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2016 J. Phys.: Conf. Ser. 670 012033

(http://iopscience.iop.org/1742-6596/670/1/012033)

Journal of Physics: Conference Series **670** (2016) 012033 doi:10.1088/1742-6596/670/1/012033

Towards the gravity/CYBE correspondence
– *the current status* –

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Abstract. We give a brief summary of Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring by focusing upon four examples, 1) gravity duals for noncommutative gauge theories, 2) γ-deformations of S^5, 3) Schrödinger spacetimes and 4) abelian twists of the global AdS_5 .

1. Introduction

The most prototypical example of AdS/CFT correspondences [1] is the conjectured equivalence between type IIB superstring on the $\mathrm{AdS}_5\times\mathrm{S}^5$ background and the four-dimensional $\mathcal{N}=4$ $SU(N)$ super Yang-Mills (SYM) theory in the large N limit. A great progress on this subject is that an integrable structure behind this duality was unveiled (For a comprehensive review, see [2]). Motivated by this discovery, we are interested in the associated classical integrability on the string-theory side. The classical string action on the $\mathrm{AdS}_5\times\mathrm{S}^5$ background was constructed by adopting the Green-Schwarz formulation [3] with the supercoset representation

$$\text{super } \mathrm{AdS}_5\times\mathrm{S}^5=\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}\,. \tag{1}$$

This supercoset enjoys the $\mathbb{Z}_4$-grading property, which ensures the classical integrability [4] in the sense of kinematical integrability that means the existence of Lax pairs (For excellent reviews, see [5, 6]). However, it should be remarked that the complete integrability in the sense of Liouville has not been shown due to the ambiguity stemming from non-ultra local terms in the Poisson structure.

As the next issue, it is intriguing to study integrable deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring. In fact, there is a long history along this direction and a vast number of the deformations are already known. Thus it is desirable to study them with the most systematic and efficient way. A promising way is to follow the Yang-Baxter sigma model description proposed by Klimcik [7]. This is such a way to study integrable deformations of 2D non-linear sigma models. By following this approach, the integrable deformations are specified by skew-symmetric linear R-operators which satisfy the modified classical Yang-Baxter equation (mCYBE). The original work [7] was invented for principal chiral models, but it was recently generalized to the symmetric cosets by Delduc-Magro-Vicedo [8][1]. Then it was straightforwardly applied to construct a q-deformed action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring [14] with the classical r-matrix of Drinfeld-Jimbo type [15]. The metric in the string frame and NS-NS two-form have been obtained in [16] (Here "NS" is an abbreviation of Neveu-Schwarz). For generalizations to $\mathrm{AdS}_n\times\mathrm{S}^n$, see [17]. For the recent progress towards the full solution, see [18–20]. This issue is closely related to our interest here.

As a possible generalization, Jordanian deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action were presented in [21] by adopting the homogeneous classical Yang-Baxter equation (CYBE). Here the Lax pair and kappa transformation are different from the ones of the work [14] and this generalization is not so difficult but not so obvious. One of the advantages of this formulation is that partial deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$ are possible, because the zero map $R = 0$ is allowed as a solution of the CYBE, but not of the mCYBE. Then, many skew-symmetric solutions of the CYBE have been identified with well-known solutions of type IIB supergravity in a series of papers [22–30]. The list of the solutions includes γ-deformations of S^5 [31, 32], gravity duals for noncommutative (NC) gauge theories [33] , Schrödinger spacetimes [34], abelian twists of the global AdS_5 [35] and further new backgrounds [22]. These identifications may be regarded as a new perspective of Yang-Baxter deformations. The conjectured relation between solutions of type IIB supergravity and classical r-matrices are called the gravity/CYBE correspondence [23] (For a short summary of the works in 2014, see [36]. This review is the update of [36] in 2015).

To establish this correspondence, it is necessary to do much effort. However, if it has been established, then it indicates that the moduli space of a certain class of solutions of type IIB supergravity can be described by the CYBE. Due to the fact that the size of classical r-matrices is finite and then the number of the solutions is also finite, the number of the associated solutions of type IIB supergravity should also be finite. It is also analogous to the bubbling scenario [37], where 1/2 BPS solutions of type IIB supergravity preserving a certain symmetry are specified by droplet configurations in a free fermion system.

Recently, Yang-Baxter deformations are further generalized to 4D Minkowski spacetime [38]. In this case, there is an obstacle that the inner product entering into the YB sigma model action is degenerate. A possible way around is to employ an embedding of 4D Minkowski spacetime into the bulk AdS_5 space. By adopting this resolution, a Yang-Baxter sigma model providing deformations of 4D Minkowski spacetime was proposed in [38]. Then classical r-matrices have been identified with a lot of gravity solutions such as Melvin backgrounds [39–42], pp-wave backgrounds [43], Hashimoto-Sethi backgrounds [44] and Spradlin-Takayanagi-Volovich backgrounds [45]. More interestingly, T-duals of dS_4 and AdS_4 also have been reproduced. In addition, new backgrounds generated by the standard q-deformation were also presented. Furthermore, the above result has an intimate connection with kappa-Minkowski spacetime [46] via preceding works e.g., [47]. For an argument with gravity duals, see [29].

It is worth noting that the gravity/CYBE correspondence may work beyond the integrability. There are many examples of non-integrable AdS/CFT correspondences. A landmark example is the $\mathrm{AdS}_5 \times T^{1,1}$ background [48], for which the complete integrability is broken because chaotic string solutions appear on the $R \times T^{1,1}$ geometry [49, 50]. Hence TsT transformations of $T^{1,1}$ [31, 51] give rise to non-integrable deformations. Interestingly enough, these deformations can be reproduced as Yang-Baxter deformations with abelian classical r-matrices [52]. This result indicates that such Yang-Baxter deformations work well even for non-integrable backgrounds of type IIB supergravity.

[1] For earlier arguments concerned with this generalization, see [9–13].

Journal of Physics: Conference Series **670** (2016) 012033 doi:10.1088/1742-6596/670/1/012033

In this article, we will give a summary for a series of our works with some updates by focusing upon four examples of classical r-matrices. The explanation includes Lax pairs as well as the metric and NS-NS two-form.

This article is organized as follows. Section 2 is a brief review of Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring. In the subsequent sections, we explain four examples of classical r-matrices: (i) gravity duals of non-commutative gauge theories [in sec. 3] , (ii) γ-deformations of S^5 [in sec. 4] , (iii) Schrödinger spacetimes [in sec. 5] , and (iv) abelian twists of the global AdS_5 [in sec. 6] . Section 7 is summary and outlooks. In Appendix A, our notation and convention are summarized.

2. Jordanian deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring

We give a brief review of Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring with the CYBE [21]. The deformations are often called Jordanian deformations.

First of all, the deformed classical action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring is given by

$$S = -\frac{\sqrt{\lambda_{\mathrm{c}}}}{4}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta})\mathrm{STr}\Big[A_\alpha\, d\circ\frac{1}{1-\eta R_g\circ d}(A_\beta)\Big]\,. \tag{2}$$

The left-invariant one-form A_α is defined as

$$A_\alpha \equiv g^{-1}\partial_\alpha g\,,\qquad g\in SU(2,2|4)\,. \tag{3}$$

Here $\alpha=(\tau,\sigma)$ is the world-sheet index. Then the world-sheet metric is taken as

$$\gamma^{\alpha\beta} = \mathrm{diag}(-1,+1) \tag{4}$$

with the conformal gauge, in which there is no coupling of the dilaton to the world-sheet scalar curvature. The anti-symmetric tensor $\epsilon^{\alpha\beta}$ is normalized as $\epsilon^{\tau\sigma}=+1$. The constant parameter λ_{c} in front of the action (2) is defined as

$$\sqrt{\lambda_{\mathrm{c}}}\equiv 2\pi L^2 T = \frac{L^2}{\alpha'}\,,\qquad T=\frac{1}{2\pi\alpha'}\,,$$

where T is the string tension and L is the curvature radius of the undeformed AdS_5 . Note that η is a deformation parameter and hence the undeformed action [3] is reproduced when $\eta=0$.

A key ingredient appearing in (2) is a chain of operations R_g , which is defined as

$$R_g(X)\equiv g^{-1}R(gXg^{-1})g\,,\qquad X\in\mathfrak{su}(2,2|4)\,. \tag{5}$$

A linear operator $R:\mathfrak{su}(2,2|4)\to\mathfrak{su}(2,2|4)$ is a solution of the CYBE[2],

$$[R(X),R(Y)]-R([R(X),Y]+[X,R(Y)])=0\,. \tag{6}$$

This R-operator is related to a skew-symmetric classical r-matrix in the tensorial notation through the following supertrace operation on the second site:

$$R(X)=\mathrm{STr}_2[r(1\otimes X)]=\sum_i\left(a_i\,\mathrm{STr}[b_iX]-b_i\,\mathrm{STr}[a_iX]\right)\,. \tag{7}$$

[2] In the original work [21], a wider class of R-operators whose image is given by $\mathfrak{gl}(4|4)$ has been proposed. The $\mathfrak{gl}(4|4)$ image is restricted on $\mathfrak{su}(2,2|4)$ in essential under the coset projection d , as pointed out in [28]. We will concentrate here on a restricted class in which the image is $\mathfrak{su}(2,2|4)$ from the beginning, so as to deal with pre-projected quantities like the deformed current J itself, without introducing extra generators. For general cases argued in [22, 25], a more detailed study would be necessary.

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Here the classical r-matrix is represented by

$$r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i) \qquad \text{with} \qquad a_i, b_i \in \mathfrak{su}(2,2|4)\,. \tag{8}$$

If a_i and b_j (do not) commute with each other, then r is called (non-)abelian.

The coset (1) enjoys the $\mathbb{Z}_4$-grading property with the projections $P_i\,(i = 0,1,2,3)$ to the graded components of the decomposition

$$\mathfrak{su}(2,2|4) = \mathfrak{su}(2,2|4)_{(0)} \oplus \mathfrak{su}(2,2|4)_{(1)} \oplus \mathfrak{su}(2,2|4)_{(2)} \oplus \mathfrak{su}(2,2|4)_{(3)}\,, \tag{9}$$

where $P_i(\mathfrak{su}(2,2|4)) \equiv \mathfrak{su}(2,2|4)_{(i)}$. Note that

$$\mathfrak{su}(2,2|4)_{(0)} = \mathfrak{so}(1,4) \oplus \mathfrak{so}(5) \tag{10}$$

is a local symmetry of the classical action (2). Then d is defined as a linear combination of P_1 , P_2 and P_3 like

$$d \;\; \equiv \;\; P_1 + 2P_2 - P_3\,. \tag{11}$$

The numerical coefficients have been fixed by requiring the kappa-symmetry [21]. This d is the same in the undeformed case [3], while it depends on η in the case of the mCYBE [14].

In the following analysis, it is convenient to introduce the light-cone expression of A_α on the world-sheet like

$$A_\pm \equiv A_\tau \pm A_\sigma\,. \tag{12}$$

In particular, it makes the expression of Lax pair much simpler.

2.1. The bosonic part of the Lagrangian

Le us study the bosonic part of the deformed action (2) , which can be rewritten as

$$L = \frac{\sqrt{\lambda_c}}{2}\,\mathrm{STr}(A_-\,P_2(J_+))\,. \tag{13}$$

Here $J_\pm$ is a deformed current defined as

$$J_\pm \equiv \frac{1}{1 \mp 2\eta R_g \circ P_2}\,A_\pm\,. \tag{14}$$

Note here that the factor 2 in front of η comes from the projection operator d given in (11) . The deformed current $J_\pm$ is determined by solving the following equations:[3]

$$(1 \mp 2\eta R_g \circ P_2)\,J_\pm = A_\pm\,. \tag{15}$$

Then the metric in the string frame and NS-NS two-form can be evaluated from the Lagrangian (13) .

[3] In order to derive the metric and NS-NS two-form, it is enough to determine the projected current $P_2(J_\pm)$ by solving the projected conditions

$$(1 \mp 2\eta P_2 \circ R_g)\,P_2(J_\pm) = P_2(A_\pm)\,.$$

On the other hand, the unprojected current $J_\pm$ itself is necessary to derive the explicit form of Lax pair.

Taking a variation of the Lagrangian (13), the equation of motion is obtained as

$$\mathcal{E} \equiv \partial_+ P_2(J_-) + \partial_- P_2(J_+) + [J_+, P_2(J_-)] + [J_-, P_2(J_+)] = 0\,. \tag{16}$$

By definition, the undeformed current $A_\pm$ satisfies the flatness condition,

$$\mathcal{Z} \equiv \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0\,. \tag{17}$$

Then, in terms of the deformed current $J_\pm$, this condition can be rewritten as

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] + 2\eta\, R_g(\mathcal{E}) + 4\eta^2\, \mathrm{CYBE}_g(P_2(J_+), P_2(J_-)) = 0\,, \tag{18}$$

where we have introduced a new quantity defined as

$$\mathrm{CYBE}_g(X, Y) \equiv [R_g(X), R_g(Y)] - R_g([R_g(X), Y] + [X, R_g(Y)])\,. \tag{19}$$

Note that $\mathrm{CYBE}_g(X, Y)$ vanishes if the R-operator satisfies the CYBE in (6). Then the relation (18) indicates that $J_\pm$ also satisfies the flatness condition with the equation of motion $\mathcal{E} = 0$. That is, $J_\pm$ is the on-shell flat current, while $A_\pm$ is the off-shell flat one.

It is helpful to decompose $J_\pm$ with the projections P_0 and P_2 like

$$J_\pm = P_0(J_\pm) + P_2(J_\pm) \equiv J_\pm^{(0)} + J_\pm^{(2)}\,, \tag{20}$$

with the completeness condition $P_0 + P_2 = 1$. The concrete expressions of the projections are given in Appendix A. Then the equation of motion (16) can be rewritten as

$$\mathcal{E} = \partial_+ J_-^{(2)} + \partial_- J_+^{(2)} + [J_+^{(0)}, J_-^{(2)}] + [J_-^{(0)}, J_+^{(2)}] = 0\,. \tag{21}$$

The flatness condition (17) can also be rewritten in a similar way:

$$\mathcal{Z} = P_0(\mathcal{Z}) + P_2(\mathcal{Z}) = 0\,. \tag{22}$$

With the help of the linear independence of the grade 0 and grade 2 parts, one can obtain the following two conditions:

$$\begin{aligned} P_0(\mathcal{Z}) &= \partial_+ J_-^{(0)} - \partial_- J_+^{(0)} + [J_+^{(0)}, J_-^{(0)}] + [J_+^{(2)}, J_-^{(2)}] + 2\eta\, P_0(R_g(\mathcal{E})) = 0\,, \\ P_2(\mathcal{Z}) &= \partial_+ J_-^{(2)} - \partial_- J_+^{(2)} + [J_+^{(0)}, J_-^{(2)}] + [J_+^{(2)}, J_-^{(0)}] + 2\eta\, P_2(R_g(\mathcal{E})) = 0\,. \end{aligned} \tag{23}$$

Note here that the terms proportional to η vanish under the equation of motion, i.e., $\mathcal{E} = 0$.

Then the three conditions in (21) and (23) can be recast into the following set of the equations $\mathcal{C}_i = 0$ $(i = 1, 2, 3)$:

$$\begin{aligned} \mathcal{C}_1 &\equiv \partial_- J_+^{(2)} - [J_+^{(2)}, J_-^{(0)}]\,, \\ \mathcal{C}_2 &\equiv \partial_+ J_-^{(2)} + [J_+^{(0)}, J_-^{(2)}]\,, \\ \mathcal{C}_3 &\equiv \partial_+ J_-^{(0)} - \partial_- J_+^{(0)} + [J_+^{(0)}, J_-^{(0)}] + [J_+^{(2)}, J_-^{(2)}]\,. \end{aligned} \tag{24}$$

Namely, the relations $\mathcal{C}_i = 0$ $(i = 1, 2, 3)$ are equivalent to the equation of motion (16) and the flatness condition (17).

2.2. Lax pair

Finally, let us introduce a Lax pair for the deformed Lagrangian (13) , which is given by

$$\mathcal{L}_\pm = J^{(0)}_\pm + \lambda^{\pm 1} J^{(2)}_\pm \tag{25}$$

with a spectral parameter $\lambda \in \mathbb{C}$. Note that the existence of the Lax pair (25) is based on the $\mathbb{Z}_2$-grading property of the bosonic $\mathrm{AdS}_5{\times}\mathrm{S}^5$ group manifold.

As usual, the flatness condition of $\mathcal{L}_\pm$

$$0 \;=\; \partial_+\mathcal{L}_- - \partial_-\mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] \tag{26}$$

is equivalent to the equation of motion $\mathcal{E} = 0$ [in (16)] and the flatness condition $\mathcal{Z} = 0$ [in (17)] . In order to confirm the equivalence, it is helpful to notice that the right-hand side of (26) can be rewritten in terms of $\mathcal{C}_i$ as follows:

$$\partial_+\mathcal{L}_- - \partial_-\mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = -\lambda\, \mathcal{C}_1 + \frac{1}{\lambda}\, \mathcal{C}_2 + \mathcal{C}_3 \,. \tag{27}$$

Thus the equivalence is now obvious.

In the following sections, we will explain classical r-matrices associated with four backgrounds, 1) gravity duals for noncommutative gauge theories, 2) γ-deformations of S^5, 3) Schrödinger spacetimes, and 4) abelian twists of the global AdS_5 . For each of the examples, we show the metric (in the string frame) and NS-NS two-form, and present an explicit form of the associated Lax pair (25) . For the conventions of the generators, see Appendix A.

3. Gravity duals of noncommutative gauge theories

Firstly, we will describe gravity duals of noncommutative gauge theories [33] from the viewpoint of Yang-Baxter deformations.

The backgrounds are associated with abelian Jordanian classical r-matrices [24]

$$r = c_1\, p_2 \wedge p_3 + c_2\, p_0 \wedge p_1 \,, \tag{28}$$

which consist of the translation generators p_μ in $\mathfrak{su}(2,2)$. The constant parameters c_1 and c_2 are related to magnetic and electric NS-NS two-forms in the solutions [33], respectively.

Note that p_μ's commute with each other and $p_\mu p_\nu = 0$ $(\mu, \nu = 0, 1, 2, 3)$. Then the square of the associated R-operator vanishes and the classical r-matrices (28) are of Jordanian type. Since the r-matrices include no $\mathfrak{su}(4)$ generator, only the AdS_5 part is deformed. Hence, we will concentrate on the AdS_5 part below, while ignoring the S^5 part.

3.1. The deformed metric and NS-NS two-form

Let us derive the metric and NS-NS two-form from the Lagrangian (13) .

We first introduce a coordinate system through a parametrization of an $SU(2,2)$ element,

$$g_a(\tau,\sigma) = \exp\Big[p_0\, x^0 + p_1\, x^1 + p_2\, x^2 + p_3\, x^3\Big]\, \exp\left[\gamma_5^a\, \frac{1}{2} \log z\right] \quad \in SU(2,2)\,. \tag{29}$$

By solving the relations in (15) , the deformed current J_α is determined as

$$J_\pm \;=\; \frac{z}{z^4 - 4{c_2}^2\eta^2}\Big[(z^2\partial_\pm x^0 \pm 2c_2\eta\partial_\pm x^1)\, p_0 + (z^2\partial_\pm x^1 \pm 2c_2\eta\partial_\pm x^0)\, p_1\Big]$$

$$+\frac{z}{z^4+4{c_1}^2\eta^2}\Big[(z^2\partial_\pm x^2\pm 2c_1\eta\partial_\pm x^3)\,p_2+(z^2\partial_\pm x^3\mp 2c_1\eta\partial_\pm x^2)\,p_3\Big]$$
$$+\frac{1}{2z}\partial_\pm z\,\gamma_5^a\,. \tag{30}$$

Then the resulting metric and NS-NS two-form are given by

$$\begin{aligned} ds^2 &= \frac{z^2[-(dx^0)^2+(dx^1)^2]}{z^4-4c_2^2\eta^2}+\frac{z^2[(dx^2)^2+(dx^3)^2]}{z^4+4c_1^2\eta^2}+\frac{dz^2}{z^2}\,, \\ B &= -\frac{2c_2\eta}{z^4-4c_2^2\eta^2}dx^0\wedge dx^1+\frac{2c_1\eta}{z^4+4c_1^2\eta^2}dx^2\wedge dx^3\,. \end{aligned}\tag{31}$$

This result exactly agrees with the solutions in [33]. When $c_1=c_2=0$, the Poincaré AdS_5 is reproduced.

The result in (31) can also be reproduced as a special limit of the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ [19]. This limit has been further confirmed at the level of Lax pairs [30].

3.2. Lax pair

Then let us see the associated Lax pair $\mathcal{L}_\pm^{\rm NC}$. It is explicitly given by [24,30]

$$\begin{aligned} \mathcal{L}_\pm^{\rm NC} &= \frac{z}{z^4-4c_2^2\eta^2}\Big[(z^2\partial_\pm x^0\pm 2c_2\eta\partial_\pm x^1)\Big(\frac{\lambda^{\pm1}}{2}\gamma_0^a-n_{05}^a\Big) \\ &\qquad\qquad +(z^2\partial_\pm x^1\pm 2c_2\eta\partial_\pm x^0)\Big(\frac{\lambda^{\pm1}}{2}\gamma_1^a-n_{15}^a\Big)\Big] \\ &\quad+\frac{z}{z^4+4c_1^2\eta^2}\Big[(z^2\partial_\pm x^2\pm 2c_1\eta\partial_\pm x^3)\Big(\frac{\lambda^{\pm1}}{2}\gamma_2^a-n_{25}^a\Big) \\ &\qquad\qquad +(z^2\partial_\pm x^3\mp 2c_1\eta\partial_\pm x^2)\Big(\frac{\lambda^{\pm1}}{2}\gamma_3^a-n_{35}^a\Big)\Big]+\frac{\lambda^{\pm1}\partial_\pm z}{2z}\gamma_5^a\,. \end{aligned}\tag{32}$$

In the undeformed limit $c_1,\,c_2\to 0$, the above expression is reduced to

$$\mathcal{L}_\pm^{\rm PAdS_5} = \frac{\partial_\pm x^\mu}{z}\Big(\frac{\lambda^{\pm1}}{2}\gamma_\mu^a-n_{\mu5}^a\Big)+\frac{\lambda^{\pm1}\partial_\pm z}{2z}\gamma_5^a\,. \tag{33}$$

This is nothing but a Lax pair for the Poincaré AdS_5.

3.3. Twisted boundary condition

In fact, the deformations with the r-matrices (28) can be regarded as twisted boundary conditions with the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$, as argued in [32].

For simplicity, suppose $c_1\neq 0$ and $c_2=0$. The analysis for the case with $c_2\neq 0$ is quite similar, though there is a subtlety for the signature of the metric (For the detail, see [33]).

After performing the Yang-Baxter deformation (equivalently the associated TsT transformation), the original coordinates $\tilde{x}^2$ and $\tilde{x}^3$ for the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ are mapped to x^2 and x^3. Then the relations between the coordinates are given by

$$\begin{aligned} \frac{1}{z^2}\,\partial_\pm\tilde{x}^2 &= \frac{z^2}{z^4+4{c_1}^2\eta^2}\left[\partial_\pm x^2\pm\frac{2c_1\eta}{z^2}\partial_\pm x^3\right]\,, \\ \frac{1}{z^2}\,\partial_\pm\tilde{x}^3 &= \frac{z^2}{z^4+4{c_1}^2\eta^2}\left[\partial_\pm x^3\mp\frac{2c_1\eta}{z^2}\partial_\pm x^2\right]\,. \end{aligned}\tag{34}$$

These relations indicate the following equivalence of Noether currents

$$\tilde{P}_i^\alpha = P_i^\alpha \qquad (i = 2,3)\,. \tag{35}$$

Here P_i^α and $\tilde{P}_i^\alpha$ are conserved currents associated with translation invariance for the x^i and $\tilde{x}^i$ directions, respectively. The τ-component of (35) indicates that the momentum $p_i \equiv P_i^\tau$ is identical to $\tilde{p}_i \equiv \tilde{P}_i^\tau$, namely $p_i = \tilde{p}_i$. Then the σ-component of (35) leads to the relations:

$$\partial_\sigma \tilde{x}^2 = \partial_\sigma x^2 + \frac{2c_1\eta}{\sqrt{\lambda_{\rm c}}}\,p_3\,, \qquad \partial_\sigma \tilde{x}^3 = \partial_\sigma x^3 - \frac{2c_1\eta}{\sqrt{\lambda_{\rm c}}}\,p_2\,. \tag{36}$$

By integrating both relations, one can realize that the deformed backgrounds with the periodic boundary condition are equivalent to the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ with twisted boundary conditions:

$$\tilde{x}^2(\sigma = 2\pi) = \tilde{x}^2(\sigma = 0) + \frac{2c_1\eta}{\sqrt{\lambda_{\rm c}}}\,P_3\,, \qquad \tilde{x}^3(\sigma = 2\pi) = \tilde{x}^3(\sigma = 0) - \frac{2c_1\eta}{\sqrt{\lambda_{\rm c}}}\,P_2\,. \tag{37}$$

Here P_i are Noether charges obtained by integrating p_i .

Thus the Yang-Baxter deformations with the classical r-matrices (28) can be reinterpreted as twisted boundary conditions with the usual $\mathrm{AdS}_5\times\mathrm{S}^5$ background.

4. γ-deformations of S^5

An example of marginal deformations of $\mathcal{N}=4$ SYM, which preserves an $\mathcal{N}=1$ superconformal symmetry, is called the β-deformation [53]. The gravity dual for the β-deformation was constructed in [31], and then it was generalized to three-parameter cases called the γ-deformations [32]. The AdS_5 part is not deformed due to the conformal symmetry, while the S^5 part is deformed due to the reduced supersymmetry.

Let us study here the γ-deformations [31,32] from the viewpoint of Yang-Baxter deformations. In the following, the AdS_5 part will be dropped off.

The backgrounds are associated with abelian classical r-matrices [23] ,

$$r = \mu_3\, h_4 \wedge h_5 + \mu_1\, h_5 \wedge h_6 + \mu_2\, h_6 \wedge h_4\,. \tag{38}$$

Here h_4 , h_5 and h_6 are the three Cartan generators of $\mathfrak{su}(4)$, and μ_i $(i = 1,2,3)$ are the deformation parameters.

4.1. The deformed metric and NS-NS two-form

It is useful to employ the following parametrization of a group element g_s of $SU(4)$,

$$g_s(\tau,\sigma) = \exp\left[\frac{i}{2}(\phi_1\, h_4 + \phi_2\, h_5 + \phi_3\, h_6)\right] \exp\left[-\zeta n_{13}^{s}\right] \exp\left[-\frac{i}{2}\, r\, \gamma_1^{s}\right]. \tag{39}$$

In the undeformed case, this describes the round S^5 with the coordinates r , ζ and ϕ_i $(i = 1,2,3)$.

By solving the equations in (15) , the deformed current $J_\alpha^{\hat\gamma_1,\hat\gamma_2,\hat\gamma_3}$ is determined as

$$J_\pm^{\hat\gamma_1,\hat\gamma_2,\hat\gamma_3} \;=\; -i\partial_\pm r\,\frac{\gamma_1^{s}}{2} - \partial_\pm\zeta\left[i\sin r\,\frac{\gamma_3^{s}}{2} + \cos r\, n_{13}^{s}\right]$$

$$
\begin{aligned}
&-G(\hat{\gamma}_i)\Big[\partial_\pm\phi_1 \pm (\hat{\gamma}_3 \sin^2 r \sin^2\zeta\partial_\pm\phi_2 - \hat{\gamma}_2\cos^2 r\partial_\pm\phi_3) \\
&\qquad\qquad + \hat{\gamma}_1 \sin^2 r\cos^2 r\sin^2\zeta\sum_{i=1}^3\hat{\gamma}_i\partial_\pm\phi_i\Big] \\
&\qquad\qquad \times\left[\cos\zeta(i\sin r\,\frac{\gamma_2^s}{2} + \cos r\, n_{12}^s) + \sin\zeta\, n_{23}^s\right] \\
&-G(\hat{\gamma}_i)\Big[\partial_\pm\phi_2 \pm (\hat{\gamma}_1 \cos^2 r\partial_\pm\phi_3 - \hat{\gamma}_3\sin^2 r\cos^2\zeta\partial_\pm\phi_1) \\
&\qquad\qquad + \hat{\gamma}_2 \sin^2 r\cos^2 r\cos^2\zeta\sum_{i=1}^3\hat{\gamma}_i\partial_\pm\phi_i\Big] \\
&\qquad\qquad \times\left[\sin\zeta(i\ \sin r\,\frac{1}{2}\gamma_4^s + \cos r\, n_{14}^s) + \cos\zeta\, n_{34}^s\right] \\
&+G(\hat{\gamma}_i)\Big[\partial_\pm\phi_3 \pm (\hat{\gamma}_2 \sin^2 r \cos^2\zeta\partial_\pm\phi_1 - \hat{\gamma}_1\sin^2 r\sin^2\zeta\partial_\pm\phi_2) \\
&\qquad\qquad + \hat{\gamma}_3 \sin^4 r\sin^2\zeta\cos^2\zeta\sum_{i=1}^3\hat{\gamma}_i\partial_\pm\phi_i\Big] \\
&\qquad\qquad \times\left[i\cos r\,\frac{\gamma_5^s}{2} - \sin r\, n_{15}^s\right]. \qquad (40)
\end{aligned}
$$

Here the constant parameters $\hat{\gamma}_i$ are defined as

$$\hat{\gamma}_i \equiv 8\,\eta\mu_i\,, \qquad (41)$$

and the scalar function $G(\hat{\gamma}_i)$ is given by

$$G^{-1}(\hat{\gamma}_i) \equiv 1 + \sin^2 r(\hat{\gamma}_1^2\cos^2 r\sin^2\zeta + \hat{\gamma}_2^2\cos^2 r\cos^2\zeta + \hat{\gamma}_3^2\sin^2 r\sin^2\zeta\cos^2\zeta)\,. \qquad (42)$$

By substituting (40) into the Lagrangian (13) , the deformed background is obtained as

$$
\begin{aligned}
ds^2 &= \sum_{i=1}^3\left(d\rho_i{}^2 + G(\hat{\gamma}_i)\rho_i{}^2 d\phi_i{}^2\right) + G(\hat{\gamma}_i)\rho_1{}^2\rho_2{}^2\rho_3{}^2\left(\sum_{i=1}^3\hat{\gamma}_i d\phi_i\right)^2, \\
B_2 &= G(\hat{\gamma}_i)\left(\hat{\gamma}_3\rho_1{}^2\rho_2{}^2 d\phi_1\wedge d\phi_2 + \hat{\gamma}_1\rho_2{}^2\rho_3{}^2 d\phi_2\wedge d\phi_3 + \hat{\gamma}_2\rho_3{}^2\rho_1{}^2 d\phi_3\wedge d\phi_1\right). \qquad (43)
\end{aligned}
$$

Here new coordinates ρ_i $(i = 1, 2, 3)$ are defined as

$$\rho_1 \equiv \sin r\cos\zeta\,, \qquad \rho_2 \equiv \sin r\sin\zeta\,, \qquad \rho_3 \equiv \cos r\,. \qquad (44)$$

The metric and NS-NS two-form in (43) agree with the ones of γ-deformations of S^5 [32].

A particular one-parameter case with

$$\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3 \equiv \hat{\gamma} \qquad (45)$$

corresponds to the metric and NS-NS two-form of the Lunin-Maldacena solution [31] ,

$$
\begin{aligned}
ds^2 &= \sum_{i=1}^3\left(d\rho_i{}^2 + G\rho_i{}^2 d\phi_i{}^2\right) + G\hat{\gamma}^2\rho_1{}^2\rho_2{}^2\rho_3{}^2\left(\sum_{i=1}^3 d\phi_i\right)^2, \\
B_2 &= G\,\hat{\gamma}(\rho_1{}^2\rho_2{}^2 d\phi_1\wedge d\phi_2 + \rho_2{}^2\rho_3{}^2 d\phi_2\wedge d\phi_3 + \rho_3{}^2\rho_1{}^2 d\phi_3\wedge d\phi_1)\,, \qquad (46)
\end{aligned}
$$

where the scalar function $G(\hat{\gamma})$ is defined as

$$G^{-1}(\hat{\gamma}) \equiv 1 + \frac{\hat{\gamma}^2}{4}(\sin^2 2r + \sin^4 r\sin^2 2\zeta)\,. \qquad (47)$$

This background is a holographic dual of the β-deformation of the $\mathcal{N} = 4$ SYM [53].

4.2. Lax pair

Next, the explicit form of the Lax pair is given by [23, 30]

$$\begin{aligned}
\mathcal{L}_{\pm}^{\hat\gamma_1,\hat\gamma_2,\hat\gamma_3} &= -i\frac{\lambda^{\pm1}}{2}\partial_\pm r\,\gamma_1^s - \partial_\pm\zeta\left[i\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_3^s + \cos r\,n_{13}^s\right] \\
&\quad -G(\hat\gamma_i)\Big[\partial_\pm\phi_1 \pm (\hat\gamma_3\sin^2 r\sin^2\zeta\partial_\pm\phi_2 - \hat\gamma_2\cos^2 r\partial_\pm\phi_3) \\
&\qquad\qquad + \hat\gamma_1\sin^2 r\cos^2 r\sin^2\zeta\sum_{i=1}^3\hat\gamma_i\partial_\pm\phi_i\Big] \\
&\qquad\qquad \times\left[\cos\zeta(i\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_2^s + \cos r\,n_{12}^s) + \sin\zeta\,n_{23}^s\right] \\
&\quad -G(\hat\gamma_i)\Big[\partial_\pm\phi_2 \pm (\hat\gamma_1\cos^2 r\partial_\pm\phi_3 - \hat\gamma_3\sin^2 r\cos^2\zeta\partial_\pm\phi_1) \\
&\qquad\qquad + \hat\gamma_2\sin^2 r\cos^2 r\cos^2\zeta\sum_{i=1}^3\hat\gamma_i\partial_\pm\phi_i\Big] \\
&\qquad\qquad \times\left[\sin\zeta(i\,\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_4^s + \cos r\,n_{14}^s) + \cos\zeta\,n_{34}^s\right] \\
&\quad +G(\hat\gamma_i)\Big[\partial_\pm\phi_3 \pm (\hat\gamma_2\sin^2 r\cos^2\zeta\partial_\pm\phi_1 - \hat\gamma_1\sin^2 r\sin^2\zeta\partial_\pm\phi_2) \\
&\qquad\qquad + \hat\gamma_3\sin^4 r\sin^2\zeta\cos^2\zeta\sum_{i=1}^3\hat\gamma_i\partial_\pm\phi_i\Big] \\
&\qquad\qquad \times\left[i\cos r\,\frac{\lambda^{\pm1}}{2}\gamma_5^s - \sin r\,n_{15}^s\right]\,.
\end{aligned} \tag{48}$$

This result is equivalent to the Lax pair obtained in [32]. Note here that, in the undeformed limit $\hat\gamma_i \to 0$, $\mathcal{L}_{\pm}^{\hat\gamma_1,\hat\gamma_2,\hat\gamma_3}$ is reduced to

$$\begin{aligned}
\mathcal{L}_{\pm}^{S} &= -i\,\partial_\pm r\frac{\lambda^{\pm1}}{2}\gamma_1^s - \partial_\pm\zeta\left[i\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_3^s + \cos r\,n_{13}^s\right] \\
&\quad -\partial_\pm\phi_1\left[\cos\zeta(i\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_2^s + \cos r\,n_{12}^s) + \sin\zeta\,n_{23}^s\right] \\
&\quad -\partial_\pm\phi_2\left[\sin\zeta(i\,\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_4^s + \cos r\,n_{14}^s) + \cos\zeta\,n_{34}^s\right] \\
&\quad +\partial_\pm\phi_3\left[i\cos r\,\frac{\lambda^{\pm1}}{2}\gamma_5^s - \sin r\,n_{15}^s\right]\,.
\end{aligned} \tag{49}$$

This is just a Lax pair for the round S^5 .

4.3. Twisted boundary condition

Again, the deformations can be regarded as the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ with twisted boundary conditions [32]. The twisted boundary conditions are given by

$$\begin{aligned}
\tilde\phi_1(\sigma=2\pi) &= \tilde\phi_1(\sigma=0) + \gamma_3 J_2 - \gamma_2 J_3 + 2\pi n_1\,, \\
\tilde\phi_2(\sigma=2\pi) &= \tilde\phi_2(\sigma=0) + \gamma_1 J_3 - \gamma_3 J_1 + 2\pi n_2\,, \\
\tilde\phi_3(\sigma=2\pi) &= \tilde\phi_3(\sigma=0) + \gamma_2 J_1 - \gamma_1 J_2 + 2\pi n_3\,,
\end{aligned} \tag{50}$$

with $\gamma_i \equiv \hat\gamma/\sqrt{\lambda_c}$. Here J_i are the Noether charges for rotation invariance in the ϕ_i directions. The integers n_i are winding numbers along the ϕ_i directions.

5. Schrödinger spacetimes

We are concerned with Schrödinger spacetimes here. Originally, 3D Schrödinger spacetimes were introduced as light-like deformations of AdS_3 [55]. Higher-dimensional Schrödinger spacetimes were constructed as holographic duals for non-relativistic conformal field theories [56]. For the coset construction of the metrics, see [54]. Then the backgrounds have been embedded into type IIB supergravity [34][4].

In the following, let us study Schrödinger spacetimes embedded in type IIB supergravity [34] from the viewpoint of Yang-Baxter deformations. Then the backgrounds are associated with the following abelian r-matrix [27]:

$$r = \frac{i}{4\sqrt{2}}\,(p_0 - p_3) \wedge (h_4 + h_5 + h_6)\,. \tag{51}$$

It contains generators of both $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ and hence deforms both AdS_5 and S^5 parts.

5.1. The deformed metric and NS-NS two-form

Let us parametrize group elements of $SU(2,2)$ and $SU(4)$ like

$$\begin{aligned} g_a(\tau,\sigma) &= \exp\left[x^0 p_0 + x^1 p_1 + x^2 p_2 + x^3 p_3\right] \exp\left[\gamma_5^a \,\frac{1}{2}\,\log z\right] \quad \in \; SU(2,2)\,, \\ g_s(\tau,\sigma) &= \exp\left[\frac{i}{2}(\psi_1 h_4 + \psi_2 h_5 + \psi_3 h_6)\right] \exp\Bigl[-\zeta n^s_{13}\Bigr] \exp\left[-\frac{i}{2} r\,\gamma_1^s\right] \quad \in \; SU(4)\,. \end{aligned} \tag{52}$$

The deformed current $J_\pm$ can be expanded in terms of the generators of $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$. Then, by solving the equations in (15) , $J_\pm$ is determined as

$$\begin{aligned} J^a_\pm &= \frac{1}{z}\partial_\pm x^1 p_1 + \frac{1}{z}\partial_\pm x^2 p_2 + \frac{1}{2z}\partial_\pm z\,\gamma_5^a \\ &\quad + \frac{1}{\sqrt{2}\,z}\partial_\pm x^+\,(p_0 + p_3) \\ &\quad + \frac{1}{\sqrt{2}\,z}\left[\partial_\pm x^- \pm \eta\partial_\pm\chi \pm \frac{\eta}{2}\sin^2\mu(\partial_\pm\psi + \cos\theta\partial_\pm\phi) + \frac{\eta^2}{z^2}\partial_\pm x^+\right](p_0 - p_3)\,, \\ J^s_\pm &= -\frac{i}{2}\partial_\pm\mu\,\gamma_1^s - \frac{1}{2}\partial_\pm\theta\left[\frac{i}{2}\sin\mu\,\gamma_3^s + \cos\mu\, n^s_{13}\right] \\ &\quad - \left[\partial_\pm\chi \pm \frac{\eta\,\partial_\pm x^+}{z^2}\right]\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma_4^s + \cos\mu\, n^s_{14} + n^s_{23}\right)\right. \\ &\qquad\qquad \left. + \cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma_2^s + \cos\mu\, n^s_{12} + n^s_{34}\right) - \frac{i}{2}\cos\mu\,\gamma_5^s + \sin\mu\, n^s_{15}\right] \\ &\quad + \frac{1}{2}\partial_\pm\phi\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma_4^s + \cos\mu\, n^s_{14} - n^s_{23}\right)\right. \\ &\qquad\qquad \left. - \cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma_2^s + \cos\mu\, n^s_{12} - n^s_{34}\right)\right] \\ &\quad - \frac{1}{2}\partial_\pm\psi\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma_4^s + \cos\mu\, n^s_{14} + n^s_{23}\right)\right. \\ &\qquad\qquad \left. + \cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma_2^s + \cos\mu\, n^s_{12} + n^s_{34}\right)\right]. \end{aligned} \tag{53}$$

[4] For another embedding into type IIB supergravity, see [57].

Here we have performed a coordinate transformation,

$$
\begin{aligned}
x^{\pm} &= \frac{x^0 \pm x^3}{\sqrt{2}}\,, \\
r &= \mu\,, \quad \zeta = \tfrac{1}{2}\theta\,, \quad \psi_1 = \chi + \tfrac{1}{2}(\psi + \phi)\,, \quad \psi_2 = \chi + \tfrac{1}{2}(\psi - \phi)\,, \quad \psi_3 = \chi\,.
\end{aligned}
$$

With the deformed current (53) , the resulting background is given by

$$
\begin{aligned}
ds^2 &= \frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \eta^2 \frac{(dx^+)^2}{z^4} + ds^2_{\mathrm{S}^5}\,, \\
B_2 &= \frac{\eta}{z^2}\, dx^+ \wedge (d\chi + \omega)\,.
\end{aligned} \tag{54}
$$

Here the S^5 metric is written as an S^1-fibration over $\mathbb{C}\mathrm{P}^2$,

$$
\begin{aligned}
ds^2_{\mathrm{S}^5} &= (d\chi + \omega)^2 + ds^2_{\mathbb{C}\mathrm{P}^2}\,, \\
ds^2_{\mathbb{C}\mathrm{P}^2} &= d\mu^2 + \sin^2\mu \left(\Sigma_1^2 + \Sigma_2^2 + \cos^2\mu\, \Sigma_3^2\right).
\end{aligned} \tag{55}
$$

Now χ is the fiber coordinate and ω is a one-form potential of the Kähler form on $\mathbb{C}\mathrm{P}^2$. The symbols Σ_i $(i = 1, 2, 3)$ and ω are defined as

$$
\begin{aligned}
\Sigma_1 &\equiv \tfrac{1}{2}(\cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi)\,, \\
\Sigma_2 &\equiv \tfrac{1}{2}(\sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi)\,, \\
\Sigma_3 &\equiv \tfrac{1}{2}(d\psi + \cos\theta\, d\phi)\,, \qquad \omega \equiv \sin^2\mu\, \Sigma_3\,.
\end{aligned} \tag{56}
$$

It is remarkable that only the AdS_5 metric is deformed while the S^5 part is not, in spite of the expression of the classical r-matrix (51) . On the other hand, the NS-NS two-form carries two indices, one of which is from AdS_5 and the other is S^5 .

So far, we have considered the one-parameter deformation, but it is easy to reproduce three-parameter deformations of [58] and non-Cartan deformations of [59] (For the detail, see [27]) .

5.2. Lax pair

In the present case, the associated Lax pair is a bit messy but given by [27, 30]

$$
\begin{aligned}
\mathcal{L}_{\pm}^{\mathrm{Sch}} = &\ \frac{1}{z}\partial_\pm x^1 \left[\frac{\lambda^{\pm 1}}{2}\gamma_1^a - n_{15}^a\right] + \frac{1}{z}\partial_\pm x^2 \left[\frac{\lambda^{\pm 1}}{2}\gamma_2^a - n_{25}^a\right] + \frac{\lambda^{\pm 1}}{2z}\partial_\pm z\, \gamma_5^a \\
&+ \frac{1}{\sqrt{2}\,z}\partial_\pm x^+ \left[\frac{\lambda^{\pm 1}}{2}\gamma_0^a + \frac{\lambda^{\pm 1}}{2}\gamma_3^a - n_{05}^a - n_{35}^a\right] \\
&+ \frac{1}{\sqrt{2}\,z}\left[\partial_\pm x^- \pm \eta\partial_\pm\chi \pm \frac{\eta}{2}\sin^2\mu(\partial_\pm\psi + \cos\theta\partial_\pm\phi) + \frac{\eta^2}{z^2}\partial_\pm x^+\right] \\
&\qquad\qquad \times \left[\frac{\lambda^{\pm 1}}{2}\gamma_0^a - \frac{\lambda^{\pm 1}}{2}\gamma_3^a - n_{05}^a + n_{35}^a\right] \\
&- \frac{i\lambda^{\pm 1}}{2}\partial_\pm\mu\, \gamma_1^s - \frac{1}{2}\partial_\pm\theta \left[\frac{i\lambda^{\pm 1}}{2}\sin\mu\, \gamma_3^s + \cos\mu\, n_{13}^s\right] \\
&- \left[\partial_\pm\chi \pm \frac{\eta\, \partial_\pm x^+}{z^2}\right] \left[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\, \gamma_4^s + \cos\mu\, n_{14}^s + n_{23}^s\right)\right. \\
&\qquad \left. + \cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\, \gamma_2^s + \cos\mu\, n_{12}^s + n_{34}^s\right) - \frac{i\lambda^{\pm 1}}{2}\cos\mu\, \gamma_5^s + \sin\mu\, n_{15}^s\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\partial_\pm\phi\Big[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_4^s+\cos\mu\,n_{14}^s-n_{23}^s\right)\\
&\qquad\qquad -\cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_2^s+\cos\mu\,n_{12}^s-n_{34}^s\right)\Big]\\
&-\frac{1}{2}\partial_\pm\psi\Big[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_4^s+\cos\mu\,n_{14}^s+n_{23}^s\right)\\
&\qquad\qquad +\cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_2^s+\cos\mu\,n_{12}^s+n_{34}^s\right)\Big]\,.
\end{aligned}
\tag{57}
$$

As $\eta \to 0$, the above Lax pair $\mathcal{L}_\pm^{\rm Sch}$ is reduced to the following:

$$
\begin{aligned}
\mathcal{L}_\pm \;=\;& \frac{1}{z}\partial_\pm x^1\left[\frac{\lambda^{\pm 1}}{2}\gamma_1^a-n_{15}^a\right]+\frac{1}{z}\partial_\pm x^2\left[\frac{\lambda^{\pm 1}}{2}\gamma_2^a-n_{25}^a\right]+\frac{\lambda^{\pm 1}}{2z}\partial_\pm z\,\gamma_5^a\\
&+\frac{1}{\sqrt{2}\,z}\partial_\pm x^+\left[\frac{\lambda^{\pm 1}}{2}\gamma_0^a+\frac{\lambda^{\pm 1}}{2}\gamma_3^a-n_{05}^a-n_{35}^a\right]\\
&+\frac{1}{\sqrt{2}\,z}\partial_\pm x^-\left[\frac{\lambda^{\pm 1}}{2}\gamma_0^a-\frac{\lambda^{\pm 1}}{2}\gamma_3^a-n_{05}^a+n_{35}^a\right]\\
&-\frac{i\lambda^{\pm 1}}{2}\partial_\pm\mu\,\gamma_1^s-\frac{1}{2}\partial_\pm\theta\left[\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_3^s+\cos\mu\,n_{13}^s\right]\\
&-\partial_\pm\chi\Big[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_4^s+\cos\mu\,n_{14}^s+n_{23}^s\right)\\
&\qquad +\cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_2^s+\cos\mu\,n_{12}^s+n_{34}^s\right)-\frac{i\lambda^{\pm 1}}{2}\cos\mu\,\gamma_5^s+\sin\mu\,n_{15}^s\Big]\\
&+\frac{1}{2}\partial_\pm\phi\Big[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_4^s+\cos\mu\,n_{14}^s-n_{23}^s\right)\\
&\qquad\qquad -\cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_2^s+\cos\mu\,n_{12}^s-n_{34}^s\right)\Big]\\
&-\frac{1}{2}\partial_\pm\psi\Big[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_4^s+\cos\mu\,n_{14}^s+n_{23}^s\right)\\
&\qquad\qquad +\cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm 1}}{2}\sin\mu\,\gamma_2^s+\cos\mu\,n_{12}^s+n_{34}^s\right)\Big]\,.
\end{aligned}
\tag{58}
$$

5.3. Twisted boundary condition

The deformations can be reinterpreted as twisted boundary conditions, again. The following twisted boundary conditions

$$
\begin{aligned}
\tilde{x}^-(\sigma=2\pi) &= \tilde{x}^-(\sigma=0)+\frac{\eta}{\sqrt{\lambda_{\rm c}}}J_\chi+2\pi n_\chi\,,\\
\tilde{\chi}(\sigma=2\pi) &= \tilde{\chi}(\sigma=0)-\frac{\eta}{\sqrt{\lambda_{\rm c}}}P_-
\end{aligned}
\tag{59}
$$

with the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ is equivalent to the deformed background with the periodic boundary condition. Here P_- and J_χ are the Noether charges for translation and rotation invariance for the x^- and χ directions, respectively. The integer n_χ is a winding number for the χ direction. It may be interesting to consider a relation between the above argument and the symmetric two-form studied in [54].

6. Abelian twists of the global AdS$_5$

In section 3, we have considered gravity duals for gauge theories on noncommutative planes. Thus the deformed backgrounds have been described as deformations of the Poincaré AdS$_5$.

On the other hand, one may consider noncommutative deformations of the global AdS$_5$ [35], in which the dual gauge theories are living on $R\times$ deformed S^3 . The deformed backgrounds can be realized by performing abelian twists (equivalently TsT transformations) as in section 4 for the global AdS$_5$.

Here, let us consider abelian twists of the global AdS$_5$ as Yang-Baxter deformations. The twists are associated with the classical r-matrix [24],

$$r = -\frac{i}{2} n_{12}^a \wedge n_{03}^a \,. \tag{60}$$

This r-matrix is composed of two Cartan generators of $\mathfrak{su}(2,2)$ and deforms only the AdS$_5$ part. Hence we will omit the S^5 part hereafter.

6.1. *The deformed metric and NS-NS two-form*

Let us consider the following parameterization of a group element of $SU(2,2)$:

$$g_a(\tau,\sigma) = \exp\left[\frac{i}{2}(\phi_1\, h_1 + \phi_2\, h_2 + \tau\, h_3)\right] \exp\left[-\theta\, n_{13}^a\right] \exp\left[-\rho\, \frac{\gamma_1^a}{2}\right] \quad \in SU(2,2)\,. \tag{61}$$

The deformed current $J_\pm$ is expanded in terms of the basis of $\mathfrak{su}(2,2)$. Then, by solving the equations in (15) , $J_\pm$ can be determined as

$$\begin{aligned}
J_\pm \;=\;& -\partial_\pm\rho\frac{1}{2}\gamma_1^a - \partial_\pm\theta\left[\frac{1}{2}\sinh\rho\,\gamma_3^a + \cosh\rho\, n_{13}^a\right] \\
& + i\partial_\pm\tau\left[\frac{1}{2}\cosh\rho\,\gamma_5^a + \sinh\rho\, n_{15}^a\right] \\
& - \hat{G}\left(\partial_\pm\phi_1 \mp \eta\sin^2\theta\sinh^2\rho\,\partial_\pm\phi_2\right) \\
& \qquad\qquad \times\left[\cos\theta\left(\frac{1}{2}\sinh\rho\,\gamma_2^a + \cosh\rho\, n_{12}^a\right) + \sin\theta\, n_{23}^a\right] \\
& + i\hat{G}\left(\partial_\pm\phi_2 \pm \eta\cos^2\theta\sinh^2\rho\,\partial_\pm\phi_1\right) \\
& \qquad\qquad \times\left[\sin\theta\left(\frac{1}{2}\sinh\rho\,\gamma_0^a - \cosh\rho\, n_{01}^a\right) - \cos\theta\, n_{03}^a\right] ,
\end{aligned} \tag{62}$$

where $\hat{G}(\eta)$ is a scalar function defined as

$$\hat{G}^{-1}(\eta) \equiv 1 + \eta^2\sin^2\theta\cos^2\theta\sinh^2\rho\,. \tag{63}$$

By using the current (62) , the deformed metric and NS-NS two-form are obtained as

$$\begin{aligned}
ds^2 &= -\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \frac{\cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2}{1+\eta^2\sin^2\theta\cos^2\theta\sinh^4\rho}\right), \\
B_2 &= -\frac{\eta\sin^2\theta\cos^2\theta\sinh^4\rho}{1+\eta^2\sin^2\theta\cos^2\theta\sinh^4\rho}\, d\phi_1\wedge d\phi_2\,.
\end{aligned} \tag{64}$$

This result precisely agrees with the ones of abelian twists of the global AdS$_5$ [35]. Incidentally, three-parameter cases have been discussed in [24].

6.2. Lax pair

The next is to determine the associated Lax pair. By using the deformed current (62) , the Lax pair can be explicitly evaluated. The resulting expression is given by [24, 30]

$$\begin{aligned}
\mathcal{L}_{\pm}^{\rm AT} &= -\partial_{\pm}\rho\frac{\lambda^{\pm 1}}{2}\gamma_1^a - \partial_{\pm}\theta\left[\frac{\lambda^{\pm 1}}{2}\sinh\rho\,\gamma_3^a + \cosh\rho\, n_{13}^a\right] \\
&\quad + i\partial_{\pm}\tau\left[\frac{\lambda^{\pm 1}}{2}\cosh\rho\,\gamma_5^a + \sinh\rho\, n_{15}^a\right] \\
&\quad - \hat{G}\left(\partial_{\pm}\phi_1 \mp \eta\sin^2\theta\sinh^2\rho\partial_{\pm}\,\phi_2\right) \\
&\qquad\qquad \times\left[\cos\theta\left(\frac{\lambda^{\pm 1}}{2}\sinh\rho\,\gamma_2^a + \cosh\rho\, n_{12}^a\right) + \sin\theta\, n_{23}^a\right] \\
&\quad + i\hat{G}\left(\partial_{\pm}\phi_2 \pm \eta\cos^2\theta\sinh^2\rho\partial_{\pm}\,\phi_1\right) \\
&\qquad\qquad \times\left[\sin\theta\left(\frac{\lambda^{\pm 1}}{2}\sinh\rho\,\gamma_0^a - \cosh\rho\, n_{01}^a\right) - \cos\theta\, n_{03}^a\right]\,. \qquad (65)
\end{aligned}$$

The existence of the Lax pair was anticipated in [35], but the explicit form has been derived in [24, 30]. In the $\eta \to 0$ limit, the above Lax pair is reduced to the following form:

$$\begin{aligned}
\mathcal{L}_{\pm}^{\rm GAdS_5} &= -\partial_{\pm}\rho\frac{\lambda^{\pm 1}}{2}\gamma_1^a - \partial_{\pm}\theta\left[\frac{\lambda^{\pm 1}}{2}\sinh\rho\,\gamma_3^a + \cosh\rho\, n_{13}^a\right] \\
&\quad + i\partial_{\pm}\tau\left[\frac{\lambda^{\pm 1}}{2}\cosh\rho\,\gamma_5^a + \sinh\rho\, n_{15}^a\right] \\
&\quad - \partial_{\pm}\phi_1\left[\cos\theta\left(\frac{\lambda^{\pm 1}}{2}\sinh\rho\,\gamma_2^a + \cosh\rho\, n_{12}^a\right) + \sin\theta\, n_{23}^a\right] \\
&\quad + i\partial_{\pm}\phi_2\left[\sin\theta\left(\frac{\lambda^{\pm 1}}{2}\sinh\rho\,\gamma_0^a - \cosh\rho\, n_{01}^a\right) - \cos\theta\, n_{03}^a\right]\,. \qquad (66)
\end{aligned}$$

This is nothing but a Lax pair for the global AdS_5 . It may be possible to consider the three-parameter generalization, up to a subtlety of the signature for the time direction.

6.3. Twisted boundary condition

As in the previous three examples, the deformations can be reinterpreted as twisted boundary conditions again. After performing a similar analysis, one can see that the deformed backgrounds with the periodic boundary condition is equivalent to the undeformed theory with the following twisted boundary conditions:

$$\begin{aligned}
\tilde{\phi}_1(\sigma = 2\pi) &= \tilde{\phi}_1(\sigma = 0) + \frac{\eta}{\sqrt{\lambda_{\rm c}}}J_2 + 2\pi n_1\,, \\
\tilde{\phi}_2(\sigma = 2\pi) &= \tilde{\phi}_2(\sigma = 0) - \frac{\eta}{\sqrt{\lambda_{\rm c}}}J_1 + 2\pi n_2\,. \qquad (67)
\end{aligned}$$

Here J_i are the Noether charges for rotation invariance in the ϕ_i directions. The integers n_i are winding numbers along the ϕ_i directions.

7. Summary and Outlooks

In this article, we have given a short summary of Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring by focusing upon four type IIB supergravity backgrounds, 1) gravity duals for noncommutative gauge theories, 2) γ-deformations of S^5, 3) Schrödinger spacetimes and 4) abelian twists of the global AdS_5 . All of them are associated with abelian classical r-matrices.

For all of the examples presented here, it has been shown that the deformed backgrounds with the periodic boundary condition are equivalent with the undeformed one with twisted boundary conditions. This result has been anticipated by the preceding works [32, 60], because these backgrounds can also be realized as TsT transformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$. Thus it seems likely that the abelian classical r-matrices can be seen as abelian twists (equivalently TsT transformations). In fact, there are general arguments supporting this statement [29, 61]. This would be the case even in non-integrable cases like $T^{1,1}$ [52].

However, in non-abelian cases, it would not be the case. Then non-local gauge transformations should enter into the argument and one needs to take account of the global structure like an integration constant matrix as argued in the case of 3D Schrödinger spacetime [12]. In particular, periodic boundary conditions are not suitable for non-local gauge fields and further the undeformed geometry cannot be reproduced even after undoing the twist for the affine extended algebra. The resulting geometry is described in terms of the dipole-like coordinates [12].

More interesting observation is that the gravity/CYBE correspondence may contain S-duality as argued in [25]. It is quite natural that the integrability survives T-dualities, but it is not certain for S-dualities. It would be very nice to reveal a relation between the integrability and S-dualities from the viewpoint of Yang-Baxter deformations.

Thus there are a lot of open problems in the case of non-abelian classical r-matrices. As a matter of course, it is of significance to study Jordanian deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring. However, as a simpler exercise, it may be interesting to consider non-abelian classical r-matrices in the context of Yang-Baxter-deformations of 4D Minkowski spacetime [38]. An intriguing example is the κ-deformations of the Poincaré algebra. For the progress along this line, see the upcoming work [62].

So far, we have concentrated on the metric and NS-NS two-form. One of the most important issues to be confirmed is the Ramond-Ramond (R-R) sector and the dilaton. These can be investigated by including the fermionic sector and then performing supercoset construction. The κ-symmetry is preserved even after the deformations have been performed. Hence, one may expect that the R-R sector and the dilaton would be reproduced as well. However, it is still necessary to be confirmed directly[5]. A nice candidate is the Schrödinger spacetimes, where the dilaton is constant and the R-R sector is the same as the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ background.

Still, one need to make much effort towards establishing the gravity/CYBE correspondence.

Acknowledgments

We are very grateful to Io Kawaguchi for collaboration at an early stage of this work. We also appreciate Ben Hoare, Takashi Kameyama, Hideki Kyono, Jerzy Lukierski, Jun-ichi Sakamoto and Stijn J. van Tongeren for useful discussions. The work of K.Y. is supported by Supporting Program for Interaction-based Initiative Team Studies (SPIRITS) from Kyoto University and by the JSPS Grant-in-Aid for Scientific Research (C) No.15K05051. This work is also supported in part by the JSPS Japan-Russia Research Cooperative Program and the JSPS Japan-Hungary Research Cooperative Program.

[5] At the present moment, there is a puzzle in the case of the q-deformation [19, 20].

Appendix A. Notation and convention

We summarize here our notation and convention of the $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ generators.

The gamma matrices

Let us first introduce the following gamma matrices:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_0 = i\gamma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

To embed $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ into $\mathfrak{su}(2,2|4)$, we follow an 8×8 matrix representation as

$$\begin{aligned} \gamma^a_\mu &= \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma^a_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with} \qquad \mu = 0,1,2,3\,, \\ \gamma^s_i &= \begin{pmatrix} 0 & 0 \\ 0 & \gamma_i \end{pmatrix}, \quad \gamma^s_5 = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_5 \end{pmatrix} \qquad \text{with} \qquad i = 1,2,3,4\,. \end{aligned} \tag{A.2}$$

Note that each block of the matrices is a 4×4 matrix.

The $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ generators

The Lie algebras $\mathfrak{su}(2,2) \sim \mathfrak{so}(2,4)$ and $\mathfrak{su}(4) \sim \mathfrak{so}(6)$ are spanned as follows:

$$\begin{aligned} \mathfrak{su}(2,2) &= \mathrm{span}_{\mathbb{R}} \left\{ \frac{1}{2}\gamma^a_\mu\,,\, \frac{1}{2}\gamma^a_5\,,\, n^a_{\mu\nu} = \frac{1}{4}[\gamma^a_\mu\,,\gamma^a_\nu]\,,\, n^a_{\mu 5} = \frac{1}{4}[\gamma^a_\mu\,,\gamma^a_5] \;\middle|\; \mu\,,\nu = 0,1,2,3 \right\}, \\ \mathfrak{su}(4) &= \mathrm{span}_{\mathbb{R}} \left\{ \frac{1}{2}\gamma^s_i\,,\, \frac{1}{2}\gamma^s_5\,,\, n^s_{ij} = \frac{1}{4}[\gamma^s_i\,,\gamma^s_j]\,,\, n^s_{i5} = \frac{1}{4}[\gamma^s_i\,,\gamma^s_5] \;\middle|\; i,j = 1,2,3,4 \right\}. \end{aligned} \tag{A.3}$$

The subalgebras $\mathfrak{so}(1,4)$ and $\mathfrak{so}(5)$ in the spinor representation are formed as

$$\begin{aligned} \mathfrak{so}(1,4) &= \mathrm{span}_{\mathbb{R}}\{\, n^a_{\mu\nu}\,,\, n^a_{\mu 5} \mid \mu\,,\nu = 0,1,2,3 \,\}\,, \\ \mathfrak{so}(5) &= \mathrm{span}_{\mathbb{R}}\{\, n^s_{ij}\,,\, n^s_{i5} \mid i,j = 1,2,3,4 \,\}\,. \end{aligned} \tag{A.4}$$

For a coset construction of Poincaré AdS_5, it is useful to employ the following basis:

$$\mathfrak{su}(2,2) = \mathrm{span}_{\mathbb{R}}\{\, p_\mu\,, k_\mu\,, h_1\,, h_2\,, h_3\,, n^a_{13}\,, n^a_{10}\,, n^a_{23}\,, n^a_{20} \mid \mu = 0,1,2,3 \,\}\,. \tag{A.5}$$

Here the generators p_μ, k_μ and the Cartan generators h_1, h_2, h_3 are defined as

$$\begin{aligned} p_\mu &\equiv \frac{1}{2}\gamma^a_\mu - n^a_{\mu 5}\,, \qquad k_\mu \equiv \frac{1}{2}\gamma^a_\mu + n^a_{\mu 5}\,, \\ h_1 &\equiv 2i\, n^a_{12} = \mathrm{diag}(-1,1,-1,1,0,0,0,0)\,, \\ h_2 &\equiv 2n^a_{30} = \mathrm{diag}(-1,1,1,-1,0,0,0,0)\,, \end{aligned}$$

$$h_3 \equiv \gamma_5^a = \mathrm{diag}(1,1,-1,-1,0,0,0,0)\,.$$

Note that the generators p_μ and k_μ commute each other,

$$[p_\mu\,,p_\nu] = [k_\mu\,,k_\nu] = [p_\mu\,,k_\nu] = 0 \qquad \text{for} \qquad \mu\,,\nu = 0,1,2,3\,. \tag{A.6}$$

For the S^5 part, the Cartan generators $h_4\,,h_5\,,h_6$ of $\mathfrak{su}(4)$ are given by

$$\begin{aligned} h_4 &\equiv 2i\,n_{12}^s = \mathrm{diag}(0,0,0,0,-1,1,-1,1)\,, \\ h_5 &\equiv 2i\,n_{34}^s = \mathrm{diag}(0,0,0,0,-1,1,1,-1)\,, \\ h_6 &\equiv \gamma_5^s = \mathrm{diag}(0,0,0,0,1,1,-1,-1)\,. \end{aligned} \tag{A.7}$$

Since non-Cartan generators of $\mathfrak{su}(4)$ are not used in our analysis here, we will not write them down explicitly.

The bosonic coset projectors

In deriving the bosonic part of Lax pairs, it is necessary to employ the coset projectors P_0 and P_2 regarding the $\mathbb{Z}_2$-grading property. The projectors P_0 and P_2 are decomposed into the AdS_5 part and the S^5 part like

$$P_0(x) = P_0^a(x) + P_0^s(x)\,, \qquad P_2(x) = P_2^a(x) + P_2^s(x)\,, \tag{A.8}$$

where $P_0^{a,s}$ and $P_2^{a,s}$ are the following coset projectors for $\mathfrak{so}(2,4)$ and $\mathfrak{su}(4)$,

$$\begin{aligned} &P_0^a\ :\quad \mathfrak{su}(2,2) \longrightarrow \mathfrak{so}(1,4)\,, \qquad && P_2^a\ :\quad \mathfrak{su}(2,2) \longrightarrow \frac{\mathfrak{su}(2,2)}{\mathfrak{so}(1,4)}\,, \\ &P_0^s\ :\quad \mathfrak{su}(4) \longrightarrow \mathfrak{so}(5)\,, && P_2^s\ :\quad \mathfrak{su}(4) \longrightarrow \frac{\mathfrak{su}(4)}{\mathfrak{so}(5)}\,. \end{aligned} \tag{A.9}$$

These coset projectors can be represented by the $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ generators as follows:

$$\begin{aligned} P_0^a(x) &= \frac{1}{2}\sum_{\mu,\nu=0}^{3}\frac{\mathrm{Tr}[n_{\mu\nu}^a x]}{\mathrm{Tr}[n_{\mu\nu}^a n_{\mu\nu}^a]}n_{\mu\nu}^a + \sum_{\mu=0}^{3}\frac{\mathrm{Tr}[n_{\mu5}^a x]}{\mathrm{Tr}[n_{\mu5}^a n_{\mu5}^a]}n_{\mu5}^a\,, \\ P_2^a(x) &= \sum_{\mu=0}^{3}\frac{\mathrm{Tr}[\gamma_\mu^a x]}{\mathrm{Tr}[\gamma_\mu^a\gamma_\mu^a]}\gamma_\mu^a + \frac{\mathrm{Tr}[\gamma_5^a x]}{\mathrm{Tr}[\gamma_5^a\gamma_5^a]}\gamma_5^a\,, \\ P_0^s(x) &= \frac{1}{2}\sum_{\mu,\nu=1}^{4}\frac{\mathrm{Tr}[n_{\mu\nu}^s x]}{\mathrm{Tr}[n_{\mu\nu}^s n_{\mu\nu}]}n_{\mu\nu}^s + \sum_{\mu=1}^{4}\frac{\mathrm{Tr}[n_{\mu5}^s x]}{\mathrm{Tr}[n_{\mu5}^s n_{\mu5}^s]}n_{\mu5}^s\,, \\ P_2^s(x) &= \sum_{\mu=1}^{4}\frac{\mathrm{Tr}[\gamma_\mu^s x]}{\mathrm{Tr}[\gamma_\mu^s\gamma_\mu^s]}\gamma_\mu^s + \frac{\mathrm{Tr}[\gamma_5^s x]}{\mathrm{Tr}[\gamma_5^s\gamma_5^s]}\gamma_5^s\,. \end{aligned} \tag{A.10}$$

The projectors are utilized in evaluating the deformed metric, NS-NS two-form and Lax pair.

Journal of Physics: Conference Series **670** (2016) 012033 doi:10.1088/1742-6596/670/1/012033

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] [hep-th/9711200].

[2] N. Beisert *et al.*, "Review of AdS/CFT Integrability: An Overview," Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982 [hep-th]].

[3] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in $\mathrm{AdS}_5\times\mathrm{S}^5$ background," Nucl. Phys. B **533** (1998) 109 [hep-th/9805028].

[4] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring," Phys. Rev. D **69** (2004) 046002 [hep-th/0305116].

[5] G. Arutyunov and S. Frolov, "Foundations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ Superstring. Part I," J. Phys. A **42** (2009) 254003 [arXiv:0901.4937 [hep-th]].

[6] M. Heinze, "Spectrum and Quantum Symmetries of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring," arXiv:1507.03005 [hep-th].

[7] C. Klimcik, "Yang-Baxter sigma models and dS/AdS T duality," JHEP **0212** (2002) 051 [hep-th/0210095]; "On integrability of the Yang-Baxter sigma-model," J. Math. Phys. **50** (2009) 043508 [arXiv:0802.3518 [hep-th]].

[8] F. Delduc, M. Magro and B. Vicedo, "On classical q-deformations of integrable σ-models," JHEP **1311** (2013) 192 [arXiv:1308.3581 [hep-th]].

[9] I. Kawaguchi and K. Yoshida, "Hybrid classical integrability in squashed sigma models," Phys. Lett. B **705** (2011) 251 [arXiv:1107.3662 [hep-th]]; "Hybrid classical integrable structure of squashed sigma models: A short summary," J. Phys. Conf. Ser. **343** (2012) 012055 [arXiv:1110.6748 [hep-th]]; "Hidden Yangian symmetry in sigma model on squashed sphere," JHEP **1011** (2010) 032. [arXiv:1008.0776 [hep-th]].

[10] I. Kawaguchi, T. Matsumoto and K. Yoshida, "The classical origin of quantum affine algebra in squashed sigma models," JHEP **1204** (2012) 115 [arXiv:1201.3058 [hep-th]]; "On the classical equivalence of monodromy matrices in squashed sigma model," JHEP **1206** (2012) 082 [arXiv:1203.3400 [hep-th]].

[11] I. Kawaguchi, D. Orlando and K. Yoshida, "Yangian symmetry in deformed WZNW models on squashed spheres," Phys. Lett. B **701** (2011) 475. [arXiv:1104.0738 [hep-th]]; I. Kawaguchi and K. Yoshida, "A deformation of quantum affine algebra in squashed WZNW models," J. Math. Phys. **55** (2014) 062302 [arXiv:1311.4696 [hep-th]].

[12] I. Kawaguchi and K. Yoshida, "Classical integrability of Schrodinger sigma models and q-deformed Poincare symmetry," JHEP **1111** (2011) 094 [arXiv:1109.0872 [hep-th]]; "Exotic symmetry and monodromy equivalence in Schrodinger sigma models," JHEP **1302** (2013) 024 [arXiv:1209.4147 [hep-th]]; I. Kawaguchi, T. Matsumoto and K. Yoshida, "Schroedinger sigma models and Jordanian twists," JHEP **1308** (2013) 013 [arXiv:1305.6556 [hep-th]].

[13] T. Kameyama and K. Yoshida, "String theories on warped AdS backgrounds and integrable deformations of spin chains," JHEP **1305** (2013) 146 [arXiv:1304.1286 [hep-th]]; "Anisotropic Landau-Lifshitz sigma models from q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstrings," JHEP **1408** (2014) 110 [arXiv:1405.4467 [hep-th]].

[14] F. Delduc, M. Magro and B. Vicedo, "An integrable deformation of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action," Phys. Rev. Lett. **112** (2014) 051601 [arXiv:1309.5850 [hep-th]]; F. Delduc, M. Magro and B. Vicedo, "Derivation of the action and symmetries of the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring," JHEP **1410**, 132 (2014) [arXiv:1406.6286 [hep-th]].

[15] V. G. Drinfel'd, "Hopf algebras and the quantum Yang-Baxter equation," Sov. Math. Dokl. **32** (1985) 254; "Quantum groups," J. Sov. Math. **41** (1988) 898 [Zap. Nauchn. Semin. **155**, 18 (1986)]; M. Jimbo, "A q difference analog of $U(g)$ and the Yang-Baxter equation," Lett. Math. Phys. **10** (1985) 63.

[16] G. Arutyunov, R. Borsato and S. Frolov, "S-matrix for strings on η-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$," JHEP **1404** (2014) 002 [arXiv:1312.3542 [hep-th]].

[17] B. Hoare, R. Roiban and A. A. Tseytlin, "On deformations of $\mathrm{AdS}_n\times\mathrm{S}^n$ supercosets," JHEP **1406** (2014) 002 [arXiv:1403.5517 [hep-th]]; B. Hoare and A. A. Tseytlin, "On integrable deformations of superstring sigma models related to $AdS_n \times S^n$ supercosets," Nucl. Phys. B **897** (2015) 448 [arXiv:1504.07213 [hep-th]].

[18] O. Lunin, R. Roiban and A. A. Tseytlin, "Supergravity backgrounds for deformations of $\mathrm{AdS}_n\times S^n$ supercoset string models," Nucl. Phys. B **891** (2015) 106 [arXiv:1411.1066 [hep-th]].

[19] G. Arutyunov, R. Borsato and S. Frolov, "Puzzles of eta-deformed $AdS_5 \times S^5$," arXiv:1507.04239 [hep-th].

[20] B. Hoare and A. A. Tseytlin, "Type IIB supergravity solution for the T-dual of the eta-deformed $AdS_5 \times S^5$ superstring," arXiv:1508.01150 [hep-th].

[21] I. Kawaguchi, T. Matsumoto and K. Yoshida, "Jordanian deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring," JHEP **1404** (2014) 153 [arXiv:1401.4855 [hep-th]];

[22] I. Kawaguchi, T. Matsumoto and K. Yoshida, "A Jordanian deformation of AdS space in type IIB supergravity," JHEP **1406** (2014) 146 [arXiv:1402.6147 [hep-th]].

[23] T. Matsumoto and K. Yoshida, "Lunin-Maldacena backgrounds from the classical Yang-Baxter equation – Towards the gravity/CYBE correspondence," JHEP **1406** (2014) 135 [arXiv:1404.1838 [hep-th]].

[24] T. Matsumoto and K. Yoshida, "Integrability of classical strings dual for noncommutative gauge theories," JHEP **1406** (2014) 163 [arXiv:1404.3657 [hep-th]].
[25] T. Matsumoto and K. Yoshida, "Yang-Baxter deformations and string dualities," JHEP **1503** (2015) 137 [arXiv:1412.3658 [hep-th]].
[26] T. Matsumoto and K. Yoshida, "Yang-Baxter sigma models based on the CYBE," Nucl. Phys. B **893** (2015) 287 [arXiv:1501.03665 [hep-th]].
[27] T. Matsumoto and K. Yoshida, "Schrödinger geometries arising from Yang-Baxter deformations," JHEP **1504** (2015) 180 [arXiv:1502.00740 [hep-th]].
[28] S. J. van Tongeren, "On classical Yang-Baxter based deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring," JHEP **1506** (2015) 048 [arXiv:1504.05516 [hep-th]].
[29] S. J. van Tongeren, "Yang-Baxter deformations, AdS/CFT, and twist-noncommutative gauge theory," arXiv:1506.01023 [hep-th].
[30] T. Kameyama, H. Kyono, J. i. Sakamoto and K. Yoshida, "Lax pairs on Yang-Baxter deformed backgrounds," arXiv:1509.00173 [hep-th].
[31] O. Lunin and J. M. Maldacena, "Deforming field theories with $U(1)\times U(1)$ global symmetry and their gravity duals," JHEP **0505** (2005) 033 [hep-th/0502086].
[32] S. Frolov, "Lax pair for strings in Lunin-Maldacena background," JHEP **0505** (2005) 069 [hep-th/0503201].
[33] A. Hashimoto and N. Itzhaki, "Noncommutative Yang-Mills and the AdS / CFT correspondence," Phys. Lett. B **465** (1999) 142 [hep-th/9907166].
J. M. Maldacena and J. G. Russo, "Large N limit of noncommutative gauge theories," JHEP **9909** (1999) 025 [hep-th/9908134].
[34] C. P. Herzog, M. Rangamani and S. F. Ross, "Heating up Galilean holography," JHEP **0811** (2008) 080 [arXiv:0807.1099 [hep-th]]; J. Maldacena, D. Martelli and Y. Tachikawa, "Comments on string theory backgrounds with non-relativistic conformal symmetry," JHEP **0810** (2008) 072 [arXiv:0807.1100 [hep-th]]; A. Adams, K. Balasubramanian and J. McGreevy, "Hot Spacetimes for Cold Atoms," JHEP **0811** (2008) 059 [arXiv:0807.1111 [hep-th]].
[35] D. Dhokarh, S. S. Haque and A. Hashimoto, "Melvin Twists of global $\mathrm{AdS}_5\times\mathrm{S}^5$ and their Non-Commutative Field Theory Dual," JHEP **0808** (2008) 084 [arXiv:0801.3812 [hep-th]]; T. McLoughlin and I. Swanson, "Integrable twists in AdS/CFT," JHEP **0608** (2006) 084 [hep-th/0605018].
[36] T. Matsumoto and K. Yoshida, "Integrable deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring and the classical Yang-Baxter equation – Towards the gravity/CYBE correspondence –," J. Phys. Conf. Ser. **563** (2014) 1, 012020 [arXiv:1410.0575 [hep-th]].
[37] H. Lin, O. Lunin and J. M. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," JHEP **0410** (2004) 025 [hep-th/0409174].
[38] T. Matsumoto, D. Orlando, S. Reffert, J. Sakamoto and K. Yoshida, "Yang-Baxter deformations of Minkowski spacetime," arXiv:1505.04553 [hep-th].
[39] M. A. Melvin, "Pure Magnetic And Electric Geons," Phys. Lett. **8** (1964) 65.
[40] G. W. Gibbons and K. i. Maeda, "Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields," Nucl. Phys. B **298** (1988) 741.
[41] A. A. Tseytlin, "Melvin solution in string theory," Phys. Lett. B **346** (1995) 55 [hep-th/9411198].
[42] A. Hashimoto and K. Thomas, "Dualities, twists, and gauge theories with non-constant non-commutativity," JHEP **0501** (2005) 033 [hep-th/0410123].
[43] C. R. Nappi and E. Witten, "A WZW model based on a nonsemisimple group," Phys. Rev. Lett. **71** (1993) 3751 [hep-th/9310112].
[44] A. Hashimoto and S. Sethi, "Holography and string dynamics in time dependent backgrounds," Phys. Rev. Lett. **89** (2002) 261601 [hep-th/0208126].
[45] M. Spradlin, T. Takayanagi and A. Volovich, "String theory in beta deformed spacetimes," JHEP **0511** (2005) 039 [hep-th/0509036].
[46] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoy, "Q deformation of Poincare algebra," Phys. Lett. B **264** (1991) 331.
J. Lukierski and A. Nowicki, "Quantum deformations of D = 4 Poincare and Weyl algebra from Q deformed D = 4 conformal algebra," Phys. Lett. B **279** (1992) 299.
[47] A. Ballesteros, F. J. Herranz and N. R. Bruno, "Quantum (anti)de Sitter algebras and generalizations of the kappa-Minkowski space," hep-th/0409295.
[48] I. R. Klebanov and E. Witten, "Superconformal field theory on three-branes at a Calabi-Yau singularity," Nucl. Phys. B **536** (1998) 199 [hep-th/9807080].
[49] P. Basu and L. A. Pando Zayas, "Chaos Rules out Integrability of Strings in $\mathrm{AdS}_5\times T^{1,1}$," Phys. Lett. B **700** (2011) 243 [arXiv:1103.4107 [hep-th]]; "Analytic Non-integrability in String Theory," Phys. Rev. D **84** (2011) 046006 [arXiv:1105.2540 [hep-th]].

[50] Y. Asano, D. Kawai, H. Kyono and K. Yoshida, "Chaotic strings in a near Penrose limit of $\mathrm{AdS}_5 \times T^{1,1}$," JHEP **1508** (2015) 060 [arXiv:1505.07583 [hep-th]].
[51] A. Catal-Ozer, "Lunin-Maldacena deformations with three parameters," JHEP **0602** (2006) 026 [hep-th/0512290].
[52] P. M. Crichigno, T. Matsumoto and K. Yoshida, "Deformations of $T^{1,1}$ as Yang-Baxter sigma models," JHEP **1412** (2014) 085 [arXiv:1406.2249 [hep-th]].
[53] R. G. Leigh and M. J. Strassler, "Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory," Nucl. Phys. B **447** (1995) 95 [arXiv:hep-th/9503121].
[54] S. Schafer-Nameki, M. Yamazaki and K. Yoshida, "Coset Construction for Duals of Non-relativistic CFTs," JHEP **0905** (2009) 038 [arXiv:0903.4245 [hep-th]].
[55] D. Israel, C. Kounnas, D. Orlando and P. M. Petropoulos, "Electric/magnetic deformations of S^3 and AdS_3, and geometric cosets," Fortsch. Phys. **53** (2005) 73 [hep-th/0405213].
[56] D. T. Son, "Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrödinger symmetry," Phys. Rev. D **78** (2008) 046003 [arXiv:0804.3972 [hep-th]];
K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," Phys. Rev. Lett. **101** (2008) 061601 [arXiv:0804.4053 [hep-th]].
[57] S. A. Hartnoll and K. Yoshida, "Families of IIB duals for nonrelativistic CFTs," JHEP **0812** (2008) 071 [arXiv:0810.0298 [hep-th]].
[58] N. Bobev and A. Kundu, "Deformations of Holographic Duals to Non-Relativistic CFTs," JHEP **0907** (2009) 098 [arXiv:0904.2873 [hep-th]].
[59] N. Bobev, A. Kundu and K. Pilch, "Supersymmetric IIB Solutions with Schrödinger Symmetry," JHEP **0907** (2009) 107 [arXiv:0905.0673 [hep-th]].
[60] L. F. Alday, G. Arutyunov and S. Frolov, "Green-Schwarz strings in TsT-transformed backgrounds," JHEP **0606** (2006) 018 [hep-th/0512253].
[61] B. Vicedo, "Deformed integrable σ-models, classical R-matrices and classical exchange algebra on Drinfel'd doubles," J. Phys. A **48** (2015) 35, 355203 [arXiv:1504.06303 [hep-th]].
[62] A. Borowiec, H. Kyono, J. Likierski, J. Sakamoto and K. Yoshida, "Yang-Baxter sigma models and Lax pairs arising from κ-Poincaré r-matrices," to appear.