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"Cardinal Utility Representation Separating Ambiguous Beliefs and Utility"

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# Cardinal Utility Representation Separating Ambiguous Beliefs and Utility

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**Abstract**

This paper proposes the weaker axioms which admit a cardinal utility representation under ambiguity separating ambiguous beliefs and utility over consequences in a purely subjective setting. The representation is obtained in an implicit form, which corresponds to the disappointment aversion utility (Gul, 1991) with respect to a non-additive measure in place of a probability measure. It includes all the properties of cardinality, ambiguity aversion, reference dependency, gain/loss asymmetry, and the distortion in probability evaluations. It enables us to capture varying attitude toward ambiguity such as the subjective common ratio effect and to explain Machina's examples (2009, 2014) in the simplest way.

**JEL Classification:** D81

**Keywords:** isometry, implicit representation, biseparable preference, rank dependent utility, disappointment aversion, common ratio effect

## 1 Introduction

The separation of subjective beliefs and utility over consequences is a significant feature in a cardinal characterization of preferences under ambiguity. Most cardinal utility representations to accommodate Ellsberg behavior (Ellsberg, 1961) share the *biseparability* characterized by the rank-dependent independence axiom imposed on two-outcome acts (Ghirardato

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and Marinacci, 2001; hereafter GM01). On the other hand, typical examples violating independence even on two-outcome acts are not only Ellsberg behavior but also the common ratio effect pioneered by Hagen (1979) and Kahnemann and Tversky (1979).

Does such common ratio effect arise under ambiguity? To see an ambiguous version of the common ratio effect, called the *Subjective Common Ratio Effect* by Machina (2014), let us examine a variant of the famous experiment in Kahnemann and Tversky (1979): There are four urns labelled A, B, C, and D containing balls as shown in Table 1. Urn A includes five balls, four out of five are unambiguously red, but the color of the rest is not informed. Urn B holds five red balls only. Similarly, urn C also contains five balls, but only one is informed red. Urn D has four balls, but only one ball in it is red.

| | urn **A** | urn **B** | urn **C** | urn **D** |
|---|---|---|---|---|
| red | 4 | 5 | 1 | 1 |
| unknown | 1 | 0 | 4 | 3 |
| total | 5 | 5 | 5 | 4 |

Table 1 Four Ambiguous Urns

As a trial, a ball is chosen from one of the four urns, there are two choice problems as follows:

**Problem 1.** urn **A**:($4000 if red; $0 otherwise), or urn **B**:($3000 if red; $0)

**Problem 2.** urn **C**:($4000 if red; $0 otherwise), or urn **D**:($3000 if red; $0)

In Problem 1, urn A gives $4000 if a red ball is chosen and $0 otherwise, and urn B gives $3000 for sure. In Problem 2, urn C gives $4000 if a red ball is chosen and $0 otherwise, and urn D gives $3000 if a red is chosen and $0 otherwise.

In most casual observations, urn B in Problem 1 and urn C in Problem 2 are more appealing than the others. This choice pattern is typical, although it violates binary comonotonic act independence. The less ambiguous balls seem to be highly evaluated in the more probable winning events. It suggests that the attitude toward ambiguity might be varying together with the associated subjective likelihood.

The more elaborated experimental results are examined by Burghart, Epper and Fehr (2016) based on the Ellsberg experiment. Their fanning-in indifference curves on a Machina-Marchak triangle are typically observed and allow the subjective common ratio effect. For capturing such properties, the most practical way is to incorporate a spirit of non-expected utility formulae.

This paper proposes a cardinal utility representation under ambiguity which separates ambiguous beliefs and utility over consequences, characterized by the weaker axioms than

GM01. The main axiom is called *binary comonotonic act isometry*, which is a weaker version of independence and a stronger version of betweenness imposed on binary comonotonic acts.

The utility representation is obtained in an implicit form, which has a disappointment aversion formula by Gul (1991) in terms of a *non-additive measure* not a *probability measure*. It is also consistent with the aforementioned ambiguous version of the common ratio effect. If there were no ambiguity, the expression would be exactly reduced to a disappointment aversion utility function, thus it affords a subjective foundation for the disappointment aversion model.

The obtained representation surprisingly includes all the properties below, since it is compatible with both Allais-type and Ellsberg-type behavior:

- Cardinality: If $V$ and $V'$ represent the same preference, there exist real numbers $a > 0$ and $b$ such that $V = aV' + b$.

- Separating beliefs and utility over outcomes

- Ambiguity aversion

- Reference dependency: The reference point is endogenously determined equal to the certainty equivalent outcome of each act.

- Gain/loss asymmetry: The gain and loss compared to the reference level are evaluated differently through a unique number named a *gain/loss ratio*.

- Relative rank-dependency: The *relative rank* including the reference point matters in evaluating acts which have more than three outcomes.

- Distortion in probability evaluations: In an explicit form, the non-additive probabilities are distorted non-linearly by a gain/loss ratio.

Based on the implicit representation specified on binary acts, the latter half of this paper also proposes the more specific formula for acts with three or more outcomes, which is designated as a *relative rank-dependent* expected utility. The characterizing axiom is called the *relative comonotonic act isometry*, which is equal to the isometry between acts whose *relative ranks* including their certainty equivalent outcomes are equivalent. Since the representation shares the above properties, it is compatible with Machina's examples (2009, 2014) in the simplest way.

The following section begins by introducing axioms and then the binary and more specified characterization results are presented in sequence. Concluding remarks discusses various further applications.

## 2 Axioms and Cardinal Representation

Let $\Omega$ be the set of *states* and $\Sigma$ be an algebra of all events over $\Omega$. $X$ is a set of *outcomes*, which is assumed to be a nonempty, separable and connected topological space. By $\mathcal{F} = \{f : \Omega \to X\}$ we denote the set of the measurable simple *acts* with respect to $\Sigma$, endowed with the product topology. For any $f$, $g \in \mathcal{F}$ and $A \in \Sigma$, $fAg$ is the act whose outcome is $f(\omega)$ if $\omega \in A$ and $g(\omega)$ if $\omega \in A^c$. $\mathcal{F}_2 = \{xAy \in \mathcal{F} \mid x, y \in X \text{ and } A \in \Sigma\}$ denotes the set of all *binary acts*. An outcome $x \in X$ also represents a constant act that yields $x$ for all $\omega \in \Omega$. A preference relation $\succcurlyeq$ is a binary relation $\succcurlyeq$ defined on $\mathcal{F}$.[1] As usual, $\succ$ and $\sim$ correspond to asymmetric and symmetric parts of $\succcurlyeq$ respectively. Two acts are *comonotonic* if there are no two states $s$ and $t$ such that $f(s) \succ f(t)$ and $g(s) \succ g(t)$. Two binary acts $xAy$ and $x'Ay'$ are *binary comonotonic* if $x \succcurlyeq y$ and $x' \succcurlyeq y'$, or $x \preccurlyeq y$ and $x' \preccurlyeq y'$.

An event $A \in \Sigma$ is called *null* if $xAy \sim y$ for all $x$, $y \in X$. An event $A^c$ is *universal* if $A$ is null. An event $A \in \Sigma$ is called *essential* if $A$ is not null nor universal.

The axioms we need are the followings:

**A1 Weak Order** A preference relation $\succcurlyeq$ is a weak order on $\mathcal{F}$: (i) for all $f, g \in \mathcal{F}$, $f \succcurlyeq g$ or $f \preccurlyeq g$, (ii) for all $f$, $g$, $h \in \mathcal{F}$, if $f \succcurlyeq g$ and $g \succcurlyeq h$, then $f \succcurlyeq h$.

**A2 Essentiality** There exist an event $A \in \Sigma$ and outcomes $x^*$, $x_* \in X$ such that $x^* \succ x^* A x_* \succ x_*$.

**A3 Monotonicity** For all acts $f$, $g \in \mathcal{F}$, if $f(\omega) \succcurlyeq g(\omega)$ for all $\omega \in \Omega$, then $f \succcurlyeq g$.

**A4 Eventwise Monotonicity** For any event $A \in \Sigma$, if for some $x \succ y$, $xAy \succ y$ (resp. $x \succ xAy$), then for all $a \succ b \succcurlyeq c$, $aAc \succ bAc$ (resp. for all $c \succcurlyeq a \succ b$, $cAa \succ cAb$).

**A5 Continuity** For all $f \in \mathcal{F}$, the sets $\{x \in X \mid f \succcurlyeq x\}$ and $\{x \in X \mid x \succcurlyeq f\}$ are closed in $X$.

Let $c : \mathcal{F} \to X$ represent the *certainty equivalent of* $f$ such that $c(f) \sim f$. Given $x$, $y \in X$ and $A \in \Sigma$, $z \in X$ is called a *preference average of* $x$ *and* $y$ *given* $A$ if, $x \succcurlyeq z \succcurlyeq y$ and $xAy \sim c(xAz)\, A\, c(zAy)$ given that such certainty equivalents exist. By $m_A(x, y)$ we denote a preference average of $x$ and $y$ given $A$. In addition, $m_A(x, y)$ is also a preference average of $y$ and $x$. Note that for some event $A$, such $m_A$ may not be unique. In that case, it is assumed that $m_A(x, y)$ chooses a specific outcome among them. Write $m_A(f, g)$ as an act $h$ such that $h(\omega) \sim m_A(f(\omega), g(\omega))$ for every $\omega \in \Omega$.

[1] Although it might assume that $\succcurlyeq$ has no extreme outcomes on $X$, hereafter we redefine $X$ as $X$ excluded the extreme outcomes if they exist with respect to $\succcurlyeq$. By $\mathrm{cl}(X)$ denote $X \cup \{\text{extreme outcomes}\}$.

**A6 Binary Comonotonic act Isometry** For all $A$, $B \in \Sigma$, all $x$, $y$, $x'$, $y' \in X$ such that $xAy$ and $x'Ay'$ are binary comonotonic and $xAy \succcurlyeq x'Ay'$,

$$m_B\left(c\left(xAy\right), c\left(x'Ay'\right)\right) \sim m_B\left(x, x'\right) A\, m_B\left(y, y'\right).$$

The axioms A1-A5 are necessary to obtain nontrivial, monotonic and continuous $V$ that represents $\succcurlyeq$. The main axiom needed to our implicit formula, A6, is a stronger version of the binary comonotonic act *betweenness*: for all binary comonotonic $xAy \succcurlyeq x'Ay'$,

$$xAy \succcurlyeq m_B\left(x, x'\right) A\, m_B\left(y, y'\right) \succcurlyeq x'Ay'.$$

The betweenness tells that the preference average of any two acts is evaluated *between* those acts, however the isometry requires more. The preference average of two binary acts is indifferent to the preference average of the certainty equivalents of those acts.

**Theorem 1** *The following statements are equivalent:*

*(i)* $\succcurlyeq$ *satisfies A1 (Weak Order), A2 (Essentiality), A3 (Monotonicity), A4 (Eventwise Monotonicity), A5 (Continuity) and A6 (Binary Comonotonic act Isometry).*

*(ii) There exist a continuous monotonic non-constant representation* $V : \mathcal{F} \to \mathbb{R}$ *of* $\succcurlyeq$*, a unique real number* $\rho > 0$ *and a unique monotone set function* $\mu : \Sigma \to [0,1]$ *such that for all* $x \succcurlyeq y$ *in* $X$ *and all* $A \in \Sigma$*,* $V\left(xAy\right)$ *is defined implicitly as a unique* $v \in \mathbb{R}$ *that solves*

$$\mu\left(A\right)\left[V\left(x\right) - V\left(xAy\right)\right] + \left(1 - \mu\left(A\right)\right)\rho\left[V\left(y\right) - V\left(xAy\right)\right] = 0. \qquad (*)$$

*Moreover, if* $V$ *and* $V'$ *represent* $\succcurlyeq$*, then there are real numbers* $a > 0$ *and* $b$ *such that* $V' = aV + b$*.*

**Cardinality and Separability** Theorem 1 clarifies the sufficient condition for a cardinal representation. The result shows that the weaker conditions than the biseparable preferences in GM01 achieve the cardinality.[2]

**Corollary 1** *A functional* $V$ *representing* $\succcurlyeq$ *on* $\mathcal{F}$ *is cardinal if* $\succcurlyeq$ *satisfies A1-A6.*

The key point for the cardinality is to separate ambiguous beliefs from outcome evaluations. In fact, $(*)$ becomes a biseparable formula since the underlying implicit form is rearranged into the explicit form: Solving for $v$, we have

$$V\left(xAy\right) = \frac{\mu\left(A\right)}{\mu\left(A\right) + \rho\left(1 - \mu\left(A\right)\right)} V\left(x\right) + \left(1 - \frac{\mu\left(A\right)}{\mu\left(A\right) + \rho\left(1 - \mu\left(A\right)\right)}\right) V\left(y\right). \qquad (1)$$

[2] I am indebted to Yutaka Nakamura for this point.

For future references, define a function $\varphi : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ as $\varphi(x;\rho) = \frac{x}{x+(1-x)\rho}$.

The expression $(*)$ and (1) correspond to the disappointment aversion utility model in Gul (1991) on $\mathcal{F}_2$ with respect to a capacity in place of a probability measure, by setting the disappointment averse coefficient $\beta = \rho - 1$. More intuitively, the rank-dependent probabilities of the winning and losing event associated to the better and worse outcomes (here, $A$ is the winning event and $A^c$ is the losing event) are distorted by a *gain-loss ratio* $\rho$.

The weighted utility by Chew (1983) and Fishburn (1983) is one of the masterpieces of the implicit representations that have explicit forms. The implicit form in our setting is

$$\mu(A)\,\phi(x)\,[V(x) - V(xAy)] + (1-\mu(A))\,\phi(y)\,[V(y) - V(xAy)] = 0, \tag{2}$$

where $\phi : X \to [0,1]$ is a weight function. The formula $(*)$ is a special case of (2) when $\phi(y)/\phi(x)$ is constant. By means of outcome mixtures, it is difficult to separate $\phi(x)$ from $V(x)$ representing cardinal utility over outcomes.

**Ambiguity Aversion** Let us examine $(*)$ in Theorem 1 by applying to the classical Ellsberg behavior.

**Example 1 (Three-Color Ellsberg Urn)** There is an urn which contains three balls. One of three is certainly red, however each of other two is whether black or yellow. A ball is chosen from the urn. There are four acts that give prizes according to the color of the chosen ball. The state space is taken as $\{R, B, Y\}$

| | Red | Black | Yellow |
|---|---|---|---|
| $f_1$ | \$100 | \$0 | \$0 |
| $f_2$ | \$0 | \$100 | \$0 |
| $f_3$ | \$100 | \$0 | \$100 |
| $f_4$ | \$0 | \$100 | \$100 |

Table 2 Three-Color Ellsberg Urn

The Ellsberg behavior, or the ambiguity averse behavior is the combination of choices: $f_1 \succ f_2$ and $f_3 \prec f_4$. Applying the expression in Theorem 1, we have

$$\begin{aligned} V(f_1) &= \frac{\frac{1}{3}}{\frac{1}{3}+\frac{2}{3}\rho} V(100)\,,\; V(f_2) = \frac{\mu(B)}{\mu(B)+(1-\mu(B))\,\rho} V(100)\,, \\ V(f_3) &= \frac{\frac{1}{3}+\mu(Y)}{\frac{1}{3}+\mu(Y)+\left(\frac{2}{3}-\mu(Y)\right)\rho} V(100)\,,\text{ and } V(f_4) = \frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{3}\rho} V(100)\,, \end{aligned}$$

where $\rho > 0$. As stated above, since $\varphi(t;\rho)$ is strictly increasing in $t \in [0,1]$ given any $\rho > 0$, $V(f_1) > V(f_2)$ iff $\mu(B) < \frac{1}{3}$, and $V(f_4) > V(f_3)$ iff $\mu(Y) < \frac{1}{3}$. The Ellsberg behavior is equivalent to the superadditivity of $\mu$. Note that the magnitude of $\rho$, interpreted as elation loving or disappointment aversion is not immediately relevant to ambiguity aversion.

**Reference Point and Gain-loss Asymmetry** In the binary expression $(*)$, the reference level is determined on its certainty equivalent outcome (CE). In any binary case, CE is always between the better and worse outcomes. The relative gain and loss compared to the CE outcome are evaluated quite differently through the gain/loss ratio $\rho$. When $\rho > 1$, the relative loss is heavily evaluated than the relative gain. It is exactly disappointment aversion called by Gul (1991). On the other hand, elation loving refers to the case where the relative gain is highly evaluated than the loss. If $\rho = 1$, then the expression is clearly equivalent to the biseparable case: $V(xAy) = \mu(A)V(x) + (1-\mu(A))V(y)$.

**Distortions in Probability Assessments** The $V$ on $\mathcal{F}_2$ is actually determined via the implicit form $(*)$, hence $\mu : \Sigma \longrightarrow [0,1]$ is a monotone set function, that is a capacity. Here we examine more about the explicit form (1).

It is straightforward $\varphi(\mu(A);\rho) = \mu(A)$ if $\rho = 1$. Since $\varphi(t;\rho)$ is strictly increasing in $t$ given any $\rho > 0$, $\varphi(\mu(\cdot);\rho)$ is monotone if $\mu$ is monotone. In addition, $\varphi(t;\rho)$ is convex given any $\rho > 1$, the superadditivity of $\mu$ together with $\rho > 1$ imply the superadditivity of $\varphi(\mu(\cdot);\rho)$. It implies that the disappointment aversion in $(*)$ implies ambiguity aversion in an explicit formula (1).

What does happen when $\mu$ is superadditive and $0 < \rho < 1$? $\varphi(\mu(\cdot);\rho)$ may be whether superadditive or not. To see a shape of $\varphi$, let us consider the case where $\varphi(x;\rho) = \frac{x^{1.6}}{x^{1.6}+(1-x^{1.6})\rho}$.

Figure 1 shows three patterns of $\varphi$ when $\rho = 1.5$, 0.9 and 0.5. Given $\rho = 1.5$ or 0.9, $\varphi$ is convex. However $\rho = 0.5$ displays an $S$-shape distortion which could not be seen even in Gul's original model.

Let $\Delta$ be the set of all probability measures on $\Omega$ and $\mathcal{C}$ be a nonempty closed convex subset of $\Delta$. $p_A$ is shorthand for $p(A)$ for any $A \in \Sigma$.

**Example 2 (Ambiguous version of common-ratio effect)** Let us consider the case of the maxmin disappointment aversion utility:

$$\begin{aligned} V(xAy) &= \min_{p \in \mathcal{C}} p \frac{V(x) + \beta v}{1+\beta} + \left[1 - \min_{p \in \mathcal{C}} p_A\right] V(y) \\ &= \frac{\min_{p \in \mathcal{C}} p}{1 + (1 - \min_{p \in \mathcal{C}} p)\beta} V(x) + \left[1 - \frac{\min_{p \in \mathcal{C}} p}{1 + (1 - \min_{p \in \mathcal{C}} p)\beta}\right] V(y). \end{aligned}$$

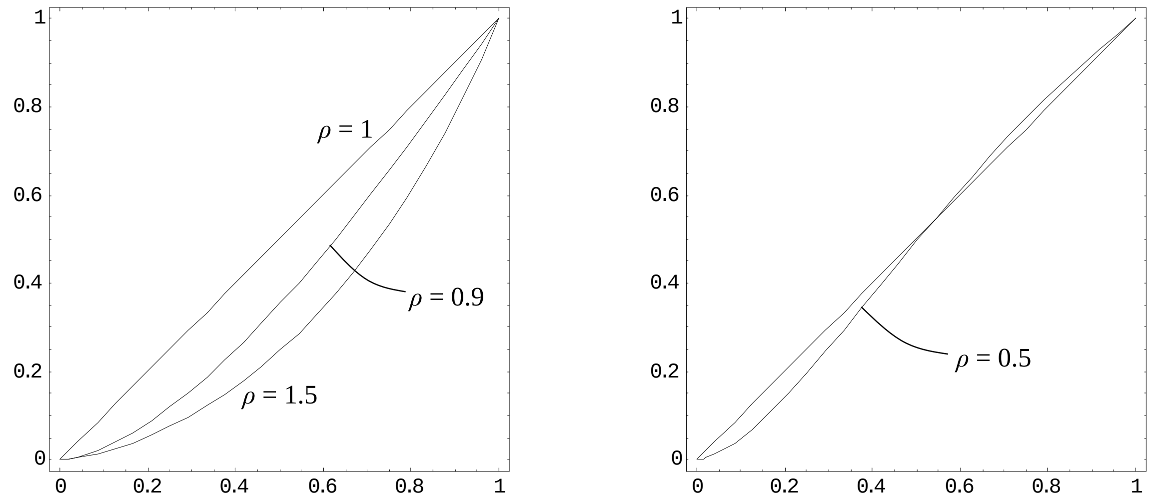

Figure 1. Distortion in Probabilities

Let $V(x) = x$ and $\beta = 0.5$ that displays disappointment aversion. In the experiment, the minimum probability of the event red is 0.8 in Urn A, 0.2 in Urn B, and 0.25 in Urn C. Evaluating every urn A to D, we have

$$\begin{aligned} V(\text{urn A}) &= 2909 < V(\text{urn D}) = 3000, \text{ and} \\ V(\text{urn B}) &= 571 > V(\text{urn C}) = 545. \end{aligned}$$

The combination of choices urn D from Problem 1 and urn B from Problem 2 is consistent with $V$.

# 3 More Specialized Representation

## 3.1 Relative Rank-dependent Representation

Let $A_i$, $i = 1, 2, \dots, k$ be the renumbered events associated with a rank-ordered outcomes of $f$ so that $f(A_1) \succ f(A_2) \succ \cdots \succ f(A_{k-1}) \succ f(A_k)$ and $B_i = \bigcup_{j=1}^{i} A_j$. Although $A_i$ and $B_i$ are determined by $f$, we use them for brevity's sake. Define *the winning event in* $f$ as $B(f) = \{\omega \in \Omega \mid f(\omega) \succcurlyeq c(f)\}$ in the sense that every outcome in those states are better than its certainty equivalent outcome. $f$ and $g$ are *relatively comonotonic* if they are comonotonic and $B(f) = B(g)$. Since the rank-ordered outcomes of $f$ in the conventional sense are extended to the rank-order in outcomes including the certainty equivalent outcome, we call it the *relative rank*, or *relative order*. Notice that the relative rank matters a *winning* or *losing* outcome relative to the certainty equivalent outcome.

**A7 Relatively Comonotonic Act Isometry** For all $A \in \Sigma$, all $f, g \in \mathcal{F}$ such that $f$ and $g$ are relatively comonotonic and $f \succcurlyeq g$,

$$m_A\left(c\left(f\right), c\left(g\right)\right) \sim m_A\left(f, g\right).$$

A7 implies A5 since in any binary act $xAy$, $x \succcurlyeq c\left(xAy\right) \succcurlyeq y$ by A3 and A4, the rank is equivalent to its relative rank. A7 also suggests that two comonotonic acts might be evaluated quite differently if they have different relative orders. Theorem 2 provides the relative rank-dependent representation.

**Theorem 2** *The following statements are equivalent:*

*(i)* $\succcurlyeq$ *satisfies A1 (Weak Order), A2 (Essentiality), A3 (Monotonicity), A4 (Eventwise Monotonicity), A5 (Continuity), A7 (Relatively Comonotonic act Isometry).*

*(ii) There exist a continuous monotonic non-constant representation* $V : \mathcal{F} \to \mathbb{R}$ *of* $\succcurlyeq$*, a unique monotone set function* $\mu : \Sigma \to [0, 1]$ *and a unique real numbers* $\rho > 0$ *such that for all* $f \in \mathcal{F}$*,* $V\left(f\right)$ *is defined implicitly as a unique* $v \in \mathbb{R}$ *that solves*

$$\begin{aligned}&\sum_{V(f(A_i)) \geqq v} \left[\mu\left(B_i\right) - \mu\left(B_{i-1}\right)\right]\left[V\left(f\left(A_i\right)\right) - V\left(f\right)\right] \\ &\qquad + \rho \sum_{V(f(A_i)) < v} \left[\mu\left(B_i\right) - \mu\left(B_{i-1}\right)\right]\left[V\left(f\left(A_i\right)\right) - V\left(f\right)\right] = v.\end{aligned} \tag{$**$}$$

*Moreover, if* $V$ *and* $V'$ *represent* $\succcurlyeq$*, then there are real numbers* $a > 0$ *and* $b$ *such that* $V' = aV + b$*.*

**Reference Dependency** The expression $(**)$ is also transformed into

$$\begin{aligned}&V\left(f\right) \\ = \ &\frac{\sum_{V(f(A_i)) \geqq v} \left[\mu\left(B_i\right) - \mu\left(B_{i-1}\right)\right] V\left(f\left(A_i\right)\right) + \rho \sum_{V(f(A_i)) < v} \left[\mu\left(B_i\right) - \mu\left(B_{i-1}\right)\right] V\left(f\left(A_i\right)\right)}{\sum_{V(f(A_i)) \geqq v} \left[\mu\left(B_i\right) - \mu\left(B_{i-1}\right)\right] + \rho \sum_{V(f(A_i)) < v} \left[\mu\left(B_i\right) - \mu\left(B_{i-1}\right)\right]} \\ = \ &v,\end{aligned}$$

which has the similar formula for the disappointment aversion utility, with respect to a non-additive measure instead of a probability measure.

The axiom of A7 means that two acts with the same relative rank are evaluated by using the same probability assessment. For illustration, consider an act $f$ such that $f\left(S_1\right) = 1$, $f\left(S_2\right) = 0$, and $f\left(S_3\right) = -1$ where $\{S_1, S_2, S_3\}$ is a partition of $\Omega$. Notice that there are two cases for the relative orders: $-1 < 0 < V\left(f\right) < 1$ and $-1 < V\left(f\right) < 0 < 1$.

Although our model is for a purely subjective setting, as a striking illustration, let us consider the case where $\mu\left(S_1\right) = \left(p_1\right)^{\alpha}$, $p$ is a probability measure on $\Omega$, and $V\left(x\right) = x$.

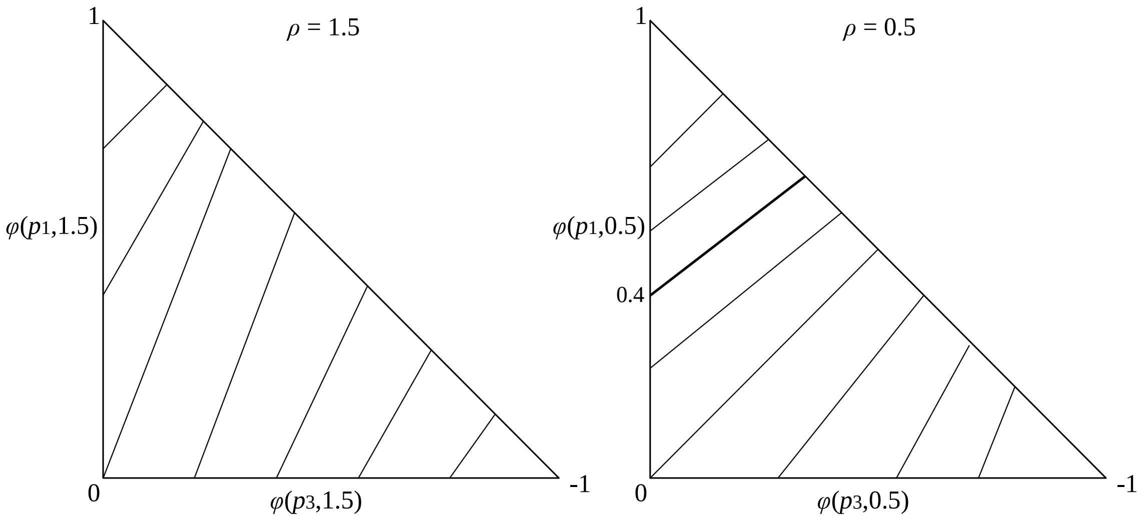


Figure 2. Indifference Curves on Probability Simplices

Figure 2 shows the indifference curves on probability simplices in case of $\varphi(\mu(S_1);\rho) = \frac{(p_1)^{1.6}}{(p_1)^{1.6}+\left(1-(p_1)^{1.6}\right)\rho}$. When $\rho = 1.5$, the indifference curves are the same as Gul's disappointment aversion utility. However, when $\rho = 0.5$, the indifference map looks quite different, since the distortion function for $p_1$ has $S$-shape as seen in Figure 1. In fact, $\frac{\partial^2 \varphi}{\partial p_1^2} = 0$ if $p_1 \doteqdot 0.3999$. The shape could not be seen in the conventional disappointment aversion model.

## 3.2 Machina's Examples

The expression $(**)$ has the property of the relative rank dependent independence. Therefore, it is consistent with every example proposed in Machina (2009). Furthermore, it is also compatible with almost all the examples in Machina (2014). Example 3 and 4 are the most difficult hurdles to overcome for many utility representations.

**Example 3 (Low versus High outcomes problem)** A three-color Ellsberg urn in Example 1, which contains one red ball and two balls, each of which is black or yellow. Both Urn I and II are such Ellsberg urns and the prizes are given according to the ball taken out of the chosen urn as in Table 3. $C$ is the certainty equivalent outcome of an objective lottery that gives \$100 with the probability of 0.5 and \$0 otherwise.

| | Red | Black | Yellow |
|---|---|---|---|
| Urn I | \$100 | \$0 | $C$ |
| Urn II | \$0 | $C$ | \$100 |

Table 3 Low vs. High outcomes problem

Machina's prediction is that, ambiguity averters prefer Urn II to Urn I. Does our formula $(**)$ conform to such Machina preferences?

To verify this, take a state space as $\{R, B, Y\}$ and let $c_i$, $i = \mathrm{I}, \mathrm{II}$ be the certainty equivalent of each Urn $i$. Assume that the certainty equivalent of Urn I or II is less than $C$, that is, $c_{\mathrm{I}} < c_{\mathrm{II}} < C$. Write $\mu_S$ instead of $\mu(S)$ for short. By setting $V(100) = 1$, we have

$$\begin{aligned} V(C) &= \tfrac{1}{1+\rho} \\ v_{II} &= \frac{\frac{1}{3}}{\frac{1}{3} + \mu_Y + \left(\frac{2}{3} - \mu_Y\right)\rho} + \frac{\mu_Y}{\frac{1}{3} + \mu_Y + \left(\frac{2}{3} - \mu_Y\right)\rho} \frac{1}{1+\rho} \\ v_I &= \frac{\mu_Y}{\frac{1}{3} + \frac{2}{3}\rho} + \frac{\frac{2}{3} - \mu_Y}{\frac{1}{3} + \frac{2}{3}\rho} \frac{1}{1+\rho}. \end{aligned}$$

As verified in Example 1, ambiguity aversion together with informational symmetry implies $\mu_Y = \mu_B < \frac{1}{3}$. By setting $\mu_Y = 0.25$, $v_{\mathrm{II}} > v_{\mathrm{I}}$ iff $\rho < 1$, and $v_{\mathrm{I}} > v_{\mathrm{II}}$ iff $\rho > 1$.

Dillenberger and Segal (2015) also examined the same Machina's example through the recursive disappointment aversion utility by setting $\rho > 1$. Their model predicts only $v_{\mathrm{I}} > v_{\mathrm{II}}$ since it has to have $\rho > 1$ to admit Ellsberg preferences. However, our model allows $\rho$ to be less than 1 since ambiguity aversion is implied by $\mu_Y < \frac{1}{3}$.

**Example 4 (Slightly bent coin problem revised)**

| Bet I | Black | White |
|---|---|---|
| Head | $1-\varepsilon$ | $c$ |
| Tail | $-1$ | $c$ |

| Bet II | Black | White |
|---|---|---|
| Head | $c$ | $c$ |
| Tail | $-1$ | $1+\varepsilon$ |

Table 4 Slightly bent coin problem revised

Bet I and Bet II has four states comprised of $\{\mathrm{Head}, \mathrm{Tail}\}$ of a slightly bent (but not informed the direction) coin flip and $\{\mathrm{Black}, \mathrm{White}\}$ of the chosen ball from two, each of which color is black or white. In Machina's version, $c$ in Bet I and Bet II $c$ and $\varepsilon$ are set to be zero, where the strict preference for Bet I to Bet II is plausibly suggested. Our utility representation predicts the indifference between two bets, same as the Choquet expected utility. However, if $c = 0$, it is difficult to interpret Bet I and Bet II as two conditional bets, since $\{WH, WT\}$ in Bet I and $\{BH, WH\}$ in Bet II cannot be *event-separable* due to ambiguity aversion.

To overcome this discrepancy, let $c$ be the certainty equivalent outcome of these two bets when $\varepsilon = 0$. Both bets gives the same value since they are indifferent. Take a sufficiently small $\varepsilon > 0$ and consider Bet I and Bet II as in Table 4. Write the certainty equivalent of each bet as $c_i$, $i = \mathrm{I}$, II and assume $c_{\mathrm{I}}^{\varepsilon} \succ c_{\mathrm{II}}^{\varepsilon}$. Write $\mu_1$ for $\mu(BH)$ and $\mu(WT)$, $\mu_3$ for $\mu(BH \cup WH \cup WT)$.

We now show that $V(\text{Bet I}) > V(\text{Bet II})$. At first $c$ satisfies

$$\mu_1 [V(1) - V(c)] + (1 - \mu_1)[V(-1) - V(c)] = 0,$$

so we have $V(c) = \frac{\mu_1 - (1-\mu_3)\rho}{\mu_1 + (1-\mu_3)\rho}$.

$$\begin{aligned} V(\text{Bet I}) &= \tfrac{\mu_1}{\mu_3 + (1-\mu_3)\rho} V(1-\varepsilon) + \tfrac{\mu_3 - \mu_1}{\mu_3 + (1-\mu_3)\rho} V(c) + \tfrac{(1-\mu_1)\rho}{\mu_3 + (1-\mu_3)\rho} V(-1), \text{ and} \\ V(\text{Bet II}) &= \tfrac{\mu_1}{\mu_1 + (1-\mu_1)\rho} V(1+\varepsilon) + \tfrac{\mu_3 - \mu_1}{\mu_1 + (1-\mu_1)\rho} V(c) + \tfrac{(1-\mu_1)\rho}{\mu_1 + (1-\mu_1)\rho} V(-1). \end{aligned}$$

Then there exists a $\rho^* \in (0,1)$ such that for every $\rho \in (\rho^*, 1)$, $V(\text{Bet I}) > V(\text{Bet II})$. By the symmetric argument, we have $V(\text{Bet II}) > V(\text{Bet I})$ when $\rho > 1$. It is because the evaluations for every event is quite different whether $c > c_{\mathrm{I}}^{\varepsilon}$ or $c < c_{\mathrm{II}}^{\varepsilon}$.

# 4 Concluding Remarks

## 4.1 Related Literature

**Connection to the non-expected utility formulae** The general approach to characterize preferences over lotteries by betweenness rather than independence is originated by Dekel (1986). The useful specific formulation is extensively investigated in the non-expected utility framework, such as the weighted utility by Chew (1983) and Fishburn (1983) and the disappointment aversion theory by Gul (1991), which possess desirable properties to conform Allais paradox behavior (Allais, 1953) and preserve linearity in probability mixtures.

The general cases of such implicit linear utility are characterized in Chew and Epstein (1989) and Chew and Epstein and Wakker (1993), or Grant, Kajii and Polak (2000). However, such a specific formula is not thoroughly examined in any conventional literature especially under subjectively ambiguous settings.

**Rank-dependent independence** The biseparability proposed by GM01 is based on binary comonotonic act independence to accommodate Ellsberg behavior (Ellsberg, 1961). The maxmin subjective expected utility by Alon and Schmeidler (2014) is also founded on this biseparability. Such binary rank-dependent independency generates a subjective probabilistic evaluation unique to any winning or losing event associated to the better or worse outcome.

The rank-dependency especially in three or more outcome acts has less behavioral foundation, as examined by Machina (2009, 2014) which motivates more elaborated utility representations without rank-dependence axioms. This paper incorporates the *relative* rank instead of the conventional rank. This relative rank is the order of outcomes including the certainty equivalent outcome, which is considered to be the reference level. Therefore, in a representation the same outcome might be evaluated quite differently depending on the winning or losing event.

**Dynamic Properties** Some dynamic properties such as the dynamic consistency or the Bayesian property restrict the nature of a preference relation. For instance, under the probabilistic sophistication in Machina and Schmeidler (1992), betweenness implies decomposability (Grant et. al, 2000), which implies dynamic consistency. Since P4* (strong comparative probability axiom) is not compatible with Ellsberg behavior, the axiom has to be reexamined. In light of extended Bayesian property, this paper proposes to utilize the certainty equivalent outcome of any act for conditioning, which implies the certainty equivalent consistency in Pires (2002), Hanany and Klibanoff (2007) and Horie (2013).

**Technique of outcome mixtures** The comonotonic outcome-mixture independence in Nakamura (1990) is enough to separate the rank-dependent probabilities from outcome evaluations, as well as to assure the probability assessments to be constant in a utility representation. However in an implicit formula, the comonotonic outcome-mixture betweenness does not imply any probability parts to be constant, which causes the stronger version of the comonotonic betweenness to be necessary. It clarifies the substantially different roles between probability and outcome mixtures in the more general utility representations.

Since outcome mixtures are the quite intelligible and tractable technique as a counterpart of probability mixtures, one may question that the outcome mixture of any two acts (hence, outcomes) is well-defined under our weaker assumption. It is quite plausible since the preference average of two acts is established through the biseparable preference in Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003). However, in our implicit representation, the preference average of any two acts is also well-defined since it has the biseparable explicit formula. In this regard, the axiom of isometry extends the scope of the preference average, thus outcome mixtures as well.

This paper proposed a cardinal representation via relative comonotonic isometry. The formula extends the rank-dependent expected utility with relative rank linearity. The binary representation provides a building block for constructing more complete representation together with other axioms. As seen in examples, maxmin or $\alpha$-MEU combined with our implicit formula enables us to capture more complicated, subtler behavior toward ambiguity. These representations have to be investigated in succession.

# Appendix

## Proof of Theorem 1

### (i)⇒(ii)

The proof consists of eight lemmas. We assume $\succcurlyeq$ satisfies A1-A6 throughout this part. Given $\succcurlyeq$ satisfying A1-A5, we obtain nontrivial, monotonic and continuous $V : \mathcal{F} \to \mathbb{R}$ that represents $\succcurlyeq$. The (sub) continuity of $V$ on $\mathcal{F}$ follows from Lemma 31 in GM01.

A set of binary acts $C_A \subset \mathcal{F}_2$ is a *binary comonotonic set given* $A$ if

$$C_A = \{xAy \in \mathcal{F}_2 \mid x \succcurlyeq y \text{ for } x,\, y \in X\}.$$

By $I_A(z) \subset C_A$ we denote a *binary comonotonic indifferent set given* $A$ *and* $z$ if $I_A(z) = \{xAy \in C_A \mid xAy \sim z\}$. Let $I_A^i(z)$, $i = 1, 2$ be

$$\begin{aligned} I_A^1(z) &= \{x \in X \mid xAy \in I_A(z) \text{ for some } y \prec x\} \\ I_A^2(z) &= \{y \in X \mid xAy \in I_A(z) \text{ for some } y \prec x\}. \end{aligned}$$

**Lemma 1** *Fix an arbitrary essential event $A \in \Sigma$ and $z \in X$. Then $\succcurlyeq$ satisfies the Hexagon condition on $I_A(z)$: for all $a,\, b,\, c \in I_A^1(z)$, all $q,\, r,\, s \in I_A^2(z)$, $aAq \sim bAr$, $aAr \sim bAs$ and $cAq \sim aAr \Rightarrow cAr \sim aAs$.*

**Proof.** Choose arbitrary $a,\, b,\, c \in I_A^1(z)$ and $q,\, r,\, s \in I_A^2(z)$ which satisfy $aAq \sim bAr$, $aAr \sim bAs$ and $cAq \sim aAr$. Without loss of generality, assume $a \succcurlyeq b$. Then by A4 (Eventwise Monotonicity), we have $q \preccurlyeq r \preccurlyeq s$. By A6

$$\begin{aligned} z &\sim aAq \sim bAr \Rightarrow m_A(a,b)\, Am_A(r,q) \sim z \\ z &\sim aAr \sim bAs \Rightarrow m_A(a,b)\, Am_A(s,r) \sim z \end{aligned}$$

Again by A6 together with A4, $m_A(r,q) \sim m_A(s,r)$. On the other hand, $q \preccurlyeq r$ implies $c \succcurlyeq a$

$$z \sim cAq \sim aAr \Rightarrow m_A(c,a)\, Am_A(r,q) \sim z$$

Then, $r \preccurlyeq s$ implies $c \succcurlyeq a \succcurlyeq b$, and $m_A(r,q) \sim m_A(s,r)$ implies $m_A(c,a)\, Am_A(s,r) \sim z$.

Assume $cAr \succ aAs$. $r \preccurlyeq s$ implies $c \succ a$. If $r \sim s$, then $q \sim r$ implies $cAq \succ aAr$, which contradicts the assumption. Thus assume $r \prec s$. However, it contradicts the fact that $q \preccurlyeq r \prec s$ and $m_A(r,q) \sim m_A(s,r)$. By the same argument, $cAr \prec aAs$ also leads to a contradiction. Therefore $cAr \sim aAs$. ■

**Lemma 2** *Fix an arbitrary essential event $A \in \Sigma$ and $z \in X$. Then there exist additive value functions $V_A^i : X \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ such that for all $xAy \in I_A(z)$ with $x \succcurlyeq y$, a* *functional* $V(xAy) = V_A^1(x;v) + V_A^1(y;v) = v$ *with* $V(z) = v$ *represents* $\succcurlyeq$ *on* $I_A(z)$. *Moreover, if both* $V = V_A^1 + V_A^2$ *and* $U = U_A^1 + U_A^2$ *represent* $\succcurlyeq$, *then there exist a unique set of real numbers* $a > 0$, $b_1$, *and* $b_2$ *such that* $V_A^i = aU_A^i + b_i$, $i = 1, 2$.

**Proof.** Based on Lemma 1, by applying Theorem 3.2 and 3.3 (a) in Wakker (1993) for $n = 2$, the local additive representation $V_A^1$ and $V_A^2$ are obtained and they are cardinal. In addition, from Assumption 2.1 in Chateauneuf and Wakker (1993), $X$ is connected, and $I_A^i(z)$, $i = 1, 2$ and $I_A(z)$ are arc connected (Chew et.al, 1993), for any $z \in X$, $V(xAy) = V_A^1(x;v) + V_A^2(y;v) = v$ represents $\succcurlyeq$ on $I_A(z)$. ■

**Lemma 3** *Fix an arbitrary essential event* $A \in \Sigma$. *Then, for every* $z \in X$ *there exist* $\mu_A^v$ *and* $u_A^i(x,v)$, $i = 1, 2$ *such that*

$$V(xAy) = \mu_A^v u_A^1(x,v) + (1 - \mu_A^v) u_A^2(y,v) = v \tag{A-1}$$

*represents* $\succcurlyeq$ *on* $C_A$.

**Proof.** The proof of lemma includes three steps (1) to (3).

(1) Construct $V(xAy)$ on $I_A(z)$.

By Lemma 2, for all $xAy \in I_A(z)$, $V(xAy) = V_A^1(x;v) + V_A^2(y;v)$ represents $\succcurlyeq$ on $I_A(z)$, where $V$, $V_A^1$ and $V_A^2$ are all continuous and unique up to positive affine transformations. From A2, there exist $x^*$ and $x_*$ such that $x^* \succ x_*$. Denote $v^* = V(x^*)$ and $v_* = V(x_*)$.

Given an essential event $A \in \Sigma$, consider the certainty equivalent outcome $z^A$ of $x^*Ax_*$, i.e., $z^A \sim x^*Ax_*$. A4 implies such $z^A$ must be in $X$. For this $z^A$, consider $I_A(z^A)$. Note that $I_A^1(z^A) = \{x \in X \mid z^A \prec x\}$ and $I_A^2(z^A) = \{y \in X \mid y \prec z^A\}$. By Lemma 2, $V(xAy) = V_A^1(x;v^A) + V_A^2(y;v^A)$ represents $\succcurlyeq$ on $I_A(z^A)$. By construction, $V(x_A^* x_*) = V_A^1(x^*;v^A) + V_A^2(x_*;v^A) = v^A$. Define

$$\mu_A^v = \frac{V_A^1(x^*;v^A)}{v^A},\ u_A^1(x,v) = \frac{V_A^1(x;v)}{\mu_A^v} \text{ and } u_A^2(x,v) = \frac{V_A^2(x;v)}{1-\mu_A^v}.$$

By definition, $\mu_A^v$ is in $(0,1)$, constant and unique to $A$. For all $xAy \in I_A(z^A)$,

$$V(xAy) = \mu_A^v u_A^1(x,v^A) + (1-\mu_A^v) u_A^2(y,v^A) = v^A,$$

which represents $\succcurlyeq$ on $I_A(z^A)$.

(2) Extend $V(xAy)$ on $C_A$.

(2-i) Consider $z \succ z^A$. Then there exists a $w \in I_A^2(z)$ such that $x^*Aw \sim z$. Since $I_A^1(z) \subset I_A^1(z^A)$, $V_A^1(x;v^A) = \mu_A^v u_A^1(x,v^A)$ represents the same order on $X$ and its cardinality, $V_A^1$ is independent of $A$, thus write $V^1$. By the same argument, $V_A^2$ is independent of $A$, so write

$V^2$. To verify the above construction works, letting $v = V(z)$,

$$\begin{aligned} V(x^* A w) &= \mu_A^v u_A^1(x^*, v) + (1-\mu_A^v) u_A^2(w, v) \\ &= V^1(x^*; v) + V^2(w; v) = v, \end{aligned}$$

which is strictly greater than $V\left(z^A\right) = v^A$.

(2-ii) Consider $z \prec z^A$. Then there exists a $b \in I_A^1(z)$ such that $bAx_* \sim z$. Since $I_A^2(z) \subset I_A^2\left(z^A\right)$, by the same argument of (2-i), letting $v = V(z)$, we have

$$\begin{aligned} V(bAx_*) &= \mu_A^v u_A^1(b, v) + (1-\mu_A^v) u_A^2(x_*, v) \\ &= V^1(b; v) + V^2(x_*; v) = v, \end{aligned}$$

which is strictly smaller than $v^A$. Therefore (A-1) represents $\succcurlyeq$ on $C_A$. ∎

**Lemma 4** *Fix an arbitrary essential event $A \in \Sigma$. Then, for every $v \in \mathbb{R}$ there exist $\mu_A^v$, $\rho > 0$ and $u : X \to \mathbb{R}$ such that $V$ represents $\succcurlyeq$ and*

$$V(xAy) = \frac{\mu_A^v}{\mu_A^v + (1-\mu_A^v)\rho} u(x) + \left[1 - \frac{\mu_A^v}{\mu_A^v + (1-\mu_A^v)\rho}\right] u(y) = v \qquad \text{(A-2)}$$

*represents $\succcurlyeq$ on $C_A$.*

**Proof.** Consider two indifferent binary acts $xAy$ and $x'Ay'$ with $x \succcurlyeq y$, $x' \succcurlyeq y'$ and $x \succcurlyeq x'$. Then $x \succ x'$ if and only if $y \prec y'$, and $x \sim x'$ if and only if $y \sim y'$ by the essentiality of $A$ and A4. It implies that

$$\begin{aligned} x &\succcurlyeq x' \Leftrightarrow y \preccurlyeq y' \\ &\Leftrightarrow u^1(x, v) \geqq u^1(x', v) \Leftrightarrow u^2(y, v) \leqq u^2(y', v). \end{aligned}$$

Thus $u^1$ and $u^2$ represents the same preference on $X$.

Take arbitrary $v \in \mathbb{R}$ and $z \in X$, normalize $u^1$ so that $u^1(z, v) - v = 0$. Since both $u^1(\cdot\,, v) - v$ and $u^2(\cdot\,, v) - v$ also represent $\succcurlyeq$ on $X$, there exists a unique set of real numbers $\rho > 0$ and $\sigma$ such that $u^2(\cdot\,, v) - v = \rho\left[u^1(\cdot\,, v) - v\right] + \sigma$. However, by Lemma 3,

$$\begin{aligned} & \mu_A^v\left[u^1(z, v) - v\right] + (1-\mu_A^v)\left\{\rho\left[u^1(z, v) - v\right] + \sigma\right\} \\ = \;& \left(\mu_A^v + (1-\mu_A^v)\rho\right)\left[u^1(z, v) - v\right] + (1-\mu_A^v)\sigma \\ = \;& 0. \end{aligned}$$

By construction, the fact $u^1(z, v) - v = 0$ and $1 - \mu_A^v > 0$ implies that $\sigma = 0$.

Now write $u(\cdot\,,v)$ instead of $u^1(\cdot\,,v)$. For all $xAy \sim z$, we have

$$\mu_A^v [u(x,v) - v] + (1-\mu_A^v)\rho [u(y,v) - v] = 0. \tag{A-3}$$

This equality holds for any $A \in \Sigma$, $x, y \in X$. Rearranging terms, we have

$$V(xAy) = \frac{\mu_A^v}{\mu_A^v + (1-\mu_A^v)\rho} u(x,v) + \left[1 - \frac{\mu_A^v}{\mu_A^v + (1-\mu_A^v)\rho}\right] u(y,v) = v.$$

Now consider $V(x)$ on $X$. Since $V$ is cardinal, if $V$ and $V'$ represent the same preference $\succcurlyeq$ on $X$, then there exists a set of real numbers $a > 0$ and $b$ such that $V' = aV + b$. Set $v' = V'(x)$ and $v = V(x)$. Then

$$V'(x) = au(x, av+b) + b = au(x,v) + b = aV(x) + b,$$

which concludes that $u(x, av+b) = u(x,v)$ for every $(x,v)$, hence $u(x,v)$ does not depend on $v$. Therefore write $u(x)$ instead of $u(x,v)$ in (A-3),

$$\mu_A^v [u(x) - v] + (1-\mu_A^v)\rho [u(y) - v] = 0. \tag{A-4}$$

Rearranging terms, we have

$$V(xAy) = \frac{\mu_A^v}{\mu_A^v + (1-\mu_A^v)\rho} u(x) + \left[1 - \frac{\mu_A^v}{\mu_A^v + (1-\mu_A^v)\rho}\right] u(y) = v.$$

■

**Lemma 5** *Fix an arbitrary essential event $A \in \Sigma$. Then for all $x \succcurlyeq y$,*

$$u(m_A(x,y)) = \tfrac{1}{2}u(x) + \tfrac{1}{2}u(y).$$

**Proof.** Given an essential event $A \in \Sigma$, $z \in X$, and $\rho > 0$, write $\pi = \frac{\mu_A^v}{\mu_A^v + (1-\mu_A^v)\rho}$ in (A2) for simplicity. Note that $0 < \pi < \mu_A^v$ since $\mu_A^v \in (0,1)$ and $\rho > 0$ for any $v \in \mathbb{R}$.

By definition of the preference average, set $m_A(x,y) \sim z$ and $v = V(z) = u(z)$. $z$ satisfies $xAy \sim c(xAz)\,Ac(zAy)$,

$$\begin{aligned} V(xAy) &= V(c(xAz)\,Ac(zAy)) \\ &\Leftrightarrow \pi u(x) + (1-\pi)u(y) = \pi[\pi u(x) + (1-\pi)u(z)] \\ &\quad + (1-\pi)[\pi u(z) + (1-\pi)u(y)] \\ &\Leftrightarrow u(z) = \tfrac{1}{2}u(x) + \tfrac{1}{2}u(y). \end{aligned}$$

The relationship holds under an arbitrary value of $\pi \in (0,1)$, hence under any arbitrary essential event. ■

**Lemma 6** *Fix an arbitrary essential event $A \in \Sigma$. Then, there exist $\mu_A \in (0,1)$ and $\rho > 0$ such that $V(xAy) = \frac{\mu_A}{\mu_A + (1-\mu_A)\rho} u(x) + \left(1 - \frac{\mu_A}{\mu_A + (1-\mu_A)\rho}\right) u(y)$ represents $\succsim$ on $C_A$.*

**Proof.** Given Lemma 1-5, we need to show that $\mu_A$ is constant for any $v \in \mathbb{R}$. Take arbitrary $x, x', y, y' \in X$ such that $x \succ y$, $x' \succ y'$ and $x \succ x'$. Write $\pi(v) = \frac{\mu_A^v}{\mu_A^v + \left(1-\mu_A^v\right)\rho}$ and consider the implicit formula (A4). Rearranging terms, we have

$$\begin{aligned} -[u(y) - v] &= \pi(v)[u(x) - v] \\ -[u(y') - v'] &= \pi(v')[u(x') - v']. \end{aligned}$$

Let $v'' = \frac{1}{2}v + \frac{1}{2}v'$. By A6, $\frac{\pi(v)[u(x)-v] + \pi(v')[u(x')-v']}{2} = \pi(v'') \frac{[u(x)-v]+[u(x')-v']}{2}$,

$$\{\pi(v) - \pi(v'')\}[u(x) - v] + \{\pi(v') - \pi(v'')\}[u(x') - v'] = 0.$$

However by construction, $u(x) - v > 0$ and $u(x') - v' > 0$, which concludes that $\pi(v) = \pi(v') = \pi(v'')$ for any $v$ and $v'$, that is $\pi$ is constant. Therefore $\mu_A^v$ is not dependent on $v$. ■

**Lemma 7** *There exist $\mu : \Sigma \to [0,1]$ and $u : X \to \mathbb{R}$ such that for all $x \succsim y$ in $X$*

$$V(xAy) = \frac{\mu(A)}{\mu(A) + (1-\mu(A))\rho} V(x) + \left(1 - \frac{\mu(A)}{\mu(A) + (1-\mu(A))\rho}\right) V(y)$$

*represents $\succsim$ on $\mathcal{F}_2$.*

**Proof.** As proved in Lemma 3 and 5, $\mu_A$ is uniquely determined for any essential event $A$ and $\mu_A \in (0,1)$.

Construct a set function $\mu : \Sigma \to [0,1]$:

$$\mu(A) = \begin{cases} \mu_A & \text{if } A \text{ is essential,} \\ 1 & \text{if } A \text{ is universal,} \\ 0 & \text{if } A \text{ is null.} \end{cases}$$

To verify monotonicity of $\mu$, suppose that there are two essential events $A$, $B \in \Sigma$ such that $B \subset A$ and $B \neq A$. Then by definition of $\mu_A$ and $\mu_B$, $\mu(A) \geqq \mu(B)$ since $x^* A x_*$ dominates $x^* B x_*$. In addtion, a common utility $u$ and $\rho$ among events are obtained in Lemma 4, hence $V(xAx) = u(x)$ for any $A \in \Sigma$. ■

**Proof of Theorem 1.** Lemma 1–8 imply that $(*)$ represents $\succsim$ on $\mathcal{F}_2$. From Lemma 31 in GM01, $V$ also represents $\succsim$ on $\mathcal{F}$, and it is continuous. The continuity here is the stronger version that results from A5. The $V$'s cardinality is straightforward given $\mu$ and $\rho$. ■

**(ii)⇒(i)**

**Proof.** Suppose that a binary relation $\succsim$ on $\mathcal{F}$ is represented by $(*)$. A1-A5 is implied by a continuous, monotonic and non-constant $V$. As for A6, take arbitrary $xAy$ and $x'Ay'$ with $V(x) > V(y)$ and $V(x') > V(y')$ so that $V(xAy) \geqq V(x'Ay')$. Let $x''$ and $y''$ satisfy $V(x'', v'') = \frac{1}{2}V(x) + \frac{1}{2}V(x')$ and $V(y'') = \frac{1}{2}V(x) + \frac{1}{2}V(x')$. Then

$$V(x''Ay'') = \tfrac{\mu(A)}{\mu(A)+(1-\mu(A))\rho}V(x'') + \left(1 - \tfrac{\mu(A)}{\mu(A)+(1-\mu(A))\rho}\right)V(y''),$$

which implies, if $xAy \succsim x'Ay'$, then $m_B(c(xAy), c(x'Ay')) \sim m_B(x, x')\, A m_B(y, y')$. ■

**Proof of Theorem 2: (i)⇒(ii)**

**Proof.** Theorem 1 proved the case where $n = 2$, the binary act case. We begin with the representation $(*)$, and then extend $V$ over the $n$ dimensional acts $\mathcal{F}$.

The proof is conducted by induction: it is assumed that the case $n = k$, $k \geqq 2$ is correct and then to be proved the case $n = k + 1$.

Take an arbitrary act $f \in \mathcal{F}$. If $f$ only has strictly less than $k+1$ distinct outcomes, the representation of $f$ is already obtained in the part of presumption of induction, therefore assume that $f$ has $k + 1$ distinct outcomes, $x_1 \succ \cdots \succ x_{k+1}$. Let $A_i = f^{-1}(x_i)$, $i = 1, \ldots, k+1$. It is also assumed that every event $A_i$ is non-null.

Define the better event of $f$, $B_0 = \varnothing$ and $B_i = \{\omega \in \Omega \mid f(\omega) \succsim x_i\}$, $i = 1, \ldots, k+1$. Given that the rank-order of outcomes in $f$, there are $k$ relative orders since the certainty equivalent outcome of $f$, $c(f)$ is between $x_j$ and $x_{j+1}$ for some $j \in \{1, \ldots, k\}$ although $c(f)$ is implicitly determined. Then the order of outcomes is rewritten so that $x_1 \succ \cdots \succ x_j \succsim c(f) \succ y_{j+1} \succ \cdots \succ y_k$. Each $x_i$ or $y_i$ respectively corresponds to a winning or losing outcome relative to $c(f)$. Notice that the winning event of $f$, $B(f) = B_j$.

Lemma 5 established the $\frac{1}{2}$-$\frac{1}{2}$ utility mixture, hence it is possible to construct $\alpha$-mixture of two acts as in Ghirardato et.al (2003). Given that two acts $f'$ and $f''$ are relatively comonotonic and $\alpha \in [0, 1]$ such that $\alpha v' + (1-\alpha)v'' = v$, the $\alpha$ mixture of $f'$ and $f''$, $\alpha f' + (1-\alpha) f''$, is the act $f$ such that $f(A_i) \sim \alpha f'(A_i) + (1-\alpha) f''(A_i)$ for every $i = 1, \ldots, k+1$.

**Case 1:** $x_j \succ c(f)$ and $j \geqq 2$

Given the chosen act $f$, construct $k$-outcome two acts $f'$ and $f''$ in the following way:

$$\begin{aligned} f' &= (x_1, A_1; x_1, A_2; \ldots; x_{j-1}, A_{j-1}; x_j, A_j; y_{j+1}, A_{j+1}; \ldots; y_k, A_k; y_{k+1}, A_{k+1}), \\ f'' &= (x_2, A_1; x_2, A_2; \ldots; x_{j-1}, A_{j-1}; x_j, A_j; y_{j+1}, A_{j+1}; \ldots; y_k, A_k; y_{k+1}, A_{k+1}). \end{aligned}$$

By A3 and A4, $c(f') \succ c(f'')$. Note that both $f'$ and $f''$ has $k$ distinct outcomes, however, by construction of $f$, there are $k-1$ relative orders $f'$ and $f''$. According to the relative ranks, they are classified into two cases.

**(a)** $f$, $f'$, and $f''$ are all relatively comonotonic.

By A7, we can find $\alpha \in (0,1)$ such that $f \sim \alpha f' + (1-\alpha) f''$, and by assumption of induction, we have $v = \alpha v' + (1-\alpha) v''$ in terms of utility representation,

$$\begin{aligned} & \alpha V(f') + (1-\alpha) V(f'') \\ = \; & \alpha \{\mu(B_1) V(x_1) + [\mu(B_2) - \mu(B_1)] V(x_1)\} \\ & \quad + (1-\alpha) \{\mu(B_1) V(x_2) + [\mu(B_2) - \mu(B_1)] V(x_2)\} \\ & \quad + \sum_{i=3}^{k+1} \alpha V^i(f(B_i)) + (1-\alpha) V^{i-1}(f(B_{i-1})) \end{aligned}$$

However

$$\begin{aligned} & \alpha \{\mu(B_1)[V(x_1) - v'] + [\mu(B_2) - \mu(B_1)][V(x_1) - v']\} \\ & \quad + (1-\alpha) \{\mu(B_1)[V(x_2) - v''] + [\mu(B_2) - \mu(B_1)][V(x_2) - v'']\} = 0. \end{aligned}$$

Rearranging terms, we have

$$\mu(B_2) \{\alpha V(x_1) + (1-\alpha) V(x_2)\} = \mu(B_1) v' + [\mu(B_2) - \mu(B_1)] v''$$

Therefore, $\alpha = \frac{\mu(B_1)}{\mu(B_2)}$, which implies $(**)$ in $k+1$ outcomes since

$$\mu(B_2) \{\alpha V(x_1) + (1-\alpha) V(x_2)\} = \mu(B_1) V(x_1) + [\mu(B_2) - \mu(B_1)] V(x_2).$$

**(b)** $f' \succcurlyeq x_j$ and/or $y_{j+1} \succcurlyeq f''$

In this case, $y_{j+1}$ is preferred to $f''$, the relative order of $f''$ is not the same as $f$ nor $f'$. Instead of them, construct $g' \sim c(f')$ and $g'' \sim c(f'')$ such that $x_m \succ c(f') \succ x_{m+1}$ for some $m \geqq j > 2$ and $y_l \succ c(f'') \succ y_{l+1}$ for some $j+1 > l \geqq k$

$$\begin{aligned} g' &= \left(x_1, B_1; x_1, B_2; \ldots; c(f'), B_m; \ldots; c(f'), B_j; y'_{j+1}, B_{j+1}; \ldots; y'_l, B_l; \ldots; y_k, B_k; y_{k+1}, B_{k+1}\right), \\ g'' &= \left(x_2, B_1; x_2, B_2; \ldots; x''_m, B_m; x''_j, B_j; c(f''), B_{j+1}; \ldots; c(f''), B_l; \ldots; y_k, B_k; y_{k+1}, B_{k+1}\right). \end{aligned}$$

where $x''_i$ or $y'_i$ is such that $x_i \sim \alpha c(g') + (1-\alpha) x''_i$, $y_i \sim \alpha y'_i + (1-\alpha) c(g'')$. Note that $g'$ and $g''$ have at most $k$ outcomes, and by construction, $f$, $g'$, and $g''$ are all relatively comonotonic.

By the same argument in (a), $f \sim \alpha g' + (1-\alpha) g''$ implies $\alpha = \frac{\mu(B_1)}{\mu(B_2)}$, thus $(**)$ holds.

**Case 2:** $x_j \sim c(f)$ and $j = 2$, or $j = 1$

**(a)** $y_l \succcurlyeq c(f'') \succ y_{l+1}$

By assumption $x_1 \succ c(f')$. Construct $h' \sim c(f')$ and $h'' \sim c(f'')$ such that $y_l \succcurlyeq c(f'') \succ y_{l+1}$ for some $j+1 \leqq l < k+1$

$$\begin{aligned} h' &= (x_1, A_1; y_2', A_2; \dots; y_l', A_l; \dots; y_k, A_k; y_k, A_{k+1}), \\ h'' &= (x_1, A_1; c(f''), A_2; \dots; c(f''), A_l; \dots; y_{k+1}, A_k; y_{k+1}, A_{k+1}), \end{aligned}$$

where $y_i'$, $i = 2, \dots, l$ is such that $y_i \sim \alpha y_i' + (1-\alpha) z''$ for some $\alpha \in (0,1)$. Assume $f \sim \alpha h' + (1-\alpha) h''$. Then

$$\begin{aligned} & \alpha V(h') + (1-\alpha) V(h'') \\ = & \sum_{i=1}^{k-1} [\mu(B_i) - \mu(B_{i-1})] V(f(B_i)) \\ & + \alpha \{[\mu(B_k) - \mu(B_{k-1})] \rho V(y_k) + [1 - \mu(B_k)] \rho V(y_k)\} \\ & + (1-\alpha) \{[\mu(B_k) - \mu(B_{k-1})] \rho V(y_{k+1}) + [1 - \mu(B_k)] \rho V(y_{k+1})\} \end{aligned}$$

However

$$\begin{aligned} 0 = & \ \alpha \{[\mu(B_k) - \mu(B_{k-1})] \rho [V(y_k) - v'] + [1 - \mu(B_k)] \rho [V(y_k) - v']\} \\ & + (1-\alpha) \{[\mu(B_k) - \mu(B_{k-1})] \rho [V(y_{k+1}) - v''] + [1 - \mu(B_k)] \rho [V(y_{k+1}) - v'']\} \end{aligned}$$

$$[1 - \mu(B_{k-1})] \{\alpha V(y_k) + (1-\alpha) V(y_{k+1})\} = [\mu(B_k) - \mu(B_{k-1})] v' + [1 - \mu(B_k)] v''$$

Therefore, we have $\alpha = \frac{\mu(B_k) - \mu(B_{k-1})}{1 - \mu(B_{k-1})}$, which implies $(**)$ since

$$[1 - \mu(B_{k-1})] \{\alpha V(y_k) + (1-\alpha) V(y_{k+1})\} = [\mu(B_k) - \mu(B_{k-1})] V(y_k) + [1 - \mu(B_k)] V(y_{k+1}).$$

■

**(ii)⇒(i)**

**Proof.** A1-A6 is already checked in the proof of Theorem 1, therefore it is to show the obtained representation for an act with three or more outcomes satisfies A7. Assume a binary relation $\succcurlyeq$ on $\mathcal{F}$ is represented by $(**)$. Take arbitrary relative comonotonic two act $f$ and $g$. Then for any essential event $A$, $m_A(f(A_i), g(A_i)) = \frac{1}{2} V(f(A_i)) + \frac{1}{2} V(g(A_i))$.

Since $f$ and $g$ are relatively comonotonic

$$\begin{aligned}
& \tfrac{1}{2}V(f)+\tfrac{1}{2}V(g) \\
= & \frac{1}{2}\sum_{i=1}^{k}\left[\mu(B_i)-\mu(B_{i-1})\right]V(f(A_i))+\frac{1}{2}\sum_{i=1}^{k}\left[\mu(B_i)-\mu(B_{i-1})\right]V(g(A_i)) \\
= & \sum_{i=1}^{k}\left[\mu(B_i)-\mu(B_{i-1})\right]\left\{\tfrac{1}{2}V(f(A_i))+\tfrac{1}{2}V(g(A_i))\right\} \\
= & \sum_{i=1}^{k}\left[\mu(B_i)-\mu(B_{i-1})\right]V(m_A(f(A_i),g(A_i))),
\end{aligned}$$

which implies $m_A(c(f),c(g))\sim m_A(f,g)$. ■

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