Parametric Bisubmodular Function Minimization and Its Associated Signed Ring Family

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Abstract

The present paper shows an extension of the theory of principal partitions for submodular functions to that for bisubmodular functions. We examine the structure of the collection of all solutions of a parametric minimization problem described by a bisubmodular function and two vectors. The bisubmodular function to be minimized for each parameter is the sum of the bisubmodular function and a parameterized box-bisubmodular function given in terms of the two vectors. We show that the collection of all the minimizers for all parameters forms a signed ring family and it thus induces a signed poset on a signed partition of the underlying set. We further examine the structure of the signed ring family and reveal the decomposition structure depending on critical values of the parameter. Moreover, we discuss the relation between the results of this paper on bisubmodular functions and the theory of principal partitions developed for submodular functions.

Keywords: Bisubmodular function, signed ring family, signed poset, principal partition
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1. Introduction

The combinatorial structure with bisubmodularity was considered in [4, 5, 7, 8, 9, 23] and was further investigated in [11, 6, 13, 14, 21] and others. Bisubmodularity is a generalization of submodularity and has been still drawing researchers' attention (see, e.g., [19, 24]).

The present work is an extension of the theory of principal partitions for submodular functions ([11, 12]) to that for bisubmodular functions. The theory of principal partitions was inaugurated with the graph tri-partition based on maximally distant pairs of spanning trees in graphs by Kishi and Kajitani in 1967 (see a survey paper [12] for the history of theoretical developments in principal partitions). The tri-partition of a connected graph $G = (V, E)$ with a vertex set V and an edge set E is given by a chain $F_1 \subseteq F_2 \subseteq E$ such that (1) (if $F_1 \neq \emptyset$) for the subgraph $G \cdot F_1$ (obtained by restriction of G on F_1 and deletion of isolated vertices) the edge set F_1 cannot be covered by any two spanning trees, (2) (if $F_1 \subset F_2$) for the subgraph $G \cdot F_2/F_1$ (obtained by restriction of G on F_2, contraction of F_1, and deletion of isolated vertices) the edge set $F_2 \setminus F_1$ can exactly be covered by two spanning trees, and (3) (if $F_2 \subset E$) for the subgraph G/F_2 (obtained by contraction of F_2) the edge set $E \setminus F_2$ can be covered by two spanning trees with at least one edge being covered by twice. The edge set F_1 is a denser part of G, $E \setminus F_2$ is a coarser part, and $F_2 \setminus F_1$ has a density $\frac{1}{2}$ (called a critical value). Historically, it has been revealed that the submodularity plays a crucial rôle in the principal partitions, so that the extensions from graph rank functions to rank functions of matrices, matroids, and polymatroids, and further to combinatorial systems with general submodular functions could have successfully been made. Through a parametric submodular function minimization defined by the associated submodular functions, we obtain a family $\mathcal{D}$ of subsets of the underlying set E which forms a distributive lattice with respect to the set union and intersection as the lattice operations. Then any maximal chain $\emptyset = F_0 \subset F_1 \subset \cdots \subset F_k \subset F_{k+1} = E$ of $\mathcal{D}$ leads us to a unique decomposition of the relevant system into subsystems on $F_{i+1} \setminus F_i$ for $i = 0, 1, \cdots, k$ (see [12] for more details). The theoretical developments along this line seems to have been completed.

But beyond the ordinary submodularity a possible extension of the theory of principal partitions to that for bisubmodular functions in particular has been an interesting open question, since bisubmodular function minimization can be made efficiently and the collection of minimizers of a bisubmodular functions forms what is called a signed ring family (a signed version of distributive lattice) (see [1, 2, 3, 14, 17, 18, 21]). However, the trouble has been that we do not know any right formulation of a parameterized bisubmodular function minimization that leads us to a nice decomposition. We will answer this question in the present paper.

We will see that what we need is a box-bisubmodular function defined by two positive vectors (considered in [15]). Given a bisubmodular function we consider a box-bisubmodular function with a parameter and investigate a parametric minimization of the sum of the given bisubmodular function and the box-bisubmodular function with a parameter (the detailed description will be given in Section 3). We examine the structure of the collection, denoted by $\mathcal{L}^*$, of all solutions of the parametric bisubmodular function minimization and reveal that the collection $\mathcal{L}^*$ nicely forms a signed ring family ([1, 2, 6]) and induces a signed poset on a signed partition of the underlying set ([1, 2, 3]).

The present paper is organized as follows. We give some definitions and preliminaries in Section 2. In Section 3 we introduce a parametric bisubmodular function minimization problem, where we consider the sum of a bisubmodular function and a parameterized box-bisubmodular function [15]. For each fixed parameter λ the minimization problem induces a signed ring family $\mathcal{L}_\lambda$ and we show that the union of all the signed ring families $\mathcal{L}_\lambda$ is a single signed ring family $\mathcal{L}^*$. In Section 4 we examine the structure of the signed ring family $\mathcal{L}^*$ in more detail. Moreover, we discuss the relation of the results of this paper to the theory of principal partitions and related topics in Section 5. Section 6 gives remarks on some problems left open.

2. Definitions and Preliminaries

Let V be a nonempty finite set and 3^V be the set of ordered pairs of disjoint subsets of V, i.e., $3^V = \{(X, Y) \mid X, Y \subseteq V,\ X \cap Y = \emptyset\}$. Define two binary operations, *reduced union* $\sqcup$ and *intersection* $\sqcap$, on 3^V as follows. For any $(X_1, Y_1), (X_2, Y_2) \in 3^V$,

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)), \tag{1}$$

$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2). \tag{2}$$

A function $f : 3^V \to \mathbb{R}$ is called *bisubmodular* ([11]) if f satisfies

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)) \tag{3}$$

for all $(X_1, Y_1), (X_2, Y_2) \in 3^V$. If inequality (3) holds with equality for a given pair of $(X_1, Y_1), (X_2, Y_2) \in 3^V$, then we call $\{(X_1, Y_1), (X_2, Y_2)\}$ an *f-bimodular pair*. Also, when $-g$ is bisubmodular, we call

g *bisupermodular*. A function that is both bisubmodular and bisupermodular is called *bimodular*. When $\mathcal{L} \subseteq 3^V$ is closed with respect to reduced union $\sqcup$ and intersection $\sqcap$, $\mathcal{L}$ is called a *signed ring family* ([2, 3]). If a function $f : \mathcal{L} \to \mathbb{R}$ on a signed ring family $\mathcal{L}$ satisfies (3) for all $(X_1, Y_1), (X_2, Y_2) \in \mathcal{L}$, we also call f a *bisubmodular function* on $\mathcal{L}$.

Consider two positive vectors $w^+, w^- : V \to \mathbb{R}_{>0}$ and define

$$\lambda_* = \max\{\lambda \in \mathbb{R} \mid \forall v \in V : \lambda w^-(v) \leq w^+(v)\} \ (> 0). \tag{4}$$

Also for any $\lambda \leq \lambda_*$ define a function $w_\lambda : 3^V \to \mathbb{R}$ by

$$w_\lambda(X, Y) = w^+(X) - \lambda w^-(Y) \qquad (\forall (X, Y) \in 3^V), \tag{5}$$

where for any $x \in \mathbb{R}^V$ and $X \subseteq V$ we define $x(X) = \sum_{v \in X} x(v)$. Then we can easily see that w_λ is a bisubmodular function, which is called a *box-bisubmodular* function in [15]. We give a proof of this bisubmodularity for the sake of completeness and to use it in the subsequent arguments.

Lemma 2.1 ([15]). *For any $\lambda \leq \lambda_*$ the function w_λ defined by* (5) *is bisubmodular.*

(Proof) For any $(X_1, Y_1), (X_2, Y_2) \in 3^V$ we have

$$\begin{aligned}
& w_\lambda(X_1, Y_1) + w_\lambda(X_2, Y_2) \\
&= w^+(X_1) - \lambda w^-(Y_1) + w^+(X_2) - \lambda w^-(Y_2) \\
&= w^+((X_1 \cup X_2) \setminus (Y_1 \cup Y_2)) + w^+(X_1 \cap X_2) + w^+((X_1 \cup X_2) \cap (Y_1 \cup Y_2)) \\
&\qquad -\lambda w^-((Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) - \lambda w^-(Y_1 \cap Y_2) - \lambda w^-((X_1 \cup X_2) \cap (Y_1 \cup Y_2)) \\
&= w_\lambda((X_1, Y_1) \sqcup (X_2, Y_2)) + w_\lambda((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\qquad +w^+((X_1 \cup X_2) \cap (Y_1 \cup Y_2)) - \lambda w^-((X_1 \cup X_2) \cap (Y_1 \cup Y_2)) \\
&\geq w_\lambda((X_1, Y_1) \sqcup (X_2, Y_2)) + w_\lambda((X_1, Y_1) \sqcap (X_2, Y_2)),
\end{aligned} \tag{6}$$

where the inequality in (6) follows from the assumption that $\lambda \leq \lambda_*$. □

Corollary 2.2. *Suppose $\lambda < \lambda_*$. Then a pair of $(X_1, Y_1), (X_2, Y_2) \in 3^V$ is a w_λ-bimodular pair if and only if $(X_1 \cup X_2) \cap (Y_1 \cup Y_2) = \emptyset$.*

(Proof) Because of the definition of λ_* and the assumption that $\lambda < \lambda_*$, (6) holds with equality if and only if $(X_1 \cup X_2) \cap (Y_1 \cup Y_2) = \emptyset$. □

3. Parametric Bisubmodular Function Minimization

Suppose that we are given a bisubmodular function $f : 3^V \to \mathbb{R}$ and a box-bisubmodular function $w_\lambda : 3^V \to \mathbb{R}$ as given in Section 2. Then let us consider the following parametric minimization problem $(\mathbf{P}_\lambda)$ of bisubmodular functions $f + w_\lambda$ for all $\lambda \leq \lambda_*$.

$$(\mathbf{P}_\lambda): \qquad \text{Minimize } \ f(X, Y) + w_\lambda(X, Y) \quad \text{subject to } \ (X, Y) \in 3^V. \tag{7}$$

Denote by p_λ the minimum value of Problem $(\mathbf{P}_\lambda)$.

For each $\lambda \leq \lambda_*$ let $\mathcal{L}_\lambda$ be the collection of all minimizers of bisubmodular function $f + w_\lambda$. As is well known (see, e.g., [6, 2]), we see the following.

Lemma 3.1. *For each $\lambda \leq \lambda_*$, $\mathcal{L}_\lambda$ is closed with respect to the binary operations $\sqcup$ and $\sqcap$. Moreover, both f and w_λ restricted on $\mathcal{L}_\lambda$ are bimodular functions.*

(Proof) The latter follows from the fact that we minimize the sum of two bisubmodular functions f and w_λ. $\square$

Any signed ring family $\mathcal{L}$ is a meet semi-lattice with respect to the partial order $\sqsubseteq$ defined by $(X, Y) \sqsubseteq (X', Y') \Leftrightarrow X \subseteq X'$ and $Y \subseteq Y'$. The minimum element of $\mathcal{L}$ is given by $\sqcap_{(X,Y)\in\mathcal{L}}(X, Y)$.

Now, we can show the following theorem.

Theorem 3.2. *For any $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2 \leq \lambda_*$ let $(X_i, Y_i) \in \mathcal{L}_{\lambda_i}$ for $i = 1, 2$. Then we have*

$$(X_1, Y_1) \sqcap (X_2, Y_2) \in \mathcal{L}_{\lambda_1}, \qquad (X_1, Y_1) \sqcup (X_2, Y_2) \in \mathcal{L}_{\lambda_2}. \tag{8}$$

Moreover,

(a) *the pair of (X_1, Y_1) and (X_2, Y_2) is an f-bimodular pair and also a w_{λ_2}-bimodular pair,*

(b) $Y_1 \subseteq Y_2$,

(c) *if $\lambda_2 < \lambda_*$, then $Y_2 \cap X_1 = \emptyset$.*

(Proof) Under the assumption of the present lemma we have

$$\begin{aligned}
& p_{\lambda_1} + p_{\lambda_2} \\
&= f(X_1, Y_1) + w_{\lambda_1}(X_1, Y_1) + f(X_2, Y_2) + w_{\lambda_2}(X_2, Y_2) \\
&= f(X_1, Y_1) + w^+(X_1) - \lambda_1 w^-(Y_1) + f(X_2, Y_2) + w^+(X_2) - \lambda_2 w^-(Y_2) \\
&\geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\quad + w^+(X_1) + w^+(X_2) - \lambda_1 w^-(Y_1) - \lambda_2 w^-(Y_2) \\
&= f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\quad + w^+(X_1) + w^+(X_2) - \lambda_2 w^-(Y_1) - \lambda_2 w^-(Y_2) + (\lambda_2 - \lambda_1) w^-(Y_1) \\
&\geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\quad + w_{\lambda_2}((X_1, Y_1) \sqcup (X_2, Y_2)) + w_{\lambda_2}((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\quad + (\lambda_2 - \lambda_1) w^-(Y_1) \\
&= f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\quad + w_{\lambda_2}((X_1, Y_1) \sqcup (X_2, Y_2)) + w_{\lambda_1}((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\quad + (\lambda_2 - \lambda_1)(w^-(Y_1) - w^-(Y_1 \cap Y_2)) \\
&\geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + w_{\lambda_2}((X_1, Y_1) \sqcup (X_2, Y_2)) \\
&\quad + f((X_1, Y_1) \sqcap (X_2, Y_2)) + w_{\lambda_1}((X_1, Y_1) \sqcap (X_2, Y_2)) \\
&\geq p_{\lambda_1} + p_{\lambda_2}.
\end{aligned} \tag{9}$$

Hence (9) must hold with equality for each inequality appearing there, which implies (8), (a), and (b). Also (c) follows from (a), (b), and Corollary 2.2. $\square$

Put

$$\mathcal{L}^* = \bigcup_{\lambda \leq \lambda_*} \mathcal{L}_\lambda. \tag{10}$$

We then have the following.

Corollary 3.3. *$\mathcal{L}^*$ is a signed ring family.*

(Proof) Immediate from Lemma 3.1 and Theorem 3.2. □

It is shown in [1] that given any signed ring family $\mathcal{L} \subseteq 3^V$, such as $\mathcal{L}^*$ here, $\mathcal{L}$ can be represented as a collection of ideals of an associated bidirected graph $G(\mathcal{L})$ with vertex set V. Roughly speaking, the underlying set V is partitioned into components, each corresponding to a strongly connected component of bidirected graph $G(\mathcal{L})$ (see [1, 2, 3] for more details). This decomposition structure should be called the *principal partition of bisubmodular function f associated with vectors $w^{\pm}$*. The relation to the (classic) theory of principal partitions ([12]) will be discussed in Section 5.

We further investigate the structure of the signed ring family $\mathcal{L}^*$ in the next section.

4. The Structure of Signed Ring Family $\mathcal{L}^*$

We first show the following lemma.

Lemma 4.1. *There exists a finite sequence of reals $\lambda_1 < \cdots < \lambda_\ell < \lambda_{\ell+1} = \lambda_*$ such that* (i) *for each $i = 1, \cdots, \ell+1$, $\mathcal{L}_\lambda$s for all $\lambda \in (\lambda_{i-1}, \lambda_i)$ are the same and* (ii) *$\mathcal{L}_{\lambda_i} \neq \mathcal{L}_\lambda$ for each $i = 1, \cdots, \ell$ and $\lambda \in (\lambda_{i-1}, \lambda_i) \cup (\lambda_i, \lambda_{i+1})$, where we define $\lambda_0 = -\infty$.*

(Proof) For each parameter λ let $\mathcal{L}_\lambda$ be the set of minimizers (X, Y) of $f + w_\lambda$. Now, for any distinct parameters $\lambda < \lambda'$, suppose that $\mathcal{L}_\lambda = \mathcal{L}_{\lambda'}$. Then we can easily see $\mathcal{L}_\mu = \mathcal{L}_\lambda (= \mathcal{L}_{\lambda'})$ for all μ such that $\lambda < \mu < \lambda'$, as follows. Choose arbitrary $(X, Y) \in \mathcal{L}_\lambda (= \mathcal{L}_{\lambda'})$ and $(Z, W) \in \mathcal{L}_\mu$. Then we have

$$f(X, Y) + w_\lambda(X, Y) \leq f(Z, W) + w_\lambda(Z, W), \tag{11}$$

$$f(X, Y) + w_{\lambda'}(X, Y) \leq f(Z, W) + w_{\lambda'}(Z, W). \tag{12}$$

Note that we have a unique convex combination representation $\mu = \alpha\lambda + \beta\lambda'$ with $\alpha > 0$ and $\beta > 0$ satisfying $\alpha + \beta = 1$. Using this convex combination coefficients α and β, $(11)\times\alpha + (12)\times\beta$ becomes

$$f(X, Y) + w_\mu(X, Y) \leq f(Z, W) + w_\mu(Z, W). \tag{13}$$

Hence (13) (as well as (11) and (12)) must hold with equality, which implies $\mathcal{L}_\mu = \mathcal{L}_\lambda (= \mathcal{L}_{\lambda'})$ due to the arbitrary choices of $(X, Y) \in \mathcal{L}_\lambda$ and $(Z, W) \in \mathcal{L}_\mu$.

Since the number of distinct $\mathcal{L}_\lambda$s for all $\lambda \leq \lambda_*$ is finite (because V is finite), the infinite interval $(-\infty, \lambda^*]$ is thus divided into a finite number of subintervals $(-\infty, \lambda_1), (\lambda_2, \lambda_3), \cdots, (\lambda_\ell, \lambda^*]$ such that on each subinterval we have a common set of minimizers $\mathcal{L}_\lambda$ for all λ in the subinterval. □

We call λ_i $(i = 1, \cdots, \ell)$ *critical values* for $f + w_\lambda$ (or for Problem $(\mathbf{P}_\lambda)$). If $\mathcal{L}_{\lambda_*} \neq \mathcal{L}_\lambda$ for $\lambda \in (\lambda_\ell, \lambda_*)$, then we also call λ_* a *critical value*. Define $\ell^* = \ell + 1$ if λ_* is a critical value and $\ell^* = \ell$ otherwise. It should be noted that Lemma 4.1 only shows that the number of critical values is finite (possibly exponential). Later, Theorem 4.3 shows that the number is at most $|V|$, based on the results in Section 3.

Define $(X^*_\lambda, Y^*_\lambda)$ to be the minimum element of $\mathcal{L}_\lambda$ for $\lambda \leq \lambda_*$. For each $i = 1, \cdots, \ell^*$ we put $(X^*_i, Y^*_i) = (X^*_{\lambda_i}, Y^*_{\lambda_i})$.

Theorem 4.2. *For each $i = 2, \cdots, \ell^*$, we have*

$$\mathcal{L}_\lambda = \mathcal{L}_{\lambda_{i-1}} \cap \mathcal{L}_{\lambda_i} \qquad (\lambda_{i-1} < \forall\lambda < \lambda_i). \tag{14}$$

Moreover, for $i = 2, \cdots, \ell^*$ and λ with $\lambda_{i-1} < \lambda < \lambda_i$ we have $Y = Y_i^*$ for all $(X, Y) \in \mathcal{L}_\lambda$ and hence $\mathcal{L}_\lambda$ is a distributive lattice with respect to the partial order $\sqsubseteq$.

(Proof) Because of the definition of critical values we have (14). Moreover, it follows from (14) and (b) of Theorem 3.2 that $Y = Y'$ for all $(X, Y), (X', Y') \in \mathcal{L}_\lambda$ with $\lambda_{i-1} < \lambda < \lambda_i$ and $i = 2, \cdots, \ell^*$. Hence, putting $\mathcal{D}_{\lambda_i} = \{X \mid (X, Y_i^*) \in \mathcal{L}_{\lambda_i}\}$, $\mathcal{D}_{\lambda_i}$ is closed with respect to $\cup$ and $\cap$. □

Denote by $\hat{X}_i$ the maximum element of $\mathcal{D}_{\lambda_i}$. Then $(\hat{X}_i, Y_i^*)$ is the maximum element and (X_i^*, Y_i^*) is the minimum element of $\mathcal{L}_\lambda$ with $\lambda_{i-1} < \lambda < \lambda_i$. It should be noted that $(\hat{X}_i, Y_i^*)$ is a maximal, but not necessarily a unique maximal, element of $\mathcal{L}_{\lambda_{i-1}}$.

Theorem 4.3. *We have $Y_1^* \subset \cdots \subset Y_\ell^*$. In particular, $\ell \leq |V|$.*

(Proof) For any $i \in \{2, \cdots, \ell\}$, by Theorem 4.2 and the definition of critical values λ_is, we have

$$\mathcal{L}_{\lambda_{i-1}} \neq \mathcal{L}_\lambda = \mathcal{L}_{\lambda_{i-1}} \cap \mathcal{L}_{\lambda_i} \qquad (\lambda_{i-1} < \forall\lambda < \lambda_i). \tag{15}$$

Hence there exists $(W, Z) \in \mathcal{L}_{\lambda_{i-1}} \setminus \mathcal{L}_{\lambda_i}$. It follows from Theorem 3.2 that

$$Y_{i-1}^* \subseteq Z \subseteq Y_i^*. \tag{16}$$

If $Z = Y_i^*$, then we have

$$\begin{aligned} &f(X_i^*, Y_i^*) + w^+(X_i^*) - \lambda_{i-1} w^-(Y_i^*) \\ &= f(W, Z) + w^+(W) - \lambda_{i-1} w^-(Y_i^*). \end{aligned} \tag{17}$$

Hence we also have

$$\begin{aligned} &f(X_i^*, Y_i^*) + w^+(X_i^*) - \lambda_i w^-(Y_i^*) \\ &= f(W, Z) + w^+(W) - \lambda_i w^-(Y_i^*). \end{aligned} \tag{18}$$

That is, $(W, Z) = (W, Y_i^*) \in \mathcal{L}_{\lambda_i}$, a contradiction. It follows that $Y_{i-1}^* \subseteq Z \subset Y_i^*$ for each $i = 2, \cdots \ell$. Hence $\ell \leq |V|$. □

Now, let us examine the structures of $\mathcal{L}_{\lambda_*}$ and $\mathcal{L}_\lambda$ for $\lambda < \lambda_1$. Every maximal element (X, Y) of ring family $\mathcal{L} \subseteq 3^V$ has the same set $X \cup Y$ ([1, 2]), which is called the *support* of $\mathcal{L}$ and is denoted by Supp($\mathcal{L}$). If Supp($\mathcal{L}$) = V, then we say $\mathcal{L}$ *spans* V.

Theorem 4.4. *Suppose that $f(\emptyset, \emptyset) = 0$. The following two statements hold:*

(i) *If $\lambda_* w^-(v) \geq f(\emptyset, \{v\})$ for all $v \in V$, then $\mathcal{L}_{\lambda_*}$ spans V.*

(ii) *If $f(X, \emptyset) + w^+(X) \geq 0$ for all $X \subseteq V$, then we have $(\emptyset, \emptyset) \in \mathcal{L}_\lambda$ for all $\lambda < \lambda_1$.*

(Proof) (i) For any $(X, Y) \in 3^V$ and $v \in V \setminus (X \cup Y)$ we have from the assumption

$$\begin{aligned} &f(X, Y) + w^+(X) - \lambda_* w^-(Y) \\ &\geq f(X, Y) + f(\emptyset, \{v\}) + w^+(X) - \lambda_*(w^-(Y) + w^-(v)) \\ &\geq f(X, Y \cup \{v\}) + w^+(X) - \lambda_*(w^-(Y \cup \{v\}). \end{aligned} \tag{19}$$

Hence every maximal $(X, Y) \in \mathcal{L}_{\lambda_*}$ must satisfy $X \cup Y = V$.

(ii) We see that for a negative real $\lambda < \lambda_1$ with sufficiently large $|\lambda|$ there exists an element $(X, \emptyset) \in \mathcal{L}_\lambda$. Then by the assumption we have $(\emptyset, \emptyset) \in \mathcal{L}_\lambda$, where recall that $\mathcal{L}_\lambda$ is the same for all $\lambda < \lambda_1$ due to the definition of λ_1. □

5. Historical Notes

We have shown how a triple (f, w^+, w^-) of a bisubmodular function $f : 3^V \to \mathbb{R}$ and positive vectors $w^+, w^- : V \to \mathbb{R}$ determines a signed ring family $\mathcal{L}^* = \bigcup_{\lambda \le \lambda_*} \mathcal{L}_\lambda$, which gives the decomposition and signed poset structure represented by a bidirected graph associated with $\mathcal{L}^*$ due to [1, 2, 3]. This result can be regarded as a theory of what is called a *principal partition* ([12]) for bisubmodular functions. We discuss how the result of the present paper extends those considered in the theory of principal partitions.

5.1. Principal partition of a polymatroid with respect to a weight vector

The principal partition of a polymatroid (V, ρ) (ρ: the rank function) with respect to a weight vector $w : V \to \mathbb{R}_{>0}$ ([10]) is based on the parametric minimization of

$$\rho(X) - \lambda w(X) \qquad (X \subseteq V)$$

for all $\lambda \in \mathbb{R}$. Define $f(X, Y) = \rho(Y) + \rho(V \setminus X) - \rho(V)$ for all $(X, Y) \in 3^V$, which is bisubmodular (see [14, p. 1066]). Also define $w^+(v)$ to be sufficiently large for all $v \in V$ and $w^- = w$. We consider λs such that $\lambda w^-(v)$ is sufficiently small compared with $w^+(v)$ for all $v \in V$. Then, every minimizer (X, Y) of $f + w_\lambda$ for such λ satisfies $X = \emptyset$, so that it is equivalent to considering the parametric minimization of $\rho - \lambda w^- (= \rho - \lambda w)$.

5.2. Principal partition of a bipartite graph

The principal partition of a bipartite graph $G = (V^+, V^-; A)$ was considered by Tomizawa.[1] Consider a parametric minimization of

$$|\Gamma(X)| - \lambda |X| \qquad (X \subseteq V^+) \tag{20}$$

for all $\lambda \ge 0$, where $\Gamma(X)$ is the set of vertices in V^- adjacent to $X \subseteq V^+$ in G. Since $|\Gamma(X)|$ as a function in $X \in 2^{V^-}$ is a polymatroid rank function, (20) can be regarded as a special case of the principal partition of a polymatroid with respect to weight vector $w = \mathbf{1}$ (a vector of all ones). This extends the Dulmage-Mendelsohn decomposition (see [20]) of bipartite graphs and gives a finer (possibly non-square) block triangularization of matrices, allowing arbitrary permutations of rows and of columns. (Note that the existence of any non-square block implies that the whole matrix is non-regular.)

It was further extended to a pair of polymatroids (V, ρ_i) $(i = 1, 2)$ by [16, 22], where they considered the parametric minimization

$$\rho_1(X) + \lambda \rho_2(V \setminus X) \qquad (X \subseteq V)$$

for $\lambda \ge 0$. This is, however, outside the framework of the present paper.

[1] Private communication in 1976.

5.3. The Dulmage-Mendelsohn type decomposition of general graphs

The partition (signed poset decomposition) of undirected graphs due to Iwata [17, 18] extends the Edmonds-Gallai decomposition (see [20]) of graphs and the Dulmage-Mendelsohn decomposition of bipartite graphs. Here he considered the minimization of

$$\rho(Y, X) + |X| - |Y| \qquad ((X, Y) \in 3^V),$$

where ρ is the rank function of the matching delta-matroid of a graph $G = (V, E)$ and is a bisubmodular function ([5, 6]). We may consider a parametric minimization

$$\rho(Y, X) + |X| - \lambda|Y| \qquad ((X, Y) \in 3^V).$$

We see that this is a special case of our parametric setting by considering $f(X, Y) = \rho(Y, X)$ for all $(X, Y) \in 3^V$ and $w^+, w^- : V \to \mathbb{R}_{>0}$ as $w^\pm(v) = 1$ for all $v \in V$. In this case we have $\lambda_* = 1$ and the conditions of (i) and (ii) in Theorem 4.4 are satisfied. Then $\mathcal{L}_{\lambda_*}$ gives the decomposition in [17]. The partition of V into $X^*_{\lambda_*}$, $Y^*_{\lambda_*}$, and $V \setminus (X^*_{\lambda_*} \cup Y^*_{\lambda_*})$ corresponds to the Edmonds-Gallai tri-partition of V. Moreover, our decomposition by $\mathcal{L}^*$ gives a (possibly non-square) block triangularization of skew-symmetric matrices. This is a skew-symmetric version of Tomizawa's block triangularization of matrices.

6. Concluding Remarks

We have shown an extension of the theory of principal partitions for submodular functions to that for bisubmodular functions.

In the theory of principal partitions ([12]) it has been revealed that there exists a universal base [10] or a universal pair of bases [22] which solves a kind of resource allocation problem associated with the principal partition. It is left open to investigate whether there exists a certain universal vector, in the *bisubmodular polyhedron* $\mathrm{P}_*(f) \equiv \{x \in \mathbb{R}^V \mid \forall (X, Y) \in 3^V : x(X) - x(Y) \leq f(X, Y)\}$ ([2, 11]), characterized by $\mathcal{L}^*$ for a given triple (f, w^+, w^-) such that it solves some (nontrivial) optimization problem associated with the triple. Here note that $\mathcal{L}^*$ determines a face of $\mathrm{P}_*(f)$ since f restricted on $\mathcal{L}^*$ is bimodular, due to Lemma 3.1 and Theorem 3.2.

Finally, we have not considered an algorithmic issue about how to compute the critical values and the signed ring family $\mathcal{L}^*$. Based on Theorems 4.3 and 4.4, we may find the critical values by the binary search and then for each critical value λ_i we can obtain the bidirected graph representation of signed ring family $\mathcal{L}_{\lambda_i}$ by bisubmodular function minimization algorithms ([14, 21]). Getting more efficient algorithms is left open.

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