WEAK CONVERGENCE OF h-TRANSFORMS FOR ONE-DIMENSIONAL DIFFUSIONS

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Abstract. It is proved that the h-transform of the killed process for one-dimensional diffusions with respect to the scale function is weakly continuous with respect to the starting point. A similar result is obtained for its bridges.

1. Introduction

Let $M = \{X, (P_x)_{x\geq 0}\}$ be the canonical representation of a one-dimensional diffusion process, where we assume that 0 is an instantaneously reflecting boundary as well as a regular recurrent state. We write S for the scale function of M, with $S(0) = 0$ and $S(\infty) = \infty$, m for the speed measure and $\mathcal{G} = \frac{d}{dm}\frac{d}{dS}$ for the canonical form of the infinitesimal generator of M. Following [1], we adopt the canonical path space as the set of real-valued continuous paths w which are defined on $[0, \zeta(w)]$ with lifetime $\zeta(w) \in (0,\infty)$ or on $[0,\infty)$ with $\zeta(w) = \infty$. Let $X = (X_t, t \geq 0)$ denote the coordinate process, $(\mathcal{F}_t, t \geq 0)$ the natural filtration and $H_x = \inf\{t : X_t = x\}$. Let $\mathbf{n}$ denote the Itô measure of M normalized by $\mathbf{n}(H_x < \zeta) = 1/S(x)$ for all $x > 0$.

We may define $M^\uparrow = \{X, (P^\uparrow_x)_{x\geq 0}\}$ as follows:

$$P^\uparrow_x(A; t < \zeta) = \frac{E_x[1_A S(X_t); t < H_0]}{S(x)} \quad (x > 0) \tag{1.1}$$

and

$$P^\uparrow_0(A; t < \zeta) = \mathbf{n}[1_A S(X_t); t < \zeta] \tag{1.2}$$

for $A \in \mathcal{F}_t$ and $t > 0$. (Note that $M^\uparrow$ is *not* conservative when $m(\infty) < \infty$; see [10].) The process $M^\uparrow = \{X, (P^\uparrow_x)_{x\geq 0}\}$ appears in certain representations of the Itô measure and in conditioning to avoid zero; see [8], [7], [9] and [10]. A natural question arises: does the following weak convergence hold?

$$P^\uparrow_x \xrightarrow{\mathrm{w}} P^\uparrow_0 \quad \text{as } x \downarrow 0. \tag{1.3}$$

We also study the $M^\uparrow$-bridges. We denote by $P^{\uparrow,u}_{x,y}$ the law of the $M^\uparrow$-bridge of duration u, starting at x and ending at y, which may be defined via h-transforms (see [5] and [2]) as follows:

$$P^{\uparrow,u}_{x,y}(A) = E^\uparrow_x\left[1_A \frac{p^\uparrow_{u-t}(X_t, y)}{p^\uparrow_u(x,y)}; t < \zeta\right] \quad (x, y \geq 0), \tag{1.4}$$

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for $A \in \mathcal{F}_t$ and $0 < t < u$, where $p_u^{\uparrow}(x,y)$ denotes the transition density of $M^{\uparrow}$ with respect to its speed measure. We may now raise a natural question: does the following weak convergence hold?

$$P_{x,y}^{\uparrow,u} \xrightarrow{\mathrm{w}} P_{0,0}^{\uparrow,u} \quad \text{as } x,y \downarrow 0. \tag{1.5}$$

The statements (1.3) and (1.5) of weak convergence have been taken for granted in [7] and [9]. The aim of this paper is to give proofs for (1.3) and (1.5). Note that similar problems have been discussed in different settings in [4] and [2].

The organization of the paper is as follows. In Section 2, we recall basic facts about $M^{\uparrow}$ and its bridges. In Section 3, we prove weak convergence results for these processes.

2. Notation and basic facts

We need several notation and basic facts about $M^{\uparrow}$ and its bridges; The proofs of all the assertions in this section can be found in [8].

We write $\widehat{M} = \{X, (\widehat{P}_x)_{x \geq 0}\}$ for the process obtained by killing M at H_0, i.e.,

$$\widehat{P}_x(A; t < \zeta) = P_x(A; t < H_0) \tag{2.1}$$

for $A \in \mathcal{F}_t$ and $t > 0$. There exists a continuous density of the transition probability:

$$\widehat{P}_x(X_t \in dy; t < \zeta) = \widehat{p}_t(x,y)m(dy). \tag{2.2}$$

We may choose

$$m^{\uparrow}(dy) = (S(y))^2 m(dy), \quad S^{\uparrow}(x) = -\frac{1}{S(x)}. \tag{2.3}$$

as the speed measure $m^{\uparrow}$ and the scale function $S^{\uparrow}$ for $M^{\uparrow}$, respectively. There exists a continuous density of the transition probability:

$$P_x^{\uparrow}(X_t \in dy; t < \zeta) = p_t^{\uparrow}(x,y)m^{\uparrow}(dy), \tag{2.4}$$

where the densities $p_t^{\uparrow}(x,y)$ and $\widehat{p}_t(x,y)$ are connected by the relation

$$p_t^{\uparrow}(x,y) = \frac{\widehat{p}_t(x,y)}{S(x)S(y)}, \quad x,y > 0. \tag{2.5}$$

Note that $p_t^{\uparrow}(x,y)$ may be continuously extended for $x, y \geq 0$ for all $t > 0$. We have the Chapman–Kolmogorov identity:

$$p_{t+s}^{\uparrow}(x,z) = \int_0^{\infty} p_t^{\uparrow}(x,y)p_s^{\uparrow}(y,z)m^{\uparrow}(dy), \quad t,s > 0, \ x,z \geq 0. \tag{2.6}$$

The boundary values of $p_t^{\uparrow}(x,y)$ for $x = 0$ or $y = 0$ play the following roles. The law of the lifetime of a generic excursion is given as

$$\mathbf{n}(\zeta \in dt) = p_t^{\uparrow}(0,0)dt. \tag{2.7}$$

If we set

$$f_{x0}(t) := p_t^{\uparrow}(0,x)S(x) = p_t^{\uparrow}(x,0)S(x) = \lim_{y \downarrow 0} \frac{\widehat{p}_t(x,y)}{S(y)}, \quad x > 0. \tag{2.8}$$

Then $f_{x0}(t)$ is a density of the hitting time of 0:

$$P_x(H_0 \in dt) = f_{x0}(t)dt \tag{2.9}$$

and at the same time it is a density of the entrance law of the Itô measure:

$$\mathbf{n}(X_t \in dx; t < \zeta) = f_{x0}(t)m(dx). \tag{2.10}$$

We make a remark on connection between $M^{\uparrow}$-bridges and $\widehat{M}$-bridges. We denote by $\widehat{P}^u_{x,y}$ for $x, y > 0$ the law of the $\widehat{M}$-bridge of duration u, starting at x and ending at y, i.e.,

$$\widehat{P}^u_{x,y}(A) = \widehat{E}_x\left[1_A \frac{\widehat{p}_{u-t}(X_t, y)}{\widehat{p}_u(x, y)}; t < \zeta\right] \tag{2.11}$$

for $A \in \mathcal{F}_t$ and $0 < t < u$. It is now obvious by definition that

$$\widehat{P}^u_{x,y} = P^{\uparrow,u}_{x,y} \quad (x, y > 0). \tag{2.12}$$

3. Weak convergence results

For $0 < u \leq \infty$, let $C([0, u))$ denote the space of continuous functions on $[0, u)$ equipped with the topology of compact uniform convergence. For probability measures on $C([0, u))$, we write $P_n \xrightarrow{\text{w}} P$ if P_n converges weakly to P, while we write $P_n \xrightarrow{\text{f.d.}} P$ if the finite-dimensional distributions of P_n converges weakly to those of P. It is well-known that the following conditions are equivalent:

(a) $P_n \xrightarrow{\text{w}} P$;
(b) $P_n \xrightarrow{\text{f.d.}} P$ and $\{P_n\}$ is tight;
(c) $P_n \circ (X|_{[0,t)})^{-1} \xrightarrow{\text{w}} P \circ (X|_{[0,t)})^{-1}$ for all $t \in [0, u)$.

For tightness, we utilize [6, Theorem VI.16], which asserts that, a sequence $\{X, P_n\}$ of canonical processes on $C([0, \infty))$ is tight if the following *Aldous condition* is satisfied: for any $t_0 > 0$,

$$X(\rho_n + \delta_n) - X(\rho_n) \to 0 \quad \text{in probability} \tag{3.1}$$

holds for all sequence $\{\delta_n\}$ of positive numbers converging to zero and all sequence $\{\rho_n\}$ of stopping times taking values in $[0, t_0]$.

We have defined $P^{\uparrow}_x$ for $x > 0$ and $P^{\uparrow}_0$ in manners which are different from each other. As it was stated in [7] and [9] without proof, we may thus concern continuity of $P^{\uparrow}_x$ at $x = 0$.

Theorem 1. $P^{\uparrow}_x \xrightarrow{\text{w}} P^{\uparrow}_0$ *as* $x \downarrow 0$.

Proof. Let $\{x_n\}$ be an arbitrary sequence such that $x_n \downarrow 0$. By the strong Markov property of $\{X, (P^{\uparrow}_x)_{x \geq 0}\}$, we have

$$E^{\uparrow}_{x_n}[F(X_t : t \geq 0)] = E^{\uparrow}_0\left[F(X_{t+H_{x_n}} : t \geq 0)\right] \tag{3.2}$$

for all bounded continuous functional F on $C([0, \infty))$. We thus see that $P^{\uparrow}_{x_n} \xrightarrow{\text{w}} P^{\uparrow}_0$ if and only if $P^{\uparrow}_0 \circ (X^n)^{-1} \xrightarrow{\text{w}} P^{\uparrow}_0$, where $X^n_t = X_{t+H_{x_n}}$.

Let $0 < t_1 < \cdots < t_m$ and let $f_1, \ldots, f_m$ be bounded continuous functions. Since

$$E^{\uparrow}_0\left[\prod_{k=1}^m f_k(X^n_{t_k})\right] = E^{\uparrow}_0\left[\prod_{k=1}^m f_k(X_{t_k+H_{x_n}})\right] \tag{3.3}$$

and since $H_{x_n} \downarrow H_0$, $P^{\uparrow}_0$-a.s., we obtain $P^{\uparrow}_0 \circ (X^n)^{-1} \xrightarrow{\text{f.d.}} P^{\uparrow}_0$ by the dominated convergence theorem.

For any $\varepsilon > 0$, we have

$$P_0^{\uparrow}\left(|X^n_{\rho_n+\delta_n} - X^n_{\rho_n}| > \varepsilon\right) = P_0^{\uparrow}\left(|X_{\rho_n+\delta_n+H_{x_n}} - X_{\rho_n+H_{x_n}}| > \varepsilon\right). \tag{3.4}$$

Since $t \mapsto X_t$ is uniformly continuous on compact intervals $P_0^{\uparrow}$-a.s., we obtain that the Aldous condition is satisfied. □

Remark 1. As another proof Theorem 1 can be deduced from a recently obtained general theorem of Dereich–Döring–Kyprianou [3, Proposition 7]. Our process $\{X, (P_x^{\uparrow})_{x\geq 0}\}$ trivially satisfies all its assumptions except the following:

$$\lim_{\varepsilon\downarrow 0}\limsup_{x\downarrow 0} E_x^{\uparrow}[H_\varepsilon] = 0. \tag{3.5}$$

Let us verify that (3.5) is satisfied. Note that we have

$$E_x^{\uparrow}[e^{-\lambda H_\varepsilon}] = \frac{R_\lambda^{\uparrow}(x,\varepsilon)}{R_\lambda^{\uparrow}(\varepsilon,\varepsilon)} = \frac{\psi_\lambda(x)/S(x)}{\psi_\lambda(\varepsilon)/S(\varepsilon)}, \quad \lambda > 0,\ 0 < x < \varepsilon, \tag{3.6}$$

where the function $R_\lambda^{\uparrow}(x,y)$ stands for the resolvent density of $\{X, (P_x^{\uparrow})_{x\geq 0}\}$ with respect to $m^{\uparrow}(dy)$, and $\psi_\lambda(x)$ for the non-negative increasing function satisfying

$$\psi_\lambda(x) = S(x) + \lambda\int_{(0,x]} (S(x) - S(y))\,\psi_\lambda(y)m(dy). \tag{3.7}$$

Note that the second identity of (3.6) comes from [8, (3.7) and (3.33)] ([8, (3.7)] must be corrected as $G(\lambda,x,y) = \psi_\lambda(x)g_\lambda(y)$ for $0 < x \leq y < l$). We thus obtain

$$E_x^{\uparrow}[H_\varepsilon] = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}E_x^{\uparrow}[1 - e^{-\lambda H_\varepsilon}] = \lim_{\lambda\downarrow 0}\frac{\frac{\psi_\lambda(\varepsilon)}{S(\varepsilon)} - \frac{\psi_\lambda(x)}{S(x)}}{\lambda\cdot\frac{\psi_\lambda(\varepsilon)}{S(\varepsilon)}} \tag{3.8}$$

$$= \int_{(0,\varepsilon]}\frac{S(\varepsilon) - S(y)}{S(\varepsilon)}S(y)m(dy) - \int_{(0,x]}\frac{S(x) - S(y)}{S(x)}S(y)m(dy), \tag{3.9}$$

which yields (3.5) by the fact that S is continuous and $S(0) = 0$.

We now concern continuity of $P_{x,y}^{\uparrow,u}$ at (x,y) with $xy = 0$.

Theorem 2. *It holds that $P_{x,y}^{\uparrow,u} \xrightarrow{\text{w}} P_{x,0}^{\uparrow,u}$ as $y \downarrow 0$ for $x > 0$, $P_{x,y}^{\uparrow,u} \xrightarrow{\text{w}} P_{0,y}^{\uparrow,u}$ as $x \downarrow 0$ for $y > 0$ and $P_{x,y}^{\uparrow,u} \xrightarrow{\text{w}} P_{0,0}^{\uparrow,u}$ as $x, y \downarrow 0$.*

Proof. We only prove the last claim; the other two claims can be proved similary.

Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences such that $x_n \downarrow 0$ and $y_n \downarrow 0$. Let $t \in [0,u)$ be fixed. By (1.4) and by the strong Markov property of $\{X, (P_x^{\uparrow})_{x\geq 0}\}$, we have

$$E_{x_n,y_n}^{\uparrow,u}[F(X_s : s\in[0,t))] = E_{x_n}^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_t, y_n)}{p_u^{\uparrow}(x_n,y_n)}F(X_s : s\in[0,t))\right] \tag{3.10}$$

$$= E_0^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_{t+H_{x_n}}, y_n)}{p_u^{\uparrow}(x_n,y_n)}F(X_{s+H_{x_n}} : s\in[0,t))\right] \tag{3.11}$$

and

$$E_{0,0}^{\uparrow,u}[F(X_s : s\in[0,t))] = E_0^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_t, 0)}{p_u^{\uparrow}(0,0)}F(X_s : s\in[0,t))\right] \tag{3.12}$$

for all bounded continuous functional F on $C([0,t))$. Let us define probability measures P_n^t and P^t on $C([0,\infty))$ by

$$E_n^t[F(X_s : s \geq 0)] = E_0^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_{t+H_{x_n}}, y_n)}{p_u^{\uparrow}(x_n, y_n)} F(X_{s+H_{x_n}} : s \geq 0)\right] \tag{3.13}$$

and

$$E^t[F(X_s : s \geq 0)] = E_0^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_t, 0)}{p_u^{\uparrow}(0,0)} F(X_s : s \geq 0)\right]. \tag{3.14}$$

We thus see that our desired convergence $P_{x_n,y_n}^{\uparrow,u} \xrightarrow{\text{w}} P_{0,0}^{\uparrow,u}$ as probability measures on $C([0,u))$ is implied by the convergence $P_n^t \xrightarrow{\text{w}} P^t$ as probability measures on $C([0,\infty))$ for all $t \in [0,u)$.

Let us prove that

$$\frac{p_{u-t}^{\uparrow}(X_{t+H_{x_n}}, y_n)}{p_u^{\uparrow}(x_n, y_n)} \to \frac{p_{u-t}^{\uparrow}(X_t, 0)}{p_u^{\uparrow}(0,0)} \quad \text{in } L^1(P_0^{\uparrow}). \tag{3.15}$$

Since $H_{x_n} \downarrow H_0$, $P_0^{\uparrow}$-a.s. and since $p_{u-t}^{\uparrow}(x,y)$ is jointly continuous in x and y, we see that the $P_0^{\uparrow}$-a.s. convergence in (3.15) holds. Since all the expectations equal to one by the Chapman–Kolmogorov identity, we obtain the L^1-convergence by Scheffé's lemma.

Let $0 < t_1 < \cdots < t_m$ and let $f_1, \ldots, f_m$ be continuous functions with compact support. Note that we have

$$E_n^t\left[\prod_{k=1}^m f_k(X_{t_k})\right] = E_0^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_{t+H_{x_n}}, y_n)}{p_u^{\uparrow}(x_n, y_n)} \prod_{k=1}^m f_k(X_{t_k+H_{x_n}})\right]. \tag{3.16}$$

Since $\prod_{k=1}^m f_k(X_{t_k+H_{x_n}})$ is uniformly bounded and converges $P_0^{\uparrow}$-a.s. to $\prod_{k=1}^m f_k(X_{t_k})$, and since we have the L^1-convergence (3.15), we obtain $P_n^t \xrightarrow{\text{f.d.}} P^t$.

For tightness, let us verify the Aldous condition. For $\varepsilon > 0$, we need to prove

$$\lim_{n\to\infty} E_0^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_{t+H_{x_n}}, y_n)}{p_u^{\uparrow}(x_n, y_n)}; |X_{\rho_n+\delta_n+H_{x_n}} - X_{\rho_n+H_{x_n}}| > \varepsilon\right] = 0. \tag{3.17}$$

Since $t \mapsto X_t$ is uniformly continuous on compact intervals $P_0^{\uparrow}$-a.s., we see that

$$\lim_{n\to\infty} E_0^{\uparrow}\left[\frac{p_{u-t}^{\uparrow}(X_t, 0)}{p_u^{\uparrow}(0,0)}; |X_{\rho_n+\delta_n+H_{x_n}} - X_{\rho_n+H_{x_n}}| > \varepsilon\right] = 0. \tag{3.18}$$

Combining this convergence together with the L^1-convergence (3.15), we obtain (3.17). The proof is now complete. □

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