PAPER

The Online Graph Exploration Problem on Restricted Graphs

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SUMMARY The purpose of the online graph exploration problem is to visit all the nodes of a given graph and come back to the starting node with the minimum total traverse cost. However, unlike the classical Traveling Salesperson Problem, information of the graph is given online. When an online algorithm (called a searcher) visits a node v, then it learns information on nodes and edges adjacent to v. The searcher must decide which node to visit next depending on partial and incomplete information of the graph that it has gained in its searching process. The goodness of the algorithm is evaluated by the *competitive analysis*. If input graphs to be explored are restricted to trees, the depth-first search always returns an optimal tour. However, if graphs have cycles, the problem is non-trivial. In this paper we consider two simple cases. First, we treat the problem on simple cycles. Recently, Asahiro et al. proved that there is a 1.5-competitive online algorithm, while no online algorithm can be $(1.25 - \epsilon)$-competitive for any positive constant ϵ. In this paper, we give an optimal online algorithm for this problem; namely, we give a $\frac{1+\sqrt{3}}{2}(\simeq 1.366)$-competitive algorithm, and prove that there is no $(\frac{1+\sqrt{3}}{2} - \epsilon)$-competitive algorithm for any positive constant ϵ. Furthermore, we consider the problem on unweighted graphs. We also give an optimal result; namely we give a 2-competitive algorithm and prove that there is no $(2 - \epsilon)$-competitive online algorithm for any positive constant ϵ.

key words: *the graph exploration problem, online algorithm, competitive analysis*

1. Introduction

In the Traveling Salesperson Problem (TSP) [14], we are given a graph and non-negative weights (lengths) on edges. Our task is to find a tour visiting all the nodes and coming back to the starting node with minimum cost. The cost of a tour is the total length of the tour. This problem is a well-known NP-hard problem, and there have been intensive studies such as heuristics and approximation algorithms. Apparently, TSP has plenty of practical applications, which includes determining a pickup or delivery tour for delivery companies or minimizing the total movement cost of robot arms in LSI wiring.

In TSP, all information on the graph is given to the algorithm *in advance*. However, in some cases of real applications, the terrain may be unknown until the algorithm visits the place, and the algorithm learns the local environment when it actually visits there. For example, suppose that we wish to gather complete information of an unknown environment using a robot searcher. At the beginning, the robot has no knowledge of the environment. It should decide where to visit next depending only on the partial information of the environment that it has gained through the exploration so far. This kind of problem is known as the *exploration* or the *map construction* problem and there are several models. In [13], the problem is formulated in an online problem on undirected edge-weighted graphs as follows: At the beginning, the searcher is at the starting node o, called the *origin*, and it knows the local information, namely, the labels of the nodes adjacent to o, and the weights of edges incident to o. When the searcher visits a node v, then it learns the labels of nodes adjacent to v and the weights of edges incident to v. When the searcher moves from u to v along the edge (u, v), it costs the weight of (u, v). The task of the searcher is to visit all the nodes and return to the origin with as small cost as possible. The goodness of the algorithm is evaluated by the *competitive analysis* [5], [10].

The most natural algorithm one may consider is the greedy type *Nearest Neighbor* algorithm (NN), which always visits a node nearest to the current node, among those that have not yet been visited. However, it has been shown that NN is not competitive even for planar graphs; there exists a planar graph G with n nodes such that the competitive ratio of NN is $\Omega(\log n)$ [15]. Kalyanasundaram and Pruhs [13] proposed a modified version of NN, called ShortCut, and proved that it is 16-competitive for planar graphs.

Note that if input graphs to be explored are restricted to trees, the depth-first search always returns an optimal tour because to visit all nodes and come back to the origin, each edge must be traversed at least twice, and the depth-first search traverses each edge exactly twice. Hence, the simplest non-trivial case is probably cycles. Recently, Asahiro et al. [2] considered the graph exploration on cycles. They proved that NN achieves the competitive ratio of 1.5 and showed that no online algorithm can have the competitive ratio better than 1.25.

Our Results.

In this paper we consider the problems on two classes of graphs and give tight bounds on the competitive ratio for both cases. First we improve both upper and lower bounds of the problem on cycles, and give a tight bound $\frac{1+\sqrt{3}}{2}(\simeq 1.366)$. For improving the upper bound, we propose a new

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algorithm called DIST, which decides the next node to visit depending on the (weighted) distances between the current node and each of the unvisited two nodes, the total length of the exploration so far, and the distance from the origin to the current node. We also consider the problem on unweighted graphs, and give a tight bound of 2; namely, we prove that algorithm DFS is 2-competitive and that no online algorithm can have the competitive ratio better than 2 (this lower bound holds even when graphs have planarity).

Related Results.

There are several variants of the problem of exploring unknown environment online. Deng and Papadimitriou [6] considered the problem of exploring a directed unweighted graph. This problem requires us to explore not only all nodes but also all edges, and the cost of the searcher is measured by the total number of edges traversed. They gave an online algorithm with $d^{O(d)}m$ edge traversals, where m is the number of edges in the graph and d is the minimum number of edges that have to be added to make the graph Eulerian. Albers and Henzinger [1] presented an algorithm that achieves an upper bound of $d^{O(\log d)}m$, and Fleischer and Trippen [9] gave an algorithm with an upper bound of $O(d^8 m)$. Fleischer and Trippen [8] also made an experimental study of major online graph traversal algorithms and evaluated their practical performance on various graph families. In the polygon exploration problem (e.g. [7], [12]), an unknown environment is modeled by a polygon. The task of a searcher is to see all the boundaries of the polygon and come back to the starting point. Ausiello et al. [3] and Ausiello et al. [4] have studied the online traveling salesperson problem in which requests are presented online, and the aim of the searcher is to visit each requested point (not necessarily in the order of requests, unlike the k-server problem).

2. Preliminaries

The purpose of the Online Graph Exploration problem is to visit all the nodes of a given graph $G = (V, E)$, where V and E denote the sets of nodes and edges, respectively. For each edge $(u, v) \in E$, a non-negative weight $\ell(u, v)$, sometimes called the *length*, is associated. Initially, the searcher is at the specified node $o \in V$, called the *origin*. It knows only the labels of the nodes adjacent to o, and the length of edges connecting o with those neighborhood nodes. Once the searcher visits a node v, it learns the labels of nodes adjacent to v and the length of edges incident to v. The searcher has a sufficiently large memory so that it can store all information obtained so far, namely, the labels of nodes, the weights of edges, and the topology of the subgraph consisting of nodes and edges it has already learned. The task of the searcher is to determine the next node to visit, using only the current knowledge. The goal of the searcher is to visit all the nodes and return to the origin. The *cost* of the searcher for the graph G is the total length of the tour made by the searcher on G.

The performance of an online algorithms is evaluated by the *competitive analysis*: Let ALG(G) denote the cost of an algorithm ALG on G, and let OPT(G) denote the cost of an optimal *offline* algorithm OPT for G. We say that ALG is *c-competitive* for a class of graphs $\mathcal{G}$ if ALG$(G)/$OPT$(G) \leq c$ for any graph $G \in \mathcal{G}$. We may write ALG and OPT instead of ALG(G) and OPT(G), respectively, when G is clear.

3. A Tight Bound on Cycles

In this section we consider the problem on cycles and give a tight bound for the competitive ratio. Here is one simple but important fact [2]. Let ℓ_{max} be the maximum length of edges and $L = \Sigma_{(u,v) \in E}\ell(u, v)$ be the sum of the length of all edges.

Fact 1: For any cycle C, OPT$(C) = L$ if $\ell_{max} \leq \frac{L}{2}$, and OPT$(C) = 2(L - \ell_{max})$ if $\ell_{max} > \frac{L}{2}$.

3.1 A Lower Bound

In this section, we give a lower bound on the competitive ratio for any online algorithm.

Theorem 1: For any positive constant ϵ, there is no $(\frac{1+\sqrt{3}}{2} - \epsilon)$-competitive online algorithm for cycles.

Proof. We will introduce an adversary giving the above mentioned lower bound. Fix an integer n and a constant μ such that $n > \frac{\sqrt{3}}{\epsilon}$ and $\mu < 1$. First, the adversary reveals two edges (o, u_1) and (o, v) incident to the origin with the equal length one. Without loss of generality, assume that the searcher moves to u_1. Then, the adversary reveals an edge (u_1, u_2) such that $\ell(u_1, u_2) = 1$. If the searcher visits u_2, then a new edge (u_2, u_3) with $\ell(u_2, u_3) = 1$ is revealed. Similarly, as long as the searcher visits a new node u_i $(i \leq n - 1)$, the adversary gives an edge (u_i, u_{i+1}) with $\ell(u_i, u_{i+1}) = 1$.

Suppose that the searcher visits the node v before visiting u_n, and suppose that this happens just after it visited u_t where $t \leq n - 1$ (i.e. it went back to v when it saw the edge (u_t, u_{t+1})) (Fig. 1 (a)). Then the edge (v, u_{t+1}) with weight t is revealed (Fig. 1 (b)). The only unvisited node is u_{t+1}, and the best way for the searcher is to go to u_{t+1} directly from

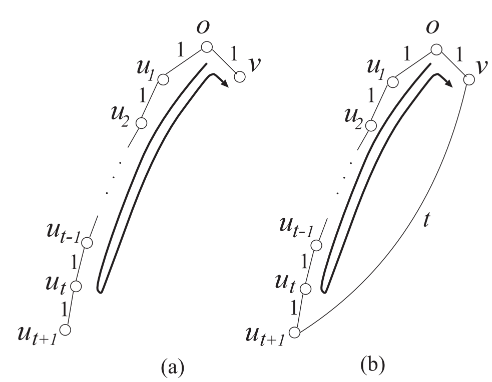

Fig. 1 Lower bound construction for cycles (I).

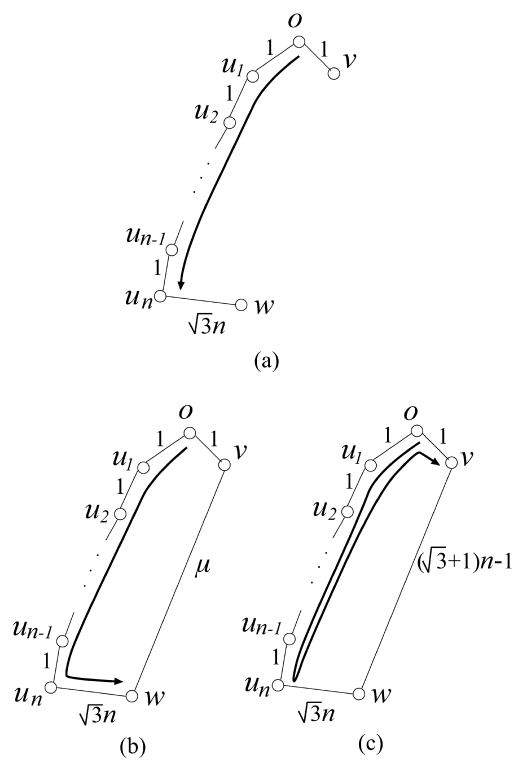

Fig. 2 Lower bound construction for cycles (II).

v, and go back to the origin by either clockwise or counter-clockwise direction. The cost of the searcher is then $4t + 2$. The optimal tour is to visit all nodes along the cycle, whose cost is $2t + 2$. The competitive ratio in this case is then $(4t + 2)/(2t + 2) \geq 1.5$ since $t \geq 1$.

Next, suppose that the searcher visits u_n before visiting v. Then the adversary gives an edge (u_n, w) with length $\sqrt{3}n$ (Fig. 2 (a)). We have two cases. First, suppose that the searcher visits w. Then the adversary reveals the edge (w, v) such that $\ell(w, v) = \mu$ (Fig. 2 (b)). The best way for the searcher is to visit v and o in this order (note that $\mu < 1$). The cost of the searcher is then $n + \sqrt{3}n + \mu + 1 = (1 + \sqrt{3})n + \mu + 1$. Note that the edge (u_n, w) has the length more than half the total length of the whole cycle. So, by Fact 1, the optimal cost is $2(n + \mu + 1)$. The competitive ratio is $\frac{(1+\sqrt{3})n+\mu+1}{2(n+\mu+1)} = \frac{1+\sqrt{3}}{2} - \frac{\sqrt{3}(\mu+1)}{2(n+\mu+1)} > \frac{1+\sqrt{3}}{2} - \frac{\sqrt{3}(\mu+1)}{2n} > \frac{1+\sqrt{3}}{2} - \frac{\sqrt{3}(1+1)}{2n} > \frac{1+\sqrt{3}}{2} - \epsilon$.

Finally, suppose that after the edge (u_n, w) is revealed, the searcher goes back to the node v. In this case, the adversary reveals the edge (v, w) with $\ell(v, w) = (\sqrt{3} + 1)n - 1$ (Fig. 2 (c)). Then the only unvisited node is w, and the best way for the searcher is now to visit w directly from v, and then go back to the origin in either clockwise or counter-clockwise direction. The total cost of the tour is $n + n + 1 + (\sqrt{3} + 1)n - 1 + (\sqrt{3} + 1)n = (2\sqrt{3} + 4)n$. The optimal tour is a one along the cycle, whose cost is $(2\sqrt{3} + 2)n$. The competitive ratio in this case is $\frac{(2\sqrt{3}+4)n}{(2\sqrt{3}+2)n} = \frac{1+\sqrt{3}}{2}$. □

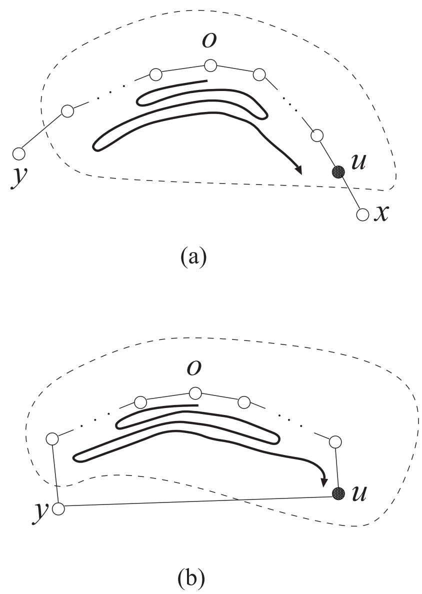

Fig. 3 Description of the algorithm.

3.2 An Upper Bound

In this section, we give an online algorithm DIST and analyze its competitive ratio.

3.2.1 Algorithm DIST

Since a given graph is a cycle, there are always two choices for the searcher: (except for the 1st step), either to go forward or to go back. (See Fig. 3 (a). The visited nodes are surrounded by a dotted curve, and the current position of the searcher is indicated by the black node.) Before presenting the algorithm, we give a few notations. Suppose that as shown in Fig. 3 (a), the searcher is currently at the node u, and is to determine which of x and y to visit. For any two nodes v_1 and v_2, let $d(v_1, v_2)$ denote the distance between v_1 and v_2 along the edges already known. Let X be the total length the searcher has traversed so far, and define $W = X - d(o, u)$. The value of W may change as time goes, so it might be appropriate to express it as e.g. W_i for Step i. However, for conciseness, we use W when there is no fear of confusion, or we sometimes say as "W-value at this moment". Now, we are ready to give our algorithm DIST:

Step 1: The searcher is at the origin o, and there are two nodes adjacent to o. It moves to the node closer to o. If both are in the same distance, it chooses arbitrary one.

Step i ($i \geq 2$): Suppose that the searcher is at a node u as shown in Fig. 3 (a). If $\ell(u, x) \leq \sqrt{3}d(u, y) - W$, then the searcher moves to x. Otherwise, i.e., if $\ell(u, x) > \sqrt{3}d(u, y) - W$, then the searcher moves to y.

Final step: The current situation is like Fig. 3 (b). When the searcher visits a node u, it learns that u is connected to the unvisited but known node y (since it has seen y when it was on the node of the other side). Now, it knows the entire graph, and there is only one unvisited node y. The searcher selects the shorter path from u to y, and then the shorter path from y to o.

3.2.2 Competitive Analysis

In this subsection, we prove the following theorem:

Theorem 2: DIST is $\frac{1+\sqrt{3}}{2}$-competitive for cycles.

Proof. Consider any time step of the online game, and suppose that the situation is like Fig. 3 (a). (In the case just before the final step, x is equal to y as Fig. 3 (b).) Let W be the current W-value, namely, the total distance the searcher has traversed so far minus $d(o, u)$. The following lemma is crucial in our analysis:

Lemma 3: For anytime before the final step, $W \le (\sqrt{3} - 1)d(o, y)$.

Note: In the case of Fig. 3 (b), $d(o, u)$ is the distance from o to u in a clockwise direction, and $d(o, y)$ is the distance from o to y in a counter-clockwise direction.

Proof. The proof is by induction. After Step 1 is performed, the situation is like Fig. 4 (a). Since $W = \ell(o, u) - \ell(o, u) = 0$, the inequality holds clearly.

Next, we assume that the inequality holds after Step i, and show that it holds after Step $i+1$. Suppose that after the execution of Step i, the situation is like Fig. 3 (a). We denote the W-value at this moment by W_i. By the induction hypothesis, the following inequality holds: $W_i \le (\sqrt{3} - 1)d(o, y)$. There are two cases to consider depending on whether the searcher moves to x or y in the next step.

Case 1. The searcher moves to x at Step $i + 1$. Then the situation is like Fig. 4 (b). Note that by definition, the current W-value, denoted by W_{i+1}, is $W_{i+1} = W_i + \ell(u, x) - \ell(u, x) = W_i$. (This means that if the searcher goes forward, then the W-value remains unchanged. This property will be sometimes used hereafter.) So $W_{i+1} \le (\sqrt{3} - 1)d(o, y)$ by the induction hypothesis, which implies that the inequality holds after Step $i + 1$.

Case 2. The searcher moves to y at Step $i + 1$. Then the situation is like Fig. 4 (c). The total length of the tour by the searcher increases by $d(y, o) + d(o, u)$ at this step. The distance from the origin was $d(o, u)$ but is now $d(o, y)$. Hence, $W_{i+1} = W_i + d(y, o) + d(o, u) - d(o, y) + d(o, u) = W_i + 2d(o, u)$. Since the searcher selected y rather than x, $\ell(u, x) > \sqrt{3}d(u, y) - W_i$ holds. Also, by the induction hypothesis, $W_i \le (\sqrt{3} - 1)d(o, y)$. From these two inequalities and the equality $d(u, y) = d(u, o) + d(o, y)$, we have $d(o, y) < \ell(u, x) - \sqrt{3}d(u, o)$. Now using the hypothesis again, we have

$$\begin{aligned} W_{i+1} &= W_i + 2d(o, u) \\ &\le (\sqrt{3} - 1)d(o, y) + 2d(o, u) \\ &< (\sqrt{3} - 1)(\ell(u, x) - \sqrt{3}d(u, o)) + 2d(o, u) \\ &= (\sqrt{3} - 1)(\ell(u, x) + d(o, u)) \\ &= (\sqrt{3} - 1)d(o, x) \end{aligned}$$

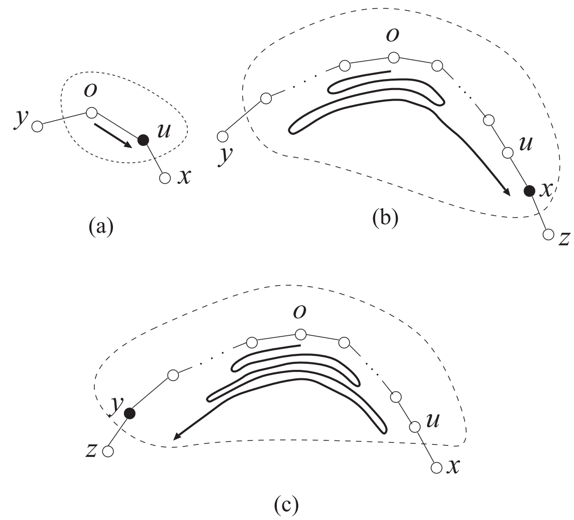

Fig. 4 Proof of Lemma 3.

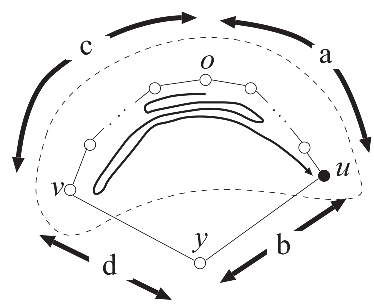

Fig. 5 Final step of DIST.

as required. □

Now, suppose that we are at the moment just before the final step. Then, the current situation looks like Fig. 5. The searcher is at the node u, and it just learned that the node y adjacent to u is the same as the one it saw from v before. Since the searcher learned $\ell(u, y)$, it came to know the whole information on the cycle.

For simplicity, let a, b, c, and d denote the lengths of paths (edges) $d(o, u)$, $\ell(u, y)$, $d(o, v)$, and $\ell(v, y)$, respectively, as depicted in Fig. 5. Let W^* be the W-value at this moment. Then, by Lemma 3, $W^* \le (\sqrt{3} - 1)d(o, y) = (\sqrt{3} - 1)(c + d)$ holds. The only unvisited node is y, and the searcher will visit y in either clockwise or counter-clockwise direction depending on which route is shorter, and then go back to o from y in either clockwise or counter-clockwise direction, again depending on which route is shorter. We will do a case analysis.

Case 1. $b > a + c + d$. In this case, $a + b > c + d$ holds. So, the searcher visits y by way of o, in a counter-clockwise direction, and goes back to o by way of v, in a clockwise direction. Since the cost of the searcher before the final step is $W^* + d(o, u) = W^* + a$ by the definition of the W-value, the final cost is DIST $= W^* + a + (a + c + d) + (c + d) = W^* + 2(a + c + d)$. Because $b > a + c + d$, $\ell_{max} = b > L/2$. So, the optimal cost is OPT $= 2(a + c + d)$ by Fact 1. The competitive ratio is

$$\frac{\mathsf{DIST}}{\mathsf{OPT}} = \frac{W^* + 2(a+c+d)}{2(a+c+d)} \le 1 + \frac{(\sqrt{3}-1)(c+d)}{2(a+c+d)} \le 1 + \frac{\sqrt{3}-1}{2} = \frac{1+\sqrt{3}}{2}.$$

Case 2. $b \le a+c+d$ and $a+b \le c+d$. The searcher visits y using the edge (u, y), and goes back to o in a counter-clockwise direction, i.e., by way of u. So, $\mathsf{DIST} = W^* + a + b + (b + a) = W^* + 2(a + b)$. Let e_{max} be an edge with maximum length, namely, $\ell(e_{max}) = \ell_{max}$. We consider subcases according to the length and position of e_{max}.

Case 2-(i). $\ell_{max} \le L/2$. By Fact 1, $\mathsf{OPT} = a+b+c+d$. Thus,

$$\frac{\mathsf{DIST}}{\mathsf{OPT}} = \frac{W^* + 2(a+b)}{a+b+c+d} \le \frac{(\sqrt{3}-1)(c+d) + 2(a+b)}{a+b+c+d} = \sqrt{3} - 1 + \frac{(3-\sqrt{3})(a+b)}{a+b+c+d} \le \sqrt{3} - 1 + \frac{(3-\sqrt{3})(a+b)}{2(a+b)} = \frac{1+\sqrt{3}}{2}.$$

Case 2-(ii). $\ell_{max} > L/2$ and $e_{max} = (u, y)$. This does not happen because $b \le a + c + d$.

Case 2-(iii). $\ell_{max} > L/2$ and $e_{max} = (v, y)$. By Fact 1, $\mathsf{OPT} = 2(a + b + c)$. Consider the time when the searcher was at v (Fig. 6 (a)), and let W' be the W-value at this moment. Let w be an unvisited node other than y (this notation is used sometimes hereafter). Then, by Lemma 3, $W' \le (\sqrt{3}-1)d(o, w) \le (\sqrt{3}-1)a$. Note that at the next step, the searcher moved to w because y is the last node visited by the searcher. Let W'' be the W-value just after the searcher moved to w. Then, the total length of the tour increased by $c + d(o, w)$, and the distance between the origin and the searcher changed from c to $d(o, w)$. Hence, by a simple calculation, $W'' = W' + c + d(o, w) + c - d(o, w) = W' + 2c$. Note that the searcher does not change the direction hereafter until it reaches u. So, the W-value remains unchanged until the searcher reaches u, namely, $W^* = W'' = W' + 2c \le (\sqrt{3}-1)a + 2c$ (recall that W^* is the W-value when the searcher is at u). Now,

$$\frac{\mathsf{DIST}}{\mathsf{OPT}} = \frac{W^* + 2(a+b)}{2(a+b+c)} \le \frac{(\sqrt{3}-1)a + 2c + 2(a+b)}{2(a+b+c)} = 1 + \frac{(\sqrt{3}-1)a}{2(a+b+c)}$$

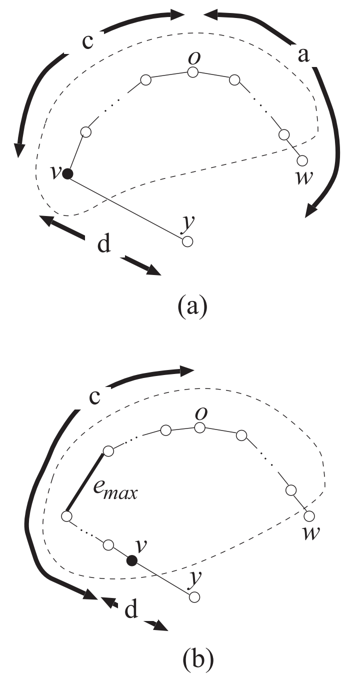

Fig. 6 Case 2-(iii) and Case 2-(iv).

$$\le 1 + \frac{\sqrt{3}-1}{2} = \frac{1+\sqrt{3}}{2}.$$

Case 2-(iv). $\ell_{max} > L/2$ and e_{max} is in the path from o to v (in a counter-clockwise direction). We can show that this case does not happen in the following way: Suppose, on the contrary, that this happens. Consider the time when the searcher was at v (Fig. 6 (b)). Since the searcher decided to move to w rather than y, it must be the case that $\ell(v, y) > \sqrt{3}d(v, w) - W'$ where W' is the W-value at this time. Also, by Lemma 3, $W' \le (\sqrt{3}-1)d(o, w)$. So, $\ell(v, y) > \sqrt{3}d(v, w) - (\sqrt{3}-1)d(o, w) = \sqrt{3}d(v, o) + d(o, w) \ge \ell_{max}$ because $d(v, o) \ge \ell_{max}$ by assumption. But this is a contradiction. So, we can conclude that this case does not happen.

Case 2-(v). $\ell_{max} > L/2$ and e_{max} is in the path from o to u (in a clockwise direction). This does not happen because $a + b \le c + d$.

Case 3. $b \le a+c+d$ and $a+b > c+d$. The searcher visits y using the edge (u, y), and goes back to o in a clockwise direction, i.e., by way of v. So, $\mathsf{DIST} = W^* + a + b + (c + d)$. Similarly, we consider the following subcases:

Case 3-(i). $\ell_{max} \le L/2$. By Fact 1, $\mathsf{OPT} = a+b+c+d$. Thus,

$$\frac{\mathsf{DIST}}{\mathsf{OPT}} = \frac{W^* + a+b+c+d}{a+b+c+d} \le 1 + \frac{(\sqrt{3}-1)(c+d)}{a+b+c+d} < 1 + \frac{(\sqrt{3}-1)(c+d)}{2(c+d)}$$

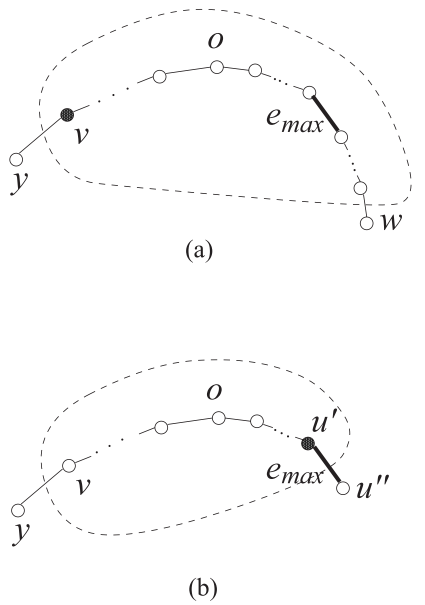

Fig. 7 Case 3-(v).

$$= \frac{1+\sqrt{3}}{2}.$$

Case 3-(ii). $\ell_{max} > L/2$ and $e_{max} = (u, y)$. This does not happen because $b \le a + c + d$.

Case 3-(iii). $\ell_{max} > L/2$ and $e_{max} = (v, y)$. This does not happen because $a + b > c + d$.

Case 3-(iv). $\ell_{max} > L/2$ and e_{max} is in the path from o to v (in a counter-clockwise direction). This does not happen because $a + b > c + d$.

Case 3-(v). $\ell_{max} > L/2$ and e_{max} is in the path from o to u (in a clockwise direction). Consider the time when the searcher was at v. We first show that the searcher had not yet traversed e_{max} at this time. On the contrary, suppose that it had already traversed e_{max} (Fig. 7 (a)). Since the searcher visits w at the next step, $\ell(v, y) > \sqrt{3}d(v, w) - W'$, where W' is the W-value at this moment. Also, by Lemma 3, $W' \le (\sqrt{3} - 1)d(o, w)$. Hence, $\ell(v, y) > \sqrt{3}d(v, w) - (\sqrt{3} - 1)d(o, w) = \sqrt{3}d(o, v) + d(o, w) \ge \ell_{max}$, a contradiction. So, the searcher traversed e_{max} for the first time after it left v.

Now, let $e_{max} = (u', u'')$ and consider the time when the searcher was at u' (Fig. 7 (b)). Let W'' be the W-value at this moment. Since the searcher visited u'' next,

$$\begin{aligned}\ell_{max} &\le \sqrt{3}d(u', y) - W'' \\ &= \sqrt{3}(d(o, u') + d(o, y)) - W'' \\ &\le \sqrt{3}(d(o, u) - \ell_{max} + d(o, y)) - W''.\end{aligned}$$

The last inequality follows from the fact that $d(o, u') + \ell_{max} \le d(o, u)$. From this inequality,

$$\begin{aligned}\ell_{max} &\le \frac{\sqrt{3}(d(o, u) + d(o, y)) - W''}{1+\sqrt{3}} \\ &= \frac{\sqrt{3}(a + c + d) - W''}{1+\sqrt{3}} \\ &= \frac{\sqrt{3}(L - b) - W''}{1+\sqrt{3}}.\end{aligned}$$

Here, recall that L is the total length of the cycle. Because y is the last node visited by the searcher, the searcher does not change the direction hereafter, until it gets u. Hence $W^* = W''$ (recall that W^* is the W-value when the searcher is at u). By Fact 1, $\mathsf{OPT} = 2(L - \ell_{max})$. Hence,

$$\begin{aligned}\frac{\mathsf{DIST}}{\mathsf{OPT}} &= \frac{W^* + a + b + c + d}{2(L - \ell_{max})} \\ &\le \frac{W^* + L}{2\left(L - \frac{\sqrt{3}(L-b)-W''}{1+\sqrt{3}}\right)} \\ &= \frac{(1+\sqrt{3})(W^* + L)}{2(L + W^* + \sqrt{3}b)} \\ &\le \frac{1+\sqrt{3}}{2}.\end{aligned}$$

□

4. A Tight Bound on Unweighted Graphs

In this section we consider the problem on graphs in which all edges have the same cost 1. Note that we do not restrict the topology of graphs.

4.1 An Upper Bound

The Depth-First Search (DFS) gives a good upper bound. When new edges and nodes are revealed, DFS chooses one of the unvisited nodes adjacent to the current node arbitrarily and visits it. If there is no such node, DFS backtracks, i.e., it goes back to the previous node through the edge used to come to the current node for the first time, and does the same procedure there.

To describe the behavior of DFS precisely, we give a recursive procedure Search. Inputs of Search are a node x and a sequence of nodes p (p could be empty). Intuitively, x is the searcher's current position and p is the record of the exploration by the searcher so far.

Procedure Search(**x: vertex, p: a sequence of nodes**)
The searcher is now at x.
If there is an unvisited node z adjacent to x, go to z and Search(z, px).
Otherwise,
  If $p \ne \phi$, let $p = p'y$ where y is the last node of p, and p' is a sequence of nodes obtained by eliminating y from p.
    Go back to y, and execute Search(y, p').
  If $p = \phi$, halt.

Algorithm DFS
Search(o, ϕ)

Theorem 4: DFS is 2-competitive for unweighted graphs.

Proof. For any given graph G, the set of the edges that algorithm DFS traverses is a spanning tree of G. Let n denote the number of nodes of G. Since DFS traverses each edge exactly twice, DFS=$2(n-1)$. On the other hand, OPT $\geq n$ holds because any algorithm should traverse at least n edges in order to visit all the nodes and return to the origin. So, $\frac{\mathsf{DFS}}{\mathsf{OPT}} \leq \frac{2(n-1)}{n} < 2$. □

4.2 A Lower Bound

In this section we prove the following theorem. Note that this theorem holds even when graphs have planarity.

Theorem 5: For any positive constant ϵ, there is no $(2-\epsilon)$-competitive online algorithm for unweighted graphs.

Proof. We will introduce an adversary giving the above mentioned lower bound. Fix an integer n such that $n > \frac{3}{\epsilon}$. For a path $v_1, v_2, \ldots, v_k$, let $\langle\langle v_1, v_2, \ldots, v_k\rangle\rangle$ denote its total length.

First, the adversary reveals two edges (o, u_1) and (o, v_1) incident to the origin. If the searcher moves to u_1, a new edge (u_1, u_2) is revealed. As long as the searcher visits a new node u_i ($i \leq n-1$), the adversary gives an edge (u_i, u_{i+1}). Similarly, if the searcher visits v_i ($i \leq n-1$), a new edge (v_i, v_{i+1}) is revealed. This procedure continues until the searcher reaches u_n or v_n. Without loss of generality we can assume he reaches u_n before v_n.

Now we assume that v_1, v_2, …, v_{t_1} have been visited, and v_{t_1+1} has not been visited. Let D_a denote the total length of the exploration so far. Because the searcher visited v_{t_1} before reaching u_n,

$$\begin{aligned} D_a &\geq 2\langle\langle o, v_1, \ldots, v_{t_1}\rangle\rangle + \langle\langle o, u_1, \ldots, u_n\rangle\rangle \\ &= n + 2t_1. \end{aligned}$$

Then, new edges (u_n, p_1) and (u_n, q_1) are revealed (Fig. 8 (a)). When the searcher visits new node p_i ($i \leq n+t_1-1$), (p_i, p_{i+1}) will be revealed, and when the searcher visits q_i ($i \leq n+t_1-1$), (q_i, q_{i+1}) will be revealed (Fig. 8 (b)). Hereafter, we will do a case analysis depending on the searcher's behavior.

First we consider the case that the searcher reaches v_{t_1+1} before visiting p_{n+t_1} or q_{n+t_1}. Let t_2 ($\leq n+t_1-1$) and t_3 ($\leq n+t_1-1$) be integers such that p_{t_3} and q_{t_2} have been visited, and p_{t_3+1} and q_{t_2+1} are unvisited (Fig. 8 (c)). The adversary does not reveal new edges anymore. Let D_c denote the total length of the exploration so far. The searcher moved from u_n to v_{t_1+1} after visiting p_{t_3} and q_{t_2}, so

$$\begin{aligned} D_c &\geq D_a + 2\langle\langle u_n, p_1, p_2, \ldots, p_{t_3}\rangle\rangle + \\ &\quad 2\langle\langle u_n, q_1, q_2, \ldots, q_{t_2}\rangle\rangle + \\ &\quad \langle\langle u_n, u_{n-1}, \ldots, u_1, o, v_1, \ldots, v_{t_1+1}\rangle\rangle \\ &= D_a + 2t_2 + 2t_3 + n + t_1 + 1 \\ &\geq 2n + 3t_1 + 2t_2 + 2t_3 + 1. \end{aligned}$$

Hereafter, the searcher visits q_{t_2+1} and p_{t_3+1}, and returns to o finishing exploration. The total distance is

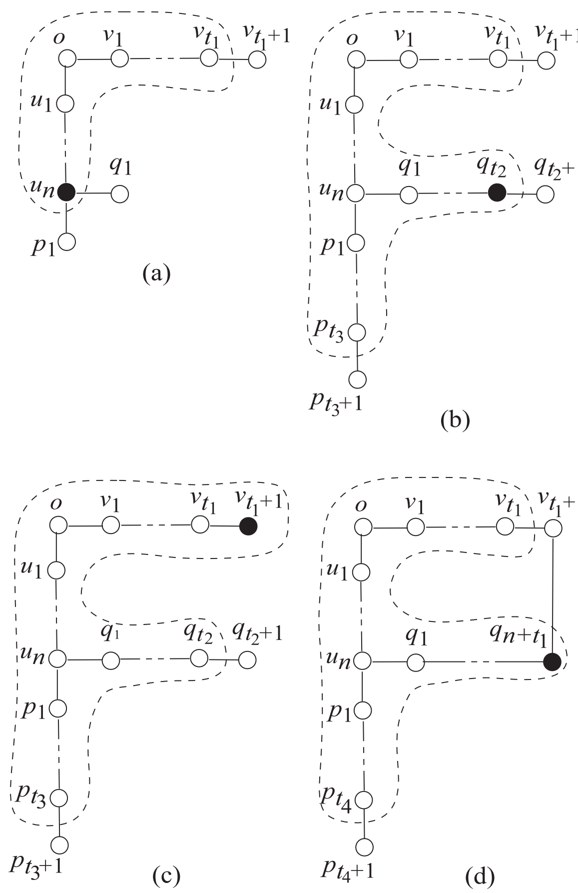

Fig. 8 Lower bound construction for unweighted graphs.

$$\begin{aligned} \mathsf{ALG} &\geq D_c + \langle\langle v_{t_1+1}, v_{t_1}, \ldots, o, u_1, \ldots, u_n\rangle\rangle + \\ &\quad 2\langle\langle u_n, q_1, q_2, \ldots, q_{t_2+1}\rangle\rangle + \\ &\quad 2\langle\langle u_n, p_1, p_2, \ldots, p_{t_3+1}\rangle\rangle + \\ &\quad \langle\langle u_n, u_{n-1}, \ldots, o\rangle\rangle \\ &= D_c + (t_1 + 1 + n) + 2(t_2 + 1) + \\ &\quad 2(t_3 + 1) + n \\ &\geq 4(n + t_1 + t_2 + t_3) + 6, \end{aligned}$$

while OPT $= 2(n + t_1 + t_2 + t_3 + 3)$. So, $\frac{\mathsf{ALG}}{\mathsf{OPT}} > 2 - \epsilon$.

Secondly, we consider the case that the searcher reaches p_{n+t_1} or q_{n+t_1} before visiting v_{t_1+1} Without loss of generality we can assume that the searcher reaches q_{n+t_1}. Now suppose that p_{t_4} has been visited and p_{t_4+1} is unvisited. Let D_d denote the total length of the exploration so far. By a similar observation as before,

$$\begin{aligned} D_d &\geq D_a + 2\langle\langle u_n, p_1, p_2, \ldots, p_{t_4}\rangle\rangle + \\ &\quad \langle\langle u_n, q_1, q_2, \ldots, q_{n+t_1}\rangle\rangle \\ &= D_a + 2t_4 + n + t_1 \\ &\geq 2n + 3t_1 + 2t_4. \end{aligned}$$

Then the adversary reveals (q_{n+t_1}, v_{t_1+1}) (Fig. 8 (d)), and finishes revealing. Hereafter the best way for the searcher is to visit v_{t_1+1} traversing (q_{n+t_1}, v_{t_1+1}), and to return to o after visiting p_{t_4+1} (the last unvisited node). The total length of the tour is

$$\begin{aligned}\mathsf{ALG} &\geq D_d + \ell(q_{n+t_1}, v_{t_1+1}) + \\ &\quad \langle\langle v_{t_1+1}, q_{n+t_1}, \ldots, u_n, p_1, \ldots, p_{t_4+1}\rangle\rangle + \\ &\quad \langle\langle p_{t_4+1}, p_{t4}, \ldots, u_n, u_{n-1}, \ldots, o\rangle\rangle \\ &= D_d + 1 + (1 + n + t_1 + t_4 + 1) + \\ &\quad (1 + t_4 + n) \\ &\geq 4(n + t_1 + t_4 + 1).\end{aligned}$$

The total length of an optimal offline tour is

$$\begin{aligned}\mathsf{OPT} &= \langle\langle o, v_1, \ldots, v_{t_1+1}\rangle\rangle + \ell(v_{t_1+1}, q_{n+t_1}) + \\ &\quad \langle\langle q_{n+t_1}, \ldots, q_1, u_n, p_1, \ldots, p_{t_4+1}\rangle\rangle + \\ &\quad \langle\langle p_{t_4+1}, \ldots, p_1, u_n, \ldots, u_1, o\rangle\rangle \\ &= (t_1 + 1) + 1 + (n + t_1 + t_4 + 1) + \\ &\quad (t_4 + 1 + n) \\ &= 2(n + t_1 + t_4 + 2).\end{aligned}$$

So, $\frac{\mathsf{ALG}}{\mathsf{OPT}} > 2 - \epsilon$. □

5. Concluding Remarks

In this paper, we have studied the online graph exploration problem on two graph classes. First, we have given a tight competitive ratio of $\frac{1+\sqrt{3}}{2}$ for the problem on cycles. We have also studied the problem on unweighted graphs and have given a tight bound of 2.

For planar graphs, the best known upper bound is 16, as mentioned in Sec. 1. Since Theorem 5 holds for planar graphs, 2 is the current best lower bound for planar graphs. There still remains a large gap between these upper and lower bounds. Narrowing the gap is a challenging problem. Another future work is to consider randomized algorithms to break deterministic lower bound.

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References

[1] S. Albers and M.R. Henzinger, "Exploring unknown environments," SIAM J. Comput., vol.29, no.4, pp.1164–1188, 2000.
[2] Y. Asahiro, E. Miyano, S. Miyazaki, and T. Yoshimuta, "Weighted nearest neighbor algorithms for the graph exploration problem on cycles," Proc. 33rd SOFSEM 2007, LNCS 4362, pp.164–175, 2007.
[3] G. Ausiello, E. Feuerstein, S. Leonardi, L. Stougie, and M. Talamo, "Algorithms for the on-line traveling salesman," Algorithmica, vol.29, no.4, pp.560–581, 2001.
[4] G. Ausiello, V. Bonifaci, and L. Laura, "The on-line asymmetric traveling salesman problem," Proc. 9th WADS, pp.306–317, 2005.
[5] A. Borodin and R. El-Yaniv, Online Computation and Competitive Analysis, Cambridge University Press, 1998.
[6] X. Deng and C.H. Papadimitriou, "Exploring an unknown graph," Proc. 31st FOCS, pp.355–361, 1990.
[7] X. Deng, T. Kameda, and C.H. Papadimitriou, "How to learn an unknown environment," Proc. 32nd FOCS, pp.298–303, 1991.
[8] R. Fleischer and G. Trippen, "Experimental studies of graph traversal algorithms," Proc. 2nd WEA, LNCS 2647, pp.120–133, 2003.
[9] R. Fleischer and G. Trippen, "Exploring an unknown graph efficiently," Proc. 13th ESA, LNCS 3669, pp.11–22, 2005.
[10] A. Fiat and G.J. Woeginger, "Competitive analysis of algorithms," Online Algorithms: The State of the Art, ed. A. Fiat and G.J. Woeginger, Springer, 1998.
[11] F. Hoffmann, C. Icking, R. Klein, and K. Kriegel, "A competitive strategy for learning a polygon," Proc. SODA, pp.166–174, 1997.
[12] F. Hoffmann, C. Icking, R. Klein, and K. Kriegel, "The polygon exploration problem," SIAM J. Comput., vol.31, no.2, pp.577–600, 2001.
[13] B. Kalyanasundaram and K.R. Pruhs, "Constructing competitive tours from local information," Theor. Comput. Sci., vol.130, pp.125–138, 1994.
[14] E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoyes, eds., The traveling salesman problem: A guided tour of combinatorial optimization, Wiley, Chichester, 1985.
[15] D.J. Rosenkrantz, R.E. Stearns, and P.M. Lewis, "An analysis of several heuristics for the traveling salesman problem," SIAM J. Comput., vol.6, no.3, pp.563–581, 1977.

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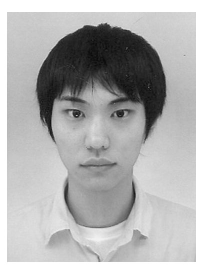

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