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# 男女平等安定マッチング問題に対する近似アルゴリズム

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**あらまし**　安定結婚問題は，Gale と Shapley によって提案されたマッチングの問題である．任意の例題について，解が存在し，それを見つける多項式時間が存在することが知られている．しかし，このアルゴリズムによって得られるマッチングは「男性最適」，つまり，男性にとっては好ましいが女性にとっては好ましくないマッチングである（逆に，男女の役割を入れ替えれば，女性最適なマッチングになる）．Gusfield と Irving によって提案された男女平等安定マッチング問題は，男女両者にとって「公平な」安定マッチングを求める，つまり，男性側の不満足度の和が女性側の不満足度の和になるべく近づくような安定マッチングを求める問題である．この問題は，強 NP 困難であることが知られている．本稿では，男女平等安定マッチング問題に対して，ほぼ最適な解を求める多項式時間アルゴリズムを与える．さらに，評価指標を一つ増やして，男女平等（sex-equality）の観点でほぼ最適なもののうち，全体の公平さ（egalitarian）が最小の安定マッチングを求める問題を考える．我々は，この問題が NP 困難であることを示し，この問題に対して近似度が 2 より良い多項式時間アルゴリズムを構築した．
**キーワード**　安定結婚問題, 男女平等安定マッチング問題, 近似アルゴリズム

# Approximation Algorithms for the Sex-Equal Stable Marriage Problem

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**Abstract**　The stable marriage problem is a classical matching problem introduced by Gale and Shapley. It is known that for any instance, there exists a solution, and there is a polynomial time algorithm to find one. However, the matching obtained by this algorithm is man-optimal, that is, the matching is preferable for men but unpreferable for women, (or, if we exchange the role of men and women, the resulting matching is woman-optimal). The sex-equal stable marriage problem, posed by Gusfield and Irving, asks to find a stable matching "fair" for both genders, namely it asks to find a stable matching with the property that the sum of the men's score is as close as possible to that of the women's. This problem is known to be strongly NP-hard. In this paper, we give a polynomial time algorithm for finding *a near optimal* solution in the sex-equal stable marriage problem. Furthermore, we consider the problem of optimizing additional criterion: among stable matchings that are near optimal in terms of the sex-equality, find a minimum egalitarian stable matching. We show that this problem is NP-hard, and give a polynomial time algorithm whose approximation ratio is less than two.
**Key words**　the stable marriage problem, the sex-equal stable marriage problem, approximation algorithms

## 1. Introduction

An instance $I$ of the stable marriage problem consists of $n$ men, $n$ women, and each person's preference list. A preference list is a totally ordered list including all members of the opposite sex depending on his/her preference. For a matching $M$ between men and women, a pair of a man $m$ and a woman $w$ is called a *blocking pair* if both prefer each other to their current partners. A matching with no blocking pair is called *stable*. Gale and Shapley showed that every instance admits at least one stable matching, and proposed a linear time algorithm to find one, which is known as the Gale-Shapley algorithm [2]. However, in general, there are many different stable matchings for a single instance, and the Gale-Shapley algorithm finds only one of them (*man-optimal* or *woman-optimal*) with an extreme property: In the man-optimal stable matching, each man is matched with his best possible partner, while each woman gets her worst possible partner, among all stable matchings. Hence, it is natural to try to obtain a matching which is not only stable but also "good" in some criterion.

There are three major optimization criteria for the quality of stable matchings. Let $p_m(w)$ ($p_w(m)$, respectively) denote the position of woman $w$ in man $m$'s preference list (the position of man $m$ in woman $w$'s preference list, respectively). For a stable matching $M$, define a *regret cost* $r(M)$ to be

$$r(M) = \max_{(m,w)\in M} \max\{p_m(w), p_w(m)\}.$$

Also, define an *egalitarian cost* $c(M)$ to be

$$c(M) = \sum_{(m,w)\in M} p_m(w) + \sum_{(m,w)\in M} p_w(m),$$

and a *sex-equalness cost* $d(M)$ to be

$$d(M) = \sum_{(m,w)\in M} p_m(w) - \sum_{(m,w)\in M} p_w(m).$$

*The minimum regret stable marriage problem* (*the minimum egalitarian stable marriage problem* and *the sex-equal stable marriage problem*, respectively) is to find a stable matching $M$ with minimum $r(M)$ ($c(M)$ and $|d(M)|$, respectively) [4]. Note that the number of stable matchings for one instance grows exponentially in general (see [6], e.g.). Nevertheless, for the first two problems, Gusfield [3], and Irving, Leather and Gusfield [7], respectively, proposed polynomial time algorithms by exploiting a lattice structure which is of polynomial size but contains information of all stable matchings.

In contrast, it is hard to obtain a sex-equal stable matching. The question of its complexity was posed by Gusfield and Irving [4], and was later proved to be strongly NP-hard by Kato [9]. Thus, the next step should be its approximability for which we have no knowledge so far.

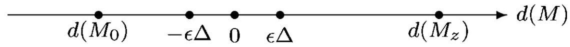


図 1 The sex-equalness costs of stable matchings

**Our Contribution.** In this paper, we consider finding *near optimal* solutions for the sex-equal stable marriage problem. Let $M_0$ and $M_z$ be the man-optimal and the woman-optimal stable matchings, respectively. Note that $d(M_0) \leqq d(M) \leqq d(M_z)$ for any stable matching $M$ (see Fig. 1). Our goal is to obtain a stable matching $M$ such that $-\epsilon\Delta \leqq d(M) \leqq \epsilon\Delta$ for a given constant $\epsilon$, where $\Delta = \min\{|d(M_0)|, |d(M_z)|\}$. Namely, we define the following problem called *Near SexEqual* (*NSE* for short). Given a stable marriage instance $I$ and a positive constant $\epsilon$, it asks to find a stable matching $M$ such that $|d(M)| \leqq \epsilon\Delta$ if such $M$ exists, or answer "No" otherwise. We give a polynomial time algorithm for NSE, which runs in time $O(n^{3+\frac{1}{\epsilon}})$.

NSE asks to find an *arbitrary* stable matching whose sex-equalness cost lies within some range. However, we may want to find a good one if there are several solutions in the range. In fact, there is an instance $I$ that has two stable matchings $M$ and $M'$ such that $d(M) = d(M') = 0$ but $c(M) \ll c(M')$. This motivates us to consider the following corresponding optimization problem MinESE (*Minimum Egalitarian Sex-Equal stable marriage problem*): Given a stable marriage instance $I$ and a positive constant $\epsilon$, find a stable matching $M$ which minimizes $c(M)$ under the condition that $|d(M)| \leqq \epsilon\Delta$, (or answer "No" if none exists). We show that MinESE is NP-hard, and give a polynomial time $(2-(\epsilon-\delta)/(2+3\epsilon))$-approximation algorithm for an arbitrary $\delta$ such that $0 < \delta < \epsilon$, whose running time is $O(n^{4+\frac{1+\epsilon}{\delta}})$. Here, Algorithm $A$ is said to be a $c$-approximation algorithm if $A(I)/OPT(I) \leqq c$ holds for any input $I$, where $A(I)$ and $OPT(I)$ are the costs of $A$'s solution and an optimal solution, respectively.

Although details are omitted, our results in this paper can be easily extended to the *weighted* versions of the above problems, in which $p_m(w)$ ($p_w(m)$, respectively) represents not simply a rank of $w$ in $m$'s preference list, but an arbitrary score of $m$ for $w$ (of $w$ for $m$), where $p_m(w) > 0$ ($p_w(m) > 0$) and $p_m(w) < p_m(w')$ if and only if $m$ prefers $w$ to $w'$ ($p_w(m) < p_w(m')$ if and only if $w$ prefers $m$ to $m'$) for all $m$ and $w$.

**Related Results.** As mentioned above, the minimum regret stable marriage problem and the minimum egalitarian stable marriage problem can be solved in polynomial time [3], [4], [7], but the sex-equal stable marriage problem is strongly NP-hard [9]. If we allow ties in preference lists, all these problems become hard, even to approximate, if we seek for optimal *weakly* stable matching: For each problem, there exists a positive constant $\delta$ such that there is no polynomial-time approximation algorithm with approximation ratio $\delta n$ unless P=NP [5].

## 2. Rotation Poset

In this section, we explain a *rotation poset (partially-ordered set)*, originally defined in [6], which is an underlying structure of stable matchings. Here, we give only a brief sketch necessary for understanding the algorithms given later. Readers can refer to [4] for further details.

We fix an instance $I$. Let $M$ be a stable matching for $I$. For each such $M$, we can associate a *reduced list*, which is obtained from the original preference lists by removing entries by some rule. One property of the reduced list associated with $M$ is that, in $M$, each man is matched with the first woman in the reduced list, and each woman is matched with the last man. A *rotation exposed in* $M$ is an ordered list of pairs $\rho = (m_0, w_0), (m_1, w_1), \ldots, (m_{r-1}, w_{r-1})$ such that, for every $i$ $(0 \leqq i \leqq r-1)$, $m_i$ and $w_i$ are matched in $M$, and $w_{i+1}$ is at the second position in $m_i$'s reduced list, where $i+1$ is taken modulo $r$. There exists at least one rotation for any stable matching except for the woman-optimal stable matching $M_z$.

For a stable matching $M$ and a rotation $\rho = (m_0, w_0), (m_1, w_1), \ldots, (m_{r-1}, w_{r-1})$ exposed in $M$, *eliminating* $\rho$ *from* $M$ means to replace $m_i$'s partner from $w_i$ to $w_{i+1}$ for each $i$ $(0 \leqq i \leqq r-1)$, (and to update a reduced list accordingly). Note that by eliminating a rotation, men become worse off while women become better off. The resulting matching is denoted by $M/\rho$. It is well known that $M/\rho$ is also stable for $I$. If a rotation is exposed in $M/\rho$, then we can similarly obtain another stable matching by eliminating it.

Now, let $\mathcal{M}$ be the set of all stable matchings for $I$, and $\Pi$ be the set of rotations $\rho$ such that $\rho$ is exposed in some stable matching in $\mathcal{M}$. Then, it is known that $|\Pi| \leqq n^2$. The *rotation poset* $(\Pi, \prec)$, which is uniquely determined for instance $I$, is the set $\Pi$ with a partial order $\prec$ defined for elements in $\Pi$. For two rotations $\rho_1$ and $\rho_2$ in $\Pi$, $\rho_1 \prec \rho_2$ intuitively means that $\rho_1$ must be eliminated before $\rho_2$, or $\rho_2$ is never exposed until $\rho_1$ is eliminated. It is known that the rotation poset can be constructed in $O(n^2)$ time.

A *closed subset* $R$ of the rotation poset $(\Pi, \prec)$ is a subset of $\Pi$ such that if $\rho \in R$ and $\rho' \prec \rho$ then $\rho' \in R$. There is a one-to-one correspondence between $\mathcal{M}$ and the set of closed subsets of $(\Pi, \prec)$: Let $R$ be a closed subset. Starting from the man-optimal stable matching $M_0$, if we eliminate all rotations in $R$ successively in any order following $\prec$, then we can obtain a stable matching. Conversely, any stable matching can be obtained by this procedure for some closed subset. We denote the stable matching corresponding to a closed subset $R$ by $M_R$. For simplicity, we sometimes write $c(R)$ and $d(R)$ instead of $c(M_R)$ and $d(M_R)$, respectively. Especially, the empty subset corresponds to the man-optimal stable matching $M_0$, and the set $\Pi$ itself corresponds to the woman-optimal stable matching $M_z$. From $M_0$, if we eliminate all rotations according to the order $\prec$, then we eventually reach $M_z$.

For a rotation $\rho = (m_0, w_0), (m_1, w_1), \ldots, (m_{r-1}, w_{r-1})$, we define $w_c(\rho)$ and $w_d(\rho)$, which represent the cost change of egalitarian and sex-equalness, respectively, by eliminating $\rho$:

$$\begin{aligned} w_c(\rho) &= \sum_{i=0}^{r-1} (p_{m_i}(w_{i+1}) - p_{m_i}(w_i)) \\ &\quad + \sum_{i=0}^{r-1} (p_{w_i}(m_{i-1}) - p_{w_i}(m_i)), \\ w_d(\rho) &= \sum_{i=0}^{r-1} (p_{m_i}(w_{i+1}) - p_{m_i}(w_i)) \\ &\quad - \sum_{i=0}^{r-1} (p_{w_i}(m_{i-1}) - p_{w_i}(m_i)). \end{aligned}$$

Here, note that $w_d(\rho) > 0$ for all $\rho$ since by eliminating a rotation, some men become worse off, some women become better off, and other people remain matched with the same partners. Now, let $\rho$ be a rotation exposed in a stable matching $M$. Then, it is obvious from the definition that $c(M/\rho) = c(M) + w_c(\rho)$ and $d(M/\rho) = d(M) + w_d(\rho)$. Also, it is easy to see that for any closed subset $R$,

$$c(M_R) = c(M_0) + \sum_{\rho \in R} w_c(\rho)$$

and

$$d(M_R) = d(M_0) + \sum_{\rho \in R} w_d(\rho).$$

Hence, the minimum egalitarian stable marriage problem (the sex-equal stable marriage problem, respectively) is equivalent to the problem of finding a closed subset $R$ such that $c(M_0) + \sum_{\rho \in R} w_c(\rho)$ ($|d(M_0) + \sum_{\rho \in R} w_d(\rho)|$, respectively) is minimum. For example, the algorithm for finding a minimum egalitarian stable matching in [7] efficiently finds such $R$ by exploiting network flow.

## 3. The Sex-Equal Stable Marriage Problem

Recall that $M_0$ is the man-optimal stable matching and $M_z$ is the woman-optimal stable matching. Note that any stable matching $M$ satisfies $d(M_0) \leqq d(M) \leqq d(M_z)$. Thus, this problem is trivial if $d(M_0) \geqq 0$ or $d(M_z) \leqq 0$, namely, if

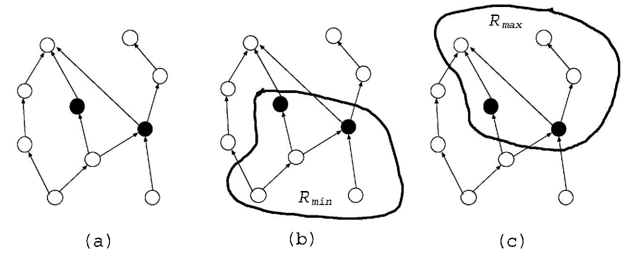

図 2 Examples of $R_{\min}$ and $R_{\max}$.

$d(M_0) \geqq 0$, $M_0$ is optimal, while if $d(M_z) \leqq 0$, $M_z$ is optimal. Therefore, we consider the case where $d(M_0) < 0 < d(M_z)$. Recall that $\Delta = \min\{|d(M_0)|, |d(M_z)|\}$. In the following, we assume without loss of generality that $|d(M_0)| \leqq |d(M_z)|$ since otherwise, we can exchange the role of men and women. Hence, $\Delta = \min\{|d(M_0)|, |d(M_z)|\} = |d(M_0)|$.

We first briefly give the underlying idea of our algorithm presented in this section. Recall that, for a given instance $I$ and $\epsilon$, we are to find a stable matching $M$ such that $-\epsilon\Delta \leqq d(M) \leqq \epsilon\Delta$ if any. As an easy case, assume that all rotations $\rho$ of $I$ satisfy $w_d(\rho) \leqq 2\epsilon\Delta$. Now, we construct a rotation poset $(\Pi, \prec)$ of $I$, and starting from $M_0$, we eliminate rotations in an order of any linear extension of $\prec$. Recall that by eliminating a rotation, the sex-equalness cost increases, but by at most $2\epsilon\Delta$ by assumption. Note that $d(M_0) < 0 < d(M_z)$, and recall that if we eliminate all rotations from $M_0$, we eventually reach $M_z$. Then, in this sequence, we certainly meet a desirable stable matching at some point.

However, this procedure fails if there is a rotation with large sex-equalness cost: If we eliminate such a rotation, then we may "jump" from $M$ to $M'$ such that $d(M) < -\epsilon\Delta$ and $d(M') > \epsilon\Delta$ even if there is a feasible solution. To resolve this problem, we will try all combinations of selecting such "large" rotations, and treat "small" rotations in the above manner. To evaluate the time complexity, we show that the number of large rotations is limited.

Before giving a description of our algorithm, we give a couple of notations. Let $R$ be any (not necessarily closed) subset of a poset $(\Pi, \prec)$. Then define $R_{\min} = R \cup \{\rho \mid \text{there exists a } \rho' \in R \text{ such that } \rho \prec \rho'\}$ and $R_{\max} = R \cup \{\rho \mid \text{there exists a } \rho' \in R \text{ such that } \rho' \prec \rho\}$. See Fig. 2 for example. Fig. 2 (a) represents a Hasse diagram of a poset, where black circles are elements of $R$. $R_{\min}$ and $R_{\max}$ are depicted in Fig. 2 (b) and Fig. 2 (c), respectively. Intuitively speaking, when constructing a closed subset $A$, if we decide to include all elements of $R$ to $A$, then $R_{\min}$ is the set of elements that must be included in $A$. Similarly, if we decide to include no elements of $R$, then $R_{\max}$ is the set of elements that must not be included in $A$.

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**Algorithm 1**

**1.** Construct the rotation poset $(\Pi, \prec)$.

**2.** Let $R^L$ be the set of rotations $\rho$ such that $w_d(\rho) > 2\epsilon\Delta$, and $R^S$ be $\Pi \setminus R^L$.

**3.** For each set $R$ in $2^{R^L}$ such that $|R| \leqq \frac{1+\epsilon}{2\epsilon}$, do,

**(a)** If $-\epsilon\Delta \leqq d(R_{\min}) \leqq \epsilon\Delta$, then output $M_{R_{\min}}$.

**(b)** Fix an arbitrary order $\rho_1, \rho_2, \cdots, \rho_k \in R^S \setminus (R_{\min} \cup (R^L \setminus R)_{\max})$ which is consistent with $\prec$.

**(c)** For $i = 1$ to $k$

if $-\epsilon\Delta \leqq d(R_{\min} \cup \{\rho_1, \rho_2, \cdots, \rho_i\}) \leqq \epsilon\Delta$,

then output $M_{R_{\min} \cup \{\rho_1, \rho_2, \cdots, \rho_i\}}$ and halt.

**4.** Output "No," and halt.

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[Theorem 3.1] There is an algorithm for NSE whose running time is $O(n^{3+\frac{1}{\epsilon}})$.

*Proof.* **Correctness Proof.** Clearly, if there is no stable matching $M$ such that $-\epsilon\Delta \leqq d(M) \leqq \epsilon\Delta$, then the algorithm answers "No." On the other hand, suppose that there is stable matching $M_X$ such that $-\epsilon\Delta \leqq d(M_X) \leqq \epsilon\Delta$, where $X$ is the set of rotations corresponding to $M_X$. Let $X^L = X \cap R^L$ and $X^S = X \cap R^S$. Then, $d(X^L) \leqq d(X) = d(M_X) \leqq \epsilon\Delta$. Because $w_d(\rho) > 2\epsilon\Delta$ for any rotation $\rho \in X^L$, $|X^L| < \frac{d(X^L) - d(M_0)}{2\epsilon\Delta} \leqq \frac{|d(M_0)| + \epsilon\Delta}{2\epsilon\Delta} = \frac{1+\epsilon}{2\epsilon}$. So, Algorithm 1 selects $X^L$ at Step 3 as $R$, and we consider this particular execution of Step 3.

First, note that $d((X^L)_{\min}) \leqq \epsilon\Delta$ since otherwise, $d(X) \geqq d((X^L)_{\min}) > \epsilon\Delta$, a contradiction. If $-\epsilon\Delta \leqq d((X^L)_{\min}) \leqq \epsilon\Delta$, then Algorithm 1 outputs $M_{(X^L)_{\min}}$ at Step 3(a). Finally, suppose that $d((X^L)_{\min}) < -\epsilon\Delta$. Note that $d((X^L)_{\min} \cup \{\rho_1, \rho_2, \cdots, \rho_k\}) \geqq d(X) \geqq -\epsilon\Delta$ and that any rotation $\rho_i$ $(1 \leqq i \leqq k)$ satisfies $w_d(\rho_i) \leqq 2\epsilon\Delta$. Hence there must be $j$ $(1 \leqq j \leqq k)$ such that $-\epsilon\Delta \leqq d((X^L)_{\min} \cup \{\rho_1, \rho_2, \cdots, \rho_j\}) \leqq \epsilon\Delta$.

**Time Complexity.** Steps 1 and 2 can be performed in $O(n^2)$. Inside the loop of Step 3 can be performed in $O(n^2)$ since the number of rotations is at most $O(n^2)$. Clearly, Step 4 can be performed in constant time.

We consider the number of repetitions of Step 3, i.e., the number of $R$ satisfying the condition at Step 3. Let this number be $t$. Recall that the number of rotations is at most $n^2$ as mentioned in Sec. 2.. So, $|R^L| \leqq n^2$. Since $|R| \leqq \frac{1+\epsilon}{2\epsilon}$,

$$t = \sum_{k=1}^{\lfloor \frac{1+\epsilon}{2\epsilon} \rfloor} \binom{n^2}{k} \leqq \sum_{k=1}^{\lfloor \frac{1+\epsilon}{2\epsilon} \rfloor} \frac{(n^2)^{\lfloor \frac{1+\epsilon}{2\epsilon} \rfloor}}{k!} = O(n^{\frac{1+\epsilon}{\epsilon}}).$$

Hence the time complexity of Algorithm 1 is $O(n^2) \cdot O(n^{\frac{1+\epsilon}{\epsilon}}) = O(n^{3+\frac{1}{\epsilon}})$. □

**Remark.** We can improve Algorithm 1 when $|d(M_0)|$ and $|d(M_z)|$ are close, more precisely, when they differ at most $\log n$ factor. Let $\Delta'$ be $\frac{1}{\log n}\max\{|d(M_0)|, |d(M_z)|\}$ and we can find a stable matching $M$ which satisfies $-\epsilon\Delta' \leqq d(M) \leqq \epsilon\Delta'$ in polynomial time by using a modified version of Algorithm 1 (Algorithm 1′). We modify Algorithm 1 so that it uses $\Delta'$ instead of $\Delta$ and executes Step 3 for all subsets of $2^{R^L}$. Note that, from the discussion in Sec. 2.,

$$d(M_z) = d(M_0) + \sum_{\rho\in\Pi} w_d(\rho).$$

Hence $|R^L| < \frac{d(M_z)-d(M_0)}{2\epsilon\Delta'} \leqq \frac{2\max\{|d(M_0)|,|d(M_z)|\}}{2\epsilon\Delta'} = \frac{\log n}{\epsilon}$. Thus the number of repetitions of Step 3 is at most $2^{|R^L|} = n^{1/\epsilon}$, which is polynomial.

**Remark.** There are several goodness measures of an approximation algorithm $A$ for a minimization problem. The usual measure is the *approximation ratio* of $A$, which is defined as $\max\{A(x)/opt(x)\}$ over all instances $x$, where $opt(x)$ and $A(x)$ are the costs of the optimal and the algorithm's solutions, respectively. However, this measure cannot be used for the sex-equal stable marriage problem, because $opt(x)$ can be zero. For such a case, there is another measure: the *relative accuracy*[1], [10], which is defined as $\max\{(\max(x) - opt(x))/(\max(x) - A(x))\}$ over all instances $x$, where $opt(x)$, $\max(x)$, and $A(x)$ are the costs of the optimal solution, the worst solution, and the algorithm's solution, respectively. By using Algorithm 1′ in the above remark, we can construct an algorithm $T$ which achieves the relative accuracy $1+\epsilon/\log n$ for an arbitrary constant $\epsilon > 0$.

Let $M_{opt}$ be an optimal solution for the sex-equal stable marriage problem. Recall that we are considering the case where $d(M_0) < 0 < d(M_z)$. If $|d(M_{opt})| > D/2$, where $D = \max\{|d(M_0)|, |d(M_z)|\}$, $M_{opt}$ can be obtained in polynomial time in the following way: Let $M_a$ be the stable matching such that no other stable matching $M$ satisfies $d(M_a) < d(M) < -D/2$ and let $M_b$ be the stable matching such that no other stable matching $M$ satisfies $D/2 < d(M) < d(M_b)$. Then, $M_{opt}$ is either $M_a$ or $M_b$. Since $d(M_b) - d(M_a) > D$, there exists a rotation $\rho_H$ such that $w_d(\rho_H) > D$ (otherwise there must be a stable matching $M_c$ such that $d(M_a) < d(M_c) < d(M_b)$). Also, this $\rho_H$ is unique because

$$\sum_{\rho\in\Pi} w_d(\rho) = d(M_z) - d(M_0) \leqq 2D.$$

It is easy to see that the maximum closed subset which does not contain $\rho_H$ corresponds to $M_a$ and that the minimum closed subset which contains $\rho_H$ corresponds to $M_b$. Thus, $M_a$ and $M_b$ can be obtained in polynomial time. Finally, assume that $|d(M_{opt})| \leqq D/2$. For each $i$ such that $i = 1, 2, \ldots, \lceil \frac{\log n}{\epsilon} \rceil$, we find a stable matching with sex-equalness cost between $-\frac{\epsilon}{2\log n}Di$ and $\frac{\epsilon}{2\log n}Di$ if any using Algorithm 1′, and output the best one. Then, it is easy to see that an output matching $M$ satisfies $|d(M)| - |d(M_{opt})| \leqq \frac{\epsilon}{2\log n}D$. Now, the relative accuracy is

$$\begin{aligned}\frac{\max(x) - opt(x)}{\max(x) - T(x)} &= 1 + \frac{T(x) - opt(x)}{\max(x) - T(x)} \\ &\leqq 1 + \frac{(\epsilon/2\log n)D}{D/2} \\ &= 1 + \frac{\epsilon}{\log n}.\end{aligned}$$

## 4. The Minimum Egalitarian Sex-Equal Stable Marriage Problem

In NSE, we are asked to find a stable matching whose sex-equalness cost is in some range close to 0. However, if there are several stable matchings satisfying the condition, there might be good ones and bad ones. In fact, there is an instance $I$ that has two stable matchings $M$ and $M'$ whose sex-equalness costs are the same (0), but egalitarian costs are significantly different. (Because of the space restriction, we omit the construction of this instance.) This motivates us to consider the following problem, *MinESE* (*the Minimum Egalitarian Sex-Equal stable marriage problem*): Given an instance $I$ and a constant $\epsilon$ such that $0 < \epsilon < 1$, find a stable matching $M$ with minimum $c(M)$, under the condition that $|d(M)| \leqq \epsilon\Delta$, (or answer "No" if no such solution exists). First, in Sec. 4. 1, we show that MinESE is NP-hard. Then, in Sec. 4. 2, we give an approximation algorithm for MinESE.

### 4. 1 NP-hardness of MinESE

It turned out that there is a polynomial-time algorithm for obtaining a stable matching $M$ such that (a) $-\epsilon\Delta \leqq d(M) \leqq \epsilon\Delta$ *or* (b) $c(M)$ is minimum. Interestingly, it is hard to obtain $M$ satisfying (a) *and* (b).

[Theorem 4.1] MinESE is NP-hard.

*Proof.* Because of the space restriction, we give only a rough idea of the proof, and omit the details.

We show that MinESE is NP-hard by a reduction from the $k$-clique problem. In this problem, we are given a graph $G(V, E)$ and an integer $k$, and asked if there exists a clique of size $k$. This problem is NP-complete.

Given a graph $G = (V, E)$ and an integer $k$, we first construct a poset $(\Pi, \prec)$ in a similar manner as the construction in [8]. Let $\Pi$ be $V \cup E$, and define the precedence relation $\prec$ as follows: $v \prec e$ if and only if $v \in V$ is incident to $e \in E$ in $G(V, E)$. We then give weights to each element in $(\Pi, \prec)$: We give some negative weight to each element corresponding to an edge, and positive weight to an element corresponding to a vertex. Note that if we want to select a closed subset with smaller weight, we want to select many elements corresponding to edges, but to make the subset closed, we need to select elements corresponding to adjacent vertices, which may increase the weight. We give weights to vertices and edges appropriately, so that only closed subsets corresponding to $k$-cliques can have a desirable (negative) weight.

Next, we construct an instance $I$ of MinESE from the poset $(\Pi, \prec)$ using a similar construction as [9], so that the rotation poset of $I$ is exactly $(\Pi, \prec)$. In the construction, we ensure that the egalitarian cost of each rotation is exactly the same to the weight of the corresponding element defined above. We also adjust sex-equalness cost of rotations so that if $R$ is a closed subset corresponding to a $k$-clique of $G$, then $d(M_R)$ lies between $-\epsilon\Delta$ and $\epsilon\Delta$. In summary, our reduction satisfies the following: $G$ has a $k$-clique if and only if there is a stable matching $M$ in $I$ such that $-\epsilon\Delta \leqq d(M) \leqq \epsilon\Delta$ and $c(M) < c(M_0)$.

We can show that MinESE is NP-hard as follows: Given an instance of the clique problem, we construct a MinESE instance $I$ by the above reduction. Then, we find an optimal solution $M$ and the man-optimal stable matching $M_0$. Finally, we compare $c(M_0)$ and $c(M)$: If $c(M) < c(M_0)$, the answer to the clique problem is "yes," otherwise, "no." □

**Remark.** Although details are omitted, the reduction in the NP-hardness proof produces an instance $(I, \epsilon)$ of MinESE such that $|d(M_0)| = |d(M_z)|$ in $I$, and $\epsilon$ is any constant such that $0 < \epsilon < 1$. Observe that if $|d(M_0)| = |d(M_z)|$ and $\epsilon = 1$, then MinESE is equivalent to the minimum egalitarian stable marriage problem, which can be solved in polynomial time.

### 4.2 Approximation Algorithms for MinESE

Here, we give a $(2 - (\epsilon - \delta)/(2 + 3\epsilon))$-approximation algorithm for MinESE for an arbitrary $\delta$ such that $0 < \delta < \epsilon$. Similarly as Sec. 3., we assume that $|d(M_0)| \leqq |d(M_z)|$. In this section, we prove two simple but important lemmas that link the egalitarian cost and the sex-equalness cost, whose proofs are given later. (i) For any stable matching $M$, $|d(M)| < c(M)$ (Lemma 4.4). (ii) For any stable matching $M$ and a rotation $\rho$ exposed in $M$, by eliminating $\rho$ from $M$, the cost change in the egalitarian cost is less than the cost change in the sex-equalness cost (Lemma 4.5).

To illustrate an idea of the algorithm, we first consider a restricted case and show that our algorithm achieves 2-approximation. For a fixed $\delta$ such that $\epsilon > \delta > 0$, suppose that all rotations satisfy $w_d(\rho) \leqq \delta\Delta$. Given $I$ and $\epsilon$, we first find a minimum egalitarian stable matching $M_{eg}$, which can be done in polynomial time. If $-\epsilon\Delta \leqq d(M_{eg}) \leqq \epsilon\Delta$, then we are done since $M_{eg}$ is an optimal solution for MinESE. If $d(M_{eg}) < -\epsilon\Delta$, then we eliminate rotations one by one as Algorithm 1 does until the sex-equalness cost first becomes $-\epsilon\Delta$ or larger. If $d(M_{eg}) > \epsilon\Delta$, then we "add" rotations one by one until the sex-equalness cost first becomes $\epsilon\Delta$ or smaller. Here, "adding a rotation" means the reverse operation of eliminating a rotation. If we do not reach a feasible solution by this procedure, then we can conclude that there is no feasible solution, by a similar argument as in Sec. 3.. If we find a stable matching $M$ such that $-\epsilon\Delta \leqq d(M) \leqq \epsilon\Delta$, then we can show that this is a 2-approximation, namely, $c(M) \leqq 2c(M_{eg})$ using (i) and (ii) above (note that the optimal cost is at least $c(M_{eg})$): Suppose, for example, that $d(M_{eg}) < -\epsilon\Delta$ (see Fig. 3). Then, by (ii), $c(M) - c(M_{eg}) < d(M) - d(M_{eg})$, and by (i), $|d(M_{eg})| < c(M_{eg})$. Also, since the costs of rotations are at most $\delta\Delta$, and since $M$ is the first feasible solution found by this procedure, $d(M) \leqq -(\epsilon - \delta)\Delta < 0$. Putting these together, we have that $c(M)/c(M_{eg}) < 2$.

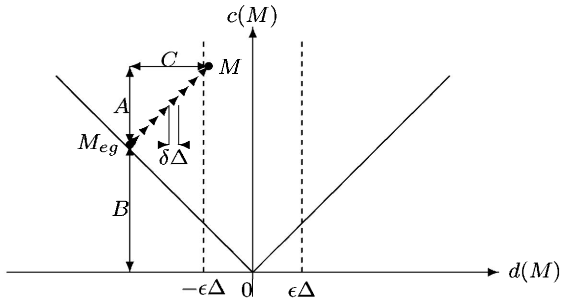


図 3 $C < B$ by (i) and $A < C$ by (ii). Hence $A+B < C+B < 2B$.

However, we may have rotations of large costs. Then we take a similar approach as in Sec. 3.: Let $R^L$ be the set of such large rotations. Then, for any partition $R_1$ and $R_2$ of $R^L$ ($R_1 \cup R_2 = R^L$ and $R_1 \cap R_2 = \emptyset$), we want to obtain a minimum egalitarian stable matching whose corresponding closed subset $A$ contains all rotations in $R_1$ but none in $R_2$. For this purpose, we need to solve the following problem: Given an instance $I$ and disjoint subsets of rotations $R_1$ and $R_2$ of $R^L$, find a minimum egalitarian stable matching $M_A$ under the condition that the corresponding closed subset $A$ satisfies $A \supseteq R_1$ and $A \cap R_2 = \emptyset$. For this problem, we can use the same algorithm for the minimum egalitarian stable marriage problem in [4]. We denote this procedure by minEgalitarian$(R_1, R_2)$. First, we review the following proposition described in [4]:

[Proposition 4.2] [4] Given a poset $(\Pi, \prec)$, there is an $O(n^4)$-time algorithm which finds a minimum-weight closed subset of $(\Pi, \prec)$ with respect to the egalitarian cost.

Our procedure minEgalitarian$(R_1, R_2)$ is as follows: Without loss of generality, assume that there are no elements such that $r_2 \prec r_1$ ($r_1 \in R_1$ and $r_2 \in R_2$) since there exists no solution in such a case. Construct the poset $(\Pi', \prec)$ by removing all the rotations in $(R_1)_{\min}$ and $(R_2)_{\max}$ from $(\Pi, \prec)$ (recall the definitions of $R_{\min}$ and $R_{\max}$ given before Algorithm 1), and let $R'$ be the subset obtained by using Proposition 4.2 to $(\Pi', \prec)$. Then, it is easy to see that $(R_1)_{\min} \cup R'$ is an optimal solution for minEgalitarian($R_1, R_2$). Now, we are ready to give the algorithm for MinESE.

---

**Algorithm 2**

**1.** Construct the rotation poset $(\Pi, \prec)$.

**2.** Let $M_{best}$ = NULL.

**3.** Let $R^L$ be the set of rotations $\rho$ such that $w_d(\rho) > \delta\Delta$, and $R^S$ be $\Pi \setminus R^L$.

**4.** For each set $R$ in $2^{R^L}$ such that $|R| \leqq \frac{1+\epsilon}{\delta}$, do,

**(a)** Let $A$ = minEgalitarian($R, R^L \setminus R$).
If $d(A) < -\epsilon\Delta$, go to **(b)**.
If $d(A) > \epsilon\Delta$, go to **(c)**.
If $-\epsilon\Delta \leqq d(A) \leqq \epsilon\Delta$, go to **(d)**.

**(b)** Fix an arbitrary order $\rho_1, \rho_2, \cdots, \rho_k \in R^S \setminus (A \cup (R^L \setminus R)_{\max})$ which is consistent with $\prec$.
For $i = 1$ to $k$,
if $-\epsilon\Delta \leqq d(A \cup \{\rho_1, \rho_2, \cdots, \rho_i\}) \leqq \epsilon\Delta$,
then let $A = A \cup \{\rho_1, \rho_2, \cdots, \rho_i\}$ and go to **(d)**.

**(c)** Fix an arbitrary order $\rho_1, \rho_2, \cdots, \rho_k \in (A \cap R^S) \setminus R_{\min}$ which is consistent with $\prec$.
For $i = k$ to 1,
if $-\epsilon\Delta \leqq d(A \setminus \{\rho_i, \rho_{i+1}, \cdots, \rho_k\}) \leqq \epsilon\Delta$,
then let $A = A \setminus \{\rho_i, \rho_{i+1}, \cdots, \rho_k\}$ and go to **(d)**.

**(d)** If $c(A) < c(M_{best})$, then let $M_{best} = M_A$.

**5.** If $M_{best} \neq$ NULL, then output $M_{best}$, otherwise output "No," and halt.

---

[Theorem 4.3] There is a $(2 - (\epsilon - \delta)/(2 + 3\epsilon))$-approximation algorithm for MinESE whose running time is $O(n^{4+\frac{1+\epsilon}{\delta}})$ for an arbitrary $\delta$ such that $0 < \delta < \epsilon$.

*Proof.* **Correctness Proof.** Clearly, if there is no stable matching $M$ such that $-\epsilon\Delta \leqq d(M) \leqq \epsilon\Delta$, then the algorithm answers "No." On the other hand, suppose that there is a feasible solution, and let $M_{opt}$ be an optimal solution. We first show that Algorithm 2 finds a feasible solution. Let $OPT$ be the rotation set corresponding to $M_{opt}$, and define $OPT^L = OPT \cap R^L$. Then, $d(OPT^L) \leqq d(OPT) = d(M_{opt}) \leqq \epsilon\Delta$. Because $w_d(\rho) > \delta\Delta$ for any rotation $\rho \in OPT^L$, $|OPT^L| < \frac{d(OPT^L) - d(M_0)}{\delta\Delta} \leqq \frac{|d(M_0)| + \epsilon\Delta}{\delta\Delta} = \frac{1+\epsilon}{\delta}$. So, Algorithm 2 selects $OPT^L$ at Step 4 as $R$, and we consider this particular execution of Step 4. We show that in this execution, Algorithm 2 finds a feasible solution. Let $A_{opt}$ = minEgalitarian($OPT^L, R^L \setminus OPT^L$). There are three cases:

**(i)** $\boldsymbol{-\epsilon\Delta \leqq d(A_{opt}) \leqq \epsilon\Delta}$**.** $M_{A_{opt}}$ is clearly a feasible solution.

**(ii)** $\boldsymbol{d(A_{opt}) < -\epsilon\Delta}$**.** Note that $d(A_{opt} \cup \{\rho_1, \rho_2, \cdots, \rho_k\}) \geqq d(M_{opt}) \geqq -\epsilon\Delta$ and that any rotation $\rho_i$ $(1 \leqq i \leqq k)$ satisfies $w_d(\rho_i) \leqq \delta\Delta$. Hence there must be $j$ $(1 \leqq j \leqq k)$ such that $-\epsilon\Delta \leqq d(A_{opt} \cup \{\rho_1, \rho_2, \cdots, \rho_j\}) \leqq -(\epsilon - \delta)\Delta$.

**(iii)** $\boldsymbol{d(A_{opt}) > \epsilon\Delta}$**.** Note that $d(A_{opt} \setminus \{\rho_1, \rho_2, \cdots, \rho_k\}) \leqq d(M_{opt}) \leqq \epsilon\Delta$ and that any rotation $\rho_i$ $(1 \leqq i \leqq k)$ satisfies $w_d(\rho_i) \leqq \delta\Delta$. Hence there must be $j$ $(1 \leqq j \leqq k)$ such that $(\epsilon - \delta)\Delta \leqq d(A_{opt} \setminus \{\rho_j, \rho_{j+1}, \cdots, \rho_k\}) \leqq \epsilon\Delta$.

Next, we analyze the approximation ratio. Let $M^*$ be the matching found in this particular execution of Step 4. We show that $c(M^*) \leqq (2 - (\epsilon - \delta)/(2 + 3\epsilon))c(M_{opt})$, which gives a proof for the approximation ratio. We first prove the following two lemmas:

[Lemma 4.4] For any stable matching $M$, $|d(M)| < c(M)$.

*Proof.* If $d(M) \geqq 0$, then $c(M) - |d(M)| = 2\sum_{(m,w)\in M} p_w(m) > 0$. Otherwise, $c(M) - |d(M)| = 2\sum_{(m,w)\in M} p_m(w) > 0$. □

[Lemma 4.5] Let $R = \{\rho_1, \ldots, \rho_{r-1}\}$ be a set of rotations and let $M_1, \cdots, M_r$ be stable matchings such that $M_{i+1} = M_i/\rho_i$ for $1 \leqq i < r$. Then, $|c(M_r) - c(M_1)| < d(M_r) - d(M_1)$.

*Proof.* Suppose that for a pair $(m, w) \in M_i$, $m$ and $w$ are included in a rotation $\rho_i$. Let $m' = M_{i+1}(w)$ and $w' = M_{i+1}(m)$. By the properties of the rotation [4], $m$ prefers $w$ to $w'$ and $w$ prefers $m'$ to $m$. Let $d(m) = p_m(w') - p_m(w)$ and $d(w) = p_w(m) - p_w(m')$. Then $d(m) > 0$ and $d(w) > 0$, and it follows that

$$\begin{aligned} |c(M_{i+1}) - c(M_i)| &= \left|\sum_m d(m) - \sum_w d(w)\right| \\ &< \left|\sum_m d(m) + \sum_w d(w)\right| \\ &= d(M_{i+1}) - d(M_i). \end{aligned}$$

By summing up the above inequality for all $i$, we have

$$\begin{aligned} |c(M_r) - c(M_1)| &\leqq \sum_{i=1}^{r-1} |c(M_{i+1}) - c(M_i)| \\ &< \sum_{i=1}^{r-1} (d(M_{i+1}) - d(M_i)) \\ &= d(M_r) - d(M_1). \end{aligned}$$

□

Note that $A_{opt}$ = minEgalitarian($OPT^L, R^L \setminus OPT^L$). So, $c(A_{opt}) \leqq c(M_{opt})$ since $OPT$, the rotation set corresponding to $M_{opt}$, is one of the candidates for $A_{opt}$. We will consider the following four cases (note that $d(A_{opt}) \geqq -\Delta$ for any stable matching $M$):

**Case (i):** $\boldsymbol{-\epsilon\Delta \leqq d(A_{opt}) \leqq \epsilon\Delta}$**.** In this case, $M^* = M_{A_{opt}}$, which is an optimal solution since $c(A_{opt}) = c(M_{opt})$.

**Case (ii):** $\boldsymbol{\epsilon\Delta < d(A_{opt}) \leqq (2 + 3\epsilon)\Delta}$**.** In this

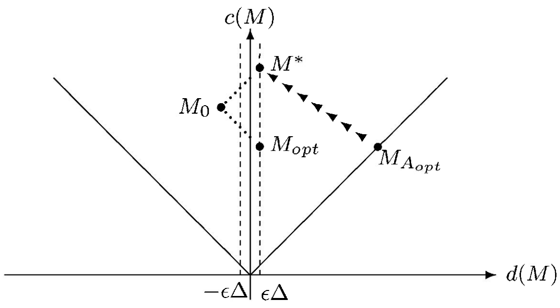

図 4 Case (iii)

case, Step 4(c) of Algorithm 2 is executed. We have $|c(A_{opt}) - c(M^*)| < d(A_{opt}) - d(M^*)$ by Lemma 4.5. Since $d(M^*) \geqq (\epsilon-\delta)\Delta$ and (ii) hold, $|c(A_{opt})-c(M^*)| < (1-(\epsilon-\delta)/(2+3\epsilon))d(A_{opt})$. Since $|d(A_{opt})| < c(A_{opt})$ by Lemma 4.4 and $c(A_{opt}) \leqq c(M_{opt})$, $c(M^*) < (2-(\epsilon-\delta)/(2+3\epsilon))c(M_{opt})$.

**Case (iii): $(2 + 3\epsilon)\Delta < d(A_{opt})$.** Since both $M_{opt}$ and $M^*$ can be obtained by repeatedly eliminating rotations from $M_0$, $|c(M_{opt}) - c(M_0)| < d(M_{opt}) - d(M_0)$ and $|c(M^*) - c(M_0)| < d(M^*) - d(M_0)$ by Lemma 4.5 (See Fig. 4). Since both $d(M_{opt})$ and $d(M^*)$ are at most $\epsilon\Delta$, $c(M^*) - c(M_{opt}) \leqq 2(1 + \epsilon)\Delta$ (note that $|d(M_0)| = \Delta$). It follows that $c(M^*) - c(M_{opt}) \leqq 2(1 + \epsilon)d(A_{opt})/(2 + 3\epsilon) = (2 - \epsilon/(2 + 3\epsilon))d(A_{opt})$. Since we have $|d(A_{opt})| < c(A_{opt})$ by Lemma 4.4 and $c(A_{opt}) \leqq c(M_{opt})$, $c(M^*) < (2 - \epsilon/(2 + 3\epsilon))c(M_{opt})$.

**Case (iv): $-\Delta \leqq d(A_{opt}) < -\epsilon\Delta$.** The same as Case (ii).

**Time Complexity.** Steps 1, 2, 3, and 5 can be executed in $O(n^2)$ time. Step 4(a) is performed in the same time complexity as finding a minimum egalitarian stable matching, namely, $O(n^4)$. We can see that Steps 4(b) through 4(d) can be performed in time $O(n^2)$ by a similar analysis of Algorithm 1. The number of repetitions of Step 4 can be analyzed in the same way as the proof of Theorem 3.1, which is $O(n^{\frac{1+\epsilon}{\delta}})$. Hence the time complexity of Algorithm 2 is $O(n^{4+\frac{1+\epsilon}{\delta}})$. □

## 5. Concluding Remarks

In this paper, we gave a polynomial time algorithm for finding near optimal sex-equal stable matching. Furthermore, we proved NP-hardness and developed a polynomial time approximation algorithm whose approximation ratio is less than 2 for MinESE. Our future work is to improve the approximation ratio of MinESE.

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