\|v_0\|_{L^2}}\right)^{6/7}\right\}\mathbb{M}_2(T)\,. \tag{7.41}$$

Combining (7.27), (7.33), (7.35), (7.40) and (7.41), we obtain

$$\mathbb{M}_2(T) \lesssim \|v_0\|_{L^2} + \|v_0\|_{H^1}^{2/7}\|v_0\|_{L^2}^{6/7}\left(\frac{\mathbb{M}_2(T)^2}{\|v_0\|_{L^2}}\right)^{6/7}\mathbb{M}_2(T)\,,$$

whence $\mathbb{M}_2(T) \lesssim \|v_0\|_{L^2}$ if δ_5 is sufficiently small. Thus we complete the proof. □

§ 8. Proof of Theorems 1.1 and 1.2

Now we are in position to complete the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By Lemma 3.3, the decomposition (3.11) satisfying the orthogonality conditions (3.7) and (3.8) exists on $[0, T]$ for a $T > 0$. Moreover, Lemma 7.1 implies

$$\mathbb{M}_{tot}(T) \lesssim \|v_0\|_{L^2} \le \frac{1}{2}\min_{0\le i\le 5}\delta_i$$

if $\|v_0\|_{L^2}$ is sufficiently small. Hence it follows from Lemma 3.3 that the decomposition (3.11) satisfying (3.7) and (3.8) persists on $[0,\infty)$. Thus we may take $T = \infty$ in Lemma 4.1 and it follows that

$$\mathbb{M}_{tot}(\infty) \lesssim \|v_0\|_{L^2}\,, \tag{8.1}$$

$$\sup_{t\ge 0}(|c(t) - c_0| + |\dot{x}(t) - c(t)|) \lesssim \|v_0\|_{L^2} \tag{8.2}$$

and

$$\begin{aligned}\|u(t,\cdot) - \varphi_{c_0}(\cdot - x(t))\|_{L^2} \le& \|\varphi_{c(t)} - \varphi_{c_0}\|_{L^2} + \|v(t,\cdot)\|_{L^2}\\ \lesssim& |c(t) - c_0| + \|v(t,\cdot)\|_{L^2} \lesssim \|v_0\|_{L^2}^{1/2}\,.\end{aligned}$$

Thus we prove (1.4).

Next we will prove (1.5) and (1.6). By Corollary 6.4,

$$\|v_1(t)\|_W^2 \lesssim \int_{\mathbb{R}} \chi_a(y) v_1^2(t,y)\,dy \to 0 \quad \text{as } t\to\infty. \tag{8.3}$$

Integrating (4.3) with respect to t and combining the resulting equation with (8.3) and the fact that

$$\int_0^\infty (\|v_1(t)\|_W^2 + \|v_2(t)\|_{L^2_a}^2)\,dt \lesssim \mathbb{M}_1(\infty)^2 + \mathbb{M}_2(\infty)^2 \lesssim \|v_0\|_{L^2}^2\,,$$

we see that $c_+ := \lim_{t\to\infty} c(t)$ exists and

$$|c_+ - c_0| \lesssim \|v_0\|_{L^2}\,. \tag{8.4}$$

Moreover, applying the Hölder inequality to (7.6) separately on the integral intervals $[0,t/2]$ and $[t/2,t]$ and using (7.7)–(7.10), we can show that

$$\lim_{t\to\infty} \|v_2(t)\|_{L^2_a} = 0\,. \tag{8.5}$$

By (4.2), (8.3) and (8.5), we see that $\dot{x}(t) - c(t) \to 0$ as $t\to\infty$. Thus we prove (1.5). Eq. (1.6) follows from (8.2) and (8.4).

Finally, we will prove (1.7). Since $\lim_{t\to\infty} \|\varphi_{c(t)} - \varphi_{c_+}\|_{L^2} = 0$, it suffices to prove that as $t\to\infty$,

$$\|u(t,\cdot) - \varphi_{c(t)}(\cdot - x(t))\|_{L^2(x\ge\sigma t)} = \|v(t,\cdot)\|_{L^2(y\ge \sigma t - x(t))} \to 0\,. \tag{8.6}$$

Note that (8.3) and (8.5) already imply

$$\lim_{t\to\infty} \int_{\mathbb{R}} \chi_a(y) v^2(t,y) = 0\,. \tag{8.7}$$

We may write (3.2) as

$$\partial_t v + \partial_y^3 v - \dot{x}\partial_y v + \ell + \partial_y f(v) + \partial_y N_3(t) = 0\,, \tag{8.8}$$

where $N_3(t) = f(\varphi_{c(t)} + v) - f(\varphi_{c(t)}) - f(v) = 6\varphi_{c(t)} v$.

Let $c_1 \in (0,\sigma)$, $t_1 > 0$ and $y_1(t) = c_1(t-t_1) - x(t) + x(t_1)$. Multiplying (8.8) by $2\chi_a(y - y_1(t))v(t,y)$ and integrating the resulting equation by parts, we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \chi_a(y-y_1(t)) v^2(t,y)\,dy + \int_{\mathbb{R}} \chi_a'(y-y_1(t))\{3(\partial_y v)^2 + c_1 v^2 - g(v)\}(t,y)\,dy \\ &= \int_{\mathbb{R}} \chi_a'''(y-y_1(t)) v^2(t,y)\,dy + J(t)\,, \end{aligned} \tag{8.9}$$

where $g(v) = 2f(v)v - 2\int_0^v f(u)\,du$ and

$$J(t) = -2\int_{\mathbb{R}} \chi_a(y-y_1(t)) v(t,y)(\ell(t) + \partial_y N_3(t))\,dy\,.$$

Lemma 9.1 implies that

$$\int_{\mathbb{R}} \chi_a'(y-y_1(t))\{3(\partial_y v)^2 + c_1 v^2 - g(v)\}(t,y)\,dy \geq \int_{\mathbb{R}} \chi_a'''(y-y_1(t))v^2(t,y)\,dy\,. \tag{8.10}$$

if a and $\|v(t)\|_{L^2} \leq \mathbb{M}_v(\infty)$ is sufficiently small. Note that $|\chi_a'''| \leq 4a^2\chi_a'$.

By (4.18) and the fact that ℓ and N_3 are exponentially localized by $\varphi_{c(t)}$ and its derivatives, we have

$$|J(t)| \lesssim \|v_1(t)\|_{W_1}^2 + \|v_2(t)\|_{H_a^1}^2\,, \tag{8.11}$$

and $J \in L^1(0,\infty)$ by (8.1) and (8.11). Integrating (8.9) over $[t_1,t]$, we obtain

$$\int_{\mathbb{R}} \chi_a(y-y_1(t))v^2(t,y)\,dy \leq \int_{\mathbb{R}} \chi_a(y)v^2(t_1,y)\,dy + \int_{t_1}^{\infty} J(s)\,ds\,.$$

Let $t \geq t_1 \to \infty$. Then by (8.7) and the fact that $J \in L^1(0,\infty)$,

$$\lim_{t\to\infty} \int_{\mathbb{R}} \chi_a(y-y_1(t))v^2(t,y)\,dy = 0\,.$$

Since $\sigma t - x(t) \geq y_1(t)$ for t sufficiently larger than t_1, we conclude (1.7). This completes the proof of Theorem 1.1. $\square$

Since Theorem 1.2 can be shown in exactly the same way as the proof of Theorem 1.1, we omit the proof.

§ 9. Appendix: Weighted Sobolev inequalities

In this section, we recollect weighted Sobolev estimates. To prove Lemma 6.3, we use the following weighted inequality as in [31].

Lemma 9.1. *Let $p = 1$, 2 or 3 and $\varepsilon > 0$. Let $\chi_\varepsilon(x) = 1 + \tanh \varepsilon x$. Then for every $v \in H^1(\mathbb{R})$ and $x_0 \in \mathbb{R}$,*

$$\left|\int_{\mathbb{R}} \chi_\varepsilon'(x+x_0)v(x)^{p+1}\,dx\right| \leq (1+2\varepsilon)^{(p-1)/2}\|v\|_{L^2}^{p-1} \int_{\mathbb{R}} \chi_\varepsilon'(x+x_0)\left(v'(x)^2 + v(x)^2\right)\,dx\,.$$

Proof. Since the case $p = 1$ is obvious and the case $p = 2$ follows from the cases $p = 1$ and $p = 3$, we only need to prove the case $p = 3$. Since $\lim_{x\to\pm\infty} v(x) = 0$ for $v \in H^1(\mathbb{R})$,

$$\begin{aligned} \chi_\varepsilon'(x+x_0)v(x)^2 &= \int_{-\infty}^{x} \left(\chi_\varepsilon'(y+x_0)v(y)^2\right)'\,dy \\ &= \int_{-\infty}^{x} \chi_\varepsilon''(y+x_0)\,v(y)^2\,dy + 2\int_{-\infty}^{x} \chi_\varepsilon'(y+x_0)v(y)v'(y)\,dy\,. \end{aligned}$$

Using the Schwarz inequality and the fact that $0 < \chi_\varepsilon'(x) < 2\varepsilon\chi_\varepsilon(x)$ and $|\chi_\varepsilon''(x)| \le 2\varepsilon\chi_\varepsilon'(x)$ for every $x \in \mathbb{R}$, we have

$$\sup_{x\in\mathbb{R}} \chi_\varepsilon'(x+x_0)v(x)^2 \le \int_{\mathbb{R}} \chi_\varepsilon'(x+x_0)\left((1+2\varepsilon)v(x)^2 + v'(x)^2\right)\,dx\,.$$

Thus we have

$$\begin{aligned}\left|\int_{\mathbb{R}} \chi_\varepsilon'(x+x_0)v^4(x)\,dx\right| \le& \|v\|_{L^2}^2 \sup_{x\in\mathbb{R}} \chi_\varepsilon'(x+x_0)v(x)^2\\ \le& \|v\|_{L^2}^2 \int_{\mathbb{R}} \chi_\varepsilon'(x+x_0)\left((1+2\varepsilon)v(x)^2 + v'(x)^2\right)\,dx\,.\end{aligned}$$

Thus we complete the proof. □

Lemma 9.2. *Let $a > 0$. Then*

$$\|w^2\|_{L_a^\infty} \le 2\|w\|_{L^2}^\theta \|\partial_x w\|_{L^2}^{1-\theta} \|\partial_x w\|_{L_a^2}^\theta \|w\|_{L_a^2}^{1-\theta} \quad \textit{for } \theta \in [0,1], \tag{9.1}$$

$$\|w\|_{L_a^\infty}^2 \lesssim \|w\|_{L_a^2}\|w\|_{H_a^1}\,, \quad \|e^{-2a|\cdot|}w^2\|_{L^\infty} \lesssim \|w\|_W \|w\|_{W_1}\,. \tag{9.2}$$

Proof of Lemma 9.2. It suffices to prove Lemma 9.2 for $w \in C_0^\infty(\mathbb{R})$. Since e^{ax} is monotone increasing,

$$e^{ax}w^2(x) = -2e^{ax}\int_x^\infty w(y)w'(y)\,dy \le 2\int_x^\infty e^{ay}|w(y)||w'(y)|\,dy\,.$$

By the Schwarz inequality, we have

$$e^{ax}w^2(x) \le 2\|w\|_{L^2}\|w'\|_{L_a^2}\,, \quad e^{ax}w^2(x) \le 2\|w\|_{L_a^2}\|w'\|_{L^2} \quad \text{for any } x \in \mathbb{R}.$$

Interpolating the above inequalities, we have (9.1). We can prove (9.2) in the similar way. □

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