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3人部屋安定ルームメイト問題のNP完全性

岡本 和也† 宮崎 修一†† 岩間 一雄†

† 京都大学 大学院情報学研究科
〒 606-8501 京都市左京区吉田本町
†† 京都大学 学術情報メディアセンター
〒 606-8501 京都市左京区吉田本町
E-mail: †{okia,iwama}@kuis.kyoto-u.ac.jp, ††shuichi@media.kyoto-u.ac.jp

あらまし 安定ルームメイト問題は，希望リストに基づいて「安定性」を満たすように，$2N$ 人の人間を N 組のペアに分割する問題である．安定ルームメイト問題は解を持たない場合もあるが，解を持つか否かを判定し，持つ場合には解を見つける多項式時間アルゴリズムが知られている．本研究ではこの問題を拡張し，$3N$ 人の人間を N 組のトリプルに分割する問題を提案する．そして，この問題において解の存在を判定する問題が NP 完全であることを示す．

キーワード 安定マッチング，安定ルームメイト問題，NP 完全

NP-Completeness of the Stable Roommates Problem with Triple Rooms

Kazuya OKAMOTO†, Shuichi MIYAZAKI††, and Kazuo IWAMA†

† Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku Kyoto 606-8501, Japan
†† Academic Center for Computing and Media Studies, Kyoto University, Yoshida-Honmachi, Sakyo-ku Kyoto 606-8501, Japan
E-mail: †{okia,iwama}@kuis.kyoto-u.ac.jp, ††shuichi@media.kyoto-u.ac.jp

Abstract In the stable roommates problem, we are given $2N$ people, each having a preference list that ranks remaining $2N-1$ people in a strict order according to his/her preferences. It asks to find a stable matching, namely, a set of N pairs that satisfies the "stability" condition. There are instances that have no solution, but Irving has developed a polynomial time algorithm that decides if there exists a solution, and finds one if exists. We extend this problem into 3-dimension. In our problem, we are given $3N$ people and are asked to partition them into N triples. We prove that the problem of deciding if a stable matching exists is NP-complete.

Key words stable matching, the stable roommates problem, NP-complete

1. Introduction

The *stable roommates problem*, which we call *SR*, is one of the stable matching problems first introduced by Gale and Shapley [5]. An instance of SR consists of $2N$ men, each having a *preference list* which is a total order of other $2N-1$ men according to his preference. A *matching* is a set of N disjoint pairs of men(1). For a matching M, a pair $\{m_1, m_2\} \notin M$ is said to be a *blocking pair for M* if the following conditions are met: $\{m_1, m_1'\}, \{m_2, m_2'\} \in M$, m_1 prefers m_2 to m_1', and m_2 prefers m_1 to m_2'. A matching M is *unstable* if there is a blocking pair for M; otherwise, M is *stable*. There are instances of SR that have no stable matching [5], but Irving has developed an $O(N^2)$ time algorithm that either finds a stable matching or reports that no stable matching exists for a given instance [8].

There are several practical applications of SR: The most natural one is to assign people to double rooms based on their preferences as the problem name suggests. Another application occurs in forming pairings of players for chess tournaments [11]. Recently, SR is studied for pairwise kidney exchange between incompatible patient-donor pairs [16].

Unlike American or European styles, it is quite common to have rooms for three or more people in Japan. For example, resort hotels in Japan, called "Ryokan", have Japanese

(1): We consider only perfect matchings in this paper, and hence, we define "matching" in this way.

style rooms that are typically used for two or more persons. As another example, in university dormitories, students are sometimes packed into 4-person rooms until they become the 3rd grade. It is then quite natural to extend 2-person rooms to k-person rooms ($\geqq 3$) in the stable roommates problem. In this paper, we consider the case of $k = 3$ and verify its computational complexity.

Our Contribution. We extend SR to 3-person rooms, which we call *3D-SR* (*3-Dimensional SR*). An instance of 3D-SR consists of $3N$ men, each having a preference list. A preference list of each man is a totally ordered list including all the other $3N - 1$ men according to his preference. A matching is now a set of N disjoint *triples*, i.e., we want to pack $3N$ men into N 3-person rooms. A matching is stable if there is no three men, each of whom becomes better off if they constitute a new triple (a formal definition will be given later). Such a triple is called a *blocking triple*. 3D-SR asks if there exists a stable matching for a given instance. Recall that in the case of the classical stable roommates problem (for 2-person rooms), the problem is solved in polynomial time. In this paper, we show that the situation changes when we consider 3-person rooms, namely, 3D-SR is NP-complete.

Related Results. SR can be considered as a matching problem in non-bipartite graphs, and its bipartite counterpart, usually interpreted as matchings between men and women, is the *stable marriage problem (SM)*, also introduced by Gale and Shapley [5]. Unlike SR, any instance of SM admits a stable matching and one can be found in $O(N^2)$ time. SM and its many-one variant, called Hospitals/Residents problem, are used in several assignment systems. For example, it is used to assign medical students to hospitals in the U.S. (NRMP) [7], [14], Canada (CaRMS) [9], Scotland (SPA) [10], and Japan (JRMS). As another example, it is used to assign students to schools in Norway [3] and in Singapore [18].

For SM, the extension to three dimension was proposed by Knuth [12]. Later, Ng and Hirschberg defined 3D-SM and proved its NP-completeness [15]. Subsequently, Subramanian gave an alternative NP-completeness proof [17]. In their definition, each preference list of a member of one party is an ordered list of pairs of the other two parties. So, if each party contains N members, the length of each preference list is N^2. Ng and Hirschberg pointed out the complexity of *3D-SM with cyclic preference lists* as an open problem [15]: Let A, B, and C be disjoint parties. In preference lists, each member of A ranks members of B, each member of B ranks members of C, and each member of C ranks members A, and the definition of stability is given in some suitable way. The complexity of this problem is still open but there are some results when the instance size is small: Boros et al. [2] and Eriksson et al. [4] showed that any instance admits a stable matching when $N \leqq 3$ and when $N \leqq 4$, respectively.

Recently, the extension to three dimension have been considered also for SR. Huang [13] defined 3D-SR in a way different from ours, and proved NP-completeness of its several variants. Arkin et al. [1] treated geometric 3D-SR, where each person is located in $\mathbf{R}^2$ and their preference lists are determined based on Euclidean distance between them. The complexity of this problem is still open, but they developed an algorithm to find a stable matching with a relaxed stability definition.

2. Preliminaries

An instance of 3D-SR consists of $3N$ men and each man's preference list. A preference list is a totally ordered list including all the other $3N-1$ men according to his preference. Fig. 1 shows an example of an instance of 3D-SR when $N = 2$.

1 : 2 3 4 5 6
2 : 4 6 1 3 5
3 : 5 1 4 2 6
4 : 3 6 2 5 1
5 : 1 3 4 6 2
6 : 5 4 3 2 1

Figure 1 An instance of 3D-SR.

1 : 2 3 4 5 6
2 : 3 4 1 5 6
3 : 4 1 2 5 6
4 : 1 2 3 5 6
5 : 1 2 3 4 6
6 : 1 2 3 4 5

Figure 2 An instance of 3D-SR that has no stable matching.

Men are denoted by numbers 1 through 6. Each row represents a preference list of each man; for example, the man 2 likes other men in an order of 4, 6, 1, 3 and 5. If man m prefers man m_2 to man m_1, we write $m_2 \succ_m m_1$. For example, $4 \succ_5 2$ in the above example. If m prefers m_2 to m_1 or m_1 and m_2 are the same man, we write $m_2 \succeq_m m_1$.

A *matching* is a set of N disjoint triples. Suppose that a matching M includes triples $\{m_1, m_1', m_1''\}$, $\{m_2, m_2', m_2''\}$ and $\{m_3, m_3', m_3''\}$. We say that the triple $\{m_1, m_2, m_3\} \notin M$

is a *blocking triple* for M if the following three conditions are met: (i) $m_2 \succeq_{m_1} m'_1$ and $m_3 \succeq_{m_1} m''_1$, or $m_3 \succeq_{m_1} m'_1$ and $m_2 \succeq_{m_1} m''_1$. (ii) $m_1 \succeq_{m_2} m'_2$ and $m_3 \succeq_{m_2} m''_2$, or $m_3 \succeq_{m_2} m'_2$ and $m_1 \succeq_{m_2} m''_2$. (iii) $m_1 \succeq_{m_3} m'_3$ and $m_2 \succeq_{m_3} m''_3$, or $m_2 \succeq_{m_3} m'_3$ and $m_1 \succeq_{m_3} m''_3$. If there is a blocking triple for M, then M is *unstable*; otherwise, M is *stable*. 3D-SR is to find a stable matching for a given instance.

For example, the matching $\{\{1,2,3\},\{4,5,6\}\}$ for the instance shown in Fig. 1 is not stable because the triple $\{3,4,5\}$ is a blocking triple. However, the matching $\{\{1,3,5\},\{2,4,6\}\}$ for the same instance is stable. There are instances of 3D-SR that have no stable matching. Fig. 2 is one such example.

3. NP-completeness of 3D-SR

We will show that 3D-SR is NP-complete using a polynomial-time reduction from PARTITION INTO TRIANGLES, which is already known to be NP-complete [6]. We first give a definition of PARTITION INTO TRIANGLES in Sec. 3.1, and show a polynomial-time reduction from PARTITION INTO TRIANGLES into 3D-SR in Sec. 3.2.

3.1 PARTITION INTO TRIANGLES

An instance of PARTITION INTO TRIANGLES consists of an undirected graph $G = (V, E)$ such that $|V| = 3q$ for some positive integer q. It asks if there exists a partition of V into q sets (sometimes called *triangles*) $T_1, T_2, \ldots, T_q$ satisfying the following properties: (i) Each T_i contains exactly three vertices. (ii) For each set $T_i = \{u_i, v_i, w_i\} (1 \leqq i \leqq q)$, $\{u_i, v_i\}, \{u_i, w_i\}, \{v_i, w_i\} \in E$.

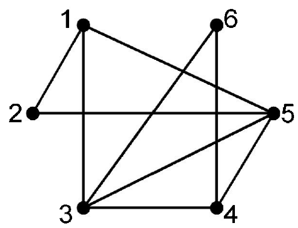

Figure 3 An instance of PARTITION INTO TRIANGLES.

Fig. 3 shows an instance of PARTITION INTO TRIANGLES. We can partition the vertices into two triangles $\{1,2,5\}$ and $\{3,4,6\}$ satisfying the conditions, and hence the answer to this instance is "YES".

3.2 Reduction from PARTITION INTO TRIANGLES to 3D-SR

Before showing a reduction, we prepare a useful gadget. Let $\mathcal{M}$ denote the set of all men in an instance. The gadget consists of four men a, b, c, and e. Fig. 4 shows the preference lists of these four men.

Each of R_a, R_b, and R_c includes men in $\mathcal{M} \setminus \{a,b,c,e\}$ in an arbitrary order. In e's list, C_e is an ordered list of the subset of men in $\mathcal{M} \setminus \{a,b,c,e\}$, and R_e is an arbitrary ordered list of the remaining men. Let us call this gadget *a 3D-SR gadget*. We denote the 3D-SR gadget consisting of a, b, c, and e, as illustlated in Fig. 4, by $[a,b,c\,|\,e]$.

a : b c e R_a

b : c e a R_b

c : e a b R_c

e : a b C_e c R_e

Figure 4 The preference lists of four men constituting a gadget.

Observe that a 3D-SR gadget has little freedom: All we can customize for a 3D-SR gadget is to specify the members of C_e and their order in the list. For a 3D-SR gadget Q, denote by m_Q the man playing a role of e above, and call him an *essential man* of Q. Also, denote by C_Q the C_e part of e's preference list, and call this part a *crucial part* of e's list. The following lemma is crucial in our reduction.

[Lemma 3.1] Let I be an instance of 3D-SR which includes a 3D-SR gadget $[a,b,c\,|\,e]$. Then, any stable matching M for I includes two triples $\{a,b,c\}$ and $\{e,x,y\}$ where $x, y \in C_e$.

Proof. To prove Lemma 3.1, we prove that M must contain the triple $\{a,b,c\}$ in Claim 3.2, and that M must contain the triple $\{e,x,y\}$ in Claim 3.3.

[Claim 3.2] If an instance I of 3D-SR includes a 3D-SR gadget $[a,b,c\,|\,e]$, then any stable matching M for I contains the triple $\{a,b,c\}$.

Proof. Suppose that $\{a,b,c\} \notin M$. Let $T(e)$ be the triple that the essential man e belongs to. We have three cases to consider: (i) None of a, b, and c is in $T(e)$. (ii) Exactly one of a, b, and c is in $T(e)$. (iii) Exactly two of a, b, and c are in $T(e)$.

Case (i): It is easy to verify that $\{a,b,c\}$ is a blocking triple for M.

Case (ii): Let m' be the one belonging to $T(e)$ among a, b, and c. If $m' = a$, a and e belong to the same triple but neither of b and c can be in $T(e)$. Then it can be seen that $\{a,b,c\}$ is a blocking triple for M. Similarly as above, we can prove that in cases of $m' = b$ and $m' = c$, $\{a,b,c\}$ and $\{a,b,e\}$ are blocking triples for M, respectively.

Case (iii): In cases of $T(e) = \{a,b,e\}$, $T(e) = \{a,c,e\}$, and $T(e) = \{b,c,e\}$, $\{a,b,c\}$, $\{a,b,e\}$, and $\{a,c,e\}$ are blocking triples for M, respectively.

So, it contradicts the stability of M in any case, which completes the proof. □

[Claim 3.3] If an instance I of 3D-SR includes 3D-SR gadget $[a, b, c \mid e]$, then any stable matching M for I includes the triple $\{e, x, y\}$, where $x, y \in C_e$.

Proof. First, by Claim 3.2, M contains the triple $\{a, b, c\}$, namely e can constitute a triple with none of a, b, and c. Now, suppose that $\{e, m', m''\} \in M$ such that $m' \in R_e$. Then $\{b, c, e\}$ is a blocking triple for M, a contradiction. □

Now the proof of Lemma 3.1 is immediate from Claims 3.2 and 3.3. □

Next, using 3D-SR gadgets, we will reduce Partition into Triangles to 3D-SR. Let $G = (V, E)(|V| = 3q = n, V = \{v_1, v_2, \cdots, v_n\})$ be an undirected graph which is an instance of Partition into Triangles. We will construct an instance $I(G)$ of 3D-SR.

First of all, we enumerate all triangles (K_3) of G. The time complexity of this enumeration is $O(n^3)$. Let $T_1, T_2, \cdots, T_m$ be all the triangles.

Next, for each triangle, we associate three 3D-SR gadgets, one for each vertex of the triangle; we call these gadgets α-gadgets. Hence, we prepare $3m$ α-gadgets. Note that if a vertex v belongs to k triangles in G, then k α-gadgets are associated with v. Suppose that a vertex $v_t (1 \leqq t \leqq n)$ belongs to the triangle $T_s (1 \leqq s \leqq m)$. Then, the α-gadget associated with v_t in T_s is denoted by $\alpha_{s,t}$. Let $\mathcal{M}_\alpha$ be the set of essential men in α-gadgets. Also, for a vertex v_t, let $\mathcal{M}_{\alpha_t}$ ($\subseteqq \mathcal{M}_\alpha$) denote the set of men in $\mathcal{M}_\alpha$ who are associated with v_t.

We further prepare other gadgets. Suppose that a vertex v_t belongs to k_t triangles. Associated with v_t, we prepare $k_t - 1$ 3D-SR gadgets $\beta_{t,1}$, $\beta_{t,2}$, $\cdots$, β_{t,k_t-1}, called β-gadgets, and $k_t - 1$ 3D-SR gadgets $\gamma_{t,1}$, $\gamma_{t,2}$, $\cdots$, γ_{t,k_t-1} called γ-gadgets. (Without loss of generality, we can assume that there is no vertex participating in no triangle, namely $k_t \geqq 1$, since otherwise, the answer to the instance G of Partition into Triangles is trivially "NO".) The numbers of β-gadgets and γ-gadgets are then $3m - n$ each. Similarly to $\mathcal{M}_{\alpha_t}$, define $\mathcal{M}_{\beta_t}$ and $\mathcal{M}_{\gamma_t}$ be the sets of essential men in β-gadgets and γ-gadgets, respectively, associated with v_t.

Finally, we will construct preference lists of men. Recall that, for each 3D-SR gadget $Q = [a, b, c \mid e]$, we have freedom only for the crucial part $C_e (= C_Q)$ in the preference list of the essential man $e (= m_Q)$. Then we will show how C_Q is constructed. Consider a triangle $T_s = \{v_{t_1}, v_{t_2}, v_{t_3}\}$ enumerated at the first step of the reduction. Recall that three α-gadgets α_{s,t_1}, α_{s,t_2}, and α_{s,t_3} are associated with T_s. Their crucial parts $C_{\alpha_{s,t_1}}$, $C_{\alpha_{s,t_2}}$, and $C_{\alpha_{s,t_3}}$ are constructed as follows:

$$C_{\alpha_{s,t_1}} : m_{\beta_{t_1,1}}\ m_{\beta_{t_1,2}}\ \cdots\ m_{\beta_{t_1,k_{t_1}-1}}\ m_{\alpha_{s,t_2}}\ m_{\alpha_{s,t_3}}\ m_{\gamma_{t_1,k_{t_1}-1}}\ m_{\gamma_{t_1,k_{t_1}-2}}\ \cdots\ m_{\gamma_{t_1,1}}$$

$$C_{\alpha_{s,t_2}} : m_{\beta_{t_2,1}}\ m_{\beta_{t_2,2}}\ \cdots\ m_{\beta_{t_2,k_{t_2}-1}}\ m_{\alpha_{s,t_1}}\ m_{\alpha_{s,t_3}}\ m_{\gamma_{t_2,k_{t_2}-1}}\ m_{\gamma_{t_2,k_{t_2}-2}}\ \cdots\ m_{\gamma_{t_2,1}}$$

$$C_{\alpha_{s,t_3}} : m_{\beta_{t_3,1}}\ m_{\beta_{t_3,2}}\ \cdots\ m_{\beta_{t_3,k_{t_3}-1}}\ m_{\alpha_{s,t_1}}\ m_{\alpha_{s,t_2}}\ m_{\gamma_{t_3,k_{t_3}-1}}\ m_{\gamma_{t_3,k_{t_3}-2}}\ \cdots\ m_{\gamma_{t_3,1}}$$

Next, we will construct crucial parts of preference lists of essential men in β- and γ-gadgets. Consider a vertex v_t participating in k_t triangles $T_{s_1}, T_{s_2}, \ldots, T_{s_{k_t}}$. Then the set of essential men in α-gadgets associated with v_t is denoted as $\mathcal{M}_{\alpha_t} = \{\alpha_{s_1,t}, \alpha_{s_2,t}, \cdots, \alpha_{s_{k_t},t}\}$. Now, associated with v_t, $C_{\beta_{t,1}}$, $C_{\beta_{t,2}}$, $\cdots$, $C_{\beta_{t,k_t-1}}$, $C_{\gamma_{t,1}}$, $C_{\gamma_{t,2}}$, $\cdots$, $C_{\gamma_{t,k_t-1}}$ of 3D-SR gadgets $\beta_{t,1}$, $\beta_{t,2}$, $\cdots$, β_{t,k_t-1}, $\gamma_{t,1}$, $\gamma_{t,2}$, $\cdots$, γ_{t,k_t-1} are defined as follows.

$$\begin{aligned}
C_{\beta_{t,1}} &: m_{\gamma_{t,1}}\ \ m_{\alpha_{s_1,t}}\ m_{\alpha_{s_2,t}}\ \cdots\ m_{\alpha_{s_{k_t},t}} \\
C_{\beta_{t,2}} &: m_{\gamma_{t,2}}\ \ m_{\alpha_{s_1,t}}\ m_{\alpha_{s_2,t}}\ \cdots\ m_{\alpha_{s_{k_t},t}} \\
&\ \vdots \\
C_{\beta_{t,k_t-1}} &: m_{\gamma_{t,k_t-1}}\ m_{\alpha_{s_1,t}}\ m_{\alpha_{s_2,t}}\ \cdots\ m_{\alpha_{s_{k_t},t}}
\end{aligned}$$

$$\begin{aligned}
C_{\gamma_{t,1}} &: m_{\beta_{t,1}}\ \ m_{\alpha_{s_1,t}}\ m_{\alpha_{s_2,t}}\ \cdots\ m_{\alpha_{s_{k_t},t}} \\
C_{\gamma_{t,2}} &: m_{\beta_{t,2}}\ \ m_{\alpha_{s_1,t}}\ m_{\alpha_{s_2,t}}\ \cdots\ m_{\alpha_{s_{k_t},t}} \\
&\ \vdots \\
C_{\gamma_{t,k_t-1}} &: m_{\beta_{t,k_t-1}}\ m_{\alpha_{s_1,t}}\ m_{\alpha_{s_2,t}}\ \cdots\ m_{\alpha_{s_{k_t},t}}
\end{aligned}$$

The reduction from Partition into Triangles to 3D-SR is now completed. It is not hard to see that the reduction can be performed in polynomial time. We will verify the correctness of the reduction in the next section.

3.3 Correctness of the Reduction

We show that an instance G of Partition into Triangles has a solution if and only if the instance $I(G)$ of 3D-SR has a solution. First, we prove the "only if" part.

[Lemma 3.4] If G has a solution, then $I(G)$ has a solution.

Proof. Let $T_1, T_2, ..., T_m$ be all triangles contained in the instance $G = (V, E)(|V| = 3q)$ of Partition into Triangles, that are enumerated in the first part of the reduction. Since G has a solution, there are q disjoint triangles. Without loss of generality, let the solution be $T = \{T_1, T_2, ..., T_q\}$. From this partition, we will construct a matching M in $I(G)$ and show that M is stable.

First of all, for each 3D-SR gadget $Q = [a, b, c \mid e]$, we construct a triple $\{a,\ b,\ c\}$ and add it to M. It then remains to show how to match essential men.

Recall that for each triangle $T_s = \{v_{t_1}, v_{t_2}, v_{t_3}\}$, we prepared 3 α-gadgets $\alpha_{s,t_1}, \alpha_{s,t_2}$, and α_{s,t_3}. If T_s is a part of the solution T, namely, if $1 \leqq s \leqq q$, then we match three essential men in these three α-gadgets to make a triple in M. It remains to show how to match essential men in β- and γ-gadgets, and in α-gadgets associated with triangles not in T.

Consider a vertex v_t participating in k_t triangles. Associated with this vertex, there are k_t essential men $m_{\alpha_{s_i},t}$ $(1 \leqq i \leqq k_t)$ in $\mathcal{M}_{\alpha_t}$, $k_t - 1$ essential men $m_{\beta_{t,i}}$ $(1 \leqq i \leqq k_t - 1)$ in $\mathcal{M}_{\beta_t}$, and $k_t - 1$ essential men $m_{\gamma_{t,i}}$ $(1 \leqq i \leqq k_t - 1)$ in $\mathcal{M}_{\gamma_t}$. Since v_t contributes to one triangle in T, one man in $\mathcal{M}_{\alpha_t}$ is already matched with some other men by the above procedure. So, remaining essential men in $\mathcal{M}_{\alpha_t}$, $\mathcal{M}_{\beta_t}$, and $\mathcal{M}_{\gamma_t}$ are $k_t - 1$ each. For each i $(1 \leqq i \leqq k_t - 1)$, we match $m_{\beta_{t,i}}$ and $m_{\gamma_{t,i}}$, thus constructing $k_t - 1$ pairs. For each such pair, add one essential man corresponding to α-gadget in an arbitrary way, making the pairs into triples. The construction of M is now completed. Note that every essential man is matched with two men in the crucial part of his list.

We then show that M constructed as above is stable. Consider an arbitrary 3D-SR gadget $Q = [a, b, c \,|\, e]$. a is matched with men at the first and the second positions of his preference list, and hence cannot be a part of a blocking triple. Since b is matched with men in the first and the third positions, he can form a blocking triple only of the form $\{b, c, e\}$. However, since e is matched with two men in the crucial part, and c is below this crucial part in e's list, e does not constitute the above blocking triple. So, b either cannot be a part of a blocking triple. Similarly, if c constitutes a blocking triple, then the triple must contain e. However, e prefers his matched mates to c and hence, c does not constitute a blocking triple. So, if there is a blocking triple for M, this triple consists of only essential men.

Then we consider men in β-gadgets and γ-gadgets. Consider essential men $m_{\beta_{t,j}}$ and $m_{\gamma_{t,j}}$ associated with vertex v_t. Recall that constructing M, we put these men into the same triple, and note that they write each other to the first position of their crucial parts. So, if $m_{\beta_{t,j}}$ constitutes a blocking triple, then $m_{\gamma_{t,j}}$ is also in the same blocking triple. Furthermore, by the construction of preference lists of $m_{\beta_{t,j}}$ and $m_{\gamma_{t,j}}$, and by the construction of the matching M, the remaining man in this blocking triple is an essential man from an α-gadget associated with v_t, say $m_{\alpha_s,t}$ for some s. By the construction of M, $m_{\alpha_s,t}$ is matched with either (i) $m_{\alpha_{s,t'}}$ and $m_{\alpha_{s,t''}}$ where v_t, $v_{t'}$, and $v_{t''}$ constitute a triangle T_s and $T_s \in T$, or (ii) $m_{\beta_{t,j'}}$ and $m_{\gamma_{t,j'}}$ associated with the same vertex v_t. In either case, it is easy to see that $\{m_{\beta_{t,j}}, m_{\gamma_{t,j}}, m_{\alpha_s,t}\}$ cannot constitute a blocking triple. So, we can conclude that essential men from β- or γ-gadgets cannot be a part of blocking triples. Thus, the only remaining case we need to consider is that a blocking triple consists of three essential men, all from α-gadgets.

As mentioned above, $m_{\alpha_s,t}$ is matched by either case (i) or (ii) above. In case (i), $m_{\alpha_s,t}$ is matched with $m_{\alpha_{s,t'}}$ and $m_{\alpha_{s,t''}}$ corresponding to the same triangle. Since in $m_{\alpha_s,t}$'s list, $m_{\alpha_{s,t'}}$ and $m_{\alpha_{s,t''}}$ are the top two among $\mathcal{M}_\alpha$, so it is impossible that $m_{\alpha_s,t}$ constitute a blocking triple. In case (ii), $m_{\alpha_s,t}$ is matched with $m_{\beta_{t,j'}}$ and $m_{\gamma_{t,j'}}$ corresponding to the same vertex v_t. Note that $m_{\alpha_s,t}$ prefers $m_{\beta_{t,j'}}$ to anyone in $\mathcal{M}_\alpha$. Hence, $m_{\alpha_s,t}$ cannot be a member of a blocking triple. □

We then proceed to the "if" part.

[Lemma 3.5] If $I(G)$ has a solution, then G has a solution.

Proof. Let M be a stable matching for $I(G)$. We will construct a solution for G. First of all, note that for a 3D-SM gadget $Q = [a, b, c \,|\, e]$, a, b, and c are in the same triple in M by Lemma 3.1. Hence, we verify how essential men are matched in M in the following claim.

Consider a vertex v_t participating in k_t triangles $T_{s_1}, T_{s_2}, \ldots, T_{s_{k_t}}$. Then there are k_t essential men $m_{\alpha_{s_i},t}$ $(1 \leqq i \leqq k_t)$ in $\mathcal{M}_{\alpha_t}$, $k_t - 1$ essential men $m_{\beta_{t,i}}$ $(1 \leqq i \leqq k_t - 1)$ in $\mathcal{M}_{\beta_t}$, and $k_t - 1$ essential men $m_{\gamma_{t,i}}$ $(1 \leqq i \leqq k_t - 1)$ in $\mathcal{M}_{\gamma_t}$.

[Claim 3.6] For any i $(1 \leqq i \leqq k_t - 1)$, $m_{\beta_{t,i}}$ and $m_{\gamma_{t,i}}$ are included in the same triple in M. Furthermore, the remaining one is from $\mathcal{M}_{\alpha_t}$, namely, $m_{\alpha_{s_i},t}$ $(1 \leqq i \leqq k_t)$.

Proof. By Lemma 3.1, $m_{\beta_{t,i}}$ is matched with men in the crucial part of his list, namely, two men in $C_{\beta_{t,i}}$. $C_{\beta_{t,i}}$ includes $m_{\gamma_{t,i}}$ and $m_{\alpha_{s_j},t}$ $(1 \leqq j \leqq k_t)$. There are two cases: (i) $m_{\beta_{t,i}}$ is matched with $m_{\gamma_{t,i}}$ and some $m_{\alpha_{s_j},t}$. (ii) $m_{\beta_{t,i}}$ is matched with two $m_{\alpha_{s_j},t}$s. By the same argument for $m_{\gamma_{t,i}}$, we can see that $m_{\gamma_{t,i}}$ is matched with either $m_{\beta_{t,i}}$ and one $m_{\alpha_{s_j},t}$, or two $m_{\alpha_{s_j},t}$s. Then it can be seen that triples containing $m_{\beta_{t,i}}$ or $m_{\gamma_{t,i}}$ $(1 \leqq i \leqq k_t - 1)$ are of the following types: (1) $(m_{\beta_{t,i}}, m_{\gamma_{t,i}}, m_{\alpha_{s_j},t})$, (2) $(m_{\beta_{t,i}}, m_{\alpha_{s_j},t}, m_{\alpha_{s_{j'}},t})$, (3) $(m_{\gamma_{t,i}}, m_{\alpha_{s_j},t}, m_{\alpha_{s_{j'}},t})$.

Let n_1, n_2, and n_3 be the numbers of triples of types (1), (2) and (3), respectively. Then, a triple of type (1) contains two men from $\mathcal{M}_{\beta_t} \cup \mathcal{M}_{\gamma_t}$, and a triple of type (2) or (3) contains one man from $\mathcal{M}_{\beta_t} \cup \mathcal{M}_{\gamma_t}$. Since $|\mathcal{M}_{\beta_t} \cup \mathcal{M}_{\gamma_t}| = 2(k_t - 1)$, $2n_1 + n_2 + n_3 = 2(k_t - 1)$. Next, a triple of type (1) ((2) and (3), respectively) contains one (two and two, respectively) man from $\mathcal{M}_{\alpha_t}$, and $|\mathcal{M}_{\alpha_t}| = k_t$. So, $n_1 + 2n_2 + 2n_3 \leqq k_t$. Since each of n_1, n_2, and n_3 is nonnegative integer, we have $n_2 = n_3 = 0$, which means that all triples under consideration are of type (1). □

Hence, currently there are $n = 3q$ unmatched men, all being in $\mathcal{M}_\alpha$, and each being associated with each vertex. So, they construct q triples. Now, let $\{m_{\alpha_{s,t}}, m_{\alpha_{s',t'}}, m_{\alpha_{s'',t''}}\}$ is an arbitrary such triple. We show that $s = s' = s''$, namely these three men correspond to the same triangle enumerated at the beginning of the reduction. For, by the construction of the list of $m_{\alpha_{s,t}}$, there are only two men from $\mathcal{M}_\alpha$ who is in the crucial part of his list; these two men are those associated with other two vertices of the same triangle. By Lemma 3.1, each man has to be matched with men in the crucial part, so the claim holds. As a consequence, for a triple $\{m_{\alpha_{s,t}}, m_{\alpha_{s',t'}}, m_{\alpha_{s'',t''}}\}$ in M, vertices v_t, $v_{t'}$ and $v_{t''}$ constitute a triangle in G. We can select q triangles in G according to these q triples of M, which completes the proof of Lemma 3.5. □

Since it is easy to see that 3D-SR is in NP, we have the following theorem:

[Theorem 3.7] 3D-SR is NP-complete.

4. Conclusions

We extended the stable roommates problem into triple rooms and proved that it is NP-complete. It may be interesting to consider an optimization problem of finding a large subset of triples containing no blocking triple, and investigate its approximability.

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