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# Analytic smoothing effect for nonlinear Schrödinger equations with quadratic interaction

By

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**Abstract**

We prove the global existence of analytic solutions to a system of nonlinear Schrödinger equations with quadratic interaction in space dimensions $n \geq 3$, under the mass resonance condition $M = 2m$ for sufficiently small Cauchy data with exponential decay. For $n \geq 4$ the smallness assumption on the data is imposed in terms of the critical Sobolev space $\dot{H}^{n/2-2}$.

## § 1. Introduction

We consider the Cauchy problem for a system of nonlinear Schrödinger equations of the form

$$(1.1) \qquad \begin{cases} i\partial_t u + \frac{1}{2m}\Delta u = \lambda v\overline{u}, \\ i\partial_t v + \frac{1}{2M}\Delta v = \mu u^2 \end{cases}$$

in space dimension $n \geq 3$, under the mass resonance condition $M = 2m$, where $u, v$ are complex valued functions of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\Delta$ is the Laplacian in $\mathbb{R}^n$, and $\lambda, \mu \in \mathbb{C}$.

The Cauchy problem for (1.1) has been studied in [11, 12, 13, 16, 20]. Particularly, Lagrangian formalism was described in [20], the existence of ground states was proved in [16], and analytic solutions were studied in [20] in terms of the generator of Galilei transforms. In this paper, we give another framework of analytic smoothing effect for (1.1).

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To state our main result precisely, we introduce the following notation. For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ is the usual Lebesgue space. For $s \in \mathbb{R}$, $H_p^s = (1-\Delta)^{-s/2}L^p$ and $\dot{H}_p^s = (-\Delta)^{-s/2}L^p$ are the Sobolev space and the homogeneous Sobolev space, respectively. $\dot{L}_s^2 = |x|^{-s}L^2$ is the weighted Lebesgue space. For $1 \le p, q \le \infty$, $\dot{B}_{p,q}^s = \dot{B}_{p,q}^s(\mathbb{R}^n)$ is the homogeneous Besov space and $L^{p,q} = L^{p,q}(\mathbb{R})$ is the Lorentz space with second exponent $q$. We put $\dot{B}_{p,2}^s = \dot{B}_p^s$. For $t \in \mathbb{R}$, $U_m(t) = e^{i\frac{t}{2m}\Delta}$ is the free propagator with mass $m$ and $J_m(t) = x + i\frac{t}{m}\nabla = U_m(t)xU_m(-t)$ is the generator of Galilei transforms. For any $t \neq 0$, $J_m(t)$ is represented as $J_m(t) = M_m(t)i\frac{t}{m}\nabla M_m(-t)$, where $M_m(t) = e^{i\frac{m}{2t}|x|^2}$. $\mathcal{F}$ denotes the Fourier transform defined by $(\mathcal{F}f)(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}f(x)dx$. For $k > 0$, $\delta \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we define the following operator [9, 15, 21, 27]:

$$P_{k\delta,m}(t) = U_m(t)e^{k\delta\cdot x}U_m(-t),$$

where $\cdot$ is the usual scalar product in $\mathbb{R}^n$. For any $t \neq 0$, $P_{k\delta,m}(t)$ is represented as

$$P_{k\delta,m}(t) = M_m(t)\mathcal{F}^{-1}e^{-t\frac{k\delta}{m}\cdot\xi}\mathcal{F}M_m(-t).$$

Let $\mathcal{B}$ be a suitable Banach space consisting of functions on space-time $\mathbb{R}\times\mathbb{R}^n$. Let $D$ be an open subset of $\mathbb{R}^n$. We introduce the following function spaces:

$$G_k^D(x;\dot{H}_2^s) = \Big\{\phi \in \dot{H}_2^s \ ; \ \|\phi; G_k^D(x;\dot{H}_2^s)\| \equiv \sup_{\delta\in D}\|e^{k\delta\cdot x}\phi;\dot{H}_2^s\| < \infty\Big\},$$
$$G_k^D(x;L^2\cap\dot{L}_s^2) = \Big\{\phi \in L^2\cap\dot{L}_s^2 \ ; \ \|\phi; G_k^D(x;L^2\cap\dot{L}_s^2)\| \equiv \sup_{\delta\in D}\|e^{k\delta\cdot x}\phi;L^2\cap\dot{L}_s^2\| < \infty\Big\},$$
$$G_k^D(J_m;\mathcal{B}) = \Big\{u \in \mathcal{B} \ ; \ \|u; G_k^D(J_m;\mathcal{B})\| \equiv \sup_{\delta\in D}\|P_{k\delta,m}u;\mathcal{B}\| < \infty\Big\}.$$

Our main purpose in this paper is twofold. One is to study analytic smoothing effect for nonlinear Schrödinger equations in terms of Hardy spaces of analytic functions on tube domains, which provides a wider framework than [21]. The other is to apply this framework to (1.1) and to improve the result in [20]. The space $G_k^D(J_m;\mathcal{B})$ is characterized by convergence of power series which respect to $k\delta\cdot J_m$ in $\mathcal{B}$ ([21]). The Cauchy problem for nonlinear Schrödinger equations in the function space in terms of absolutely convergence of power series with respect to $k\delta\cdot J_m$ is studied by ([19, 20, 33, 34]).

For $n \ge 4$ we introduce the following function spaces:

$$\mathcal{X}_0 = L^\infty(\mathbb{R};L^2)\cap L^2(\mathbb{R};L^{2^*}),$$
$$\dot{\mathcal{X}} = L^\infty(\mathbb{R};\dot{H}_2^{n/2-2})\cap L^2(\mathbb{R};\dot{B}_{2^*}^{n/2-2})$$

with associated norm

$$\begin{aligned}\|u; \mathcal{X}_0\| &= \|u; L^\infty(\mathbb{R}; L^2)\| + \|u; L^2(\mathbb{R}; L^{2^*})\|,\\ \|u; \dot{\mathcal{X}}\| &= \|u; L^\infty(\mathbb{R}; \dot{H}_2^{n/2-2})\| + \|u; L^2(\mathbb{R}; \dot{B}_{2^*}^{n/2-2})\|,\end{aligned}$$

where $2^* = 2n/(n-2)$ is the Sobolev critical exponent. For $n = 3$ we introduce the following function space:

$$\mathcal{Y}_0 = L^\infty(\mathbb{R}; L^2) \cap L^{4,2}(\mathbb{R}; L^3)$$

with associated norm

$$\|u; \mathcal{Y}_0\| = \|u; L^\infty(\mathbb{R}; L^2)\| + \|u; L^{4,2}(\mathbb{R}; L^3)\|.$$

We define $|J_m|^s$ by

$$|J_m|^s(t) = U_m(t)|x|^s U_m(-t)$$

for $t \in \mathbb{R}$. For any $t \neq 0$, $|J_m|^s(t)$ is represented as [14]

$$|J_m|^s(t) = M_m(t)\Big(-\frac{t^2}{m^2}\Delta\Big)^{s/2} M_m(-t).$$

We put

$$\dot{\mathcal{Y}}_m = |J_m|^{-1/2}\mathcal{Y}_0.$$

We treat (1.1) in the following function spaces:

$$\begin{aligned}G_k^D(J_m) &= \mathcal{X}_0 \cap \dot{G}_k^D(J_m) && \text{for } n \geq 4,\\ G_k^D(J_m) &= G_{k,0}^D(J_m) \cap \dot{G}_k^D(J_m) && \text{for } n = 3,\end{aligned}$$

where

$$\begin{aligned}&\dot{G}_k^D(J_m) = G_k^D(J_m; \dot{H}_2^{n/2-2}) && \text{for } n \geq 4,\\ &G_{k,0}^D(J_m) = G_k^D(J_m; \mathcal{Y}_0),\ \dot{G}_k^D(J_m) = G_k^D(J_m; \dot{\mathcal{Y}}_m) && \text{for } n = 3.\end{aligned}$$

We define $B_k^D(\rho)$ as follows

$$\begin{aligned}B_k^D(\rho) &= \Big\{\phi \in G_k^D(x; \dot{H}_2^{n/2-2})\ ;\ \sup_{\delta\in D}\|e^{\delta\cdot x}\phi; \dot{H}_2^{n/2-2}\| \leq \rho\Big\} && \text{for } n \geq 4,\\ B_k^D(\rho) &= \Big\{\phi \in G_k^D(x; L^2 \cap \dot{L}_{1/2}^2)\ ;\ \sup_{\delta\in D}\|e^{\delta\cdot x}\phi; \dot{L}_{1/2}^2\| \leq \rho\Big\} && \text{for } n = 3.\end{aligned}$$

We consider the following integral equations of (1.1) with the Cauchy data $(\phi, \psi)$ at $t = 0$ :

$$\begin{cases} u(t) = U_m(t)\phi - i\int_0^t U_m(t-s)\lambda\big(v\overline{u}\big)(s)ds, \\ v(t) = U_{2m}(t)\psi - i\int_0^t U_{2m}(t-s)\mu u^2(s)ds. \end{cases}$$

We state our main result.

**Theorem 1.1.** *Let $n \geq 3$. There exists $\rho > 0$ such that for any domain $D \subset \mathbb{R}^n$ satisfying $0 \in D$, $-D = D$ and any $(\phi, \psi) \in B_1^D(\rho) \times B_2^D(\rho)$ (1.1) has a unique pair of solutions $(u, v) \in G_1^D(J_m) \times G_2^D(J_{2m})$.*

*Remark* 1. Under the assumptions of Theorem 1.1, if $\phi \in \dot{H}_2^{n/2-2}$ satisfies $\sup_{\delta \in D} \|e^{k\delta \cdot x}\phi; \dot{H}_2^{n/2-2}\| < \infty$, then $\phi \in L^1 \cap L^{n/2}$ ([21, 40]).

*Remark* 2. Under the assumptions of Theorem 1.1, let $u \in G_k^D(J_{lm})$, $l > 0$. Then for any $t \neq 0$, $u(t)$ is real analytic and has an analytic continuation to $\mathbb{R}^n + i\frac{t}{lm}kD$ ([17, 18, 21, 27, 35, 40]).

## § 2. Preliminaries

In this section we collect some basic lemmas. We define the following integral operator

$$\big(\Xi_m f\big)(t) = \int_0^t U_m(t-s)f(s)ds.$$

For $n \geq 3$, we say that $(q, r)$ is an admissible pair if

$$\frac{2}{q} = n\Big(\frac{1}{2} - \frac{1}{r}\Big),$$

where $r \in [2, \frac{2n}{n-2}]$.

**Lemma 2.1.** *([3, 22, 38, 42, 44]) Let $n \geq 1$, $s \in \mathbb{R}$ and let $(q, r)$ and $(q_j, r_j)$ be admissible pairs, $j = 1, 2$. Then the following estimates hold:*

$$\begin{aligned} &\|U_m(\cdot)\phi; L^q(\mathbb{R}; L^r)\| \leq C\|\phi; L^2\|, \\ &\|\Xi_m f; L^{q_2}(\mathbb{R}; L^{r_2})\| \leq C\|f; L^{q_1'}(\mathbb{R}; L^{r_1'})\|, \\ &\|U_m(\cdot)\phi; L^q(\mathbb{R}; \dot{B}_r^s)\| \leq C\|\phi; \dot{H}_2^s\|, \\ &\|\Xi_m f; L^{q_2}(\mathbb{R}; \dot{B}_{r_2}^s)\| \leq C\|f; L^{q_1'}(\mathbb{R}; \dot{B}_{r_1'}^s)\|, \end{aligned}$$

*where $p'$ is the dual exponent to $p$ defined by $1/p + 1/p' = 1$.*

**Lemma 2.2.** *([30, 31]) Let $n \geq 1$ and let $(q,r)$ and $(q_j, r_j)$ be admissible pairs, $j = 1, 2$. Then the following estimates hold:*

$$\|U_m(\cdot)\phi; L^{q,2}(\mathbb{R}; L^r)\| \leq C\|\phi; L^2\|,$$
$$\|\Xi_m f; L^\infty(\mathbb{R}; L^2)\| + \|\Xi_m f; L^{q_2,2}(\mathbb{R}; L^{r_2})\| \leq C\|f; L^{q_1',2}(\mathbb{R}; L^{r_1'})\|.$$

**Lemma 2.3.** *([2, 26]) Let $n \geq 1$. Let $1 \leq p, q \leq \infty, 1/p_1 + 1/p_2 = 1/p$ and $s > 0$. Then there exists $C > 0$ such that the following estimate*

$$\|uv; \dot{B}^s_{p,q}\| \leq C\Big(\|u; \dot{B}^s_{p_1,q}\|\|v; L^{p_2}\| + \|u; L^{p_2}\|\|v; \dot{B}^s_{p_1,q}\|\Big)$$

*holds.*

**Lemma 2.4.** *([13]) Let $n = 3$. Then there exists $C > 0$ such that the following estimates hold:*

$$\|uv; L^{3/2}\| \leq C|t|^{-1/2}\Big\||J_m|^{1/2}(t)u; L^2\Big\|\|v; L^3\|,$$
$$\Big\||J_m|^{1/2}(t)\big(v\overline{u}\big); L^{3/2}\Big\| \leq C|t|^{-1/2}\Big(\Big\||J_m|^{1/2}(t)u; L^2\Big\|\Big\||J_{2m}|^{1/2}(t)v; L^3\Big\|$$
$$+ \Big\||J_m|^{1/2}(t)u; L^3\Big\|\Big\||J_{2m}|^{1/2}(t)v; L^2\Big\|\Big),$$
$$\Big\||J_{2m}|^{1/2}(t)u^2; L^{3/2}\Big\| \leq C|t|^{-1/2}\Big\||J_m|^{1/2}(t)u; L^2\Big\|\Big\||J_m|^{1/2}(t)u; L^3\Big\|$$

*for all $t \neq 0$ and $m > 0$.*

## § 3. Proof of Theorem 1.1.

Let $(\phi, \psi) \in B_1^D(\rho) \times B_2^D(\rho)$. We consider the mapping $(\Phi, \Psi) : (u, v) \longmapsto \big(\Phi(u,v), \Psi(u,v)\big)$ defined by

$$\begin{cases} \Phi(u,v)(t) = U_m(t)\phi - i\lambda\Xi_m\big(v\overline{u}\big)(t), \\ \Psi(u,v)(t) = U_{2m}(t)\psi - i\mu\Xi_{2m}\big(u^2\big)(t) \end{cases}$$

on $G_1^D(J_m) \times G_2^D(J_{2m})$. For $R, \varepsilon > 0$ we define the metric spaces:

$$X(R,\varepsilon) = \Big\{(u,v) \in G_1^D(J_m) \times G_2^D(J_{2m}) \ ;$$
$$\|(u,v); \mathcal{X}_0 \times \mathcal{X}_0\| \leq R, \|(u,v); \dot{G}_1^D(J_m) \times \dot{G}_2^D(J_{2m})\| \leq \varepsilon\Big\} \quad \text{for} \quad n \geq 4,$$

$$X(R,\varepsilon) = \Big\{(u,v) \in G_1^D(J_m) \times G_2^D(J_{2m}) \ ;$$
$$\|(u,v); G_{1,0}^D(J_m) \times G_{2,0}^D(J_{2m})\| \leq R, \|(u,v); \dot{G}_1^D(J_m) \times \dot{G}_2^D(J_{2m})\| \leq \varepsilon\Big\} \quad \text{for} \quad n = 3$$

with metric

$$d\big((u,v),(u',v')\big) = \|(u-u',v-v'); G_1^D(J_m)\times G_2^D(J_{2m})\|.$$

We see that $\big(X(R,\varepsilon),d\big)$ is a complete metric space.
Let $(u,v)\in G_1^D(J_m)\times G_2^D(J_{2m})$. We have

$$\begin{aligned}
P_{\delta,m}\Phi(u,v) &= U_m e^{\delta\cdot x}U_m^{-1}\Phi(u,v)\\
&= U_m e^{\delta\cdot x}\phi - i\lambda \Xi_m\Big(U_m e^{\delta\cdot x}U_m^{-1}(v\overline{u})\Big)\\
&= U_m e^{\delta\cdot x}\phi - i\lambda \Xi_m\Big((U_{2m}e^{2\delta\cdot x}U_{2m}^{-1}v)\overline{(U_m e^{-\delta\cdot x}U_m^{-1}u)}\Big)\\
&= U_m e^{\delta\cdot x}\phi - i\lambda \Xi_m\Big((P_{2\delta,2m}v)(\overline{P_{-\delta,m}u})\Big),\\
P_{2\delta,2m}\Psi(u,v) &= U_{2m}e^{2\delta\cdot x}U_{2m}^{-1}\Psi(u,v)\\
&= U_{2m}e^{2\delta\cdot x}\psi - i\mu \Xi_{2m}\Big(U_{2m}e^{2\delta\cdot x}U_{2m}^{-1}u^2\Big)\\
&= U_{2m}e^{2\delta\cdot x}\psi - i\mu \Xi_{2m}\Big((U_m e^{\delta\cdot x}U_m u)^2\Big)\\
&= U_{2m}e^{2\delta\cdot x}\psi - i\mu \Xi_{2m}\Big((P_{\delta,m}u)^2\Big).
\end{aligned}$$

For $n\geq 4$, by Lemmas 2.1 and 2.4, the Hölder's inequality with $1/2 = 1/2^* + 1/n$, and the Sobolev embeddings $\dot B_{2^*}^{n/2-2}\subset \dot H_{2^*}^{n/2-2}\subset L^n$, we estimate

$$\begin{aligned}
\|\Phi(u,v);\mathcal{X}_0\| &\leq C\|\phi;L^2\| + C\|v\overline{u};L^1(\mathbb{R};L^2)\|\\
&\leq C\|\phi;L^2\| + C\|v;L^2(\mathbb{R};L^n)\|\|u;L^2(\mathbb{R};L^{2^*})\|\\
&\leq C\|\phi;L^2\| + C\|v;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|\|u;L^2(\mathbb{R};L^{2^*})\|,\\
\|\Phi(u,v);\dot{\mathcal{X}}\| &\leq C\|\phi;\dot H_2^{n/2-2}\| + C\|v\overline{u};L^1(\mathbb{R};\dot B_2^{n/2-2})\|\\
&\leq C\|\phi;\dot H_2^{n/2-2}\| + C\Big(\|v;L^2(\mathbb{R};L^n)\|\|u;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|\\
&\qquad\qquad + \|u;L^2(\mathbb{R};L^n)\|\|v;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|\Big)\\
&\leq C\|\phi;\dot H_2^{n/2-2}\| + C\|v;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|\|u;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|,\\
\|\Psi(u,v);\mathcal{X}_0\| &\leq C\|\psi;L^2\| + C\|u;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|\|u;L^2(\mathbb{R};L^{2^*})\|,\\
\|\Psi(u,v);\dot{\mathcal{X}}\| &\leq C\|\psi;\dot H_2^{n/2-2}\| + C\|u;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|\|u;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|.
\end{aligned}$$

Similarly, we estimate $P_{\delta,m}\Phi(u,v)$ and $P_{2\delta,2m}\Psi(u,v)$ in $\dot{\mathcal{X}}$

$$\|P_{\delta,m}\Phi(u,v);\dot{\mathcal{X}}\| \leq C\|e^{\delta\cdot x}\phi;\dot H_2^{n/2-2}\| + C\|P_{2\delta,2m}v;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|\|P_{-\delta,m}u;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|,$$

$$\|P_{2\delta,2m}\Psi(u,v);\dot{\mathcal{X}}\| \leq C\|e^{2\delta\cdot x}\psi;\dot H_2^{n/2-2}\| + C\|P_{\delta,m}u;L^2(\mathbb{R};\dot B_{2^*}^{n/2-2})\|^2,$$

where $C$ is independent of $\delta$. We take supremum over $\delta \in D$ on both sides of the above inequalities to obtain

$$\sup_{\delta\in D}\|P_{\delta,m}\Phi(u,v);\dot{\mathcal{X}}\| \le C\sup_{\delta\in D}\|e^{\delta\cdot x}\phi;\dot H_2^{n/2-2}\| + C\sup_{\delta\in D}\|P_{2\delta,2m}v;\dot{\mathcal{X}}\|\sup_{\delta\in D}\|P_{-\delta,m}u;\dot{\mathcal{X}}\|,$$
$$\sup_{\delta\in D}\|P_{2\delta,2m}\Psi(u,v);\dot{\mathcal{X}}\| \le C\sup_{\delta\in D}\|e^{2\delta\cdot x}\psi;\dot H_2^{n/2-2}\| + C\Big(\sup_{\delta\in D}\|P_{\delta,m}u;\dot{\mathcal{X}}\|\Big)^2.$$

For $(u,v),(u',v') \in G_1^D(J_m)\times G_2^D(J_{2m})$, similarly we estimate

$$\|\Phi(u,v)-\Phi(u',v');\mathcal{X}_0\| \le C\big(\|u;\dot{\mathcal{X}}\|+\|v';\dot{\mathcal{X}}\|\big)\Big(\|u-u';\mathcal{X}_0\|+\|v-v';\mathcal{X}_0\|\Big),$$
$$\|\Psi(u,v)-\Psi(u',v');\mathcal{X}_0\| \le C\big(\|u;\dot{\mathcal{X}}\|+\|u';\dot{\mathcal{X}}\|\big)\Big(\|u-u';\mathcal{X}_0\|+\|v-v';\mathcal{X}_0\|\Big),$$
$$\|\Phi(u,v)-\Phi(u',v');\dot G_1^D(J_m)\|$$
$$\le C\big(\|u;\dot G_1^D(J_m)\|+\|v';\dot G_2^D(J_{2m})\|\big)\Big(\|u-u';\dot G_1^D(J_m)\|+\|v-v';\dot G_2^D(J_{2m})\|\Big),$$
$$\|\Psi(u,v)-\Psi(u',v');\dot G_2^D(J_{2m})\|$$
$$\le C\big(\|u;\dot G_1^D(J_m)\|+\|u';\dot G_1^D(J_m)\|\big)\Big(\|u-u';\dot G_1^D(J_m)\|+\|v-v';\dot G_2^D(J_{2m})\|\Big).$$

For $n=3$, by Lemmas 2.2 and 2.4 and the Hölder inequality on Lorentz spaces with two exponents $3/4 = 1/2+1/\infty+1/4$, $1/2 = 1/\infty+1/\infty+1/2$, we estimate

$$\begin{aligned}\|\Phi(u,v);\mathcal{Y}_0\| &\le C\|\phi;L^2\| + C\|v\overline{u};L^{4/3,2}(\mathbb{R};L^{3/2})\|\\
&\le C\|\phi;L^2\| + C\Big\||t|^{-1/2}\Big\||J_{2m}|^{1/2}(t)v(t);L^2\Big\|\|u(t);L^3\|;L_t^{4/3,2}\Big\|\\
&\le C\|\phi;L^2\| + C\||t|^{-1/2};L_t^{2,\infty}\|\Big\||J_{2m}|^{1/2}v;L^\infty(\mathbb{R};L^2)\Big\|\|u;L^{4,2}(\mathbb{R};L^3)\|\\
&\le C\|\phi;L^2\| + C\Big\||J_{2m}|^{1/2}v;L^\infty(\mathbb{R};L^2)\Big\|\|u;L^{4,2}(\mathbb{R};L^3)\|\\
&\le C\|\phi;L^2\| + C\|v;\dot{\mathcal{Y}}_{2m}\|\|u;\mathcal{Y}_0\|,\end{aligned}$$

$$\begin{aligned}\|\Phi(u,v);\dot{\mathcal{Y}}_m\| &\le C\|\phi;\dot L^2_{1/2}\| + C\||J_m|^{1/2}v\overline{u};L^{4/3,2}(\mathbb{R};L^{3/2})\|\\
&\le C\|\phi;\dot L^{1/2}_{1/2}\| + C\Big\||t|^{-1/2}\Big\||J_m|^{1/2}(t)u(t);L^2\Big\|\Big\||J_{2m}|^{1/2}(t)v(t);L^3\Big\|\\
&\qquad + |t|^{-1/2}\Big\||J_m|^{1/2}(t)u(t);L^3\Big\|\Big\||J_{2m}|^{1/2}(t)v(t);L^2\Big\|;L_t^{4/3,2}\Big\|\\
&\le C\|\phi;\dot L^2_{1/2}\| + C\Big(\Big\||J_m|^{1/2}u;L^\infty(\mathbb{R};L^2)\Big\|\Big\||J_{2m}|^{1/2}v;L^{4,2}(\mathbb{R};L^3)\Big\|\\
&\qquad + \Big\||J_m|^{1/2}u;L^{4,2}(\mathbb{R};L^3)\Big\|\Big\||J_{2m}|^{1/2}v;L^\infty(\mathbb{R};L^2)\Big\|\Big)\\
&\le C\|\phi;\dot L^2_{1/2}\| + C\|v;\dot{\mathcal{Y}}_{2m}\|\|u;\dot{\mathcal{Y}}_m\|,\end{aligned}$$

and we also have

$$\|\Psi(u,v);\mathcal{Y}_0\| \le C\|\psi;L^2\| + C\|u;\dot{\mathcal{Y}}_m\|\|u;\mathcal{Y}_0\|,$$
$$\|\Psi(u,v);\dot{\mathcal{Y}}_{2m}\| \le C\|\psi;\dot L^2_{1/2}\| + C\|u;\dot{\mathcal{Y}}_m\|^2.$$

Similarly, we estimate $P_{\delta,m}\Phi(u,v)$ and $P_{2\delta,2m}\Psi(u,v)$ in $\mathcal{Y}_0$, $\dot{\mathcal{Y}}_m$ and $\dot{\mathcal{Y}}_{2m}$ as follows

$$\|P_{\delta,m}\Phi(u,v);\mathcal{Y}_0\| \le C\|e^{\delta\cdot x}\phi;L^2\| + C\|P_{2\delta,2m}v;\dot{\mathcal{Y}}_{2m}\|\|P_{-\delta,m}u;\mathcal{Y}_0\|,$$
$$\|P_{2\delta,2m}\Psi(u,v);\mathcal{Y}_0\| \le C\|e^{2\delta\cdot x}\psi;L^2\| + C\|P_{\delta,m}u;\dot{\mathcal{Y}}_m\|\|P_{\delta,m}u;\mathcal{Y}_0\|,$$

and

$$\|P_{\delta,m}\Phi(u,v);\dot{\mathcal{Y}}_m\| \le C\|e^{\delta\cdot x}\phi;\dot{L}^2_{1/2}\| + C\|P_{2\delta,2m}v;\dot{\mathcal{Y}}_{2m}\|\|P_{-\delta,m}u;\dot{\mathcal{Y}}_m\|,$$
$$\|P_{2\delta,2m}\Psi(u,v);\dot{\mathcal{Y}}_{2m}\| \le C\|e^{2\delta\cdot x}\psi;\dot{L}^2_{1/2}\| + C\|P_{\delta,m}u;\dot{\mathcal{Y}}_m\|^2,$$

where $C$ is independent of $\delta$. We take supremum over $\delta \in D$ on both sides of the above inequalities to obtain

$$\sup_{\delta\in D}\|P_{\delta,m}\Phi(u,v);\mathcal{Y}_0\| \le C\sup_{\delta\in D}\|e^{\delta\cdot x}\phi;L^2\| + C\sup_{\delta\in D}\|P_{2\delta,2m}v;\dot{\mathcal{Y}}_{2m}\|\sup_{\delta\in D}\|P_{-\delta,m}u;\mathcal{Y}_0\|,$$
$$\sup_{\delta\in D}\|P_{2\delta,2m}\Psi(u,v);\mathcal{Y}_0\| \le C\sup_{\delta\in D}\|e^{2\delta\cdot x}\psi;L^2\| + C\sup_{\delta\in D}\|P_{\delta,m}u;\dot{\mathcal{Y}}_m\|\sup_{\delta\in D}\|P_{\delta,m}u;\mathcal{Y}_0\|,$$
$$\sup_{\delta\in D}\|P_{\delta,m}\Phi(u,v);\dot{\mathcal{Y}}_m\| \le C\sup_{\delta\in D}\|e^{\delta\cdot x}\phi;\dot{L}^2_{1/2}\| + C\sup_{\delta\in D}\|P_{2\delta,2m}v;\dot{\mathcal{Y}}_{2m}\|\sup_{\delta\in D}\|P_{-\delta,m}u;\dot{\mathcal{Y}}_m\|,$$
$$\sup_{\delta\in D}\|P_{2\delta,2m}\Psi(u,v);\dot{\mathcal{Y}}_{2m}\| \le C\sup_{\delta\in D}\|e^{2\delta\cdot x}\psi;\dot{L}^2_{1/2}\| + C\Big(\sup_{\delta\in D}\|P_{\delta,m}u;\dot{\mathcal{Y}}_m\|\Big)^2.$$

For $(u,v),(u',v') \in G_1^D(J_m)\times G_2^D(J_{2m})$, similarly we estimate

$$\begin{aligned}
&\|\Phi(u,v)-\Phi(u',v');G^D_{1,0}(J_m)\|\\
&\le C\big(\|u;\dot{G}^D_m(J_m)\| + \|v';\dot{G}^D_{2m}(J_{2m})\|\big)\Big(\|u-u';G^D_{1,0}(J_m)\| + \|v-v';G^D_{2,0}(J_{2m})\|\Big),\\
&\|\Psi(u,v)-\Psi(u',v');G^D_{2,0}(J_{2m})\|\\
&\le C\big(\|u;\dot{G}^D_m(J_m)\| + \|u';\dot{G}^D_m(J_m)\|\big)\Big(\|u-u';G^D_{1,0}(J_m)\| + \|v-v';G^D_{2,0}(J_{2m})\|\Big),\\
&\|\Phi(u,v)-\Phi(u',v');\dot{G}^D_1(J_m)\|\\
&\le C\big(\|u;\dot{G}^D_1(J_m)\| + \|v';\dot{G}^D_2(J_{2m})\|\big)\Big(\|u-u';\dot{G}^D_1(J_m)\| + \|v-v';\dot{G}^D_2(J_{2m})\|\Big),\\
&\|\Psi(u,v)-\Psi(u',v');\dot{G}^D_2(J_{2m})\|\\
&\le C\big(\|u;\dot{G}^D_1(J_m)\| + \|u';\dot{G}^D_1(J_m)\|\big)\Big(\|u-u';\dot{G}^D_1(J_m)\| + \|v-v';\dot{G}^D_2(J_{2m})\|\Big).
\end{aligned}$$

Therefore for any $(u,v),(u',v') \in X(R,\varepsilon)$ with $n \ge 4$, we have

$$\begin{aligned}
&\Big\|\Big(\Phi(u,v),\Psi(u,v)\Big);\mathcal{X}_0\times\mathcal{X}_0\Big\| \le C\|(\psi,\phi);L^2\times L^2\| + CR\varepsilon,\\
&\Big\|\Big(\Phi(u,v),\Psi(u,v)\Big);\dot{G}^D_1(J_m)\times\dot{G}^D_2(J_{2m})\Big\| \le C\rho + CR\varepsilon,\\
&\Big\|\Big(\Phi(u,v)-\Phi(u',v'),\Psi(u,v)-\Psi(u',v')\Big);\mathcal{X}_0\times\mathcal{X}_0\Big\| \le C\varepsilon\|(u-u',v-v');\mathcal{X}_0\times\mathcal{X}_0\|,\\
&\Big\|\Big(\Phi(u,v)-\Phi(u',v'),\Psi(u,v)-\Psi(u',v')\Big);\dot{G}^D_1(J_m)\times\dot{G}^D_2(J_{2m})\Big\|\\
&\le C\varepsilon\|(u-u',v-v');\dot{G}^D_1(J_m)\times\dot{G}^D_2(J_{2m})\|,
\end{aligned}$$

and with $n = 3$, we have

$$
\begin{aligned}
&\left\|\Big(\varPhi(u,v),\varPsi(u,v)\Big);G^D_{1,0}(J_m)\times G^D_{2,0}(J_{2m})\right\| \le C\|(\phi,\psi);G^D_1(x;L^2)\times G^D_2(x;L^2)\| + CR\varepsilon,\\
&\left\|\Big(\varPhi(u,v),\varPsi(u,v)\Big);\dot G^D_1(J_m)\times \dot G^D_2(J_{2m})\right\| \le C\rho + CR\varepsilon,\\
&\left\|\Big(\varPhi(u,v)-\varPhi(u',v'),\varPsi(u,v)-\varPsi(u',v')\Big);G^D_{1,0}(J_m)\times G^D_{2,0}(J_{2m})\right\|\\
&\le C\varepsilon\|(u-u',v-v');G^D_{1,0}(J_m)\times G^D_{2,0}(J_{2m})\|,\\
&\left\|\Big(\varPhi(u,v)-\varPhi(u',v'),\varPsi(u,v)-\varPsi(u',v')\Big);\dot G^D_1(J_m)\times \dot G^D_2(J_{2m})\right\|\\
&\le C\varepsilon\|(u-u',v-v');\dot G^D_1(J_m)\times \dot G^D_2(J_{2m})\|.
\end{aligned}
$$

Therefore for any $(\phi,\psi)\in B^D_1(\rho)\times B^D_2(\rho)$ with $R,\varepsilon$ and $\rho$ satisfying

$$
\begin{cases}
C\|(\phi,\psi);L^2\times L^2\| + CR\varepsilon \le R,\\
C\rho + CR\varepsilon \le \varepsilon,\\
C\varepsilon < 1,
\end{cases}
$$

for $n \ge 4$

$$
\begin{cases}
C\|(\phi,\psi);G^D_1(x;L^2)\times G^D_2(x;L^2)\| + CR\varepsilon \le R,\\
C\rho + CR\varepsilon \le \varepsilon,\\
C\varepsilon < 1,
\end{cases}
$$

for $n = 3$, the mapping $(\varPhi,\varPsi):(u,v)\longmapsto(\varPhi(u,v),\varPsi(u,v))$ is a contraction in $G^D_1(J_m)\times G^D_2(J_{2m})$ and has a unique fixed point in $G^D_1(J_m)\times G^D_2(J_{2m})$. This completes the proof of Theorem 1.1.

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