Studies on Non-autonomous Discrete Hungry Integrable Systems Associated with Some Eigenvalue Problems

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Chapter 1

Introduction

The main theme of this thesis is to analyze, as the discrete-time goes to infinity, asymptotic behaviors of the solutions to the non-autonomous discrete integrable systems associated with some eigenvalue problems. In scientific computing, eigenvalue problems of matrix are principal and important subjects. Several solvers of matrix eigenvalue problems have been developed in the studies of numerical linear algebra. For example, the QR algorithm is one of famous algorithms for computing eigenvalues of matrices [41], which is also known as an analytically stable method based on the QR decomposition, which writes a matrix as a product of an orthogonal matrix and an upper triangular matrix. Furthermore, in order to reduce the number of iterations of the QR algorithm, the QR algorithm employs an implicit shift [11]. However, their convergence theorem for unsymmetric matrix claims that the QR algorithm "essentially" converges, namely, the diagonal entries tend to eigenvalues of unsymmetric matrices, and the (i, j) entries go to 0 if $i > j$ and oscillate if $i < j$. Another famous algorithm is the quotient-difference (qd) algorithm proposed by Rutishauser [32] for computing eigenvalues of symmetric tridiagonal matrices. Both the QR algorithm and qd algorithm make symmetric tridiagonal matrices converge to some upper bidiagonal matrices without causing oscillation after the diagonal entries have converged to eigenvalues. The diagonal and the off-diagonal entries are eigenvalues of the symmetric tridiagonal matrices and some constants, respectively. Since the convergence of the qd algorithm may be slow if two eigenvalues lie very close together, it is necessary to introduce an implicit shift for promoting the convergence. The qd algorithm with implicit shift accomplishes convergence of acceleration if no divisions by zero occur in computation.

Many integrable systems first appeared in mathematical physics or in mathematical biology for real-life problems. In the studies of integrable system, the Lax representation and determinant solutions are important mathematical concepts as integrability properties. Time discretizations of such integrable systems can be useful for understanding physical and biological phenomena numerically. Surprisingly, discrete-time evolutions in discrete integrable systems also contribute to scientific computing as key components of numerical algorithms. This is because the solution to the Lax representation in discrete integrable systems, which can be regarded as a similarity LR transformation in numerical computation, converges to an upper triangular matrix associated with them. The existence of determinant solution enables us to prove the convergence theorems by using asymptotic expansion of determinant, as the discrete-time goes to infinity, of numerical algorithms for computing matrix eigenvalues based on discrete integrable systems.

The Toda lattice equation is one of the most famous integrable systems, and describes the motion governed by a nonlinear spring [36], but studies have branched into many fields, for example, nonlinear electric circuits [15], explicit soliton solutions [25], and the connection with simple Lie algebra [2]. Interestingly, the Toda equation and its time discretization both have relationships to well-known numerical algorithms for computing eigenvalues of tridiagonal matrices. The continuous-time evolution in the Toda equation corresponds to the 1-step of the QR algorithm for tridiagonal matrix exponentials [35]. The discrete Toda (dToda) equation is equivalent to the recursion formula that generates the similarity LR transformations of tridiagonal matrices in the qd algorithm [31]. The qd algorithm is also used to compute singular values of bidiagonal matrices [28].

In mathematical physics based on the theory of orthogonal polynomials, the compatibility condition of transformations for discrete-time evolution and its inverse yields the dToda equation with arbitrary functions [16, 22, 23], which often is called the non-autonomous dToda equation. A non-autonomous dToda equation also gives the LR transformations with implicit shifts of tridiagonal matrices. In other words, the non-autonomous dToda equation can be considered a recursion formula of the shifted qd algorithm that may converge faster than the original algorithm converges.

The integrable Lotka-Volterra (LV) system is a simple biological model describing a predator-prey interaction of several species [43]. For $(2m-1)$ species, the LV system assumes that the 1st, 2nd, $\ldots$, $(2m-2)$th species prey on only the 2nd, 3rd, $\ldots$, $(2m-1)$th species, respectively. There is no predator for the 1st species and no prey for the $(2m-1)$th species. The discrete LV (dLV) system is a time-discretization of the integrable LV system, which describes the population of each species in predator-prey relationships at any time. The dLV system also generates the LR and a shifted LR transformations of positive-definite tridiagonal matrices for computing singular values of bidiagonal matrices [19, 20, 38]. The relationship between the dToda equations and the dLV system is so-called the Bäcklund transformation which used to be a important tool in several integrable systems.

As described above, the numerical algorithm associated with the dToda equations and the dLV system just compute tridiagonal matrix eigenvalues and bidiagonal matrix singular values, respectively. To develop the numerical algorithms computing eigenvalues of band matrices with larger bandwidth, the extensions of dToda equations and dLV system are associated with eigenvalue problems. The discrete hungry integrable systems, which can be regarded as one of generalization of discrete integrable systems, are the discrete hungry Toda (dhToda) equations and the discrete hungry Lotka-Volterra (dhLV) systems. While the famous discrete Toda (dToda) equation was considered first in studies of discrete integrable systems, the discrete Lotka-Volterra (dLV) system was extended first to a discrete hungry integrable system. The two extended LV systems, where each species may prey on one or more species, are called hungry LV (hLV) systems [3, 18, 26]. Time-discretizations of these systems are known as dhLV systems, where the simple dLV system is a special case of the dhLV systems. Hama et al. [13] distinguished the two systems by designating the $\mathrm{dhLV_I}$ and $\mathrm{dhLV_{II}}$ systems. The species appearing in the $\mathrm{dhLV_{II}}$ system survive much more than those in the $\mathrm{dhLV_I}$ system over the passage of time.

One of the dhToda equations first appeared in the study of box and ball systems [37]. In particular, it is derived from an inverse ultra-discretization of the ultra-discrete integrable system that describes a numbered box and ball system. Since this dhToda equation can be transformed to the $\mathrm{dhLV_I}$ system using the Bäcklund transformation [8], it is sometimes called the $\mathrm{dhToda_I}$ equation. The other, known as the $\mathrm{dhToda_{II}}$ equation, is defined as the corresponding dhToda equation to the $\mathrm{dhLV_{II}}$ system.

These four systems are all applicable to computing eigenvalues of totally nonnegative (TN) matrices, where TN matrices are square matrices with all minors nonnegative [7, 10, 33, 42]. A tridiagonal matrix having positive diagonal and off-diagonal entries is a typical example of TN matrices. Under the initial settings given from entries of the TN matrices, as the discrete-time variable goes to infinity, some of the variables of the discrete hungry integrable systems or their combinations converge to TN eigenvalues, and the others converge to 0.

Fukuda et al. [10] and Sumikura et al. [33] introduced shifts of origin into algorithms based on the $\mathrm{dhToda_I}$ and $\mathrm{dhToda_{II}}$ equations, respectively, to accelerate the rate of convergence. That is, to design effective TN eigenvalue algorithms, these equations included parameters depend on independent variables, which lead to the so-called non-autonomous $\mathrm{dhToda_I}$ and $\mathrm{dhToda_{II}}$ equations, respectively. The non-autonomous $\mathrm{dhToda_I}$ and $\mathrm{dhToda_{II}}$ variables given from the TN matrices are all nonnegative at the initial discrete time. The positivity of the non-autonomous $\mathrm{dhToda_I}$ and $\mathrm{dhToda_{II}}$ variables at any discrete time then holds under a suitable choice of arbitrary parameters. Fukuda et al. [10] and Sumikura et al. [33] thus focused on the case where the non-autonomous $\mathrm{dhToda_I}$ and $\mathrm{dhToda_{II}}$ variables are always positive. From the view point of integrability properties, not numerical algorithm, it is remarkable here that the non-autonomous $\mathrm{dhToda_I}$ and $\mathrm{dhToda_{II}}$ equations can generate similarity transformations involving implicit shifts of more general matrices, not just TN matrices.

Despite this, the convergence in the non-autonomous $\mathrm{dhToda_{II}}$ equation has not been discussed without requiring the positivity of the non-autonomous $\mathrm{dhToda_{II}}$ variables. The positivity of the non-autonomous $\mathrm{dhToda_I}$ and $\mathrm{dhToda_{II}}$ variables contribute to the derivation of the convergence theorems in Fukuda et al. [10] and Sumikura et al. [33], so the theorems are not easily extended. In this thesis, through investigating integrability properties, we derive the eigenvalue problems associated with $\mathrm{dhToda_{II}}$ and non-autonomous one. Although the examination using orthogonal polynomials immediately leads to non-autonomous discrete integrable systems, the non-autonomous hungry scheme does not appear naturally under the same methodology. Then, we observe the asymptotic behavior of the non-autonomous $\mathrm{dhToda_{II}}$ variables from the viewpoint of studies of discrete integrable systems as the discrete-time variable goes to infinity. That is, we give determinant expressions of the non-autonomous $\mathrm{dhToda_{II}}$ variables, and then present a extended convergence theorem for the non-autonomous $\mathrm{dhToda_{II}}$ variables assuming the band matrices which can be decomposed into a product of lower and upper triangular factors instead of the positivity of variables. Since the positivity of variables is a sufficient condition for this assumption, the resulting convergence theorem implies that the application range of matrix using the algorithms based on discrete hungry integrable systems are extended from TN matrices to band matrices decomposable into a product of lower and upper triangular factors. We also clarify asymptotic convergence in terms of $\mathrm{dhLV_I}$ variables and the Bäcklund transformation between non-autonomous discrete hungry integrable systems.

Finally, we show that, if the arbitrary parameters appearing in non-autonomous discrete hungry integrable systems are fixed as some special values, asymptotic behavior of them, as the discrete-time goes to infinity, is dominated by the extended Fibonacci numbers. That is, under some suitable initial settings with the extended Fibonacci numbers, one of variables of the non-autonomous discrete hungry integrable systems tends to an extended golden ratio as the discrete-time goes to infinity.

The thesis is organized as follows. In Chapter 2 [A1,A3,A4], we first introduce the infinite sequence, which is sometimes so-called the moment sequence, with two types of discrete-time variables. Next, we give expansion technique in terms of the poles of formal power series with entries of the Casorati determinants appearing in the solutions to discrete integrable systems as their coefficients. Finally, we then present, referring the resulting expressions of moments, asymptotic expansions for the extended Hankel determinants associated with discrete hungry integrable systems. In Chapter 3 [A3], we clarify the determinant solution and a Lax pair of the autonomous $\mathrm{dhToda}_{\mathrm{II}}$ variables by focusing on the moment sequence. We show that the resulting determinant solution firmly covers the general solution to the autonomous $\mathrm{dhToda}_{\mathrm{II}}$ variables, and provide an asymptotic analysis of the general solution as the discrete-time variable goes to infinity. In Chapter 4 [A4], we show the convergence of the solution to the non-autonomous discrete hungry integrable systems to matrix eigenvalues with assuming not the positivity but the LR decomposability. Thus, we present convergence theorems, which can contribute to computing eigenvalues of more general matrices, not just TN matrices but matrices which can be decomposed into a product of lower and upper triangular factors. To this end, we show determinant solutions to two non-autonomous discrete hungry integrable systems. We also clarify a Bäcklund transformation, which mutually relates these two non-autonomous discrete hungry integrable systems. In Chapter 5 [A2], we show that, if the entries of the determinant associated with dhLV system become an extended Fibonacci sequence at the initial discrete-time, then those are also an extended Fibonacci sequence at any discrete-time. In other words, the extended Fibonacci sequence always appears in the entries of the determinant under the time evolution of the dhLV system with a suitable initial setting. We also show that one of the dhLV variables converges to the ratio of two successive extended Fibonacci numbers as the discrete-time goes to infinity. Chapter 6 is devoted to concluding remarks of the thesis.

Chapter 2

Asymptotic expansions of the determinants associated with discrete integrable systems

The Casorati determinants appear in representations of solutions to several discrete integrable systems [21]. Asymptotic expansions of Casorati determinants and their other expressions thus play key roles in the investigation of asymptotic analysis of such systems. In this chapter, we define the Casorati determinants and the extended Hankel determinants associated with discrete hungry integrable systems, and then present asymptotic expansion techniques for them.

2.1 Moment sequence with two types of discrete-time variables

In this section, we first introduce an infinite complex sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ with respect to two types of discrete-time variables s and t. Solutions to several discrete integrable systems are known to be expressed using determinants associated with infinite sequences. Elements of the infinite sequence are often defined by certain inner products such as expected values of powers of differences between random variables and their statistical or probabilistic averages. Thus, the elements of the infinite sequences are sometimes called moments, hereinafter denoted by $f_s^{(t)}$.

For arbitrary distinct constants $\lambda_1, \lambda_2, \ldots, \lambda_m$, let us introduce an m-degree polynomial with respect to $z \in \mathbb{C}$,

$$p(z) := (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m). \tag{2.1}$$

The polynomial $p(z)$ can then be regarded as the characteristic polynomial of an m-by-m matrices with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Obviously, the polynomial

$p(z)$ can be expanded as

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \tag{2.2}$$

where $a_1, a_2, \ldots, a_m$ are complex constants.

Let us consider the case where the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ satisfies linear equations with respect to the coefficients $a_1, a_2, \ldots, a_m$ appearing in (2.2),

$$f_{s+Mm}^{(t)} + \sum_{i=1}^{m} a_i f_{s+(m-i)M}^{(t)} = 0, \quad s, t = 0, 1, \ldots. \tag{2.3}$$

Furthermore, let us assume another linear relation which gives transformation from $f_s^{(t)}$ to $f_s^{(t+1)}$ as

$$f_s^{(t+1)} = f_{s+M}^{(t)} - \mu^{(t)} f_s^{(t)}, \quad s, t = 0, 1, \ldots, \tag{2.4}$$

where $\{\mu^{(t)}\}_{t=0}^{\infty}$ are arbitrary complex sequence. Obviously, if $f_0^{(0)}$, $f_1^{(0)}$, $\ldots$, $f_{Mm-1}^{(0)}$ are given, then any moments of the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ is uniquely fixed.

2.2 Asymptotic expansions of the Casorati determinants

For $k = 1, 2, \ldots, m+1$, let us consider determinants of square matrices of order k involving the moments $f_s^{(t)}$,

$$C_k^{(s,t)} := \begin{vmatrix} f_s^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-1}^{(t)} \\ f_s^{(t+1)} & f_{s+1}^{(t+1)} & \cdots & f_{s+k-1}^{(t+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_s^{(t+k-1)} & f_{s+1}^{(t+k-1)} & \cdots & f_{s+k-1}^{(t+k-1)} \end{vmatrix}, \quad s, t = 0, 1, \ldots, \tag{2.5}$$

where $C_{-1}^{(s,t)} := 0$ and $C_0^{(s,t)} := 1$. The determinants (2.5) are called the Casorati determinants or Casoratian. The Casorati determinants are useful in the theory of difference equations, particularly in mathematical physics, and play a role similar to the Wronskian in the theory of differential equations [40].

In this section, we first give an expression of the entries of the Casorati determinants $C_k^{(s,t)}$ in terms of poles of the formal power series $F_s(z)$ associated with $C_k^{(s,t)}$. Referring to the theorem on analyticity for the Hankel determinants given in [14], we present an asymptotic expansion of the Casorati determinants $C_k^{(s,t)}$ as $t \to \infty$ by using the poles of $F_s(z)$. We also describe the case where some restrictions are imposed on the poles of $F_s(z)$.

Let $F_s(z) = \sum_{t=0}^{\infty} f_s^{(t)} z^t$, which is the formal power series associated with $C_k^{(s,t)}$ for $s = 0, 1, \ldots$, be analytic at $z = 0$ and meromorphic in the disk $D = \{z \mid |z| < \zeta\}$. Furthermore, let $r_{s,1}^{-1}, r_{s,2}^{-1}, \ldots$, denote the poles of $F_s(z)$ such that $|r_{s,1}^{-1}| < |r_{s,2}^{-1}| < \cdots < \zeta$. By extracting the principal parts in $F_s(z)$, we derive

$$F_s(z) = \frac{\alpha_{s,1}}{r_{s,1}^{-1} - z} + \frac{\alpha_{s,2}}{r_{s,2}^{-1} - z} + \cdots + \frac{\alpha_{s,p}}{r_{s,p}^{-1} - z} + \sum_{t=0}^{\infty} g_s^{(t)} z^t, \tag{2.6}$$

where p is an arbitrary positive integer, $\alpha_{s,1}, \alpha_{s,2}, \ldots, \alpha_{s,p}$ are some nonzero constants, and $g_s^{(t)}$, which contains the terms with respect to $r_{s,p+1}^{-1}, r_{s,p+2}^{-1}, \ldots$, satisfies

$$|g_s^{(t)}| \leq b_s \rho_s^t \tag{2.7}$$

for some nonzero positive constants b_s and ρ_s with $|r_{s,p+1}| < \rho_s < |r_{s,p}|$. The proof of (2.7) is given in [14] utilizing the Cauchy coefficient estimate. We now give a lemma for an expression of $f_s^{(t)}$ appearing in $F_s(z) = \sum_{t=0}^{\infty} f_s^{(t)} z^t$.

Lemma 2.2.1. *Let us assume that the poles $r_{s,1}^{-1}, r_{s,2}^{-1}, \ldots, r_{s,p}^{-1}$ of $F_s(z)$ are not multiple. Then, the $f_s^{(t)}$ can be expressed using $r_{s,1}, r_{s,2}, \ldots, r_{s,p}$ as*

$$f_s^{(t)} = \sum_{i=1}^{p} c_{s,i} r_{s,i}^t + g_s^{(t)}, \tag{2.8}$$

where $c_{s,1}, c_{s,2}, \ldots, c_{s,p}$ are some nonzero constants.

Proof. The crucial element is the replacement $\alpha_{s,1} = c_{s,1} r_{s,1}^{-1}, \alpha_{s,2} = c_{s,2} r_{s,2}^{-1}, \ldots, \alpha_{s,p} = c_{s,p} r_{s,p}^{-1}$ in (2.6), namely,

$$F_s(z) = \frac{c_{s,1} r_{s,1}^{-1}}{r_{s,1}^{-1} - z} + \frac{c_{s,2} r_{s,2}^{-1}}{r_{s,2}^{-1} - z} + \cdots + \frac{c_{s,p} r_{s,p}^{-1}}{r_{s,p}^{-1} - z} + \sum_{t=0}^{\infty} g_s^{(t)} z^t. \tag{2.9}$$

Since each $c_{s,i} r_{s,i}^{-1} / (r_{s,i}^{-1} - z)$ in (2.9) can be regarded as the summation of a geometric series, we obtain

$$\begin{aligned} F_s(z) &= \sum_{t=0}^{\infty} c_{s,1} r_{s,1}^t z^t + \sum_{t=0}^{\infty} c_{s,2} r_{s,2}^t z^t + \cdots + \sum_{t=0}^{\infty} c_{s,p} r_{s,p}^t z^t + \sum_{t=0}^{\infty} g_s^{(t)} z^t \\ &= \sum_{t=0}^{\infty} \left[\left(\sum_{i=1}^{p} c_{s,i} r_{s,i}^t \right) + g_s^{(t)} \right] z^t, \end{aligned}$$

which implies (2.8). □

Similarly to the asymptotic expansion as $t \to \infty$ of the Hankel determinants given in [14], we have the following theorem for the Casorati determinants $C_k^{(s,t)}$.

Theorem 2.2.2. *Let us assume that the poles* $r_{s,1}^{-1}, r_{s,2}^{-1}, \dots, r_{s,p}^{-1}$ *of* $F_s(z)$ *are not multiple. Then, there exists some constant* $c_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}$ *independently of* t *such that, as* $t \to \infty$,

$$C_k^{(s,t)} = \sum_\sigma \left[c_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)} (r_{s,\kappa_1} r_{s+1,\kappa_2} \dots r_{s+k-1,\kappa_k})^t \left(1 + \sum_{i=1}^{k} O\left(\left(\frac{\rho_{s+i-1}}{|r_{s+i-1,\kappa_i}|} \right)^t \right) \right) \right], \tag{2.10}$$

where σ *denotes the mapping from* $\{\kappa_1, \kappa_2, \dots, \kappa_k\}$ *to* $\{1, 2, \dots, p\}$ *and* ρ_{s+i-1} *is some constant such that* $|r_{s+i-1,p+1}| < \rho_{s+i-1} < |r_{s+i-1,p}|$.

Proof. By applying Lemma 2.2.1 and the addition formula of determinants to the Casorati determinants $C_k^{(s,t)}$, we derive

$$C_k^{(s,t)} = \sum_\sigma D_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}^{(t)} + \sum_\sigma \hat{D}_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}^{(t)}, \tag{2.11}$$

where in the first summation

$$D_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}^{(t)} := \begin{vmatrix} c_{s,\kappa_1} r_{s,\kappa_1}^t & c_{s+1,\kappa_2} r_{s+1,\kappa_2}^t & \cdots & c_{s+k-1,\kappa_k} r_{s+k-1,\kappa_k}^t \\ c_{s,\kappa_1} r_{s,\kappa_1}^{t+1} & c_{s+1,\kappa_2} r_{s+1,\kappa_2}^{t+1} & \cdots & c_{s+k-1,\kappa_k} r_{s+k-1,\kappa_k}^{t+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s,\kappa_1} r_{s,\kappa_1}^{t+k-1} & c_{s+1,\kappa_2} r_{s+1,\kappa_2}^{t+k-1} & \cdots & c_{s+k-1,\kappa_k} r_{s+k-1,\kappa_k}^{t+k-1} \end{vmatrix},$$

and $\hat{D}_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}^{(t)}$ in the second summation denotes a determinant of the same form as $D_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}^{(t)}$ except that the ith column is replaced with $\boldsymbol{g}_i := (g_{s+i-1}^{(t)}, g_{s+i-1}^{(t+1)}, \dots, g_{s+i-1}^{(t+k-1)})^\top$ for at least one of i. Evaluating the first summation in (2.11), we obtain

$$\sum_\sigma D_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}^{(t)} = \sum_\sigma c_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)} (r_{s,\kappa_1} r_{s+1,\kappa_2} \dots r_{s+k-1,\kappa_k})^t, \tag{2.12}$$

where

$$c_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)} := \begin{vmatrix} c_{s,\kappa_1} & c_{s+1,\kappa_2} & \cdots & c_{s+k-1,\kappa_k} \\ c_{s,\kappa_1} r_{s,\kappa_1} & c_{s+1,\kappa_2} r_{s+1,\kappa_2} & \cdots & c_{s+k-1,\kappa_k} r_{s+k-1,\kappa_k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s,\kappa_1} r_{s,\kappa_1}^{j-1} & c_{s+1,\kappa_2} r_{s+1,\kappa_2}^{k-1} & \cdots & c_{s+k-1,\kappa_k} r_{s+k-1,\kappa_k}^{k-1} \end{vmatrix}. \tag{2.13}$$

To estimate the second summation in (2.11), for example, we consider the case where the 1st column is replaced with $\boldsymbol{g}_1$. It immediately follows from (2.7) that

$$\begin{vmatrix} g_s^{(t)} & c_{s+1,\kappa_2}r_{s+1,\kappa_2}^t & \cdots & c_{s+k-1,\kappa_k}r_{s+k-1,\kappa_k}^t \\ g_s^{(t+1)} & c_{s+1,\kappa_2}r_{s+1,\kappa_2}^{t+1} & \cdots & c_{s+k-1,\kappa_k}r_{s+k-1,\kappa_k}^{t+1} \\ \vdots & \vdots & \ddots & \vdots \\ g_s^{(t+k-1)} & c_{s+1,\kappa_2}r_{s+1,\kappa_2}^{t+k-1} & \cdots & c_{s+k-1,\kappa_k}r_{s+k-1,\kappa_k}^{t+k-1} \end{vmatrix} = O((\rho_s r_{s+1,\kappa_2}\dots r_{s+k-1,\kappa_k})^t).$$

It is also easy to check $O((r_{s,\kappa_1}r_{s+1,\kappa_2}\dots r_{s+i-2,\kappa_{i-1}}\rho_{s+i-1}r_{s+i,\kappa_{i+1}}\dots r_{s+k-1,\kappa_k})^t)$ if the ith column is replaced with $\boldsymbol{g}_i$. Similarly, by examining the case where some columns are replaced with some of $\boldsymbol{g}_1, \boldsymbol{g}_2, \dots, \boldsymbol{g}_k$, we can see that

$$\begin{aligned} &\sum_\sigma \hat{D}_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)} \\ &\quad = \sum_\sigma c_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)}(r_{s,\kappa_1}r_{s+1,\kappa_2}\cdots r_{s+k-1,\kappa_k})^t \sum_{i=1}^k O\left(\left(\frac{\rho_{s+i-1}}{|r_{s+i-1,\kappa_i}|}\right)^t\right). \end{aligned} \tag{2.14}$$

Thus, from (2.12)–(2.14), we obtain (2.10). □

Now, let us consider the restriction $r_{s,1} = r_1, r_{s,2} = r_2, \dots, r_{s,k} = r_k$ in $F_s(z)$. Then, by replacing $r_{s,i}$ with r_i in (2.8), we easily obtain

$$f_s^{(t)} = \sum_{i=1}^p c_{s,i}r_i^t + g_s^{(t)}. \tag{2.15}$$

As a specialization of Theorem 2.2.2, we derive the following theorem for an asymptotic expansion of the Casorati determinants $C_k^{(s,t)}$ with restricted $f_s^{(t)}$ as $t \to \infty$.

Theorem 2.2.3. *Let us assume that the poles $r_1^{-1}, r_2^{-1}, \dots, r_k^{-1}$ of $F_s(z)$ are not multiple. Then, there exists some constant $c_{s,k} \neq 0$ independently of t such that, for $|r_{k+1}| < \rho_s < |r_k|$, as $t \to \infty$,*

$$C_k^{(s,t)} = c_{s,k}(r_1r_2\dots r_k)^t\left(1 + \sum_{i=1}^k O\left(\left(\frac{\rho_{s+i-1}}{|r_k|}\right)^t\right)\right). \tag{2.16}$$

Proof. Replacing $r_{s,1} = r_1, r_{s,2} = r_2, \dots, r_{s,p} = r_p$ in (2.13) gives

$$c_{s,\sigma(\kappa_1,\kappa_2,\dots,\kappa_k)} = c_{s,\kappa_1}c_{s+1,\kappa_2}\dots c_{s+k-1,\kappa_k}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_{\kappa_1} & r_{\kappa_2} & \cdots & r_{\kappa_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{\kappa_1}^{k-1} & r_{\kappa_2}^{k-1} & \cdots & r_{\kappa_k}^{k-1} \end{vmatrix}. \tag{2.17}$$

Thus, by taking into account that $c_{s,\sigma(\kappa_1,\kappa_2,\ldots,\kappa_k)} \neq 0$ only in the case where $\kappa_1, \kappa_2, \ldots, \kappa_k$ are distinct to each other, we can simplify (2.12) as

$$\sum_{\pi} D^{(t)}_{s,\pi(\kappa_1,\kappa_2,\ldots,\kappa_k)}$$
$$= (r_1 r_2 \ldots r_k)^t \sum_{\pi} c_{s,\kappa_1} c_{s+1,\kappa_2} \ldots c_{s+k-1,\kappa_k} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_{\kappa_1} & r_{\kappa_2} & \cdots & r_{\kappa_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{\kappa_1}^{k-1} & r_{\kappa_2}^{k-1} & \cdots & r_{\kappa_k}^{k-1} \end{vmatrix}, \tag{2.18}$$

where π denotes the bijection from $\{\kappa_1, \kappa_2, \ldots, \kappa_k\}$ to $\{1, 2, \ldots, k\}$. It is noted here that the bijection π is equal to the mapping σ with $p = k$. Furthermore, there exists a constant ρ_s, which is not equal to one in Theorem 2.2.2, such that $|r_{k+1}| < \rho_s < |r_k|$. This is because ρ_s and ρ_{s+1} do not always satisfy $\rho_s = \rho_{s+1}$ even if $r_{s,1} = r_1, r_{s,2} = r_2, \ldots, r_{s,k} = r_k$ in Theorem 2.2.2. Thus, (2.14) becomes

$$\sum_{i=1}^{k} O\left((r_1 r_2 \ldots r_{k-1} \rho_{s+i-1})^t\right). \tag{2.19}$$

Therefore, from (2.18) and (2.19), we obtain (2.16). □

Theorem 2.2.3 covers an asymptotic expansion of the Hankel determinants in [14]. Theorems 2.2.2 and 2.2.3 should be useful for the asymptotic analysis of dynamical systems with solutions expressed in terms of the Casorati determinants $C_k^{(s,t)}$.

2.3 The extended Hankel determinants and their asymptotic expansions

In this section, we firstly examine properties of the extended Hankel determinants associated with the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$, and finally present an expansion of the extended Hankel determinants $H_k^{(s,t)}$ in terms of the general term of the miment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$.

The Caorati determinants with linear dependence (2.4) immediately lead to

$$H_k^{(s,t)} := \begin{vmatrix} f_s^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-1}^{(t)} \\ f_{s+M}^{(t)} & f_{s+M+1}^{(t)} & \cdots & f_{s+M+k-1}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s+(k-1)M}^{(t)} & f_{s+(k-1)M+1}^{(t)} & \cdots & f_{s+(k-1)(M+1)}^{(t)} \end{vmatrix}, \quad s, t = 0, 1, \ldots, \tag{2.20}$$

where $H_{-1}^{(s,t)} := 0$ and $H_0^{(s,t)} := 1$. Since $H_k^{(s,t)}$ with $M = 1$ are just standard Hankel determinants appearing in solution expressions of the dToda equations and the dLV system, we can regard $H_k^{(s,t)}$ as an extension of the Hankel determinants. Hereinafter, $H_k^{(s,t)}$ refers to extended Hankel determinants. It is worth noting that both standard Hankel determinants and their extended ones appearing in several discrete integrable systems are equivalent to the Casorati determinants $C_k^{(s,t)}$.

Considering the linear dependence (2.3), we derive the following proposition for the extended Hankel determinants $H_k^{(s,t)}$ with the case $k = m + 1$.

Proposition 2.3.1. *It holds that*

$$H_{m+1}^{(s,t)} = 0, \quad s, t = 0, 1, \ldots. \tag{2.21}$$

Proof. Multiplying the 1st, 2nd, ..., mth rows in the extended Hankel determinants $H_{m+1}^{(s,t)}$ by a_m, a_{m-1}, ..., a_1, respectively, and adding these to the $(m+1)$th row, we obtain

$$H_{m+1}^{(s,t)} = \begin{vmatrix} f_s^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+m}^{(t)} \\ f_{s+M}^{(t)} & f_{s+M+1}^{(t)} & \cdots & f_{s+M+m}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{m} a_i f_{s+(m-i)M}^{(t)} & \sum_{i=0}^{m} a_i f_{s+(m-i)M+1}^{(t)} & \cdots & \sum_{i=0}^{m} a_i f_{s+(m-i)M+m}^{(t)} \end{vmatrix},$$

where $a_0 := 1$. Since it is obvious from (2.3) that the $(m+1, 1)$, $(m+1, 2)$, ..., $(m+1, m+1)$ entries become 0, we thus have (2.21). □

Then, we derive the following proposition concerning expansions of the extended Hankel determinants $H_k^{(s,t)}$ involving the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$.

Proposition 2.3.2. *Let us assume that the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ with initial settings of $f_0^{(t)}, f_1^{(t)}, \ldots, f_{Mm-1}^{(t)}$ satisfies the linear difference equations* (2.3). *Then, for $s = 0, 1, \ldots$, the extended Hankel determinants $H_k^{(s,t)}$ can be expressed as*

$$H_k^{(M\ell+j,t)} = \sum_{1 \le i_1 \le \cdots \le i_k \le m} D_{i_1,i_2,\ldots,i_k}^{(j,t)} (\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k})^{\ell} \prod_{1 \le k_1 \le k_2 \le m} (\lambda_{k_2} - \lambda_{k_1}),$$
$$j = 0, 1, \ldots, M-1, \quad \ell = 0, 1, \ldots, \tag{2.22}$$

where $D_{i_1,i_2,\ldots,i_k}^{(j,t)}$ are determinants of k-by-k matrices with some constants $d_1^{(j,t)}$,

$d_2^{(j,t)}$, ..., $d_m^{(j,t)}$ *satisfying* $d_i^{(j+M,t)} = \lambda_i d_i^{(j,t)}$ *defined by*

$$D_{i_1,i_2,\dots,i_k}^{(j,t)} := \begin{vmatrix} d_{i_1}^{(j,t)} & d_{i_1}^{(j+1,t)} & \cdots & d_{i_1}^{(j+k-1,t)} \\ d_{i_2}^{(j,t)} & d_{i_2}^{(j+1,t)} & \cdots & d_{i_2}^{(j+k-1,t)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i_k}^{(j,t)} & d_{i_k}^{(j+1,t)} & \cdots & d_{i_k}^{(j+k-1,t)} \end{vmatrix}, \quad j = 0, 1, \dots, M-1. \tag{2.23}$$

Proof. The general terms of the linear difference equations (2.3) can be expressed as

$$f_{M\ell+j}^{(t)} = \sum_{i=1}^{m} d_i^{(j,t)} \lambda_i^\ell, \quad j = 0, 1, \dots, M-1, \quad \ell = 0, 1, \dots. \tag{2.24}$$

This proof is easily obtained by substituting (2.24) to the linear difference equations (2.3). The replacement of the entries in the extended Hankel determinants $H_k^{(M\ell+j,t)}$ with (2.24) immediately leads to

$$H_k^{(M\ell+j,t)} = \begin{vmatrix} \sum_{i=1}^{m} d_i^{(j,t)} \lambda_i^\ell & \sum_{i=1}^{m} d_i^{(j+1,t)} \lambda_i^\ell & \cdots & \sum_{i=1}^{m} d_i^{(j+k-1,t)} \lambda_i^\ell \\ \sum_{i=1}^{m} d_i^{(j,t)} \lambda_i^{\ell+1} & \sum_{i=1}^{m} d_i^{(j+1,t)} \lambda_i^{\ell+1} & \cdots & \sum_{i=1}^{m} d_i^{(j+k-1,t)} \lambda_i^{\ell+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{m} d_i^{(j,t)} \lambda_i^{\ell+k-1} & \sum_{i=1}^{m} d_i^{(j+1,t)} \lambda_i^{\ell+k-1} & \cdots & \sum_{i=1}^{m} d_i^{(j+k-1,t)} \lambda_i^{\ell+k-1} \end{vmatrix}. \tag{2.25}$$

By applying the Cauchy-Binet formula to the determinants on the right-hand side of (2.25), we obtain

$$H_k^{(M\ell+j,t)} = \sum_{1 \le i_1 \le \cdots \le i_k \le m} \begin{vmatrix} \lambda_{i_1}^\ell & \lambda_{i_2}^\ell & \cdots & \lambda_{i_k}^\ell \\ \lambda_{i_1}^{\ell+1} & \lambda_{i_2}^{\ell+1} & \cdots & \lambda_{i_k}^{\ell+1} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_{i_1}^{\ell+k-1} & \lambda_{i_2}^{\ell+k-1} & \cdots & \lambda_{i_k}^{\ell+k-1} \end{vmatrix} \begin{vmatrix} d_{i_1}^{(j,t)} & d_{i_1}^{(j+1,t)} & \cdots & d_{i_1}^{(j+k-1,t)} \\ d_{i_2}^{(j,t)} & d_{i_2}^{(j+1,t)} & \cdots & d_{i_2}^{(j+k-1,t)} \\ \vdots & \vdots & \cdots & \vdots \\ d_{i_k}^{(j,t)} & d_{i_k}^{(j+1,t)} & \cdots & d_{i_k}^{(j+k-1,t)} \end{vmatrix}. \tag{2.26}$$

By removing the scalars $\lambda_{i_1}^\ell, \lambda_{i_2}^\ell, \dots, \lambda_{i_k}^\ell$ from the left determinants and noting that the right determinants are equal to the determinants $D_{i_1,i_2,\dots,i_k}^{(j,t)}$ on the right-hand side of (2.26), we can rewrite the right-hand side of (2.26) as

$$\sum_{1\le i_1\le\cdots\le i_k\le m} D^{(j,t)}_{i_1,i_2,\dots,i_k}(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k})^\ell \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_k} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_{i_1}^{k-1} & \lambda_{i_2}^{k-1} & \cdots & \lambda_{i_k}^{k-1} \end{vmatrix}. \tag{2.27}$$

Taking into account that the determinants appearing in (2.27) are just the Vandermonde determinants, we thus have (2.22). □

With the help of Proposition 2.3.2 gives expansions of the extended Hankel determinants $H_k^{(s,t)}$. Using this Proposition, we thus derive asymptotic expansions of the extended Hankel determinants $H_k^{(s,t)}$ as $s\to\infty$.

Proposition 2.3.3. *Let us assume that $\lambda_1,\lambda_2,\dots,\lambda_m$ are nonzero constants such that $|\lambda_1|>|\lambda_2|>\cdots>|\lambda_m|$ and the extended Hankel determinants $H_k^{(s,t)}$ are always nonzero. Then, for any ρ_k satisfying $\rho_k<|\lambda_{k+1}|/|\lambda_k|$, it holds that, as $\ell\to\infty$,*

$$H_k^{(M\ell+j,t)} = K^{(j,t)}_{1,2,\dots,k}(\lambda_1\lambda_2\cdots\lambda_k)^\ell(1+O(\rho_k^\ell)), \tag{2.28}$$

where $K^{(j,t)}_{1,2,\dots,k} := D^{(j,t)}_{i_1,i_2,\dots,i_k}\prod_{1\le k_1<k_2\le m}(\lambda_{k_2}-\lambda_{k_1})$.

Proof. If $|\lambda_1|>|\lambda_2|>\cdots>|\lambda_m|$, then it is clear from Proposition 2.3.2 that, as $\ell\to\infty$, the leading term of $H_k^{(M\ell+j,t)}$ can be expressed as $K^{(j,t)}_{1,2,\dots,k}(\lambda_1\lambda_2\cdots\lambda_k)^\ell$. Note that $\max_{i_1,i_2,\dots,i_k}[(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k})/(\lambda_1\lambda_2\cdots\lambda_k)]^\ell = O(\rho_k^\ell)$ for $i_1,i_2,\dots,i_k$ satisfying $\{i_1,i_2,\dots,i_k\}\ne\{1,2,\dots,k\}$. We thus have (2.28). □

The following proposition gives an expression concerning general terms of the moments $f_s^{(t)}$ satisfying the linear dependences (2.3) and (2.4).

Proposition 2.3.4. *Let us assume that the moments $f_s^{(t)}$ satisfy the linear dependence (2.3) and (2.4). Then, the moments $f_s^{(t)}$ can be expressed using λ_1, λ_2, …, λ_m as, for $\ell,t=0,1,\dots$,*

$$f^{(t)}_{M\ell+j} = \sum_{i=1}^m c_i^{(j)}\rho_i^{(t)}\lambda_i^\ell, \quad j=0,1,\dots,M-1, \tag{2.29}$$

where $c_1^{(j)},c_2^{(j)},\dots,c_m^{(j)}$ are constants given using $f_0^{(0)},f_1^{(0)},\dots,f_{Mm-1}^{(0)}$ by

$$\begin{pmatrix} c_1^{(j)} \\ c_2^{(j)} \\ \vdots \\ c_m^{(j)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix}^{-1} \begin{pmatrix} f_j^{(0)} \\ f_{j+M}^{(0)} \\ \vdots \\ f_{j+(m-1)M}^{(0)} \end{pmatrix},$$

$$j=0,1,\dots,M-1, \tag{2.30}$$

and $\rho_1^{(t)}$, $\rho_2^{(t)}$, ..., $\rho_m^{(t)}$ *are constants such that*

$$\begin{aligned} &\rho_1^{(0)} := 1, \quad \rho_2^{(0)} := 1, \quad \ldots, \quad \rho_m^{(0)} := 1, \\ &\rho_i^{(t)} = (\lambda_i - \mu^{(0)})(\lambda_i - \mu^{(1)}) \cdots (\lambda_i - \mu^{(t-1)}), \quad i = 1, 2, \ldots, m, \quad t = 1, 2 \ldots. \end{aligned} \tag{2.31}$$

Proof. Using (2.29), we can rewrite the left-hand side of the linear dependence (2.3) with $s \mapsto M\ell + j$ as

$$f_{M(m+\ell)+j}^{(t)} + \sum_{i=1}^{m} a_i f_{M(m+\ell-i)+j}^{(t)} = \sum_{i=1}^{m} c_i^{(j)} \rho_i^{(t)} \left(\lambda_i^{\ell+m} + \sum_{j=1}^{m} a_j \lambda_i^{\ell+m-j} \right). \tag{2.32}$$

From (2.2), it is obvious that

$$\lambda_i^{\ell+m} + \sum_{j=1}^{m} a_j \lambda_i^{\ell+m-j} = \lambda_i^{\ell} p(\lambda_i). \tag{2.33}$$

Combining (2.32) and (2.33) with $p(\lambda_i) = 0$, we thus see that the moments $f_s^{(t)}$ in (2.29) satisfy the linear dependence (2.3). Setting $\ell = 0, 1, \ldots, m-1$ in (2.29) with $t = 0$ and considering $\rho_i^{(0)} := 1$, we derive

$$\begin{pmatrix} f_j^{(0)} \\ f_{j+M}^{(0)} \\ \vdots \\ f_{j+(m-1)M}^{(0)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix} \begin{pmatrix} c_1^{(j)} \\ c_2^{(j)} \\ \vdots \\ c_m^{(j)} \end{pmatrix}, \quad j = 0, 1, \ldots, M-1. \tag{2.34}$$

On the right-hand side of (2.34), the coefficient matrix involving distinct λ_1, λ_2, ..., λ_m is nonsingular because the coefficient matrix is the Vandermonde matrix. Therefore, by considering the inverse of the coefficient matrix in (2.34), we have (2.30). From (2.4) and (2.29), we easily obtain

$$\rho_i^{(t+1)} = (\lambda_i - \mu^{(t)}) \rho_i^{(t)}, \quad i = 1, 2, \ldots, m, \tag{2.35}$$

which immediately leads to (2.31). □

Proposition 2.3.4 claims that the moments $f_s^{(t)}$ have $(M+1)m$ arbitrary constants λ_1, λ_2, ..., λ_m and $c_1^{(j)}, c_2^{(j)}, \ldots, c_m^{(j)}$ for $j = 0, 1, \ldots, M-1$. The general term of the moments sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ involving λ_1, λ_2, ..., λ_m and $c_1^{(j)}$, $c_2^{(j)}$, ..., $c_m^{(j)}$ varies from previously used.

With the help of Proposition 2.3.4, we also obtain the following proposition concerning expansions of the extended Hankel determinants $H_k^{(s,t)}$.

Proposition 2.3.5. *Let us assume the moments $f_s^{(t)}$ satisfy the linear dependences* (2.3) *and* (2.4)*. Then, for $\ell = 0, 1, \dots$ and $j = 0, 1, \dots, M-1$, the extended Hankel determinants $H_k^{(s,t)}$ can be expanded as,*

$$H_k^{(M\ell+j,t)} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} K_{i_1,i_2,\dots,i_k}^{(j)} (\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k})^\ell \rho_{i_1}^{(t)}\rho_{i_2}^{(t)}\dots\rho_{i_k}^{(t)}, \tag{2.36}$$

where

$$K_{i_1,i_2,\dots,i_k}^{(j)} := \begin{vmatrix} \tilde{c}_{i_1}^{(j)} & \tilde{c}_{i_2}^{(j)} & \cdots & \tilde{c}_{i_k}^{(j)} \\ \tilde{c}_{i_1}^{(j+1)} & \tilde{c}_{i_2}^{(j+1)} & \cdots & \tilde{c}_{i_k}^{(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{i_1}^{(j+k-1)} & \tilde{c}_{i_2}^{(j+k-1)} & \cdots & \tilde{c}_{i_k}^{(j+k-1)} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1}^{k-1} & \lambda_{i_2}^{k-1} & \cdots & \lambda_{i_k}^{k-1} \end{vmatrix}, \tag{2.37}$$

$$\tilde{c}_i^{(M\ell+j)} := \lambda_i^\ell c_i^{(j)}, \quad i = 1, 2, \dots, m, \quad \ell = 0, 1, \dots. \tag{2.38}$$

Proof. We first investigate the case where $M > k-1$. Combining Proposition 2.3.4 with (2.20), we can express the extended Hankel determinants $H_k^{(M\ell+j,t)}$ as

$$H_k^{(M\ell+j,t)} = \begin{vmatrix} \sum_{i=1}^m c_i^{(j)}\rho_i^{(t)}\lambda_i^\ell & \sum_{i=1}^m c_i^{(j+1)}\rho_i^{(t)}\lambda_i^\ell & \cdots & \sum_{i=1}^m c_i^{(j+k-1)}\rho_i^{(t)}\lambda_i^\ell \\ \sum_{i=1}^m c_i^{(j)}\rho_i^{(t)}\lambda_i^{\ell+1} & \sum_{i=1}^m c_i^{(j+1)}\rho_i^{(t)}\lambda_i^{\ell+1} & \cdots & \sum_{i=1}^m c_i^{(j+k-1)}\rho_i^{(t)}\lambda_i^{\ell+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^m c_i^{(j)}\rho_i^{(t)}\lambda_i^{\ell+k-1} & \sum_{i=1}^m c_i^{(j+1)}\rho_i^{(t)}\lambda_i^{\ell+k-1} & \cdots & \sum_{i=1}^m c_i^{(j+k-1)}\rho_i^{(t)}\lambda_i^{\ell+k-1} \end{vmatrix}.$$

Applying the Cauchy-Binet formula to the determinants and using the elementary transformations of the determinants , we derive

$$H_k^{(M\ell+j,t)} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} (\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k})^\ell \rho_{i_1}^{(t)}\rho_{i_2}^{(t)}\dots\rho_{i_k}^{(t)} \times \begin{vmatrix} c_{i_1}^{(j)} & c_{i_2}^{(j)} & \cdots & c_{i_k}^{(j)} \\ c_{i_1}^{(j+1)} & c_{i_2}^{(j+1)} & \cdots & c_{i_k}^{(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_1}^{(j+k-1)} & c_{i_2}^{(j+k-1)} & \cdots & c_{i_k}^{(j+k-1)} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1}^{k-1} & \lambda_{i_2}^{k-1} & \cdots & \lambda_{i_k}^{k-1} \end{vmatrix}. \tag{2.39}$$

Thus, we have (2.36) and (2.37).

For the case where $M \leq k-1$, the proof is given similarly to the case where $M > k-1$. □

With the help of Proposition 2.3.5, we can easily derive asymptotic expansions as $t \to \infty$ of the extended Hankel determinants $H_k^{(s,t)}$.

Lemma 2.3.6. *Let us assume that $\lambda_1, \lambda_2, \dots, \lambda_m$ are constants such that $|\lambda_1 - \mu^{(t)}| > |\lambda_2 - \mu^{(t)}| > \cdots > |\lambda_m - \mu^{(t)}|$. Then, for $k = 1, 2, \dots, m$, it holds that*

$$H_k^{(M\ell+j,t)} = K_{1,2,\dots,k}^{(j)} (\lambda_1 \lambda_2 \cdots \lambda_k)^\ell \rho_1^{(t)} \rho_2^{(t)} \cdots \rho_k^{(t)} \left(1 + O(\varrho_k^t)\right), \quad t \to \infty, \tag{2.40}$$

where ϱ_k are constants satisfying $\varrho_k < |\lambda_{k+1} - \mu^{(t)}|/|\lambda_k - \mu^{(t)}|$.

Proof. Observing the expansion of the extended Hankel determinants $H_k^{(M\ell+j,t)}$ appearing in Proposition 2.3.5, we easily see that the dominant term as $t \to \infty$ is $K_{1,2,\dots,k}^{(j)} (\lambda_1 \lambda_2 \cdots \lambda_k)^\ell \rho_1^{(t)} \rho_2^{(t)} \cdots \rho_k^{(t)}$ if it holds

$$|\lambda_1 - \mu^{(t)}| > |\lambda_2 - \mu^{(t)}| > \cdots > |\lambda_m - \mu^{(t)}|. \tag{2.41}$$

Thus, we immediately have (2.40). □

Chapter 3

Discrete analysis for the autonomous discrete hungry integrable systems

The dhToda_I and dhToda_II equations are respectively written with the involvement of an arbitrary integer M as

$$\begin{cases} q_k^{(s+M)} + e_{k-1}^{(s+1)} = q_k^{(s)} + e_k^{(s)}, \quad k = 1, 2, \ldots, m, \quad s = 0, 1, \ldots, \\ e_k^{(s+1)} q_k^{(s+M)} = q_{k+1}^{(s)} e_k^{(s)}, \quad k = 1, 2, \ldots, m-1, \quad s = 0, 1, \ldots, \\ e_0^{(s)} = 0, \quad e_m^{(s)} = 0, \quad s = 0, 1, \ldots, \end{cases} \tag{3.1}$$

and

$$\begin{cases} q_k^{(s+1)} + e_{k-1}^{(s+M)} = q_k^{(s)} + e_k^{(s)}, \quad k = 1, 2, \ldots, m, \quad s = 0, 1, \ldots, \\ e_k^{(s+M)} q_k^{(s+1)} = q_{k+1}^{(s)} e_k^{(s)}, \quad k = 1, 2, \ldots, m-1, \quad s = 0, 1, \ldots, \\ e_0^{(s)} = 0, \quad e_m^{(s)} = 0, \quad s = 0, 1, \ldots, \end{cases} \tag{3.2}$$

where the subscripts and superscripts with parentheses denote discrete-spatial and discrete-time variables, respectively. Both the dhToda equations with $M = 1$ become the simple dToda equation. However, the dhToda_II equation has not been examined yet from the viewpoint of discrete integrable systems.

Both the dhToda equations have been shown to be applicable to the computation of eigenvalues of TN matrices, where all the minors are nonnegative [9, 33]. It is well-known that symmetric positive-definite tri-diagonal matrices are specializations of TN matrices. Thus, the qd algorithm can be considered to be a specialization of two algorithms designed from the dhToda equations. We hereinafter focus only on the dhToda_II equation (3.2). The dhToda_II equation (3.2) admits a matrix representation called the Lax representation,

$$L^{(s+M)} R^{(s+1)} = R^{(s)} L^{(s)}, \quad s = 0, 1, \ldots, \tag{3.3}$$

where $L^{(s)}$ and $R^{(s)}$ are bidiagonal matrices given using the dhToda$_{\text{II}}$ variables $q_k^{(s)}$ and $e_k^{(s)}$ as

$$L^{(s)} := \begin{pmatrix} 1 & & & & \\ e_1^{(s)} & 1 & & & \\ & e_2^{(s)} & \ddots & & \\ & & \ddots & 1 & \\ & & & e_{m-1}^{(s)} & 1 \end{pmatrix}, \quad R^{(s)} := \begin{pmatrix} q_1^{(s)} & 1 & & \\ & q_2^{(s)} & \ddots & \\ & & \ddots & 1 \\ & & & q_m^{(s)} \end{pmatrix}. \tag{3.4}$$

The Lax representation (3.3) suggests that the dhToda$_{\text{II}}$ equation (3.2) provides a similarity transformation from $A^{(s)} := L^{(s)}L^{(s+1)}\cdots L^{(s+M-1)}R^{(s)}$ to $A^{(s+1)}$. This is because the Lax representation (3.3) immediately leads to $A^{(s+1)} = (L^{(s)})^{-1}A^{(s)}L^{(s)}$. Furthermore, it is shown that $q_k^{(s)}$ and $e_k^{(s)}$ with initial settings $q_k^{(0)} > 0$ and $e_k^{(0)} > 0$, $e_k^{(1)} > 0, \ldots, e_k^{(M-1)} > 0$ converge to some positive constants and zeros, respectively, as $s \to \infty$. If $q_k^{(0)} > 0$ and $e_k^{(0)} > 0$, $e_k^{(1)} > 0, \ldots, e_k^{(M-1)} > 0$, then it is obvious that $L^{(0)}, L^{(1)}, \ldots, L^{(M-1)}, R^{(0)}$ and their products are TN matrices [29]. Thus, eigenvalues of $\lim_{s\to\infty} A^{(s)}$ coincide with those of $A^{(0)}$, and $\lim_{s\to\infty} A^{(s)}$ has the diagonal form as $\text{diag}(\lim_{s\to\infty} q_1^{(s)}, \lim_{s\to\infty} q_2^{(s)}, \ldots, \lim_{s\to\infty} q_m^{(s)})$ if $A^{(0)}$ is a TN matrix. In other words, $\lim_{s\to\infty} q_1^{(s)}$, $\lim_{s\to\infty} q_2^{(s)}, \ldots$, and $\lim_{s\to\infty} q_m^{(s)}$ are eigenvalues of the TN matrix $A^{(0)}$. This convergence theorem covers only the case where $A^{(0)}$ is a TN matrix. However, the dhToda$_{\text{II}}$ equation can generate a sequence of similarity transformations of $A^{(0)}$ even if $A^{(0)}$ is not TN. In this chapter we therefore develop an unlimited asymptotic analysis of the dhToda$_{\text{II}}$ equation as $s \to \infty$, after presenting the general solution and a Lax pair of dhToda$_{\text{II}}$ equation (3.2).

3.1 Determinant solution to the discrete hungry Toda equation

In this section, by focusing on an moment sequence and its associated determinants, we derive the determinant solution to the dhToda$_{\text{II}}$ equation (3.2).

The following proposition gives identities for the extended Hankel determinants.

Proposition 3.1.1 ([37, Proposition 4]). *The extended Hankel determinants $H_k^{(s,t)}$ associated with the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ satisfy*

$$H_k^{(s+M+1,t)}H_k^{(s,t)} - H_{k-1}^{(s+M+1,t)}H_{k+1}^{(s,t)} = H_k^{(s+M,t)}H_k^{(s+1,t)}, \tag{3.5}$$
$$k = 0, 1, \ldots, \quad s, t = 0, 1, \ldots.$$

With the help of Proposition 3.1.1 and Lemma 2.3.1, we thus have an expression for the solution to the dhToda$_{\mathrm{II}}$ equation (3.2) using the extended Hankel determinants $H_k^{(s,t)}$ as follows.

Theorem 3.1.2. *Let us assume that the moments sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *with initial settings of* $f_0^{(t)}$, $f_1^{(t)}$, $\ldots$, $f_{Mm-1}^{(t)}$ *satisfiy the linear relation* (2.3) *and the extended Hankel determinants* $H_k^{(s,t)}$ *satisfy*

$$H_k^{(s,t)} \neq 0, \quad k = 1, 2, \ldots, m, \quad s, t = 0, 1, \ldots. \tag{3.6}$$

Then, the dhToda$_{II}$ variables $q_k^{(s,t)}$ *and* $e_k^{(s,t)}$ *can be expressed as*

$$q_k^{(s,t)} := \frac{H_{k-1}^{(s,t)} H_k^{(s+M,t)}}{H_k^{(s,t)} H_{k-1}^{(s+M,t)}}, \quad k = 1, 2, \ldots, m, \quad s, t = 0, 1, \ldots, \tag{3.7}$$

$$e_k^{(s,t)} := \frac{H_{k+1}^{(s,t)} H_{k-1}^{(s+1,t)}}{H_k^{(s,t)} H_k^{(s+1,t)}}, \quad k = 0, 1, \ldots, m, \quad s, t = 0, 1, \ldots. \tag{3.8}$$

Proof. Using (3.7) and (3.8), we can rewrite $(q_k^{(s+1,t)} + e_{k-1}^{(s+M,t)}) - (q_k^{(s,t)} + e_k^{(s,t)})$ as

$$\begin{aligned} &\frac{H_{k-1}^{(s+1,t)}}{H_k^{(s+1,t)} H_{k-1}^{(s+M+1,t)} H_k^{(s,t)}} \left(H_k^{(s+M+1,t)} H_k^{(s,t)} - H_{k-1}^{(s+M+1,t)} H_{k+1}^{(s,t)} \right) \\ &\quad - \frac{H_k^{(s+M,t)}}{H_{k-1}^{(s+M,t)} H_{k-1}^{(s+M+1,t)} H_k^{(s,t)}} \left(H_{k-1}^{(s+M+1,t)} H_{k-1}^{(s,t)} - H_{k-2}^{(s+M+1,t)} H_k^{(s,t)} \right). \end{aligned} \tag{3.9}$$

From Proposition 3.1.1, we can see that (3.9) becomes 0 which implies that $q_k^{(s,t)}$ in (3.7) and $e_k^{(s,t)}$ in (3.8) satisfy

$$q_k^{(s+1,t)} + e_{k-1}^{(s+M,t)} = q_k^{(s,t)} + e_k^{(s,t)}. \tag{3.10}$$

We can also easily check

$$e_k^{(s+M,t)} q_k^{(s+1,t)} = q_{k+1}^{(s,t)} e_k^{(s,t)} \tag{3.11}$$

For any fixed t, equations (3.10) and (3.11) respectively coincide with the 1st equation and the 2nd equation of the dhToda$_{\mathrm{II}}$ equation (3.2). Since $H_{-1}^{(s,t)} = 0$ and $H_0^{(s,t)} = 1$, it is obvious from (3.8) with $k = 0$ that $e_0^{(s,t)} = H_1^{(s,t)} H_{-1}^{(s+1,t)} / (H_0^{(s,t)} H_0^{(s+1,t)}) = 0$. Similarly, by considering (3.8) with $k = m$, we can observe that $e_m^{(s,t)} = H_{m+1}^{(s,t)} H_{m-1}^{(s+1,t)} / (H_m^{(s,t)} H_m^{(s+1,t)}) = 0$. □

At each t, to obtain $(2m-1)$ infinite sequences of $\{q_1^{(s,t)}\}_{s=0}^{\infty}$, $\{q_2^{(s,t)}\}_{s=0}^{\infty}$, $\ldots$, $\{q_m^{(s,t)}\}_{s=0}^{\infty}$ and $\{e_1^{(s,t)}\}_{s=0}^{\infty}, \{e_2^{(s,t)}\}_{s=0}^{\infty}, \ldots, \{e_{m-1}^{(s,t)}\}_{s=0}^{\infty}$, the dhToda$_{\mathrm{II}}$ equation (3.2) requires initial settings of $q_1^{(0,t)}$, $q_2^{(0,t)}$, $\ldots$, $q_m^{(0,t)}$ and $\{e_1^{(s,t)}\}_{s=0}^{M-1}$, $\{e_2^{(s,t)}\}_{s=0}^{M-1}$, $\ldots$, $\{e_{m-1}^{(s,t)}\}_{s=0}^{M-1}$. On the other hand, the linear difference equations (2.3) with arbitrary constants a_1, a_2, $\ldots$, a_m and initial settings of $f_0^{(t)}$, $f_1^{(t)}$, $\ldots$, $f_{Mm-1}^{(t)}$ uniquely generates the moment sequence $\{f_s^{(t)}\}_{s=0}^{\infty}$ whose elements appear in the extended Hankel determinants $H_k^{(s,t)}$. The arbitrary degree of the moment sequence $\{f_s^{(t)}\}_{s=0}^{\infty}$ is thus larger than that of the $(2m-1)$ infinite sequences $\{q_1^{(s,t)}\}_{s=0}^{\infty}, \{q_2^{(s,t)}\}_{s=0}^{\infty}, \ldots, \{q_m^{(s,t)}\}_{s=0}^{\infty}$ and $\{e_1^{(s,t)}\}_{s=0}^{\infty}, \{e_2^{(s,t)}\}_{s=0}^{\infty}, \ldots, \{e_{m-1}^{(s,t)}\}_{s=0}^{\infty}$. Thus, Theorem 3.1.2 therefore covers the general solution to the dhToda$_{\mathrm{II}}$ equation (3.2).

3.2 The related eigenvalue problem

In this section, by examining determinant polynomials based on the extended Hankel determinants $H_k^{(s,t)}$, we associate the dhToda$_{\mathrm{II}}$ equation (3.2) with an eigenvalue problem of band matrices.

First, we define $H_k^{(s,t)}(z)$ as determinants obtained by replacing $\boldsymbol{f}_{k+1}^{(s+k,t)} := (f_{s+k}^{(t)}, f_{s+M+k}^{(t)}, \ldots, f_{s+(k+1)M)}^{(t)})^\top$ with $\boldsymbol{z}_k := (1, z, \ldots, z^k)^\top$ in the $(k+1)$th row of the extended Hankel determinants $H_{k+1}^{(s,t)}$. The determinants $H_k^{(s,t)}(z)$ are obviously kth degree polynomials with respect to z. Furthermore, we introduce the kth degree monic polynomials

$$\begin{cases} \mathcal{H}_{-1}^{(s,t)}(z) := 0, \quad \mathcal{H}_0^{(s,t)}(z) := 1, \quad s,t = 0,1,\ldots, \\ \mathcal{H}_k^{(s,t)}(z) := \dfrac{H_k^{(s,t)}(z)}{H_k^{(s,t)}}, \quad k = 1,2,\ldots,m, \quad s,t = 0,1,\ldots. \end{cases} \tag{3.12}$$

Since $\mathcal{H}_k^{(s,t)}(z)$ with $M = 1$ are just the simple Hadamard polynomials [14], we hereinafter refer to $\mathcal{H}_k^{(s,t)}(z)$ as the extended Hadamard polynomials. The following proposition gives the relationships of elements of the infinite polynomial sequence $\{\mathcal{H}_k^{(s,t)}(z)\}_{s,t=0}^{\infty}$.

Proposition 3.2.1. *Let us assume that the moment sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *with initial settings of* $f_0^{(t)}, f_1^{(t)}, \ldots, f_{Mm-1}^{(t)}$ *satisfies the linear difference equations* (2.3) *and the extended Hankel determinants* $H_k^{(s,t)}$ *are always nonzero. Then, the extended Hadamard polynomials* $\mathcal{H}_k^{(s,t)}(z)$ *satisfy*

$$z\mathcal{H}_k^{(s+M,t)}(z) = \mathcal{H}_{k+1}^{(s,t)}(z) + q_{k+1}^{(s,t)}\mathcal{H}_k^{(s,t)}(z), \quad k = 0,1,\ldots,m-1, \tag{3.13}$$

$$\mathcal{H}_k^{(s,t)}(z) = \mathcal{H}_k^{(s+1,t)}(z) + e_k^{(s,t)}\mathcal{H}_{k-1}^{(s+1,t)}(z), \quad k = 0,1,\ldots,m. \tag{3.14}$$

Proof. It is shown in [17] that, for any determinant D,

$$DD\begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} = D\begin{bmatrix} i_1 \\ j_1 \end{bmatrix} D\begin{bmatrix} i_2 \\ j_2 \end{bmatrix} - D\begin{bmatrix} i_1 \\ j_2 \end{bmatrix} D\begin{bmatrix} i_2 \\ j_1 \end{bmatrix}, \tag{3.15}$$

where $D\begin{bmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{bmatrix}$ denotes the determinant obtained from D by deleting the i_1th, the i_2th,...,the i_nth rows and the j_1th, the j_2th,..., the j_nth columns. Equation (3.15) is called the Jacobi's identities. Considering Jacobi's identities (3.15) with $i_1 = j_1 = k+1, i_2 = 1, j_2 = k$ for the polynomials $H_k^{(s,t)}(z)$, we can derive

$$zH_k^{(s,t)} H_{k-1}^{(s+M,t)}(z) = H_k^{(s+M,t)} H_{k-1}^{(s,t)}(z) + H_{k-1}^{(s+M,t)} H_k^{(s,t)}(z), \quad k = 1, 2, \dots, m. \tag{3.16}$$

By dividing the both side of (3.16) by $H_k^{(s,t)} H_{k-1}^{(s+M,t)}$, we immediately obtain (3.13).

For $(k+1)$-by-$(k+3)$ matrices $X_k := (\boldsymbol{x}_1\ \boldsymbol{x}_2\ \cdots\ \boldsymbol{x}_{k+3})$ with $(k+1)$-dimensional vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{k+3}$, it is shown in reference [12] that determinants of the submatrices satisfy

$$\begin{aligned} &\left|\boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \cdots\ \ \boldsymbol{x}_{k-1}\ \ \boldsymbol{x}_k\ \ \boldsymbol{x}_{k+1}\right| \left|\boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \cdots\ \ \boldsymbol{x}_{k-1}\ \ \boldsymbol{x}_{k+2}\ \ \boldsymbol{x}_{k+3}\right| \\ &- \left|\boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \cdots\ \ \boldsymbol{x}_{k-1}\ \ \boldsymbol{x}_k\ \ \boldsymbol{x}_{k+2}\right| \left|\boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \cdots\ \ \boldsymbol{x}_{k-1}\ \ \boldsymbol{x}_{k+1}\ \ \boldsymbol{x}_{k+3}\right| \\ &+ \left|\boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \cdots\ \ \boldsymbol{x}_{k-1}\ \ \boldsymbol{x}_k\ \ \boldsymbol{x}_{k+3}\right| \left|\boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \cdots\ \ \boldsymbol{x}_{k-1}\ \ \boldsymbol{x}_{k+1}\ \ \boldsymbol{x}_{k+2}\right| = 0. \end{aligned} \tag{3.17}$$

Equation (3.17) is the so-called Plüker relations for X_k. The replacements $\boldsymbol{x}_1 = \boldsymbol{f}_{k+1}^{(s+1,t)}$, $\boldsymbol{x}_2 = \boldsymbol{f}_{k+1}^{(s+2,t)}$, ..., $\boldsymbol{x}_k = \boldsymbol{f}_{k+1}^{(s+k-1,t)}$, $\boldsymbol{x}_{k+1} = \boldsymbol{f}_{k+1}^{(s,t)}$, $\boldsymbol{x}_{k+2} = \boldsymbol{z}_k$ and $\boldsymbol{x}_{k+3} = \boldsymbol{e}_{k+1} := (0, 0, \dots, 0, 1)^\top$ in (3.17) yield

$$H_k^{(s+1,t)} H_k^{(s,t)}(z) = H_{k+1}^{(s,t)} H_{k-1}^{(s+1,t)}(z) + H_k^{(s,t)} H_k^{(s+1,t)}(z). \tag{3.18}$$

By dividing the both sides of (3.18) by $H_k^{(s,t)} H_k^{(s+1,t)}$, we thus have (3.14). □

According to the theory of orthogonal polynomials [5], we can recognize the identities (3.13) with $M = 1$ as the Christoffel transformations for the simple Hadamard polynomials. Since the identities (3.14) generate inverse discrete-time evolutions for the identities (3.13), we see that (3.14) with $M = 1$ are just the Geronimus transformations for the simple Hadamard polynomials. The Christoffel and Geronimus transformations are useful in the study of integrable discrete systems [44]. Of course, by considering compatibility condition (3.13) and (3.14) with fixed t, we can derive the dhToda$_{\mathrm{II}}$ equation (3.2).

In the case where $k = m$, the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$ have the following relationships to the characteristic polynomial of matrices with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$, namely, $p(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)$.

Lemma 3.2.2. *Let us assume that the moment sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *with initial settings of* $f_0^{(t)}$, $f_1^{(t)}$, $\ldots$, $f_{Mm-1}^{(t)}$ *satisfies the linear difference equations* (2.3) *and the extended Hankel determinants* $H_k^{(s,t)}$ *are always nonzero. Then, the extended Hadamard polynomials* $\mathcal{H}_k^{(s,t)}(z)$ *with* $k = m$ *satisfy*

$$\mathcal{H}_m^{(s,t)}(z) = p(z), \quad s, t = 0, 1, \ldots. \tag{3.19}$$

Proof. By multiplying the 1st, 2nd, $\ldots$, mth rows of the polynomial $H_m^{(s,t)}(z)$ by a_m, a_{m-1}, $\ldots$, a_1, respectively, and then adding these to the $(m+1)$th row, we can obtain

$$H_m^{(s,t)}(z) = \begin{vmatrix} \boldsymbol{f}_m^{(s)} & \boldsymbol{f}_m^{(s+1)} & \cdots & \boldsymbol{f}_m^{(s+m-1)} & \boldsymbol{z}_m \\ \sum_{i=0}^{m} a_i f_{s+M(m-i)}^{(t)} & \sum_{i=0}^{m} a_i f_{s+M(m-i)+1}^{(t)} & \cdots & \sum_{i=0}^{m} a_i f_{s+M(m-i)+m-1}^{(t)} & \sum_{i=0}^{m} a_i z^{m-i} \end{vmatrix},$$

where $\boldsymbol{f}_m^{(s+j-1,t)} := (f_{s+j-1}^{(t)}, f_{s+M+j-1}^{(t)}, \ldots, f_{s+M(m-1)+j-1}^{(t)})^\top$ for $j = 1, 2, \ldots, m$. From the characteristics polynomial $p(z)$ in (2.2) and the difference equations (2.3), we see that the $(m+1, 1)$, $(m+1, 2)$, $\ldots$, $(m+1, m)$ entries become 0 and the $(m+1, m+1)$ entry is just $p(z)$. Thus, it follows that $H_m^{(s,t)}(z) = H_m^{(s,t)} p(z)$. By combining this with $\mathcal{H}_m^{(s,t)}(z) = H_m^{(s,t)}(z)/H_m^{(s,t)}$, we therefore have (3.19). □

Let us introduce m-dimensional column vectors

$$\mathcal{H}_i^{(s,t)} := \begin{pmatrix} \mathcal{H}_0^{(s,t)}(\lambda_i) \\ \mathcal{H}_1^{(s,t)}(\lambda_i) \\ \vdots \\ \mathcal{H}_{m-1}^{(s,t)}(\lambda_i) \end{pmatrix}, \quad i = 1, 2, \ldots, m, \quad s, t = 0, 1, \ldots. \tag{3.20}$$

Furthermore, for $s, t = 0, 1, \ldots$, let us introduce m-by-m bidiagonal matrices involving the non-autonomous dhToda variables $e_1^{(s,t)}$, $e_2^{(s,t)}$, $\ldots$, $e_{m-1}^{(s,t)}$ and $q_1^{(s,t)}$, $q_2^{(s,t)}$, $\ldots$, $q_m^{(s,t)}$,

$$L^{(s,t)} := \begin{pmatrix} 1 & & & \\ e_1^{(s,t)} & 1 & & \\ & \ddots & \ddots & \\ & & e_{m-1}^{(s,t)} & 1 \end{pmatrix}, \quad R^{(s,t)} := \begin{pmatrix} q_1^{(s,t)} & 1 & & \\ & q_2^{(s,t)} & \ddots & \\ & & \ddots & 1 \\ & & & q_m^{(s,t)} \end{pmatrix}. \tag{3.21}$$

Using the bidiagonal matrices $L^{(s,t)}$ and $R^{(s,t)}$ with the dhToda$_{\mathrm{II}}$ variables $q_k^{(s,t)}$ and $e_k^{(s,t)}$, we present the following theorem concerning a Lax pair for the dhToda$_{\mathrm{II}}$ equation (3.2).

Theorem 3.2.3. *Let us assume that the moment sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *satisfies the linear equations* (2.3) *and the extended Hankel determinants* $H_k^{(s,t)}$ *are always nonzero. Then, for m-dimensional vectors* $\mathcal{H}_i^{(s,t)}$*, it holds that*

$$R^{(s,t)}\mathcal{H}_i^{(s,t)} = \lambda_i \mathcal{H}_i^{(s+M,t)}, \quad i = 1, 2, \ldots, m, \quad s, t = 0, 1, \ldots, \tag{3.22}$$

$$L^{(s,t)}\mathcal{H}_i^{(s+1,t)} = \mathcal{H}_i^{(s,t)}, \quad i = 1, 2, \ldots, m, \quad s, t = 0, 1, \ldots. \tag{3.23}$$

Proof. We can easily check that the identities in the matrix entries on both sides of (3.22) and (3.23) coincide with those in Proposition 3.2.1 with $z = \lambda_i$. □

From the viewpoint of the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$, we then obtain a proposition concerning the eigenpairs of $A^{(s,t)} := L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1,t)} R^{(s,t)}$.

Proposition 3.2.4. *Eigenvalues of* $A^{(s,t)}$ *coincide with the roots* λ_i *of the polynomial* $p(z)$*. Furthermore, eigenvectors corresponding to* λ_i *are* $\mathcal{H}_i^{(s,t)}$*. Namely,*

$$A^{(s,t)}\mathcal{H}_i^{(s,t)} = \lambda_i \mathcal{H}_i^{(s,t)}, \quad i = 1, 2, \ldots, m. \tag{3.24}$$

Proof. From the identities (3.13) with $z = \lambda_i$, it follows that

$$\begin{cases} \lambda_i \mathcal{H}_0^{(s+M,t)}(\lambda_i) = \mathcal{H}_1^{(s,t)}(\lambda_i) + q_1^{(s,t)}\mathcal{H}_0^{(s,t)}(\lambda_i), \\ \lambda_i \mathcal{H}_1^{(s+M,t)}(\lambda_i) = \mathcal{H}_2^{(s,t)}(\lambda_i) + q_2^{(s,t)}\mathcal{H}_1^{(s,t)}(\lambda_i), \\ \quad \vdots \\ \lambda_i \mathcal{H}_{m-1}^{(s+M,t)}(\lambda_i) = \mathcal{H}_m^{(s,t)}(\lambda_i) + q_m^{(s,t)}\mathcal{H}_{m-1}^{(s,t)}(\lambda_i). \end{cases} \tag{3.25}$$

Since $\mathcal{H}_m^{(s,t)}(\lambda_i) = 0$, we can express (3.25) using the upper bidiagonal matrix $R^{(s,t)}$ as

$$\lambda_i \mathcal{H}_i^{(s+M,t)} = R^{(s,t)}\mathcal{H}_i^{(s,t)}, \quad i = 1, 2, \ldots, m. \tag{3.26}$$

Similarly, the identities (3.14) with $z = \lambda_i$ lead to

$$\mathcal{H}_i^{(s,t)} = L^{(s,t)}\mathcal{H}_i^{(s+1,t)}, \quad i = 1, 2, \ldots, m. \tag{3.27}$$

Multiplying both sides of (3.26) by $L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1,t)}$ from the left, we obtain

$$\lambda_i L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1,t)}\mathcal{H}_i^{(s+M,t)} = A^{(s,t)}\mathcal{H}_i^{(s,t)}, \quad i = 1, 2, \ldots, m. \tag{3.28}$$

Using (3.27) repeatedly, we can rewrite the left-hand side of (3.28) as $\lambda_i \mathcal{H}_i^{(s,t)}$. Thus, we have (3.24). □

With the help of (3.26) and (3.27) in the proof of Proposition 3.2.4, we derive a proposition for the two types of similarity transformations of $A^{(s,t)}$.

Proposition 3.2.5. *The similarity transformations for* $A^{(s,t)}$ *with respect to* s *are given as*

$$A^{(s+M,t)} = R^{(s,t)}A^{(s,t)}(R^{(s,t)})^{-1}, \tag{3.29}$$

$$A^{(s+1,t)} = (L^{(s,t)})^{-1}A^{(s,t)}L^{(s,t)}. \tag{3.30}$$

Proof. For $s, t = 0, 1, \ldots,$ by multiplying both sides of (3.27) by $R^{(s,t)}$ from the left and considering (3.26) and (3.27) on the left-hand side, we obtain

$$\lambda_i L^{(s+M,t)}\mathcal{H}_i^{(s+M+1,t)} = R^{(s,t)}L^{(s,t)}\mathcal{H}_i^{(s+1,t)}, \quad i = 1, 2, \ldots, m. \tag{3.31}$$

Using (3.26) again, we can rewrite the left-hand side of (3.31) as $L^{(s+M,t)}R^{(s+1,t)}$ $\mathcal{H}_i^{(s+1,t)}$. Recalling that $\mathcal{H}_i^{(s+1,t)}$ are eigenvector corresponding to λ_i, we see that $\mathcal{H}_i^{(s+1,t)} \neq 0$. Thus, it follows that

$$L^{(s+M,t)}R^{(s+1,t)} = R^{(s,t)}L^{(s,t)}, \quad s, t = 0, 1, \ldots. \tag{3.32}$$

Using (3.32) repeatedly, we can gradually transform $A^{(s+M,t)}$ as follows

$$\begin{aligned}
A^{(s+M,t)} &= L^{(s+M,t)}L^{(s+M+1,t)} \cdots L^{(s+2M-1,t)}R^{(s+M,t)} \\
&= L^{(s+M,t)}L^{(s+M+1,t)} \cdots L^{(s+2M-2,t)}R^{(s+M-1,t)}L^{(s+M-1,t)} \\
&\;\;\vdots \\
&= R^{(s,t)}L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1,t)} \\
&= R^{(s,t)}L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1,t)}R^{(s,t)}(R^{(s,t)})^{-1}.
\end{aligned}$$

Therefore, by noting that $L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1,t)}R^{(s,t)} = A^{(s,t)}$, we have (3.29).

Similarly, by using (3.32) for $A^{(s+1,t)}$, we obtain

$$A^{(s+1,t)} = (L^{(s,t)})^{-1}L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1)}R^{(s,t)}L^{(s,t)}. \tag{3.33}$$

Applying (3.33) to $A^{(s,t)} = L^{(s,t)}L^{(s+1,t)} \cdots L^{(s+M-1,t)}R^{(s,t)}$, we thus derive (3.30). □

3.3 Initial settings

In this section, by relating the initial settings of the dhToda$_{\text{II}}$ equation (3.2) to the entries of the band matrix $A^{(0,t)}$, we show that the determinant solution shown in

Section 3.1 is the general solution to the dhToda$_{\rm II}$ equation (3.2) when providing a sequence of similarity transformation of $A^{(0,t)}$.

Reconsidering Proposition 3.2.1, we can relate the extended Hadamard polynomials in sequences $\{\mathcal{H}_k^{(0,t)}(z)\}_{k=1}^m$, $\{\mathcal{H}_k^{(1,t)}(z)\}_{k=1}^{m-1}$, ..., $\{\mathcal{H}_k^{(M,t)}(z)\}_{k=1}^{m-1}$ to the band matrix $A^{(0,t)}$ as follows.

Lemma 3.3.1. *For a given band matrix $A^{(0,t)}$ consisting of nonzero $q_1^{(0,t)}$, $q_2^{(0,t)}$, ..., $q_m^{(0,t)}$ and nonzero sequences $\{e_1^{(s,t)}\}_{s=0}^{M-1}$, $\{e_2^{(s,t)}\}_{s=0}^{M-1}$, ..., $\{e_{m-1}^{(s,t)}\}_{s=0}^{M-1}$, the extended Hadamard polynomials in sequences $\{\mathcal{H}_k^{(0,t)}(z)\}_{k=1}^m$, $\{\mathcal{H}_k^{(1,t)}(z)\}_{k=1}^{m-1}$, ..., $\{\mathcal{H}_k^{(M,t)}(z)\}_{k=1}^{m-1}$ are uniquely determined.*

Proof. We may assume that $\{q_k^{(0,t)}\}_{k=1}^m$ and $\{e_k^{(0,t)}\}_{k=1}^{m-1}, \{e_k^{(1,t)}\}_{k=1}^{m-1}, \dots, \{e_k^{(M-1,t)}\}_{k=1}^{m-1}$ are given because they generate all the non-trivial entries of $A^{(0,t)}$. Let us recall here that $\mathcal{H}_0^{(s,t)}(z) := 1$ for $s, t = 0, 1, \dots$. Then, it is obvious from (3.13) with $k = 0$ and $s = 0$ that

$$\mathcal{H}_1^{(0,t)}(z) = z - q_1^{(0,t)}. \tag{3.34}$$

Furthermore, it follows from (3.14) with $k = 1$ and $s = 0, 1, \dots, M-1$ that

$$\mathcal{H}_1^{(s+1,t)}(z) = \mathcal{H}_1^{(s,t)}(z) - e_1^{(s,t)}. \tag{3.35}$$

Equations (3.34) and (3.35) imply that $\mathcal{H}_1^{(0,t)}(z)$, $\mathcal{H}_1^{(1,t)}(z)$, ..., $\mathcal{H}_1^{(M,t)}(z)$ are uniquely determined if $q_1^{(0,t)}$ and $e_1^{(0,t)}, e_1^{(1,t)}, \dots, e_1^{(M-1,t)}$ are given. Similarly, by observing the case where $k = 1, 2, \dots, m-1$ in (3.13) and $k = 2, 3, \dots, m-1$ in (3.14), we see that $\{\mathcal{H}_k^{(0,t)}(z)\}_{k=2}^m$, $\{\mathcal{H}_k^{(1,t)}(z)\}_{k=2}^{m-1}$, ..., $\{\mathcal{H}_k^{(M,t)}(z)\}_{k=2}^{m-1}$ are uniquely determined if $A^{(0,t)}$ is given. ☐

With the help of Lemma 3.3.1, we derive the following theorem for initial settings of the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$.

Theorem 3.3.2. *For a given band matrix $A^{(0,t)}$ consisting of nonzero $q_1^{(0,t)}$, $q_2^{(0,t)}$, ..., $q_m^{(0,t)}$ and nonzero sequences $\{e_1^{(s,t)}\}_{s=0}^{M-1}$, $\{e_2^{(s,t)}\}_{s=0}^{M-1}$, ..., $\{e_{m-1}^{(s,t)}\}_{s=0}^{M-1}$, initial settings of the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ with $f_0^{(t)} = 1$, $f_1^{(t)} = 1$, ..., $f_{M-1}^{(t)} = 1$ are uniquely determined.*

Proof. Let $f_0^{(t)} = 1, f_1^{(t)} = 1, \dots, f_{M-1}^{(t)} = 1$. Recalling the definition of the Hadamard polynomials $\mathcal{H}_1^{(s,t)}(z)$ in (3.12), we obtain

$$\mathcal{H}_1^{(s,t)}(z) = \begin{vmatrix} 1 & 1 \\ f_{s+M}^{(t)} & z \end{vmatrix}, \quad s = 0, 1, \dots, M-1. \tag{3.36}$$

From (3.36), it is obvious that $f_M^{(t)}$, $f_{M+1}^{(t)}$, ..., $f_{2M}^{(t)}$ are uniquely determined from $\mathcal{H}_1^{(0,t)}(z)$, $\mathcal{H}_1^{(1,t)}(z)$, ..., $\mathcal{H}_1^{(M,t)}(z)$, respectively. Let us assume that $f_0^{(t)}$, $f_1^{(t)}$, ..., $f_{(\ell+1)M+\ell-1}^{(t)}$ for some $\ell < m$ are uniquely given from the Hadamard polynomial sequences $\{\mathcal{H}_1^{(s,t)}(z)\}_{s=0}^M$, $\{\mathcal{H}_2^{(s,t)}(z)\}_{s=0}^M$, ..., $\{\mathcal{H}_\ell^{(s,t)}(z)\}_{s=0}^M$. Then, by observing the Hadamard polynomials in $\{\mathcal{H}_{\ell+1}^{(s,t)}(z)\}_{s=0}^M$, we can see that the elements of the initial sequence $\{f_s^{(t)}\}_{s=0}^{Mm-1}$ except for $f_0^{(t)}$, $f_1^{(t)}$, ..., $f_{(\ell+1)M+\ell-1}^{(t)}$ appear in only the $(\ell+2, \ell+1)$ entries in the numerators. Such entries are $f_{(\ell+1)M+\ell}^{(t)}$, $f_{(\ell+1)M+\ell+1}^{(t)}$, ..., $f_{(\ell+2)M+\ell}^{(t)}$, and are uniquely determined if $\{\mathcal{H}_{\ell+1}^{(s,t)}(z)\}_{s=0}^M$ is given. From Lemma 3.3.1, $\{\mathcal{H}_{\ell+1}^{(s,t)}(z)\}_{s=0}^M$ is also uniquely determined if $A^{(0,t)}$ is given. By induction for ℓ, we thus conclude that the initial sequence $\{f_s^{(t)}\}_{s=0}^{Mm-1}$ is uniquely governed corresponding to the band matrix $A^{(0,t)}$. □

Theorem 3.3.2 suggests that Theorem 3.1.2 with $f_0^{(t)} = 1$, $f_1^{(t)} = 1$, ..., $f_{M-1}^{(t)} = 1$ eventually gives the general solution to the dhToda$_{\mathrm{II}}$ equation (3.2) for generating similarity transformations of the band matrix $A^{(0,t)}$.

3.4 Asymptotic behavior

In this section, we present an asymptotic analysis as $s \to \infty$ of the dhToda$_{\mathrm{II}}$ equation (3.2) that generates similarity transformations of the band matrix $A^{(0,t)}$.

Combining Proposition 2.3.3 with Theorem 3.1.2, we obtain the following theorem concerning asymptotic convergence of the dhToda$_{\mathrm{II}}$ variables $q_k^{(s,t)}$ to eigenvalues λ_1, λ_2, ..., λ_m of $A^{(0,t)}$ as $s \to \infty$.

Theorem 3.4.1. *Let us assume that $\lambda_1, \lambda_2, \dots, \lambda_m$ are nonzero constants such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|$. At each t, if $q_1^{(0,t)}$, $q_2^{(0,t)}$, ..., $q_m^{(0,t)}$ are nonzero and $\{e_1^{(s,t)}\}_{s=0}^{M-1}$, $\{e_2^{(s,t)}\}_{s=0}^{M-1}$, ..., $\{e_{m-1}^{(s,t)}\}_{s=0}^{M-1}$ are nonzero sequences, and if $\{q_1^{(s,t)}\}_{s=0}^\infty$, $\{q_2^{(s,t)}\}_{s=0}^\infty$, ..., $\{q_m^{(s,t)}\}_{s=0}^\infty$ are nonzero sequnces, it then holds that*

$$\lim_{s\to\infty} q_k^{(s,t)} = \lambda_k, \quad k = 1, 2, \dots, m, \tag{3.37}$$

$$\lim_{s\to\infty} e_k^{(s,t)} = 0, \quad k = 1, 2, \dots, m-1. \tag{3.38}$$

Proof. From Theorem 3.1.2, it is obvious that

$$q_k^{(M\ell+j,t)} = \frac{H_{k-1}^{(M\ell+j,t)} H_k^{(M(\ell+1)+j,t)}}{H_k^{(M\ell+j,t)} H_{k-1}^{(M(\ell+1)+j,t)}}, \tag{3.39}$$

$$e_k^{(M\ell+j,t)} = \frac{H_{k+1}^{(M\ell+j,t)} H_{k-1}^{(M\ell+j+1,t)}}{H_k^{(M\ell+j,t)} H_k^{(M\ell+j+1,t)}}. \tag{3.40}$$

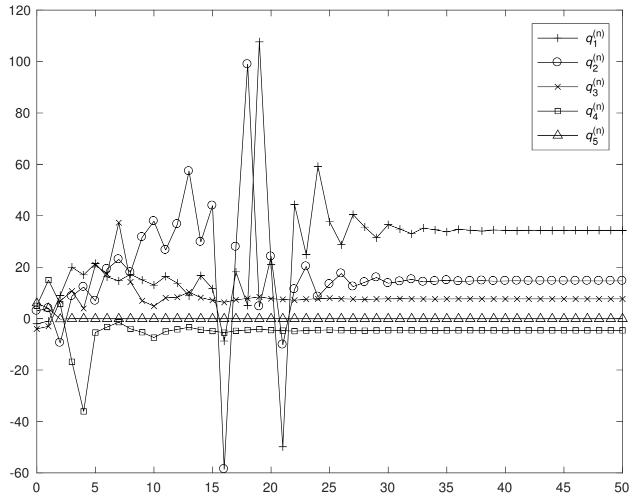

Figure 3.1: A graph of the discrete-time s (x-axis) and the values $q_1^{(s,t)}$, $q_2^{(s,t)}$, $q_3^{(s,t)}$, $q_4^{(s,t)}$ and $q_5^{(s,t)}$ (y-axis) in the dhToda$_{\text{II}}$ equation with $M = 3$ and $m = 5$ under the initial settings $q_1^{(0,t)} = -2$, $q_2^{(0,t)} = 3$, $q_3^{(0,t)} = -4$, $q_4^{(0,t)} = 5$, $q_5^{(0,t)} = 6$, $e_1^{(0,t)} = 1$, $e_2^{(0,t)} = -2$, $e_3^{(0,t)} = 3$, $e_4^{(0,,t)} = 5$, $e_1^{(1,t)} = 10$, $e_2^{(1,t)} = -9$, $e_3^{(1,t)} = 7$, $e_4^{(1,t)} = 6$, $e_1^{(2,t)} = 11$, $e_2^{(2,t)} = 13$, $e_3^{(2,t)} = 14$ and $e_4^{(2,t)} = -15$.

Applying Proposition 2.3.3 to (3.39) and (3.40), we thus have, as $\ell \to \infty$,

$$q_k^{(M\ell+j,t)} = \lambda_k \frac{(1 + O(\rho_k^\ell))(1 + O(\rho_k^{\ell+1}))}{(1 + O(\rho_k^\ell))(1 + O(\rho_k^{\ell+1}))}, \tag{3.41}$$

$$e_k^{(M\ell+j)} = \frac{O((\lambda_1\lambda_2 \cdots \lambda_{k+1})^\ell)O((\lambda_1\lambda_2 \cdots \lambda_{k-1})^\ell)}{O((\lambda_1\lambda_2 \cdots \lambda_k)^\ell)O((\lambda_1\lambda_2 \cdots \lambda_k)^\ell)}. \tag{3.42}$$

Equations (3.41) and (3.42) therefore immediately lead to (3.37) and (3.38), respectively. □

A convergence theorem similar to Theorem 3.4.1 was found in Sumikura et al. [33] without considering the solution to the dhToda$_{\text{II}}$ equation (3.2). Theorem 3.4.1 differs from the pervious theorem in that the initial settings of the dhToda$_{\text{II}}$ equation (3.2) and eigenvalues of the band matrix $A^{(0,t)}$ are not limited to be positive, i.e., $A^{(0,t)}$ is not required to be TN. The determinant solution plays a key role in providing an unlimited asymptotic analysis of the dhToda$_{\text{II}}$ equation (3.2).

We now give a numerical example to demonstrate asymptotic convergence of the dhToda$_{\rm II}$ variables to eigenvalues of not TN matrix. We used a computer with Mac OS Sierra (ver. 10.12.4) and CPU: 3.1 GHz Intel Core i7 CPU. We employed floating-point arithmetic in the numerical computing software MATLAB (ver. 9.1.0.441655 (R2016b)). Let us consider the case where $q_1^{(0,t)} = -2$, $q_2^{(0,t)} = 3$, $q_3^{(0,t)} = -4$, $q_4^{(0,t)} = 5$, $q_5^{(0,t)} = 6$, $e_1^{(0,t)} = 1$, $e_2^{(0,t)} = -2$, $e_3^{(0,t)} = 3$, $e_4^{(0,t)} = 5$, $e_1^{(1,t)} = 10$, $e_2^{(1,t)} = -9$, $e_3^{(1,t)} = 7$, $e_4^{(1,t)} = 6$, $e_1^{(2,t)} = 11$, $e_2^{(2,t)} = 13$, $e_3^{(2,t)} = 14$ and $e_4^{(2,t)} = -15$ in the dhToda$_{\rm II}$ equation (3.2) with $M = 3$ and $m = 5$. Then, it follows that

$$A^{(0,t)} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ -44 & 25 & 1 & 0 & 0 \\ 282 & -135 & -2 & 1 & \\ 594 & 12 & 7 & 29 & 1 \\ 0 & 1365 & -301 & 169 & 2 \end{pmatrix}.$$

It is clear that the band matrix $A^{(0,t)}$ is not TN matrix but a band matrix which can be decomposed into a product of lower and upper triangular factors, namely, an LR decomposable matrix. Figure 3.1 shows that $q_1^{(s,t)}$, $q_2^{(s,t)}$, $q_3^{(s,t)}$, $q_4^{(s,t)}$ and $q_5^{(s,t)}$ tend to λ_1, λ_2, λ_3, λ_4 and λ_5, respectively, as s grows larger. The relative gaps between the dhToda$_{\rm II}$ variables and eigenvalues of $A^{(0,t)}$ are $|(q_1^{(300,t)} - \lambda_1)/\lambda_1| = 1.242218385413229 \times 10^{-15}$, $|(q_2^{(300,t)} - \lambda_2)/\lambda_2| = 1.576637814285385 \times 10^{-15}$, $|(q_3^{(300,t)} - \lambda_3)/\lambda_3| = 5.799826421245403 \times 10^{-15}$, $|(q_4^{(300,t)} - \lambda_4)/\lambda_4| = 2.325793669966101 \times 10^{-15}$ and $|(q_5^{(300,t)} - \lambda_5)/\lambda_5| = 4.419587434294721 \times 10^{-14}$, where λ_1, λ_2, λ_3, λ_4 and λ_5 were computed using the Matlab function **eig** in 32-digits floating-point arithmetic.

Chapter 4

Discrete analysis for the non-autonomous discrete hungry integrable systems

In this chapter focuses only on the non-autonomous type of the $\mathrm{dhToda}_{\mathrm{II}}$ equation and the $\mathrm{dhLV}_{\mathrm{I}}$ system, between which we obtain a Bäcklund transformation. For simplicity and following some conventions, we hereinafter refer to these as the non-autonomous dhToda equation and the dhLV system, respectively.

4.1 The non-autonomous discrete hungry Toda equation

In this section, through deriving the non-autonomous dhToda equation from the polynomial identities, we finally clarify the solution to the non-autonomous dhToda equation in terms of the extended Hankel determinants $H_k^{(s,t)}$.

Focusing on identities for the minors of the polynomials $H_k^{(s,t)}(z)$, we obtain the following proposition for discrete-time evolutions concerning the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$ with respect to not s but t.

Proposition 4.1.1. *Let us assume that the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ makes the extended Hankel determinants $H_k^{(s,t)}$ be all nonzero. Then, for $s, t = 0, 1, \ldots$, the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$ satisfy*

$$(z - \mu^{(t)})\mathcal{H}_{k-1}^{(s,t+1)}(z) = \mathcal{H}_k^{(s,t)}(z) + Q_k^{(s,t)}\mathcal{H}_{k-1}^{(s,t)}(z), \quad k = 1, 2, \ldots, m, \qquad (4.1)$$

where $Q_k^{(s,t)}$ *are given using the extended Hankel determinants* $H_k^{(s,t)}$ *as*

$$Q_k^{(s,t)} := \frac{H_{k-1}^{(s,t)} H_k^{(s,t+1)}}{H_k^{(s,t)} H_{k-1}^{(s,t+1)}}, \quad k = 1, 2, \ldots, m. \tag{4.2}$$

Proof. By multiplying the kth row by $-\mu^{(t)}$ in the $H_k^{(s,t)}(z)$, and adding this to the $(k+1)$th row and using (2.4), we derive

$$H_k^{(s,t)}(z) = \begin{vmatrix} f_s^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-1}^{(t)} & 1 \\ f_{s+M}^{(t)} & f_{s+M+1}^{(t)} & \cdots & f_{s+M+k-1}^{(t)} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{s+(k-1)M}^{(t)} & f_{s+(k-1)M+1}^{(t)} & \cdots & f_{s+(k-1)(M+1)}^{(t)} & z^{k-1} \\ f_{s+(k-1)M}^{(t+1)} & f_{s+(k-1)M+1}^{(t+1)} & \cdots & f_{s+(k-1)(M+1)}^{(t+1)} & (z-\mu^{(t)})z^{k-1} \end{vmatrix}.$$

Applying a process which is similar to the kth , $(k-1)$th, $\ldots$, 2nd rows, we obtain

$$H_k^{(s,t)}(z) = \begin{vmatrix} f_s^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-1}^{(t)} & 1 \\ \boldsymbol{f}_k^{(s,t+1)} & \boldsymbol{f}_k^{(s+1,t+1)} & \cdots & \boldsymbol{f}_k^{(s+k-1,t+1)} & (z-\mu^{(t)})\boldsymbol{z}_k \end{vmatrix}. \tag{4.3}$$

Let $H_k^{(s,t)}(z)\begin{bmatrix} i_1, i_2, \ldots \\ j_1, j_2, \ldots \end{bmatrix}$ denote determinants obtained by deleting $i_1, i_2, \ldots$ rows and $j_1, j_2, \ldots$ columns of the $H_k^{(s,t)}(z)$. Then, it follows that

$$H_k^{(s,t)}(z)\begin{bmatrix} k+1 \\ k+1 \end{bmatrix} = \begin{vmatrix} f_s^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-1}^{(t)} \\ \boldsymbol{f}_{k-1}^{(s,t+1)} & \boldsymbol{f}_{k-1}^{(s+1,t+1)} & \cdots & \boldsymbol{f}_{k-1}^{(s+k-1,t+1)} \end{vmatrix} = H_k^{(s,t)},$$

$$H_k^{(s,t)}(z)\begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{vmatrix} \boldsymbol{f}_k^{(s,t+1)} & \boldsymbol{f}_k^{(s+1,t+1)} & \cdots & \boldsymbol{f}_k^{(s+k-2,t+1)} & (z-\mu^{(t)})\boldsymbol{z}_k \end{vmatrix} = (z-\mu^{(t)})H_{k-1}^{(s,t+1)}(z),$$

$$H_k^{(s,t)}(z)\begin{bmatrix} 1 & k+1 \\ k & k+1 \end{bmatrix} = \begin{vmatrix} \boldsymbol{f}_{k-1}^{(s,t+1)} & \boldsymbol{f}_{k-1}^{(s+1,t+1)} & \cdots & \boldsymbol{f}_{k-1}^{(s+k-2,t+1)} \end{vmatrix} = H_{k-1}^{(s,t+1)},$$

$$H_k^{(s,t)}(z)\begin{bmatrix} k+1 \\ k \end{bmatrix} = \begin{vmatrix} f_s^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-2}^{(t)} & 1 \\ \boldsymbol{f}_{k-1}^{(s,t+1)} & \boldsymbol{f}_{k-1}^{(s+1,t+1)} & \cdots & \boldsymbol{f}_{k-1}^{(s+k-2,t+1)} & (z-\mu^{(t)})\boldsymbol{z}_{k-1} \end{vmatrix} = H_{k-1}^{(s,t)}(z),$$

$$H_k^{(s,t)}(z)\begin{bmatrix} 1 \\ k+1 \end{bmatrix} = \begin{vmatrix} \boldsymbol{f}_k^{(s,t+1)} & \boldsymbol{f}_k^{(s+1,t+1)} & \cdots & \boldsymbol{f}_k^{(s+k-1,t+1)} \end{vmatrix} = H_k^{(s,t+1)}.$$

Thus, we can rewrite Jacobi's identities [17] as

$$(z-\mu^{(t)})H_k^{(s,t)}H_{k-1}^{(s,t+1)}(z) = H_{k-1}^{(s,t+1)}H_k^{(s,t)}(z) + H_k^{(s,t+1)}H_{k-1}^{(s,t)}(z).$$

Dividing both sides by $H_k^{(s,t)}H_{k-1}^{(s,t+1)}$ and recalling that $\mathcal{H}_k^{(s,t)}(z) = H_k^{(s,t)}(z)\,/H_k^{(s,t)}$, we therefore have (4.1) and (4.2). □

With the help of Proposition 4.1.1, we have the following theorem that gives the non-autonomous dhToda equation involving the parameter $\mu^{(t)}$.

Theorem 4.1.2. *Let us assume that the moment sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *makes the extended Hankel determinants* $H_k^{(s,t)}$ *be all nonzero. Then, the variables* $q_k^{(s,t)}$, $e_k^{(s,t)}$, *and* $Q_k^{(s,t)}$ *satisfy, for* $s, t = 0, 1, \ldots,$

$$\begin{cases} q_k^{(s,t+1)} + Q_{k+1}^{(s,t)} = q_{k+1}^{(s,t)} + Q_k^{(s+M,t)}, & k = 1, 2, \ldots, m-1, \\ q_k^{(s,t+1)} Q_k^{(s,t)} = q_k^{(s,t)} Q_k^{(s+M,t)}, & k = 1, 2, \ldots, m, \end{cases} \tag{4.4}$$

and

$$\begin{cases} e_k^{(s,t)} + Q_k^{(s,t)} = e_{k-1}^{(s,t+1)} + Q_k^{(s+1,t)}, & k = 1, 2, \ldots, m, \\ e_k^{(s,t)} Q_{k+1}^{(s,t)} = e_k^{(s,t+1)} Q_k^{(s+1,t)}, & k = 1, 2, \ldots, m-1, \end{cases} \tag{4.5}$$

where $Q_1^{(s,t)} = q_1^{(s,t)} - \mu^{(t)}$.

Proof. Using (3.13) with $t \mapsto t+1$, we can rewrite (4.1) as

$$\begin{aligned} &z(z - \mu^{(t)})\mathcal{H}_{k-1}^{(s+M,t+1)}(z) \\ &\quad = \mathcal{H}_{k+1}^{(s,t)}(z) + (q_k^{(s,t+1)} + Q_{k+1}^{(s,t)})\mathcal{H}_k^{(s,t)}(z) + q_k^{(s,t+1)} Q_k^{(s,t)} \mathcal{H}_{k-1}^{(s,t)}(z). \end{aligned} \tag{4.6}$$

Furthermore, it follows from (3.13) and (4.1) with $s \mapsto s + M$ that

$$\begin{aligned} &z(z - \mu^{(t)})\mathcal{H}_{k-1}^{(s+M,t+1)}(z) \\ &\quad = \mathcal{H}_{k+1}^{(s,t)}(z) + (q_{k+1}^{(s,t)} + Q_k^{(s+M,t)})\mathcal{H}_k^{(s,t)}(z) + q_k^{(s,t)} Q_k^{(s+M,t)} \mathcal{H}_{k-1}^{(s,t)}(z). \end{aligned} \tag{4.7}$$

Observing (4.6) and (4.7), and considering that $\mathcal{H}_k^{(s,t)}(z) \neq 0$ for any s and t, we thus obtain (4.4). Similarly, by using (4.1) with s and (4.1) with $s \mapsto s + 1$, we can rewrite (3.14) with t and (3.14) with $t \mapsto t + 1$ as, respectively,

$$\begin{aligned} &(z - \mu^{(t)})\mathcal{H}_{k-1}^{(s,t+1)}(z) \\ &\quad = \mathcal{H}_k^{(s+1,t)}(z) + (e_k^{(s,t)} + Q_k^{(s,t)})\mathcal{H}_{k-1}^{(s+1,t)}(z) + e_{k-1}^{(s,t)} Q_k^{(s,t)} \mathcal{H}_{k-2}^{(s+1,t)}(z), \end{aligned} \tag{4.8}$$

$$\begin{aligned} &(z - \mu^{(t)})\mathcal{H}_k^{(s,t+1)}(z) \\ &\quad = \mathcal{H}_{k+1}^{(s+1,t)}(z) + (e_k^{(s,t+1)} + Q_{k+1}^{(s+1,t)})\mathcal{H}_k^{(s+1,t)}(z) + e_k^{(s,t+1)} Q_k^{(s+1,t)} \mathcal{H}_{k-1}^{(s+1,t)}(z), \end{aligned} \tag{4.9}$$

which immediately lead to (4.5).

Since $Q_1^{(s,t)}$ are expressed as (4.2), we easily see $Q_1^{(s,t)} = H_1^{(s,t+1)}/H_1^{(s,t)} = f_s^{(t+1)}/f_s^{(t)}$. Combining this with the linear dependence (2.4), we thus have $Q_1^{(s,t)} = q_1^{(s,t)} - \mu^{(t)}$. □

It is easy to check that (4.4) with $s = 0$ and (4.5) with $s = 0, 1, \dots, M-1$ are mathematically equivalent to the non-autonomous dhToda equation appearing in Sumikura et al. [33]. We thus conclude that the solution to the non-autonomous dhToda equation is given using the extended Hankel determinants $H_k^{(s,t)}$ as (3.7) with $s = 0$ and (3.8) with $s = 0, 1, \dots, M-1$.

From the viewpoint of numerical stability, we employ the auxiliary variables

$$D_k^{(0,t)} := Q_k^{(M,t)} - q_k^{(0,t+1)}, \tag{4.10}$$

$$F_k^{(s,t)} := Q_k^{(s,t)} - e_{k-1}^{(s,t+1)}, \quad s = 0, 1, \dots, M-1. \tag{4.11}$$

With the help of (4.10) and (4.11), we can reduce subtractions in the non-autonomous dhToda equation (4.4) with $s = 0$ and (4.5) with $s = 0, 1, \dots, M-1$ via the differential form proposed by Sumikura et al. [33]. Combining (3.7) and (4.2) with (4.10), we obtain the determinant expression of the auxiliary variables $D_k^{(0,t)}$,

$$D_k^{(0,t)} = \frac{H_k^{(M,t+1)}}{H_{k-1}^{(M,t+1)}} \left(\frac{H_{k-1}^{(M,t)}}{H_k^{(M,t)}} - \frac{H_{k-1}^{(0,t+1)}}{H_k^{(0,t+1)}} \right). \tag{4.12}$$

Similarly, from (3.8), (4.2), and (4.11), it follows that

$$F_k^{(s,t)} = \frac{H_k^{(s,t+1)}}{H_{k-1}^{(s,t+1)}} \left(\frac{H_{k-1}^{(s,t)}}{H_k^{(s,t)}} - \frac{H_{k-2}^{(s+1,t+1)}}{H_{k-1}^{(s+1,t+1)}} \right). \tag{4.13}$$

4.2 The related eigenvalue problem

In this section, by examining the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$, we describe the eigenvalue problem of Hessenberg matrices associated with the non-autonomous dhToda equation.

Now, let us introduce m-by-m upper bidiagonal matrices involving $Q_1^{(s,t)}$, $Q_2^{(s,t)}$, $\dots$, $Q_m^{(s,t)}$, not $q_1^{(s,t)}$, $q_2^{(s,t)}$, $\dots$, $q_m^{(s,t)}$, like so

$$\mathcal{R}^{(s,t)} := \begin{pmatrix} Q_1^{(s,t)} & 1 & & \\ & Q_2^{(s,t)} & \ddots & \\ & & \ddots & 1 \\ & & & Q_m^{(s,t)} \end{pmatrix}. \tag{4.14}$$

Considering Proposition 4.1.1 with $z = \lambda_i$, we obtain

$$\mathcal{R}^{(s,t)} \mathcal{H}_i^{(s,t)} = (\lambda_i - \mu^{(t)}) \mathcal{H}_i^{(s,t+1)}. \tag{4.15}$$

Equation (4.15) with (3.26) and (3.27) gives relationships of $\mathcal{R}^{(s,t)}$ to $L^{(s,t)}$ and $R^{(s,t)}$.

Theorem 4.2.1. *The similarity transformation for $A^{(s,t)}$ with respect to t is given by*

$$A^{(s,t+1)} = \mathcal{R}^{(s,t)} A^{(s,t)} (\mathcal{R}^{(s,t)})^{-1}. \tag{4.16}$$

Proof. Combining (3.27) with (4.15), we obtain $\mathcal{R}^{(s,t)} L^{(s,t)} \mathcal{H}_i^{(s+1,t)} = (\lambda_i - \mu^{(t)}) \mathcal{H}_i^{(s,t+1)}$. Using (3.27) and (4.15) again, we can rewrite this as $\mathcal{R}^{(s,t)} L^{(s,t)}\ \mathcal{H}_i^{(s+1,t)} = L^{(s,t+1)}$ $\mathcal{R}^{(s+1,t)} \mathcal{H}_i^{(s+1,t)}$. Since $\mathcal{H}_i^{(s+1,t)} \neq 0$, we thus have

$$\mathcal{R}^{(s,t)} L^{(s,t)} = L^{(s,t+1)} \mathcal{R}^{(s+1,t)}. \tag{4.17}$$

Similarly, it follows from (3.26) with (4.15), that $\mathcal{R}^{(s+M,t)} R^{(s,t)} \mathcal{H}_i^{(s,t)} = \lambda_i(\lambda_i - \mu^{(t)}) \mathcal{H}_i^{(s+M,t+1)} = R^{(s,t+1)} \mathcal{R}^{(s,t)} \mathcal{H}_i^{(s,t)}$, which immediately leads to

$$R^{(s,t+1)} \mathcal{R}^{(s,t)} = \mathcal{R}^{(s+M,t)} R^{(s,t)}. \tag{4.18}$$

Applying (4.18) to $A^{(s,t+1)} = L^{(s,t+1)} L^{(s,t+1)} \cdots L^{(s+M-1,t+1)} R^{(s,t+1)}$, we derive

$$A^{(s,t+1)} = L^{(s,t+1)} L^{(s,t+1)} \cdots L^{(s+M-1,t+1)} \mathcal{R}^{(s+M,t)} R^{(s,t)} (\mathcal{R}^{(s,t)})^{-1}.$$

Along a similar line as a part of the proof of Proposition 3.2.5, we can rewrite $A^{(s,t+1)}$ using (4.17) repeatedly as

$$A^{(s,t+1)} = \mathcal{R}^{(s,t)} L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)} R^{(s,t)} (\mathcal{R}^{(s,t)})^{-1},$$

which implies (4.16). □

Proposition 3.2.5 and Theorem 4.2.1 imply that the eigenvalues of the Hessenberg matrices $A^{(s,t)}$ do not vary for any s and t. We here describe how to apply the evolution from t to $t+1$ in the non-autonomous dhToda equation to generate similarity transformations of $A^{(0,0)} = L^{(0,0)} L^{(1,0)} \cdots L^{(M-1,0)} R^{(0,0)}$. The sequences $\{q_k^{(0,0)}\}_{k=1}^m$, $\{e_k^{(0,0)}\}_{k=1}^{m-1}$, $\{e_k^{(1,0)}\}_{k=1}^{m-1}$, $\ldots$, $\{e_k^{(M-1,0)}\}_{k=1}^{m-1}$ consisting of $A^{(0,0)}$ yield the auxiliary sequences $\{Q_k^{(0,0)}\}_{k=1}^m$, $\{Q_k^{(1,0)}\}_{k=1}^m$, $\ldots$, $\{Q_k^{(M,0)}\}_{k=1}^m$ and $\{q_k^{(0,1)}\}_{k=1}^m$, $\{e_k^{(0,1)}\}_{k=1}^{m-1}$, $\{e_k^{(1,1)}\}_{k=1}^{m-1}$, $\ldots$, $\{e_k^{(M-1,1)}\}_{k=1}^{m-1}$ as follows.

1: Compute $Q_1^{(0,0)} = q_1^{(0,0)} - \mu^{(0)}$.

2: Compute $Q_1^{(1,0)}$, $Q_1^{(2,0)}$, $\ldots$, $Q_1^{(M,0)}$ recursively using the 1st equation of (4.5) with $k = 1$.

3: Compute $q_1^{(0,1)}$ using the 2nd equation of (4.4) with $k = 1$.

4: Compute $Q_2^{(0,0)}$ using the 1st equation in (4.4) with $k = 1$.

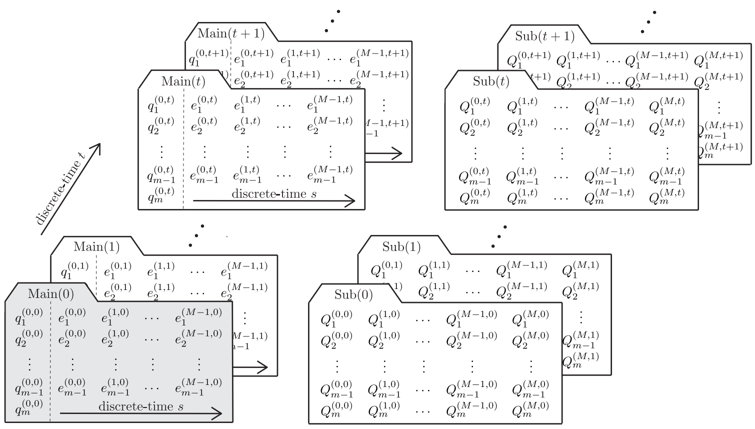

Figure 4.1: Discrete-time evolutions in the non-autonomous dhToda equation.

5: Compute $e_1^{(0,1)}$ recursively using the 2nd equation of (4.5) with $k = 1$.

6: Compute $Q_2^{(1,0)}$ recursively using the 1st equation of (4.5) with $k = 1$.

7: Compute $e_1^{(1,1)}$, $e_1^{(2,1)}$, $\ldots$, $e_1^{(M-1,1)}$ and $Q_2^{(2,0)}$, $Q_2^{(3,0)}$, $\ldots$, $Q_2^{(M,0)}$ as per steps 5–6.

8: Compute the sequence $\{Q_k^{(0,0)}\}_{k=3}^m$, $\{Q_k^{(1,0)}\}_{k=3}^m$, $\ldots$, $\{Q_k^{(M,0)}\}_{k=3}^m$ and $\{q_k^{(0,1)}\}_{k=2}^{m-1}$, $\{e_k^{(0,1)}\}_{k=2}^{m-1}$, $\{e_k^{(1,1)}\}_{k=2}^{m-1}$, $\ldots$, $\{e_k^{(M-1,1)}\}_{k=2}^{m-1}$ as per steps 2–7.

9: Compute $q_m^{(0,1)}$ using the 2nd equation of (4.4) with $k = m$.

Figure 5.1 shows the discrete-time evolutions with respect to s and t in the non-autonomous dhToda equation. Similarly to the above procedure for deriving the sequences in Sub(0) and Main(1) from those in Main(0), we can recursively generate the sequences in Sub(t) and Main($t+1$) from those in Main(t) for $t = 1, 2, \ldots$. Thus, we can obtain $A^{(0,1)}$, $A^{(0,2)}$, $\ldots$, whose eigenvalues are all equal to $A^{(0,0)}$ by computing the sequences in Main($t + 1$) from those in Main(0) using the non-autonomous dhToda equation.

4.3 The discrete hungry Lotka-Volterra system

In this section, we first reconsider Propositions 3.2.1 and 4.1.1 by symmetrizing with the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$. Next, similarly to work in [34, 39], we derive the dhLV system by observing the resulting propositions for the symmetric extended Hadamard polynomials. We then describe the solution to the dhLV system in terms of the extended Hankel determinants $H_k^{(s,t)}$.

Let us define the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_k^{(s,t)}(z)$ by

$$\tilde{\mathcal{H}}_{(M+1)k+j}^{(s,t)}(z) := z^j \mathcal{H}_k^{(s+j,t)}(z^{M+1}), \quad j = 0, 1, \ldots, M, \quad k = 0, 1, \ldots, m, \tag{4.19}$$

where $\tilde{\mathcal{H}}_{-1}^{(s,t)}(z) := 0$, $\tilde{\mathcal{H}}_{-2}^{(s,t)}(z) := 0$, $\ldots$, $\tilde{\mathcal{H}}_{-M}^{(s,t)}(z) := 0$. Proposition 3.2.1 then gives three-term recurrence relations with respect to the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_k^{(s,t)}(z)$.

Proposition 4.3.1. *For $t = 0, 1, \ldots,$ the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_k^{(s,t)}(z)$ satisfy*

$$z\tilde{\mathcal{H}}_{k-1}^{(s,t)}(z) = \tilde{\mathcal{H}}_k^{(s,t)}(z) + v_{k-M}^{(s,t)} \tilde{\mathcal{H}}_{k-M-1}^{(s,t)}(z), \quad k = 1, 2, \ldots, (M+1)m + M, \tag{4.20}$$

where $v_{(M+1)k-M}^{(s,t)} := q_k^{(s,t)}$ and $v_{(M+1)k-M+j+1}^{(s,t)} := e_k^{(s+j,t)}$ for $j = 0, 1, \ldots, M-1$.

Proof. Replacing z with z^{M+1} in the three-term recurrence relations (3.13) and (3.14), we obtain

$$z^{M+1}\mathcal{H}_{k-1}^{(s+M,t)}(z^{M+1}) = \mathcal{H}_k^{(s,t)}(z^{M+1}) + q_k^{(s,t)}\mathcal{H}_{k-1}^{(s,t)}(z^{M+1}), \tag{4.21}$$

$$\mathcal{H}_k^{(s,t)}(z^{M+1}) = \mathcal{H}_k^{(s+1,t)}(z^{M+1}) + e_k^{(s,t)}\mathcal{H}_{k-1}^{(s+1,t)}(z^{M+1}). \tag{4.22}$$

Multiplying both sides of (4.22) with $s \mapsto s, s+1, \ldots, s+M-1$ by z, z^2, $\ldots$, z^M, respectively, we derive, for $j = 0, 1, \ldots, M-1$,

$$z^{j+1}\mathcal{H}_k^{(s+j,t)}(z^{M+1}) = z^{j+1}\mathcal{H}_k^{(s+j+1,t)}(z^{M+1}) + e_k^{(s+j,t)} z^{j+1}\mathcal{H}_{k-1}^{(s+j+1,t)}(z^{M+1}). \tag{4.23}$$

Using $v_{(M+1)k-M}^{(s,t)} = q_k^{(s,t)}$, $v_{(M+1)k-M+j+1}^{(s,t)} = e_k^{(s+j,t)}$, and (4.19), we can rewrite (4.21) and (4.23), respectively, as

$$z\tilde{\mathcal{H}}_{(M+1)(k-1)+M}^{(s,t)}(z) = \tilde{\mathcal{H}}_{(M+1)k}^{(s,t)}(z) + v_{(M+1)k-M}^{(s,t)}\tilde{\mathcal{H}}_{(M+1)(k-1)}^{(s,t)}(z), \tag{4.24}$$

$$z\tilde{\mathcal{H}}_{(M+1)k+j}^{(s,t)}(z) = \tilde{\mathcal{H}}_{(M+1)k+j+1}^{(s,t)}(z) + v_{(M+1)k-M+j+1}^{(s,t)}\tilde{\mathcal{H}}_{(M+1)(k-1)+j+1}^{(s,t)}(z). \tag{4.25}$$

Equations (4.24) and (4.25) are just identities in the cases where k is replaced with $(M+1)k$ and $(M+1)k+j+1$ in (4.20), respectively. □

Proposition 4.1.1 also yields the time evolution from t to $t+1$ in the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_k^{(s,t)}(z)$.

Proposition 4.3.2. *For $t = 0, 1, \ldots,$ the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_k^{(s,t)}(z)$ satisfy*

$$(z^{M+1} - \mu^{(t)})\tilde{\mathcal{H}}_{k-1}^{(s,t+1)}(z) = \tilde{\mathcal{H}}_{k+M}^{(s,t)}(z) + V_k^{(s,t)}\tilde{\mathcal{H}}_{k-1}^{(s,t)}(z), \quad k = 1, 2, \ldots, (M+1)m, \tag{4.26}$$

where $V_{(M+1)(k-1)+j+1}^{(s,t)} := Q_k^{(s+j,t)}$ for $j = 0, 1, \ldots, M$.

Proof. Replacing s and z with $s+j$ and z^{M+1}, respectively, in the three-term recurrence relation (4.1) of Proposition 4.1.1, and multiplying both sides with z^j, we obtain, for $j = 0, 1, \dots, M$,

$$\begin{aligned} &z^j(z^{M+1} - \mu^{(t)})\mathcal{H}_{k-1}^{(s+j,t+1)}(z^{M+1}) \\ &\quad = z^j\mathcal{H}_k^{(s+j,t)}(z^{M+1}) + Q_k^{(s+j,t)} z^j \mathcal{H}_{k-1}^{(s+j,t)}(z^{M+1}). \end{aligned} \tag{4.27}$$

Considering $Q_k^{(s+j,t)} = V_{(M+1)(k-1)+j+1}^{(s,t)}$ and (4.19), we thus have (4.26). □

Propositions 4.3.1 and 4.3.2 lead to relationships of the variables $v_k^{(s,t)}$ and $V_k^{(s,t)}$.

Lemma 4.3.3. *For $s, t = 0, 1, \dots,$ the variables $v_k^{(s,t)}$ and $V_k^{(s,t)}$ satisfy*

$$v_k^{(s,t+1)} + V_{k+M+1}^{(s,t)} = v_{k+M+1}^{(s,t)} + V_{k+M}^{(s,t)}, \quad k = 0, 1, \dots, (M+1)(m-1), \tag{4.28}$$

$$v_k^{(s,t+1)} V_k^{(s,t)} = v_k^{(s,t)} V_{k+M}^{(s,t)}, \quad k = 1, 2, \dots, (M+1)m - M. \tag{4.29}$$

Proof. From Propositions 4.3.1 and 4.3.2, it immediately follows that

$$\begin{aligned} &(v_{k-M}^{(s,t+1)} + V_{k+1}^{(s,t)} - v_{k+1}^{(s,t)} - V_k^{(s,t)})\tilde{\mathcal{H}}_k^{(s,t)}(z) \\ &\quad + (v_{k-M}^{(s,t+1)} V_{k-M}^{(s,t)} - v_{k-M}^{(s,t)} V_k^{(s,t)})\tilde{\mathcal{H}}_{k-M-1}^{(s,t)}(z) = 0. \end{aligned}$$

Since the extended Hadamard polynomials $\mathcal{H}_0^{(s,t)}(z)$, $\mathcal{H}_1^{(s,t)}(z)$, …, $\mathcal{H}_{(M+1)m+M}^{(s,t)}(z)$ have different degrees, they are linearly independent. The symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_k^{(s,t)}(z)$ and $\tilde{\mathcal{H}}_{k-M-1}^{(s,t)}(z)$ are thus linearly independent. Therefore, we derive

$$v_{k-M}^{(s,t+1)} + V_{k+1}^{(s,t)} = v_{k+1}^{(s,t)} + V_k^{(s,t)}, \quad k = M, M+1, \dots, (M+1)(m-1) + M,$$

$$v_{k-M}^{(s,t+1)} V_{k-M}^{(s,t)} = v_{k-M}^{(s,t)} V_k^{(s,t)}, \quad k = M+1, M+2, \dots, (M+1)m,$$

which are equivalent to (4.28) and (4.29), respectively. □

Now, let us introduce an infinite sequence $\{\kappa^{(t)}\}_{t=0}^{\infty}$ given using the infinite sequence $\{\mu^{(t)}\}_{t=0}^{\infty}$ as

$$(\kappa^{(t)})^{M+1} := \mu^{(t)}, \quad t = 0, 1, \dots. \tag{4.30}$$

Furthermore, let $u_k^{(s,t)}$ be new variables defined by

$$u_k^{(s,t)} := v_k^{(s,t)} \frac{\tilde{\mathcal{H}}_{k-1}^{(s,t)}(\kappa^{(t)})}{\tilde{\mathcal{H}}_{k+M-1}^{(s,t)}(\kappa^{(t)})}. \tag{4.31}$$

Then, we obtain relationships of the variables $v_k^{(s,t)}$ and $V_k^{(s,t)}$ to the new variables $u_k^{(s,t)}$.

Theorem 4.3.4. *The variables $v_k^{(t)}$ and $V_k^{(t)}$ are expressed using the variables $u_k^{(t)}$ and the parameters $\kappa_k^{(t)}$ as*

$$v_k^{(s,t)} = u_k^{(s,t)} \prod_{\ell=1}^{M} (\kappa^{(t)} - u_{k-\ell}^{(s,t)}), \quad k = -M+1, -M+2, \dots, (M+1)m, \tag{4.32}$$

$$V_k^{(s,t)} = -\prod_{\ell=0}^{M} (\kappa^{(t)} - u_{k-\ell}^{(s,t)}), \quad k = 1, 2, \dots, (M+1)m. \tag{4.33}$$

Furthermore, it holds that

$$u_k^{(s,t+1)} \prod_{\ell=1}^{M} (\kappa^{(t+1)} - u_{k-\ell}^{(s,t+1)}) = u_k^{(s,t)} \prod_{\ell=1}^{M} (\kappa^{(t)} - u_{k+\ell}^{(s,t)}). \tag{4.34}$$

Proof. Dividing both sides of the three-term recurrence relation (4.20) with $z = \kappa^{(t)}$ in Proposition 4.3.1 by $\tilde{\mathcal{H}}_{k-1}^{(s,t)}(\kappa^{(t)})$, we derive

$$\frac{\tilde{\mathcal{H}}_k^{(s,t)}(\kappa^{(t)})}{\tilde{\mathcal{H}}_{k-1}^{(s,t)}(\kappa^{(t)})} = \kappa^{(t)} - u_{k-M}^{(s,t)}. \tag{4.35}$$

Using (4.31), we can rewrite (4.35) as (4.32). From the three-term recurrence relation (4.26) with $z = \kappa^{(t)}$ in Proposition 4.3.2, we see that $V_k^{(s,t)} = -\tilde{\mathcal{H}}_{k+M}^{(s,t)}(\kappa^{(t)})/\tilde{\mathcal{H}}_{k-1}^{(s,t)}(\kappa^{(t)})$. Combining this with (4.35), we thus obtain (4.33). Furthermore, from (4.32) and (4.33) with (4.29) in Lemma 4.3.3, we find that

$$v_k^{(s,t+1)} = u_k^{(s,t)} \prod_{\ell=0}^{M-1} (\kappa^{(t)} - u_{k+M-\ell}^{(s,t)}). \tag{4.36}$$

Since it is obvious that $\prod_{\ell=0}^{M-1}(\kappa^{(t)} - u_{k+M-\ell}^{(s,t)}) = \prod_{\ell=1}^{M}(\kappa^{(t)} - u_{k+\ell}^{(s,t)})$, we therefore have (4.34). We can easily check that (4.32) and (4.33) give (4.28). □

Equation (4.34) of Theorem 4.3.4 is essentially equivalent to the dhLV system

$$\begin{cases} U_k^{(s,t+1)} \prod_{\ell=1}^{M} (1 + \delta^{(t+1)} U_{k-\ell}^{(s,t+1)}) = U_k^{(s,t)} \prod_{\ell=1}^{M} (1 + \delta^{(t)} U_{k+\ell}^{(s,t)}), \\ \quad k = 1, 2, \dots, (M+1)m - M, \\ U_{j-M}^{(s,t)} := 0, \quad U_{(M+1)m+j-M}^{(s,t)} := 0, \quad j = 1, 2, \dots, M, \\ t = 0, 1, \dots, \end{cases} \tag{4.37}$$

This is because the dhLV system is actually derived by replacing $u_k^{(s,t)}$ and $(\kappa^{(t)})^{M+1}$ with $U_k^{(s,t)}/(\kappa^{(t)})^M$ and $-1/\delta^{(t)}$, respectively, in (4.34). Considering that $e_0^{(s,t)} = 0$

and $e_m^{(s,t)} = 0$ for $s, t = 0, 1, \dots$, we also obtain the boundary conditions $U_{1-M}^{(s,t)} = 0$, $U_{2-M}^{(s,t)} = 0$, $\dots$, $U_0^{(s,t)} = 0$ and $U_{(M+1)m-M+1}^{(s,t)} = 0$, $U_{(M+1)m-M+2}^{(s,t)} = 0$, $\dots$, $U_{(M+1)m}^{(s,t)} = 0$.

Here, let $\tilde{H}_k^{(s,t)} := (\delta^{(0)}\delta^{(1)} \cdots \delta^{(t-1)})^k H_k^{(s,t)}$, namely,

$$\tilde{H}_k^{(s,t)} = \begin{vmatrix} \tilde{f}_s^{(t)} & \tilde{f}_{s+1}^{(t)} & \cdots & \tilde{f}_{s+k-1}^{(t)} \\ \tilde{f}_{s+M}^{(t)} & \tilde{f}_{s+M+1}^{(t)} & \cdots & \tilde{f}_{s+M+k-1}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{f}_{s+(k-1)M}^{(t)} & \tilde{f}_{s+(k-1)M+1}^{(t)} & \cdots & \tilde{f}_{s+M_{k-1}}^{(t)} \end{vmatrix}, \quad s, t = 0, 1, \dots, \tag{4.38}$$

where $\tilde{f}_s^{(t)} := \delta^{(0)}\delta^{(1)} \cdots \delta^{(t-1)} f_s^{(t)}$. The following theorem then achieves the determinant solution to the dhLV system (4.37).

Theorem 4.3.5. *Let us assume that the moment sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *makes* $H_k^{(s,t)}$ *be all nonzero. Then, the dhLV variables* $U_k^{(s,t)}$ *can be expressed as*

$$U_{(M+1)k+1}^{(s,t)} = \frac{\tilde{H}_{k+1}^{(s+M,t)}\tilde{H}_k^{(s,t+1)}}{\tilde{H}_{k+1}^{(s,t)}\tilde{H}_k^{(s+M,t+1)}}, \quad k = 0, 1, \dots, m-1, \tag{4.39}$$

$$U_{(M+1)(k-1)+j+1}^{(s,t)} = \frac{\tilde{H}_{k+1}^{(s+j-1,t)}\tilde{H}_{k-1}^{(s+j,t+1)}}{\tilde{H}_k^{(s+j,t)}\tilde{H}_k^{(s+j-1,t+1)}}, \quad j = 1, 2, \dots, M, \quad k = 1, 2, \dots, m. \tag{4.40}$$

Proof. From (4.31), it is obvious that $u_{(M+1)k+1}^{(s,t)} = v_{(M+1)k+1}^{(s,t)}\tilde{\mathcal{H}}_{(M+1)k}^{(s,t)}(\kappa^{(t)})\,/\tilde{\mathcal{H}}_{(M+1)k+M}^{(s,t)}(\kappa^{(t)})$. Using (4.19), we can rewrite this as

$$u_{(M+1)k+1}^{(s,t)} = q_{k+1}^{(s,t)}\frac{\mathcal{H}_k^{(s,t)}(\mu^{(t)})}{(\kappa^{(t)})^M\mathcal{H}_k^{(s+M,t)}(\mu^{(t)})}, \quad k = 0, 1, \dots, m-1. \tag{4.41}$$

Reconsidering the $H_k^{(s,t)}(z)$ in (4.3), we observe that $H_k^{(s,t)}(\mu^{(t)}) = (-1)^k H_k^{(s,t+1)}$. Since it holds that $\mathcal{H}_k^{(s,t)}(\mu^{(t)}) = H_k^{(s,t)}(\mu^{(t)})/H_k^{(s,t)}$, we also see that $\mathcal{H}_k^{(s,t)}(\mu^{(t)}) = (-1)^k H_k^{(s,t+1)}/H_k^{(s,t)}$. Thus, by combining this with (4.41), we obtain

$$u_{(M+1)k+1}^{(s,t)} = \frac{q_{k+1}^{(s,t)}}{(\kappa^{(t)})^M}\frac{H_k^{(s,t+1)}H_k^{(s+M,t)}}{H_k^{(s,t)}H_k^{(s+M,t+1)}}. \tag{4.42}$$

Using $q_k^{(s,t)} = H_{k-1}^{(s,t)}H_k^{(s+M,t)}/(H_k^{(s,t)}H_{k-1}^{(s+M,t)})$ in (4.42), we derive

$$u_{(M+1)k+1}^{(s,t)} = \frac{1}{(\kappa^{(t)})^M}\frac{H_{k+1}^{(s+M,t)}H_k^{(s,t+1)}}{H_{k+1}^{(s,t)}H_k^{(s+M,t+1)}}.$$

Recalling that $u_k^{(s,t)} = U_k^{(s,t)}/(\kappa^{(t)})^M$ and $H_k^{(s,t)}(z) = \tilde{H}_k^{(s,t)}/(\delta^{(0)}\delta^{(1)}\cdots\delta^{(t-1)})^k$, we thus have (4.39). Similarly, it follows that

$$u_{(M+1)(k-1)+j+1}^{(s,t)} = e_k^{(s+j-1,t)} \frac{\kappa^{(t)}\mathcal{H}_{k-1}^{(s+j,t)}(\mu^{(t)})}{\mathcal{H}_k^{(s+j-1,t)}(\mu^{(t)})}, \quad j = 1, 2, \dots, M. \tag{4.43}$$

Since $e_k^{(s+j-1,t)} = H_{k+1}^{(s+j-1,t)} H_{k-1}^{(s+j,t)} / (H_k^{(s+j-1,t)} H_k^{(s+j,t)})$ and $\mathcal{H}_k^{(s,t)}(\mu^{(t)}) = (-1)^k H_k^{(s,t+1)}/H_k^{(s,t)}$, we can thus rewrite (4.43) as

$$u_{(M+1)(k-1)+j+1}^{(s,t)} = -\kappa^{(t)} \frac{H_{k+1}^{(s+j-1,t)} H_{k-1}^{(s+j,t+1)}}{H_k^{(s+j,t)} H_k^{(s+j-1,t+1)}}.$$

Therefore, by letting $U_k^{(s,t)} = (\kappa^{(t)})^M u_k^{(s,t)}$ and $H_k^{(s,t)} = \tilde{H}_k^{(s,t)}/(\delta^{(0)}\delta^{(1)}\cdots\delta^{(t-1)})^k$, we obtain (4.40). □

Theorem 4.3.5 shows the solution to the dhLV system in terms of the determinants $\tilde{H}_k^{(s,t)} = (\delta^{(0)}\delta^{(1)}\cdots\delta^{(t-1)})^k H_k^{(s,t)}$ associated with the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ involving two types of discrete-time variables s and t. It is worth noting here that the dhLV variables $U_k^{(s,t)}$ and the non-autonomous dhToda variables $q_k^{(s,t)}$, $e_k^{(s,t)}$, and $Q_k^{(s,t)}$ can both be expressed using the extended Hankel determinants $H_k^{(s,t)}$. The following theorem thus gives relationships of the dhLV variables $U_k^{(s,t)}$ to the non-autonomous dhToda variables $q_k^{(s,t)}$, $e_k^{(s,t)}$, and $Q_k^{(s,t)}$.

Theorem 4.3.6. *For $k = 1, 2, \dots, m$, the non-autonomous dhToda variables $q_k^{(s,t)}$, $e_k^{(s+j-1,t)}$, and $Q_k^{(s+j,t)}$, and the dhLV variables $U_k^{(s,t)}$ satisfy*

$$q_k^{(s,t)} = U_{(M+1)(k-1)+1}^{(s,t)} \prod_{\ell=1}^{M} (1 + \delta^{(t)} U_{(M+1)(k-1)+1-\ell}^{(s,t)}), \tag{4.44}$$

$$e_k^{(s+j-1,t)} = U_{(M+1)(k-1)+j+1}^{(s,t)} \prod_{\ell=1}^{M} (1 + \delta^{(t)} U_{(M+1)(k-1)+j+1-\ell}^{(s,t)}), \quad j = 1, 2, \dots, M, \tag{4.45}$$

$$Q_k^{(s+j,t)} = \frac{1}{\delta^{(t)}} \prod_{\ell=0}^{M} (1 + \delta^{(t)} U_{(M+1)(k-1)+j+1-\ell}^{(s,t)}), \quad j = 0, 1, \dots, M. \tag{4.46}$$

Proof. Replacing $v_{(M+1)(k-1)+1}^{(s,t)} = q_k^{(s,t)}$, $u_k^{(s,t)} = U_k^{(s,t)}/(\kappa^{(t)})^M$, and $\delta^{(t)} = -1/(\kappa^{(t)})^{M+1}$ in (4.32) with $k = (M+1)(k-1)+1$ of Theorem 4.3.4, we easily obtain (4.44). Similarly, since $e_k^{(s+j-1,t)} = v_{(M+1)(k-1)+1+j}^{(s,t)}$ and $Q_k^{(s+j,t)} = V_{(M+1)(k-1)+j+1}^{(s,t)}$, we have (4.45) and (4.46), respectively. □

Equations (4.44)–(4.46) in Theorem 4.3.6 can be regarded as the Bäcklund transformation between the non-autonomous dhToda equation and the dhLV system. This Bäcklund transformation differs from already known ones [8, 13] in that the dhLV system is related to the non-autonomous dhToda equation rather than the autonomous one. Focusing on determinant solutions not the associated LR transformations is also a different point

4.4 Convergence of the determinant solution

In this section, with the help of the asymptotic expansions as $t \to \infty$ of the extended Hankel determinants $H_k^{(s,t)}$, we present asymptotic analysis as $t \to \infty$ for solutions to the non-autonomous dhToda equation and the dhLV system.

Using Lemma 2.3.6, we obtain asymptotic behavior as $t \to \infty$ of the non-autonomous dhToda variables $q_k^{(s,t)}$, $e_k^{(s,t)}$, and $Q_k^{(s,t)}$.

Theorem 4.4.1. *Let us assume that the moment sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *satisfies* $H_k^{(s,t)} \neq 0$. *Furthermore, let* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *be constants such that* $|\lambda_1 - \mu^{(t)}| > |\lambda_2 - \mu^{(t)}| > \cdots > |\lambda_m - \mu^{(t)}|$. *Then, it holds that*

$$\lim_{t\to\infty} q_k^{(s,t)} = \lambda_k, \quad k = 1, 2, \ldots, m, \tag{4.47}$$

$$\lim_{t\to\infty} e_k^{(s,t)} = 0, \quad k = 1, 2, \ldots, m-1, \tag{4.48}$$

$$\lim_{t\to\infty} Q_k^{(s,t)} = \lambda_k - \mu^*, \quad k = 1, 2, \ldots, m, \tag{4.49}$$

where $\mu^* := \lim_{t\to\infty} \mu^{(t)}$.

Proof. Applying Lemma 2.3.6 to the non-autonomous dhToda variables $q_k^{(s,t)} = H_{k-1}^{(s,t)} H_k^{(s+M,t)} / (H_k^{(s,t)} H_{k-1}^{(s+M,t)})$, we immediately obtain (4.47). Using Lemma 2.3.6, we can rewrite $e_k^{(s,t)} = H_{k+1}^{(s,t)} H_{k-1}^{(s+1,t)} / (H_k^{(s,t)} H_k^{(s+1,t)})$ as

$$e_k^{(s,t)} = \frac{K_{1,2,\ldots,k+1}^{(j)} K_{1,2,\ldots,k-1}^{(j+1)}}{K_{1,2,\ldots,k}^{(j)} K_{1,2,\ldots,k}^{(j+1)}} \cdot \frac{\lambda_{k+1}}{\lambda_k} \times \frac{(\rho_1^{(t)} \rho_2^{(t)} \cdots \rho_{k+1}^{(t)})(\rho_1^{(t)} \rho_2^{(t)} \cdots \rho_{k-1}^{(t)})}{(\rho_1^{(t)} \rho_2^{(t)} \cdots \rho_k^{(t)})^2} \times \frac{(1 + O(\varrho_{k+1}^t))(1 + O(\varrho_{k-1}^t))}{(1 + O(\varrho_k^t))(1 + O(\varrho_k^t))}. \tag{4.50}$$

Recalling $\rho_i^{(t)} = (\lambda_i - \mu^{(0)})(\lambda_i - \mu^{(1)}) \cdots (\lambda_i - \mu^{(t-1)})$ in (2.31), we see that

$$\frac{(\rho_1^{(t)} \rho_2^{(t)} \cdots \rho_{k+1}^{(t)})(\rho_1^{(t)} \rho_2^{(t)} \cdots \rho_{k-1}^{(t)})}{(\rho_1^{(t)} \rho_2^{(t)} \cdots \rho_k^{(t)})^2} = \frac{(\lambda_{k+1} - \mu^{(0)})(\lambda_{k+1} - \mu^{(1)}) \cdots (\lambda_{k+1} - \mu^{(t-1)})}{(\lambda_k - \mu^{(0)})(\lambda_k - \mu^{(1)}) \cdots (\lambda_k - \mu^{(t-1)})}. \tag{4.51}$$

Thus, by combining (4.51) with (4.50), we have (4.48) under the assumption (2.41). Similarly, by considering Lemma 2.3.6 with (4.51) in the non-autonomous dhToda variables $Q_k^{(s,t)} = H_{k-1}^{(s,t)} H_k^{(s,t+1)}/(H_k^{(s,t)} H_{k-1}^{(s,t+1)})$, we derive

$$\begin{aligned}\lim_{t\to\infty} Q_k^{(s,t)} &= \lim_{t\to\infty} \frac{\rho_k^{(t+1)}}{\rho_k^{(t)}} \cdot \frac{(1+O(\varrho_{k+1}^t))(1+O(\varrho_{k-1}^t))}{(1+O(\varrho_k^t))(1+O(\varrho_k^t))} \\ &= \lim_{t\to\infty} (\lambda_k - \mu^{(t)}) \cdot \frac{(1+O(\varrho_{k+1}^t))(1+O(\varrho_{k-1}^t))}{(1+O(\varrho_k^t))(1+O(\varrho_k^t))},\end{aligned}$$

which leads to (4.49). □

The following theorem shows asymptotic convergence of the auxiliary variables $D_k^{(0,t)}$, $F_k^{(0,t)}$, $F_k^{(1,t)}$, ..., $F_k^{(M-1,t)}$ in the differential non-autonomous dhToda equation.

Theorem 4.4.2. *As $n \to \infty$, the auxiliary non-autonomous dhToda variables $D_k^{(0,t)} = H_k^{(M,t+1)}/H_{k-1}^{(M,t+1)}(H_{k-1}^{(M,t)}/H_k^{(M,t)} - H_{k-1}^{(0,t+1)}/H_k^{(0,t+1)})$ and $F_k^{(s,t)} = H_k^{(s,t+1)}/H_{k-1}^{(s,t+1)}(H_{k-1}^{(s,t)}/H_k^{(s,t)} - H_{k-2}^{(s+1,t+1)}/H_{k-1}^{(s+1,t+1)})$ converge to $-\mu^*$ and $\lambda_k - \mu^*$, respectively.*

Proof. From Lemma 2.3.6 with $\rho_k^{(t)} = (\lambda_k - \mu^{(0)})(\lambda_k - \mu^{(1)}) \cdots (\lambda_k - \mu^{(t-1)})$, it follows that

$$
\begin{aligned}
\lim_{t\to\infty} D_k^{(0,t)} &= \frac{K_{1,2,\ldots,k}^{(0)}}{K_{1,2,\ldots,k-1}^{(0)}}\lambda_k \lim_{t\to\infty} \rho_k^{(t+1)} \frac{(1+O(\varrho_k^{t+1})}{(1+O(\varrho_{k-1}^{t+1})} \\
&\quad \times \left(\frac{K_{1,2,\ldots,k-1}^{(0)}}{K_{1,2,\ldots,k}^{(0)}} \frac{1}{\lambda_k} \lim_{t\to\infty} \frac{1}{\rho_k^{(t)}} \frac{1+O(\varrho_{k-1}^{t})}{1+O(\varrho_k^{t})} \right. \\
&\qquad \left. - \frac{K_{1,2,\ldots,k-1}^{(0)}}{K_{1,2,\ldots,k}^{(0)}} \lim_{t\to\infty} \frac{1}{\rho_k^{(t+1)}} \frac{1+O(\varrho_{k-1}^{t+1})}{1+O(\varrho_k^{t+1})} \right) \\
&= \left(\lim_{t\to\infty} \frac{\rho_k^{(t+1)}}{\rho_k^{(t)}} \frac{(1+O(\varrho_k^{t+1})(1+O(\varrho_{k-1}^{t}))}{(1+O(\varrho_{k-1}^{t+1}))(1+O(\varrho_k^{t}))} \right. \\
&\qquad \left. -\lambda_k \frac{(1+O(\varrho_k^{t+1}))(1+O(\varrho_{k-1}^{t+1}))}{(1+O(\varrho_{k-1}^{t+1}))(1+O(\varrho_k^{t+1}))} \right) \\
&= -\mu^*, \\
\lim_{t\to\infty} F_k^{(s,t)} &= \frac{K_{1,2,\ldots,k}^{(s)}}{K_{1,2,\ldots,k-1}^{(s)}} \lim_{t\to\infty} \rho_k^{(t+1)} \frac{1+O(\varrho_k^{t+1})}{1+O(\varrho_{k-1}^{t+1})} \\
&\quad \times \left(\frac{K_{1,2,\ldots,k-1}^{(s)}}{K_{1,2,\ldots,k}^{(s)}} \lim_{t\to\infty} \frac{1}{\rho_k^{(t)}} \frac{1+O(\varrho_{k-1}^{t})}{1+O(\varrho_k^{t})} \right. \\
&\qquad \left. - \frac{K_{1,2,\ldots,k-2}^{(s)}}{K_{1,2,\ldots,k-1}^{(s)}} \lim_{t\to\infty} \frac{1}{\rho_{k-1}^{(t+1)}} \frac{1+O(\varrho_{k-1}^{t+1})}{1+O(\varrho_k^{t+1})} \right) \\
&= \lambda_k - \mu^*, \quad s = 0, 1, \ldots, M-2, \\
\lim_{t\to\infty} F_k^{(M-1,t)} &= \frac{K_{1,2,\ldots,k}^{(M-1)}}{K_{1,2,\ldots,k-1}^{(M-1)}} \lim_{t\to\infty} \rho_k^{(t+1)} \frac{1+O(\varrho_k^{t+1})}{1+O(\varrho_{k-1}^{t+1})} \\
&\quad \times \left(\frac{K_{1,2,\ldots,k-1}^{(M-1)}}{K_{1,2,\ldots,k}^{(M-1)}} \lim_{t\to\infty} \frac{1}{\rho_k^{(t)}} \frac{1+O(\varrho_{k-1}^{t})}{1+O(\varrho_k^{t})} \right. \\
&\qquad \left. - \frac{K_{1,2,\ldots,k-2}^{(0)}}{K_{1,2,\ldots,k-1}^{(0)}} \frac{1}{\lambda_{k-1}} \lim_{t\to\infty} \frac{1}{\rho_{k-1}^{(t+1)}} \frac{1+O(\varrho_{k-1}^{t+1})}{1+O(\varrho_k^{t+1})} \right) \\
&= \lambda_k - \mu^*.
\end{aligned}
$$

□

Combining Theorem 4.3.5 with Lemma 2.3.6, we can identify the asymptotic convergence of the non-autonomous dhLV variables $U_k^{(s,t)}$ as $t \to \infty$.

Theorem 4.4.3. *Let us assume that the moment sequence* $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ *satisfies* $H_k^{(s,t)} \neq 0$. *Furthermore, let* $\lambda_1, \lambda_2, \dots, \lambda_m$ *be constants such that* $|\lambda_1 - \mu^{(t)}| > |\lambda_2 - \mu^{(t)}| > \cdots > |\lambda_m - \mu^{(t)}|$. *Then, asymptotic behavior as* $t \to \infty$ *of the dhLV variables* $U_k^{(t)}$ *is given by*

$$\lim_{t\to\infty} U_{(M+1)k+1}^{(s,t)} = \lambda_{k+1}, \quad k = 0, 1, \dots, m-1, \tag{4.52}$$

$$\lim_{t\to\infty} U_{(M+1)(k-1)+j+1}^{(s,t)} = 0, \quad j = 1, 2, \dots, M, \quad k = 1, 2, \dots, m. \tag{4.53}$$

Proof. Noting that $\tilde{H}_k^{(s,t)} = (\delta^{(0)}\delta^{(1)}\cdots\delta^{(t-1)})^k H_k^{(s,t)}$ in Theorem 4.3.5, we see that $U_{(M+1)k+1}^{(s,t)} = H_{k+1}^{(s+M,t)} H_k^{(s,t+1)} / (H_{k+1}^{(s,t)} H_k^{(s+M,t+1)})$. Applying Lemma 2.3.6, we derive

$$U_{(M+1)k+1}^{(s,t)} = \frac{\lambda_1\lambda_2\cdots\lambda_{k+1}}{\lambda_1\lambda_2\cdots\lambda_k} \cdot \frac{(1+O(\varrho_{k+1}^t))(1+O(\varrho_k^{t+1}))}{(1+O(\varrho_{k+1}^t))(1+O(\varrho_k^{t+1}))}, \tag{4.54}$$

which immediately leads to (4.52). Let us consider the case $j'+j < M$, where j' are integers such that $s = M\ell + j'$. Using Theorem 4.3.5 and Lemma 2.3.6, as $t \to \infty$, we can also rewrite $U_{(M+1)(k-1)+j+1}^{(s,t)} = H_{k+1}^{(s+j-1,t)} H_{k-1}^{(s+j,t+1)} / (\delta^{(t)} H_k^{(s+j,t)} H_k^{(s+j-1,t+1)})$ for $j = 1, 2, \dots, M$ as

$$\begin{aligned} U_{(M+1)(k-1)+j+1}^{(M\ell+j',t)} &= \frac{1}{\delta^{(t)}} \cdot \frac{K_{1,2,\dots,k+1}^{(j'+j-1)} K_{1,2,\dots,k-1}^{(j'+j)}}{K_{1,2,\dots,k}^{(j'+j)} K_{1,2,\dots,k}^{(j'+j-1)}} \qquad (4.55)\\ &\times \frac{(\lambda_1\lambda_2\cdots\lambda_{k+1})^\ell (\lambda_1\lambda_2\cdots\lambda_{k-1})^\ell}{(\lambda_1\lambda_2\cdots\lambda_k)^\ell (\lambda_1\lambda_2\cdots\lambda_k)^\ell} \\ &\times \frac{(\rho_1^{(t)}\rho_2^{(t)}\cdots\rho_{k+1}^{(t)})(\rho_1^{(t+1)}\rho_2^{(t+1)}\cdots\rho_{k-1}^{(t+1)})}{(\rho_1^{(t)}\rho_2^{(t)}\cdots\rho_k^{(t)})(\rho_1^{(t+1)}\rho_2^{(t+1)}\cdots\rho_k^{(t+1)})} \\ &\times \frac{(1+O(\varrho_{k+1}^t))(1+O(\varrho_{k-1}^{t+1}))}{(1+O(\varrho_k^t))(1+O(\varrho_k^{t+1}))}, \quad t \to \infty. \qquad (4.56) \end{aligned}$$

Considering $\rho_k^{(t)} = (\lambda_k - \mu^{(0)})(\lambda_k - \mu^{(1)})\cdots(\lambda_k - \mu^{(t-1)})$ in (4.56), we obtain, for $j = 1, 2, \dots, M$,

$$\begin{aligned} U_{(M+1)(k-1)+j+1}^{(M\ell+j',t)} &= \frac{1}{\delta^{(t)}} \cdot \frac{K_{1,2,\dots,k+1}^{(j'+j-1)} K_{1,2,\dots,k-1}^{(j'+j)}}{K_{1,2,\dots,k}^{(j'+j)} K_{1,2,\dots,k}^{(j'+j-1)}} \cdot \frac{\lambda_{k+1}^\ell}{\lambda_k^\ell(\lambda_k - \mu^{(t)})} \\ &\times \frac{(\lambda_{k+1}-\mu^{(0)})(\lambda_{k+1}-\mu^{(1)})\cdots(\lambda_{k+1}-\mu^{(t-1)})}{(\lambda_k-\mu^{(0)})(\lambda_k-\mu^{(1)})\cdots(\lambda_k-\mu^{(t-1)})} \\ &\times \frac{(1+O(\varrho_{k+1}^t))(1+O(\varrho_{k-1}^{t+1}))}{(1+O(\varrho_k^t))(1+O(\varrho_k^{t+1}))}, \quad t \to \infty, \qquad (4.57) \end{aligned}$$

which leads to (4.53). Similarly, in the cases where $j'+j=M$ and $j'+j>M$, from Theorem 4.3.5 and Lemma 2.3.6, it respectively follows that

$$
\begin{aligned}
U_{(M+1)(k-1)+j+1}^{(M\ell+j',t)} &= \frac{1}{\delta^{(t)}}\cdot\frac{K_{1,2,\ldots,k+1}^{(M-1)}K_{1,2,\ldots,k-1}^{(0)}}{K_{1,2,\ldots,k}^{(0)}K_{1,2,\ldots,k}^{(M-1)}}\cdot\frac{\lambda_{k+1}^{\ell}}{\lambda_k^{\ell+1}(\lambda_k-\mu^{(t)})}\\
&\times\frac{(\lambda_{k+1}-\mu^{(0)})(\lambda_{k+1}-\mu^{(1)})\cdots(\lambda_{k+1}-\mu^{(t-1)})}{(\lambda_k-\mu^{(0)})(\lambda_k-\mu^{(1)})\cdots(\lambda_k-\mu^{(t-1)})}\\
&\times\frac{(1+O(\varrho_{k+1}^t))(1+O(\varrho_{k-1}^{t+1}))}{(1+O(\varrho_k^t))(1+O(\varrho_k^{t+1}))},\quad t\to\infty,\quad j'+j=M, \quad (4.58)
\end{aligned}
$$

and

$$
\begin{aligned}
U_{(M+1)(k-1)+j+1}^{(M\ell+j',t)} &= \frac{1}{\delta^{(t)}}\cdot\frac{K_{1,2,\ldots,k+1}^{(j'+j-M-1)}K_{1,2,\ldots,k-1}^{(j'+j-M)}}{K_{1,2,\ldots,k}^{(j'+j-M)}K_{1,2,\ldots,k}^{(j'+j-M-1)}}\cdot\frac{\lambda_{k+1}^{\ell+1}}{\lambda_k^{\ell+1}(\lambda_k-\mu^{(t)})}\\
&\times\frac{(\lambda_{k+1}-\mu^{(0)})(\lambda_{k+1}-\mu^{(1)})\cdots(\lambda_{k+1}-\mu^{(t-1)})}{(\lambda_k-\mu^{(0)})(\lambda_k-\mu^{(1)})\cdots(\lambda_k-\mu^{(t-1)})}\\
&\times\frac{(1+O(\varrho_{k+1}^t))(1+O(\varrho_{k-1}^{t+1}))}{(1+O(\varrho_k^t))(1+O(\varrho_k^{t+1}))},\quad t\to\infty,\quad j'+j>M, \quad (4.59)
\end{aligned}
$$

which imply (4.53). □

According to Proposition 3.2.4, $\lambda_1,\lambda_2,\ldots,\lambda_m$ are eigenvalues of the Hessenberg matrix $A^{(0,0)}=L^{(0,0)}L^{(1,0)}\cdots L^{(M-1,0)}R^{(0,0)}$ involving $\{q_k^{(0,0)}\}_{k=1}^m$, $\{e_k^{(0,0)}\}_{k=1}^{m-1}$, $\{e_k^{(1,0)}\}_{k=1}^{m-1}$, $\ldots$, $\{e_k^{(M-1,0)}\}_{k=1}^{m-1}$. Theorems 4.4.1 and 4.4.3 thus suggest that the non-autonomous dhToda equation with initial settings $\{q_k^{(0,0)}\}_{k=1}^m$, $\{e_k^{(0,0)}\}_{k=1}^{m-1}$, $\{e_k^{(1,0)}\}_{k=1}^{m-1}$, $\ldots$, $\{e_k^{(M-1,0)}\}_{k=1}^{m-1}$ and the dhLV system with initial settings $\{U_{j-M}^{(0,0)}\}_{j=1}^M$ and $\{U_{(M+1)m+j-M}^{(0,0)}\}_{j=1}^M$ are both applicable to computing eigenvalues of the Hessenberg matrix $A^{(0,0)}$. This observation differs from Sumikura et al. [33] in that $\{q_k^{(0,0)}\}_{k=1}^m$, $\{e_k^{(0,0)}\}_{k=1}^{m-1}$, $\{e_k^{(1,0)}\}_{k=1}^{m-1}$, $\ldots$, $\{e_k^{(M-1,0)}\}_{k=1}^{m-1}$ are not restricted to be positive and so $A^{(0,0)}$ is not restricted to be TN. In other words, algorithms based on the non-autonomous dhToda equation and the dhLV system can generate eigenvalues of even the Hessenberg matrix $A^{(0,0)}$ involving some negative q's and e's values.

Here, we give an example to demonstrate asymptotic convergence without requiring positivity of some of the non-autonomous dhToda variables. We used a computer with Mac OS Sierra (ver. 10.12.4) and CPU: 3.1 GHz Intel Core i7 CPU. We employed floating-point arithmetic in the numerical computing software MATLAB (ver. 9.1.0.441655 (R2016b)). Let us consider the case where $M=3$, $m=3$, $q_1^{(0,0)}=-6$, $q_2^{(0,0)}=-6$, $q_3^{(0,0)}=-0.5$, $e_1^{(0,0)}=2$, $e_2^{(0,0)}=-6$, $e_1^{(1,0)}=1$, $e_2^{(1,0)}=5$, $e_1^{(2,0)}=-4$, and $e_2^{(2,0)}=-6.5$ in the non-autonomous dhToda equation.

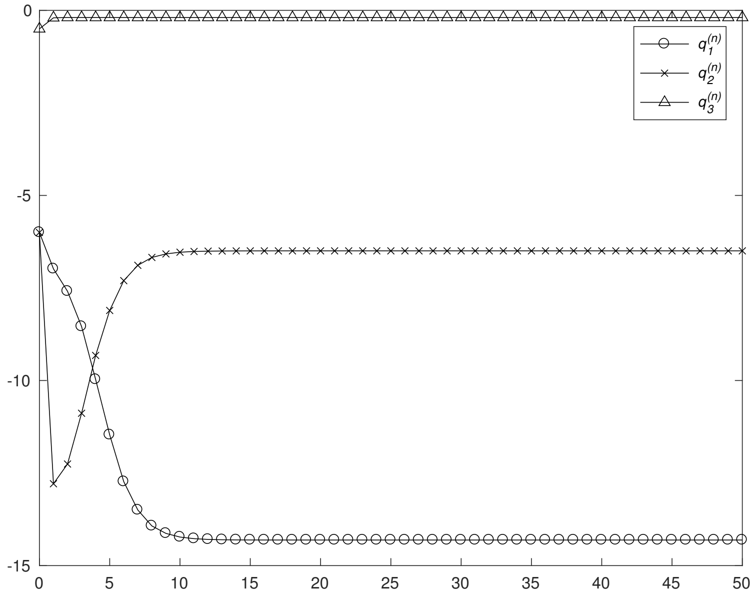

Figure 4.2: A graph of the discrete-time t (x-axis) and the logarithmical values of $q_1^{(0,t)}$, $q_2^{(0,t)}$ and $q_3^{(0,t)}$ (y-axis) in the non-autonomous dhToda equation with $M = 3$ and $m = 3$ under the initial settings $q_1^{(0,0)} = -6$, $q_2^{(0,0)} = -6$, $q_3^{(0,0)} = -0.5$, $e_1^{(0,0)} = 2$, $e_2^{(0,0)} = -6$, $e_1^{(1,0)} = 1$, $e_2^{(1,0)} = 5$, $e_1^{(2,0)} = -4$, and $e_2^{(2,0)} = -6.5$.

Furthermore, let $\mu^{(t)} = F_3^{(2,t)}$ where $\mu^{(0)} = 0.1$. Then the non-autonomous dhToda equation gives similarity transformations of the Hessenberg matrix

$$A^{(0,0)} = L^{(0,0)} L^{(1,0)} L^{(2,0)} R^{(0,0)} \\ = -\begin{pmatrix} 6 & -1 & 0 \\ -6 & 7 & -1 \\ -12 & -43 & 8 \end{pmatrix}.$$

It is remarkable that $\hat{A}^{(0,0)} := -A^{(0,0)}$ is not a TN matrix but an LR decomposable matrix. Square matrices are called M-matrices if they form $\alpha I - B$ for nonnegative matrices B and real numbers α that are larger than the spectrum radius of B [27]. Therefore, the $\hat{A}^{(0,0)}$ is also classified as a M-matrix. The M-matrix appears in, for example, gross substitutability, stability of a general equilibrium and Leontief's input-output analysis in economic systems [30]. Computed eigenvalues of the M-matrix $\hat{A}^{(0,0)}$ using the MATLAB function **eig** were $\hat{\lambda}_1 = 14.3089310318622$, $\hat{\lambda}_2 = 6.49746173237448$, and $\hat{\lambda}_3 = 0.193607235763316$. Figure 4.2 shows the convergence of $q_1^{(0,t)}$, $q_2^{(0,t)}$, and $q_3^{(0,t)}$ to $-\hat{\lambda}_1$, $-\hat{\lambda}_2$, and $-\hat{\lambda}_3$, respectively, as t grows larger. Since $q_1^{(0,50)} = -14.3089310318622$, $q_2^{(0,50)} = -6.49746173237441$, and $q_3^{(0,50)} = -0.193607235763316$, the relative differences from $-\hat{\lambda}_1$, $-\hat{\lambda}_2$ and $-\hat{\lambda}_3$ are considerably small.

Chapter 5

An extended Fibonacci sequence associated with the discrete hungry integrable systems

5.1 An extension of Fibonacci sequence

In this section, we clarify an unexpected relationship of an extension of the m-step Fibonacci sequence to the entries of the extended Hankel determinants in (2.20) under the discrete-time evolution from t to $t+1$ in the dhLV system (4.37) with the fixed parameters $\delta^{(t)}$ as suitable values.

The case of the dhLV system with $M = 1$, namely, the simple dLV system has been already discussed in [1]. The determinant solution to the dLV system is associated with the m-step Fibonacci sequence $\{F_k\}_{k=0}^{\infty}$ whose elements satisfy

$$F_{k+m} = F_k + F_{k+1} + \cdots + F_{k+m-1}, \quad k = 0, 1, \ldots, \tag{5.1}$$

for arbitrary integers $F_0, F_1, \ldots, F_{m-1}$. The 2-step Fibonacci sequence is often called the Fibonacci sequence simply and the 3-step Fibonacci sequence is also referred to as the Tribonacci sequence. From the viewpoint of an extension of the Fibonacci sequence, Brezinski and Lembarki [4] found interesting relationships between continued fractions and quadratic equations.

As an extension of the m-step Fibonacci sequence, let us introduce a new sequence $\{G_k\}_{k=0}^{\infty}$, which satisfies

$$G_{k+mM} = G_k + G_{k+M} + \cdots + G_{k+(m-1)M}, \quad k = 0, 1, \ldots, \tag{5.2}$$

for arbitrary integers $G_0, G_1, \ldots, G_{mM-1}$. In this paper, let us call $\{G_k\}_{k=0}^{\infty}$ the (m, M)-step Fibonacci sequence.

Hereinafter, let us consider the case where $\delta^{(t)} = -(\mu^{(t)})^{-1} = 1$. Then, it is obvious from (2.4) that

$$f_s^{(t+1)} = f_{s+M}^{(t)} + f_s^{(t)}. \tag{5.3}$$

The following lemma gives the evolution from t to $t+1$ in the extended Hankel determinant $H_m^{(s,t)}$ under the assumption that $\{f_s^{(t)}\}_{s=0}^{(M+1)m-1}$ in $H_k^{(s,t)}$ with $s = 0, 1, \ldots, M$ and $k = 1, 2, \ldots, m$ is the (m, M)-step Fibonacci sequence for some t.

Lemma 5.1.1. *For some t, let us assume that $\{f_s^{(t)}\}_{s=0}^{(M+1)m-1}$ is the (m, M)-step Fibonacci sequence. Then, it holds that, for even m,*

$$H_m^{(0,t+1)} = H_m^{(0,t)}, \tag{5.4}$$

and, for odd m,

$$H_m^{(0,t+1)} = 2H_m^{(0,t)}. \tag{5.5}$$

Proof. For even m, let $m = 2\ell$ for some positive integer ℓ. Then, the left hand side on (5.4) becomes

$$H_{2\ell}^{(0,t+1)} = \begin{vmatrix} f_0^{(t+1)} & f_1^{(t+1)} & \cdots & f_{2\ell-1}^{(t+1)} \\ f_M^{(t+1)} & f_{M+1}^{(t+1)} & \cdots & f_{M+2\ell-1}^{(t+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{(2\ell-1)M}^{(t+1)} & f_{(2\ell-1)M+1}^{(t+1)} & \cdots & f_{(2\ell-1)(M+1)}^{(t+1)} \end{vmatrix}. \tag{5.6}$$

It is obvious from (5.3) that $f_{(2\ell-1)M}^{(t+1)} = f_{(2\ell-1)M}^{(t)} + f_{2\ell M}^{(t)}$ in the $(2\ell, 1)$ entry of $H_{2\ell}^{(0,t+1)}$. Under the assumption, it holds that $f_{2\ell M}^{(t)} = f_0^{(t)} + f_M^{(t)} + \cdots + f_{(2\ell-1)M}^{(t)}$. Furthermore, by taking account that $f_0^{(t)} + f_M^{(t)} = f_0^{(t+1)}$, $f_{2M}^{(t)} + f_{3M}^{(t)} = f_{2M}^{(t+1)}, \ldots$, $f_{(2\ell-2)M}^{(t)} + f_{(2\ell-1)M}^{(t)} = f_{(2\ell-2)M}^{(t+1)}$, we get

$$f_{(2\ell-1)M}^{(t+1)} = f_{(2\ell-1)M}^{(t)} + f_0^{(t+1)} + f_{2M}^{(t+1)} + \cdots + f_{(2\ell-2)M}^{(t+1)}. \tag{5.7}$$

Similarly, in the $(2\ell, 2)$, the $(2\ell, 3)$, ..., the $(2\ell, 2\ell)$ entries, it follows that

$$\begin{aligned} f_{(2\ell-1)M+1}^{(t+1)} &= f_{(2\ell-1)M+1}^{(t)} + f_1^{(t+1)} + f_{2M+1}^{(t+1)} + \cdots + f_{(2\ell-2)M+1}^{(t+1)}, \\ f_{(2\ell-1)M+2}^{(t+1)} &= f_{(2\ell-1)M+2}^{(t)} + f_2^{(t+1)} + f_{2M+2}^{(t+1)} + \cdots + f_{(2\ell-2)M+2}^{(t+1)}, \\ &\ \ \vdots \\ f_{(2\ell-1)(M+1)}^{(t+1)} &= f_{(2\ell-1)(M+1)}^{(t)} + f_{2\ell-1}^{(t+1)} + f_{2M+2\ell-1}^{(t+1)} + \cdots + f_{(2\ell-2)M+2\ell-1}^{(t+1)}. \end{aligned} \tag{5.8}$$

It turns out from (5.6)–(5.8) that the 2ℓth row of $H_{2\ell}^{(0,t+1)}$ is equal to the sum of the 1st, the 3rd, …, the $(2\ell-1)$th rows and $(f_{(2\ell-1)M}^{(t)}, f_{(2\ell-1)M+1}^{(t)}, \dots, f_{(2\ell-1)M+2\ell-2}^{(t)}, f_{(2\ell-1)(M+1)}^{(t)})$. Thus, we can rewrite (5.6) as

$$H_{2\ell}^{(0,t+1)} = \begin{vmatrix} f_0^{(t+1)} & f_1^{(t+1)} & \cdots & f_{2\ell-2}^{(t+1)} & f_{2\ell-1}^{(t+1)} \\ f_M^{(t+1)} & f_{M+1}^{(t+1)} & \cdots & f_{M+2\ell-2}^{(t+1)} & f_{M+2\ell-1}^{(t+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{(2\ell-2)M}^{(t+1)} & f_{(2\ell-2)M+1}^{(t+1)} & \cdots & f_{(2\ell-2)(M+1)}^{(t+1)} & f_{(2\ell-2)M+2\ell-1}^{(t+1)} \\ f_{(2\ell-1)M}^{(t)} & f_{(2\ell-1)M+1}^{(t)} & \cdots & f_{(2\ell-1)M+2\ell-2}^{(t)} & f_{(2\ell-1)(M+1)}^{(t)} \end{vmatrix}$$

By subtracting the 2ℓth row from the $(2\ell-1)$th row in $H_{2\ell}^{(0,t+1)}$ and by taking account of (5.3), we may replace the $(2\ell-1)$th row of $H_{2\ell}^{(0,t+1)}$ with that of $H_{2\ell}^{(0,t)}$. Similarly, the value of $H_{2\ell}^{(0,t+1)}$ is not changed by replacing the $(2\ell-2)$th, the $(2\ell-3)$th, …, the 1st rows of in $H_{2\ell}^{(0,t+1)}$ with those of $H_{2\ell}^{(0,t)}$.

Next, let us consider the case where m is odd, namely, $m = 2\ell+1$. Then, the left hand side of (5.5) becomes

$$H_{2\ell+1}^{(0,t+1)} = \begin{vmatrix} f_0^{(t+1)} & f_1^{(t+1)} & \cdots & f_{2\ell}^{(t+1)} \\ f_M^{(t+1)} & f_{M+1}^{(t+1)} & \cdots & f_{M+2\ell}^{(t+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{2\ell M}^{(t+1)} & f_{2\ell M+1}^{(t+1)} & \cdots & f_{2\ell(M+1)}^{(t+1)} \end{vmatrix}. \tag{5.9}$$

Similar to the even case, by using (5.3) under the assumption, we derive

$$f_{2\ell M+k}^{(t+1)} = 2f_{2\ell M+k}^{(t)} + f_k^{(t+1)} + f_{2M+k}^{(t+1)} + \cdots + f_{(2\ell-2)M+k}^{(t+1)}, \quad k = 0, 1, \dots, 2\ell, \tag{5.10}$$

which differs from (5.7) in that the coefficient of $f_{2\ell M+k}^{(t)}$ is 2. Equation (5.10) suggests that the $(2\ell-1)$th row of $H_{2\ell+1}^{(0,t+1)}$ can be expressed by the sum of the 1st, the 3rd, …, the $(2\ell-3)$th rows and $(f_{2\ell M}^{(t)}, f_{2\ell M+1}^{(t)}, \dots, f_{2\ell(M+1)}^{(t)})$. Thus, it follows that

$$H_{2\ell+1}^{(0,t+1)} = 2\begin{vmatrix} f_0^{(t+1)} & f_1^{(t+1)} & \cdots & f_{2\ell-1}^{(t+1)} & f_{2\ell}^{(t+1)} \\ f_M^{(t+1)} & f_{M+1}^{(t+1)} & \cdots & f_{M+2\ell-1}^{(t+1)} & f_{M+2\ell}^{(t+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{(2\ell-1)M}^{(t)} & f_{(2\ell-1)M+1}^{(t)} & \cdots & f_{(2\ell-1)(M+1)}^{(t)} & f_{(2\ell-1)(M+1)+1}^{(t)} \\ f_{2\ell M}^{(t)} & f_{2\ell M+1}^{(t)} & \cdots & f_{2\ell(M+1)-1}^{(t)} & f_{2\ell(M+1)}^{(t)} \end{vmatrix}. \tag{5.11}$$

By using (5.3), we can easily check that the right hand side of (5.11) is equal to $2H_{2\ell+1}^{(0,t)}$. Therefore, we have (5.5). □

Let us recall here that $H_k^{(s,t)} = C_k^{(s,t)}$. Thus, in the case where $\delta^{(t)} = 1$, it immediately follows that, for even m,

$$C_m^{(0,t+1)} = C_m^{(0,t)}, \tag{5.12}$$

and, for odd m,

$$C_m^{(0,t+1)} = 2C_m^{(0,t)}. \tag{5.13}$$

The following lemma gives relationships among the Casorati determinants $\{C_k^{(s,t)}\}$ under the evolution from t to $t+1$.

Lemma 5.1.2. *The elements of the sequences* $\{C_k^{(s,t)}\}_{s=0}^M$ *and* $\{C_k^{(s,t+1)}\}_{s=0}^M$ *satisfy*

$$C_{k+1}^{(s,t)} C_{k-1}^{(s+1,t+1)} = C_k^{(s,t)} C_k^{(s+1,t+1)} - C_k^{(s,t+1)} C_k^{(s+1,t)}. \tag{5.14}$$

Proof. By letting $i_1 = j_1 = 1$, $i_2 = j_2 = k+1$ and $D = C_{k+1}^{(s,t)}$ in (3.15), we easily get

$$\begin{aligned} &C_{k+1}^{(s,t)} C_{k+1}^{(s,t)} \begin{bmatrix} 1 & k+1 \\ 1 & k+1 \end{bmatrix} \\ &= C_{k+1}^{(s,t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} C_{k+1}^{(s,t)} \begin{bmatrix} k+1 \\ k+1 \end{bmatrix} - C_{k+1}^{(s,t)} \begin{bmatrix} k+1 \\ 1 \end{bmatrix} C_{k+1}^{(s,t)} \begin{bmatrix} 1 \\ k+1 \end{bmatrix}. \end{aligned} \tag{5.15}$$

The left hand side and the right hand side of (5.15) can be rewritten as, respectively,

$$C_{k+1}^{(s,t)} \begin{vmatrix} f_{s+1}^{(t+1)} & f_{s+2}^{(t+1)} & \cdots & f_{s+k-1}^{(t+1)} \\ f_{s+1}^{(t+2)} & f_{s+2}^{(t+2)} & \cdots & f_{s+k-1}^{(t+2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s+1}^{(t+k-1)} & f_{s+2}^{(t+k-1)} & \cdots & f_{s+k-1}^{(t+k-1)} \end{vmatrix} = C_{k+1}^{(s,t)} C_{k-1}^{(s+1,t+1)},$$

$$\begin{aligned} &\begin{vmatrix} f_{s+1}^{(t+1)} & f_{s+2}^{(t+1)} & \cdots & f_{s+k}^{(t+1)} \\ f_{s+1}^{(t+2)} & f_{s+2}^{(t+2)} & \cdots & f_{s+k}^{(t+2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s+1}^{(t+k-1)} & f_{s+2}^{(t+k-1)} & \cdots & f_{s+k}^{(t+k-1)} \end{vmatrix} \begin{vmatrix} f_{s}^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-1}^{(t)} \\ f_{s}^{(t+1)} & f_{s+1}^{(t+1)} & \cdots & f_{s+k-1}^{(t+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s}^{(t+k-1)} & f_{s+1}^{(t+k-1)} & \cdots & f_{s+k-1}^{(t+k-1)} \end{vmatrix} \\ &\quad - \begin{vmatrix} f_{s+1}^{(t)} & f_{s+2}^{(t)} & \cdots & f_{s+k}^{(t)} \\ f_{s+1}^{(t+1)} & f_{s+2}^{(t+1)} & \cdots & f_{s+k}^{(t+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s+1}^{(t+k-1)} & f_{s+2}^{(t+k-1)} & \cdots & f_{s+k}^{(t+k-1)} \end{vmatrix} \begin{vmatrix} f_{s}^{(t+1)} & f_{s+1}^{(t+1)} & \cdots & f_{s+k-1}^{(t+1)} \\ f_{s}^{(t+2)} & f_{s+1}^{(t+2)} & \cdots & f_{s+k-1}^{(t+2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s}^{(t+k)} & f_{s+1}^{(t+k)} & \cdots & f_{s+k-1}^{(t+k)} \end{vmatrix} \\ &= C_k^{(s,t)} C_k^{(s+1,t+1)} - C_k^{(s,t+1)} C_k^{(s+1,t)}. \end{aligned}$$

□

With the help of (5.12), (5.13) and Lemma 5.1.2, we derive a lemma for the evolution from $C_m^{(s,t)}$ to $C_m^{(s,t+1)}$ for $i = 1, 2, \ldots, M$.

Lemma 5.1.3. *Let us assume that* $\{f_s^{(t)}\}_{s=0}^{(M+1)m-1}$ *is the* (m, M)*-step Fibonacci sequence. Furthermore, let* $C_m^{(s,t)}$ *for* $s = 1, 2, \ldots, M$ *be nonzero constants. Then, it holds that, for even* m*,*

$$C_m^{(s,t+1)} = C_m^{(s,t)}, \quad s = 1, 2, \ldots, M, \tag{5.16}$$

and, for odd m*,*

$$C_m^{(s,t+1)} = 2C_m^{(s,t)}, \quad s = 1, 2, \ldots, M. \tag{5.17}$$

Proof. By letting $k = m$ in Lemma 5.1.2 and by using $C_{m+1}^{(s,t)} = 0$, we get

$$C_m^{(s,t)} C_m^{(s+1,n+1)} - C_m^{(s,t+1)} C_m^{(s+1,t)} = 0. \tag{5.18}$$

Let us consider two cases where $m = 2\ell$ and $m = 2\ell + 1$ for some positive integer ℓ. Equations (5.12), (5.13) and (5.18) with $s = 0$ lead to

$$C_{2\ell}^{(0,t)} C_{2\ell}^{(1,t+1)} = C_{2\ell}^{(0,t)} C_{2\ell}^{(1,t)}, \tag{5.19}$$
$$C_{2\ell+1}^{(0,t)} C_{2\ell+1}^{(1,t+1)} = 2C_{2\ell+1}^{(0,t)} C_{2\ell+1}^{(1,t)}. \tag{5.20}$$

Under the assumption that $C_{2\ell}^{(0,t)} \neq 0$ in (5.19) and $C_{2\ell+1}^{(0,t)} \neq 0$ in (5.20), we derive $C_{2\ell}^{(1,t+1)} = C_{2\ell}^{(1,t)}$ and $C_{2\ell+1}^{(1,t+1)} = 2C_{2\ell+1}^{(1,t)}$. Let us assume that (5.16) and (5.17) hold for $s = s' - 1$, namely, $C_{2\ell}^{(s'-1,t+1)} = C_{2\ell}^{(s'-1,t)}$ and $C_{2\ell+1}^{(s'-1,t+1)} = 2C_{2\ell+1}^{(s'-1,t)}$. Then, (5.18) with $s = s' - 1$ gives

$$C_{2\ell}^{(s'-1,t)} C_{2\ell}^{(s',t+1)} = C_{2\ell}^{(s'-1,t)} C_{2\ell}^{(s',t)}, \tag{5.21}$$
$$C_{2\ell+1}^{(s'-1,t)} C_{2\ell+1}^{(s',t+1)} = 2C_{2\ell+1}^{(s'-1,t)} C_{2\ell+1}^{(s',t)}. \tag{5.22}$$

Since $C_{2\ell}^{(s'-1,t)} \neq 0$ and $C_{2\ell+1}^{(s'-1,t)} \neq 0$, it follows that $C_{2\ell}^{(s',t+1)} = C_{2\ell}^{(s',t)}$ and $C_{2\ell+1}^{(s',t+1)} = 2C_{2\ell+1}^{(s',t)}$. Thus, by induction for $s = 0, 1, \ldots,$ we have (5.16) and (5.17). □

Lemma 5.1.3 yields the following proposition concerning the entries of $C_k^{(s,t+1)}$ for $s = 0, 1, \ldots, M$ and $k = 1, 2, \ldots, m$.

Proposition 5.1.4. *Let us assume that* $\{f_s^{(t)}\}_{s=0}^{(M+1)m-1}$ *is the* (m, M)*-step Fibonacci sequence. Furthermore, let* $C_m^{(s,t)}$ *for* $s = 1, 2, \ldots, M$ *be nonzero constants. Then,* $\{f_s^{(t+1)}\}_{s=0}^{(M+1)m-1}$ *is also the* (m, M)*-step Fibonacci sequence, namely,*

$$f_{i+mM}^{(t+1)} = f_i^{(t+1)} + f_{i+M}^{(t+1)} + \cdots + f_{i+(m-1)M}^{(t+1)}, \quad i = 0, 1, \ldots, m-1. \tag{5.23}$$

Proof. Under the assumption that $\{f_s^{(t)}\}_{s=0}^{(M+1)m-1}$ is the (m, M)-step Fibonacci sequence, the cases where $i = 0, 1, \ldots, m-M-1$ in (5.23) are easily shown by using (5.3) as

$$\begin{aligned}
f_{i+mM}^{(t+1)} &= f_{i+mM}^{(t)} + f_{i+(m+1)M}^{(t)} \\
&= (f_i^{(t)} + f_{i+M}^{(t)} + \cdots + f_{i+(m-1)M}^{(t)}) + (f_{i+M}^{(t)} + f_{i+2M}^{(t)} + \cdots + f_{i+mM}^{(t)}) \\
&= (f_i^{(t)} + f_{i+M}^{(t)}) + (f_{i+M}^{(t)} + f_{i+2M}^{(t)}) + \cdots + (f_{i+(m-1)M}^{(t)} + f_{i+mM}^{(t)}) \\
&= f_i^{(t+1)} + f_{i+M}^{(t+1)} + \cdots + f_{i+(m-1)M}^{(t+1)}.
\end{aligned}$$

The remainder is to prove the cases where $i = m-M, m-M+1, \ldots, m-1$ in (5.23). The proof is hereinafter divided into two cases where $M \leq m-1$ and $M > m-1$. The case where $M = 1$ in [1] is a special case of the former. The latter case is essentially new. Furthermore, it is necessary to consider whether m is even or odd.

We firstly investigate the case where $M \leq m-1$ with $m = 2\ell$ for some positive integer ℓ. By adding the 2nd to the 1st, the 3rd to the 2nd, …, the 2ℓth to the $(2\ell-1)$th, in the rows of $H_{2\ell}^{(1,t)}$ and by applying (5.3) to it, we get

$$H_{2\ell}^{(1,t)} = \begin{vmatrix}
f_1^{(t+1)} & f_2^{(t+1)} & \cdots & f_{2\ell}^{(t+1)} \\
f_{M+1}^{(t+1)} & f_{M+2}^{(t+1)} & \cdots & f_{M+2\ell}^{(t+1)} \\
\vdots & \vdots & \ddots & \vdots \\
f_{(2\ell-2)M+1}^{(t+1)} & f_{(2\ell-2)M+2}^{(t+1)} & \cdots & f_{(2\ell-2)M+2\ell}^{(t+1)} \\
f_{(2\ell-1)M+1}^{(t)} & f_{(2\ell-1)M+2}^{(t)} & \cdots & f_{(2\ell-1)M+2\ell}^{(t)}
\end{vmatrix}. \tag{5.24}$$

Furthermore, by adding the 1st, the 3rd, …, the $(2\ell-1)$th rows to the 2ℓth one in (5.24), by using (5.3) and by taking account of the assumption, we can rewrite the $(2\ell, 1)$, the $(2\ell, 2)$, …, the $(2\ell, 2\ell-1)$ entries of $H_{2\ell}^{(1,t)}$ as

$$\begin{aligned}
&f_{(2\ell-1)M+j}^{(t)} + (f_j^{(t+1)} + f_{2M+j}^{(t+1)} + \cdots + f_{(2\ell-2)M+j}^{(t+1)}) \\
&\quad = f_{(2\ell-1)M+j}^{(t)} + \Big[(f_j^{(t)} + f_{M+j}^{(t)}) + (f_{2M+j}^{(t)} + f_{3M+j}^{(t)}) \\
&\qquad + \cdots + (f_{(2\ell-2)M+j}^{(t)} + f_{(2\ell-1)M+j}^{(t)})\Big] \\
&\quad = f_{(2\ell-1)M+j}^{(t)} + f_{2\ell M+j}^{(t)} \\
&\quad = f_{(2\ell-1)M+j}^{(t+1)}, \quad j = 1, 2, \ldots, 2\ell-1.
\end{aligned}$$

Simultaneously, the $(2\ell, 2\ell)$ entry can be transformed into

$$\begin{aligned}
&f^{(t)}_{(2\ell-1)M+2\ell} + (f^{(t+1)}_{2\ell} + f^{(t+1)}_{2M+2\ell} + \cdots + f^{(t+1)}_{(2\ell-2)M+2\ell}) \\
&\quad = \Big[(f^{(t)}_{-M+2\ell} + f^{(t)}_{2\ell}) + (f^{(t)}_{M+2\ell} + f^{(t)}_{2M+2\ell}) \\
&\qquad + \cdots + (f^{(t)}_{(2\ell-3)M+2\ell} + f^{(t)}_{(2\ell-2)M+2\ell})\Big] \\
&\qquad + (f^{(t+1)}_{2\ell} + f^{(t+1)}_{2M+2\ell} + \cdots + f^{(t+1)}_{(2\ell-2)M+2\ell}) \\
&\quad = (f^{(t+1)}_{-M+2\ell} + f^{(t+1)}_{M+2\ell} + \cdots + f^{(t+1)}_{(2\ell-3)M+2\ell}) \\
&\qquad + (f^{(t+1)}_{2\ell} + f^{(t+1)}_{2M+2\ell} + \cdots + f^{(t+1)}_{(2\ell-2)M+2\ell}) \\
&\quad = f^{(t+1)}_{-M+2\ell} + f^{(t+1)}_{2\ell} + \cdots + f^{(t+1)}_{(2\ell-2)M+2\ell}.
\end{aligned}$$

Thus, it follows that

$$H^{(1,t)}_{2\ell} = \begin{vmatrix}
f^{(t+1)}_{1} & f^{(t+1)}_{2} & \cdots & f^{(t+1)}_{2\ell-1} & f^{(t+1)}_{2\ell} \\
f^{(t+1)}_{M+1} & f^{(t+1)}_{M+2} & \cdots & f^{(t+1)}_{M+2\ell-1} & f^{(t+1)}_{M+2\ell} \\
\vdots & \vdots & \ddots & \vdots & \\
f^{(t+1)}_{(2\ell-2)M+1} & f^{(t+1)}_{(2\ell-2)M+2} & \cdots & f^{(t+1)}_{(2\ell-2)M+2\ell-1} & f^{(t+1)}_{(2\ell-2)M+2\ell} \\
f^{(t+1)}_{(2\ell-1)M+1} & f^{(t+1)}_{(2\ell-1)M+2} & \cdots & f^{(t+1)}_{(2\ell-1)M+2\ell-1} & f^{(t+1)}_{-M+2\ell} + f^{(t+1)}_{2\ell} + \cdots + f^{(t+1)}_{(2\ell-2)M+2\ell}
\end{vmatrix}. \tag{5.25}$$

According to Lemma 5.1.3, $C^{(1,t)}_{2\ell}$ is also equal to $C^{(1,t+1)}_{2\ell}$ under the assumption $C^{(1,t)}_{2\ell}$ is nonzero constant. By taking account that $C^{(s,t)}_{k} = H^{(s,t)}_{k}$, we find $H^{(1,t)}_{2\ell} = H^{(1,t+1)}_{2\ell}$. Therefore, by combining it with (5.25), we derive $f^{(t+1)}_{2\ell-M+2\ell M} = f^{(t+1)}_{2\ell-M} + f^{(t+1)}_{2\ell} + \cdots + f^{(t+1)}_{2\ell-M+(2\ell-1)M}$ which coincides with the case where $i = 2\ell - M$ in (5.23). By observing the $(2\ell, 2\ell)$ entries of the equalities $C^{(2,t)}_{2\ell} = C^{(2,t+1)}_{2\ell}$, $C^{(3,t)}_{2\ell} = C^{(3,t+1)}_{2\ell}, \ldots, C^{(M,t)}_{2\ell} = C^{(M,t+1)}_{2\ell}$, we inductively have (5.23) with the cases where $i = 2\ell - M + 1, 2\ell - M + 2, \ldots, 2\ell - 1$.

Similarly, in the case where $m = 2\ell + 1$, it follows that

$$
\begin{aligned}
&H_{2\ell+1}^{(1,t)} \\
&= \begin{vmatrix}
f_1^{(t+1)} & f_2^{(t+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(t+1)} \\
f_{M+1}^{(t+1)} & f_{M+2}^{(t+1)} & \cdots & f_{M+2\ell}^{(t+1)} & f_{M+2\ell+1}^{(t+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f_{(2\ell-1)M+1}^{(t+1)} & f_{(2\ell-1)M+2}^{(t+1)} & \cdots & f_{(2\ell-1)M+2\ell}^{(t+1)} & f_{(2\ell-1)M+2\ell+1}^{(t+1)} \\
f_{2\ell M+1}^{(t+1)} & f_{2\ell M+2}^{(t+1)} & \cdots & f_{2\ell M+2\ell}^{(t+1)} & f_{-M+2\ell}^{(t+1)} + f_{2\ell}^{(t+1)} + \cdots + f_{(2\ell-2)M+2\ell}^{(t+1)}
\end{vmatrix} \\
&\quad - \begin{vmatrix}
f_1^{(t+1)} & f_2^{(t+1)} & \cdots & f_{2\ell}^{(t+1)} & f_{2\ell+1}^{(t+1)} \\
f_{M+1}^{(t+1)} & f_{M+2}^{(t+1)} & \cdots & f_{M+2\ell}^{(t+1)} & f_{M+2\ell+1}^{(t+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f_{(2\ell-1)M+1}^{(t+1)} & f_{(2\ell-1)M+2}^{(t+1)} & \cdots & f_{(2\ell-1)M+2\ell}^{(t+1)} & f_{(2\ell-1)M+2\ell+1}^{(t+1)} \\
f_{2\ell M+1}^{(t)} & f_{2\ell+M+2}^{(t)} & \cdots & f_{2\ell M+2\ell}^{(t)} & f_{2\ell M+2\ell+1}^{(t)}
\end{vmatrix}. \qquad (5.26)
\end{aligned}
$$

By taking account that the 2nd term on the right hand side is equal to $-H_{2\ell+1}^{(1,t)}$, we get

$$
\begin{aligned}
&2H_{2\ell+1}^{(1,t)} \\
&= \begin{vmatrix}
f_1^{(t+1)} & f_2^{(t+1)} & \cdots & f_{2\ell}^{(t+1)} & f_{2\ell+1}^{(t+1)} \\
f_{M+1}^{(t+1)} & f_{M+2}^{(t+1)} & \cdots & f_{M+2\ell}^{(t+1)} & f_{M+2\ell+1}^{(t+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f_{(2\ell-1)M+1}^{(t+1)} & f_{(2\ell-1)M+2}^{(t+1)} & \cdots & f_{(2\ell-1)M+2\ell}^{(t+1)} & f_{(2\ell-1)M+2\ell+1}^{(t+1)} \\
f_{2\ell M+1}^{(t+1)} & f_{2\ell M+2}^{(t+1)} & \cdots & f_{2\ell M+2\ell}^{(t+1)} & f_{-M+2\ell}^{(t+1)} + f_{2\ell}^{(t+1)} + \cdots + f_{(2\ell-2)M+2\ell}^{(t+1)}
\end{vmatrix}.
\end{aligned}
\qquad (5.27)
$$

Equation (5.27) with the equality $2C_{2\ell}^{(1,t)} = C_{2\ell}^{(1,t+1)}$ in Lemma 5.1.3 leads to the case where $i = 2\ell - M + 1$ in (5.23). By grasping the $(2\ell+1, 2\ell+1)$ entries of $C_{2\ell+1}^{(2,t+1)}, C_{2\ell+1}^{(3,t+1)}, \ldots, C_{2\ell+1}^{(M,t+1)}$, we complete the proof for the case where $M \leq m-1$ with $m = 2\ell + 1$.

We next discuss the case where $M > m-1$. The proof is essentially given along the line similar to the case where $M \leq m-1$. The different point is to observe the (m, m) entries of not $C_m^{(1,t)}, C_m^{(2,t)}, \ldots, C_m^{(M,t)}$ but $C_m^{(M-m+1,t)}$, $C_m^{(M-m+2,t)}, \ldots, C_m^{(M,t)}$. Thus, by starting proof from $C_m^{(M-m+1,t)}$, we inductively have (5.23) for the case $M > m-1$. □

Let us turn to a conserved quantity of the dhLV system (4.37),

$$\sum_{k=1}^{(M+1)m-M} v_k^{(s,t)} = \sum_{k=1}^{(M+1)m-M} v_k^{(s,t+1)}. \tag{5.28}$$

The replacement of $v_k^{(s,t)}$ for $w_k^{(t)}$ in (5.28) coincides with the conserved quantity in [7]. From the viewpoint of (5.28), we next give a lemma concerning the nonzero boundedness of $C_k^{(s,t)}$ with $s = 0, 1, \dots, M$ and $k = 1, 2, \dots, m$.

Lemma 5.1.5. *Let us assume that $\{f_s^{(0)}\}_{s=0}^{(M+1)m-1}$ are the (m, M)-step Fibonacci sequence. Furthermore, let $C_k^{(s,0)}$ with $s = 0, 1, \dots, M$ and $k = 1, 2, \dots, m$ be nonzero constants. Then, $C_k^{(s,1)}, C_k^{(s,2)}, \dots$ with $s = 0, 1, \dots, M$ and $k = 1, 2, \dots, m$ are also nonzero constants.*

Proof. For some t, let us assume that $C_k^{(s,t)}$ with $s = 0, 1, \dots, M$ and $k = 1, 2, \dots, m$ are nonzero constants. Then, it is obvious that the dhLV variable $v_k^{(0,t)}$ and the sum $\sum_{k=1}^{(M+1)m-M} v_k^{(0,t)}$ are both nonzero constants. Thus, by combining it with (5.28), we see that the sum $\sum_{k=1}^{(M+1)m-M} v_k^{(0,t+1)}$ is also nonzero constant. It is worth noting that $\sum_{k=1}^{(M+1)m-M} v_k^{(0,t+1)}$ is expressed in terms of the Casorati determinants as

$$\sum_{k=1}^{(M+1)m-M} v_k^{(0,t+1)}$$
$$= \sum_{k=1}^{m-1} \left(\frac{C_k^{(M,t+1)} C_{k-1}^{(0,t+1)}}{C_{k-1}^{(M,t+1)} C_k^{(0,t+1)}} + \sum_{i=0}^{M-1} \frac{C_{k+1}^{(i,t+1)} C_{k-1}^{(i+1,t+1)}}{C_k^{(i,t+1)} C_k^{(i+1,t+1)}} \right) + \frac{C_m^{(M,t+1)} C_{m-1}^{(0,t+1)}}{C_{m-1}^{(M,t+1)} C_m^{(0,t+1)}}. \tag{5.29}$$

By taking account of the boundedness of the Casorati determinants, we therefore conclude that the Casorati determinants in the denominators are nonzero constant. The proof is completed by induction for $t = 0, 1, \dots$. □

By combining Proposition 5.1.4 with Lemma 5.1.5, we have the main theorem in this section.

Theorem 5.1.6. *Let us assume that $\{f_s^{(0)}\}_{s=0}^{(M+1)m-1}$ is the (m, M)-step Fibonacci sequence. Furthermore, let $C_k^{(s,0)}$ with $s = 0, 1, \dots, M$ and $k = 1, 2, \dots, m$ be nonzero constants. Then, $\{f_s^{(t)}\}_{s=0}^{(M+1)m-1}$ for $t = 1, 2, \dots$ are also the (m, M)-step Fibonacci sequences, namely,*

$$f_{i+mM}^{(t)} = f_i^{(t)} + f_{i+M}^{(t)} + \cdots + f_{i+(m-1)M}^{(t)}, \quad i = 0, 1, \dots, m-1, \quad t = 1, 2, \dots. \tag{5.30}$$

5.2 Convergence to the ratio of two successive extended Fibonacci numbers

It is well known that the ratio F_{j+1}/F_j tends to the golden ratio as $j \to \infty$ in the standard Fibonacci sequence $\{F_j\}_{j=0}^{\infty}$ [6]. In the case of the m-step Fibonacci sequence, it is shown in [24] that, as $j \to \infty$, the ratio F_{j+1}/F_j converges to a real solution c_m to the algebraic equation $x^m - x^{m-1} - \cdots - x - 1 = 0$. The convergence of one of the dLV variables to c_m as $t \to \infty$ is also proved in [1] under a suitable initial setting of the dLV system. In this section, we investigate the convergence associated with the (m, M)-step Fibonacci sequence in the dhLV system (4.37) after finding an algebraic equation associated with the (m, M)-step Fibonacci sequence.

Let us begin with giving a lemma for a relationship between the (m, M)-step Fibonacci sequence $\{G_k\}_{k=0}^{\infty}$ and an algebraic equation.

Lemma 5.2.1. *Let us assume that the ratio G_{k+1}/G_k converges to a constant d_m as $k \to \infty$. Then, d_m is one of the real solutions to the mM-degree algebraic equation*

$$x^{mM} - x^{(m-1)M} - \cdots - x^M - 1 = 0. \tag{5.31}$$

Proof. The assumption $G_{k+1}/G_k \to d_m$ as $k \to \infty$ immediately leads to

$$\begin{aligned}\lim_{k\to\infty} \frac{G_{k+\ell}}{G_k} &= \lim_{k\to\infty} \frac{G_{k+1}}{G_k} \lim_{k\to\infty} \frac{G_{k+2}}{G_{k+1}} \cdots \lim_{k\to\infty} \frac{G_{k+\ell}}{G_{k+\ell-1}} \\ &= d_m^{\ell}. \end{aligned} \tag{5.32}$$

The power of d_m corresponds to the subscript ℓ in the ratio $G_{k+\ell}/G_k$. By taking account that $G_{k+mM} = G_k + G_{k+M} + \cdots + G_{k+(m-1)M}$, we derive

$$\lim_{k\to\infty} \frac{G_{k+mM}}{G_k} = \lim_{k\to\infty} \left(1 + \frac{G_{k+M}}{G_k} + \cdots + \frac{G_{k+(m-1)M}}{G_k}\right). \tag{5.33}$$

Thus, by combining (5.32) with (5.33), we thus have

$$d_m^{mM} - d_m^{(m-1)M} - \cdots - d_m^M - 1 = 0,$$

which implies that d_m is one of the real solutions to the mM-degree algebraic equation (5.31). □

The replacement of x^M for x in the mM-degree algebraic equation (5.31) generates the m-degree algebraic equation $x^m - x^{m-1} - \cdots - 1 = 0$ associated with the m-step Fibonacci sequence $\{F_k\}_{k=1}^{\infty}$. So, it turns out that d_m is the Mth root of $c_m = \lim_{j\to\infty} F_{j+1}/F_j$.

The dhLV system (4.37) with $\delta^{(t)} = 1$ and $s = 0$ can be rewritten as

$$\begin{cases} U_k^{(0,t+1)} = \displaystyle\prod_{\ell=1}^{M} \frac{1 + U_{k+\ell}^{(0,t)}}{1 + U_{k-\ell}^{(0,t+1)}} U_k^{(0,t)}, \quad k = 1, 2, \ldots, (M+1)m - M, \\ U_{j-M}^{(0,t)} := 0, \quad U_{(M+1)m+j-M}^{(0,t)} := 0, \quad j = 1, 2, \ldots, M, \\ t = 0, 1, \ldots, \end{cases} \tag{5.34}$$

The variable $v_k^{(s,t)}$ in (4.32) with the replacements $u_k^{(s,t)} = U_k^{(s,t)}/(\kappa^{(t)})^M$ and $(\kappa^{(t)})^{M+1} = -1/\delta^{(t)}$ can be also expressed as

$$v_k^{(s,t)} = U_k^{(s,t)} \prod_{\ell=1}^{M} (1 + U_{k-\ell}^{(s,t)}), \quad k = -M+1, -M+2, \ldots, (M+1)m. \tag{5.35}$$

It is of significance to note that $U_1^{(0,t)} = v_1^{(0,t)} = f_M^{(t)}/f_0^{(t)}$. With the help of Theorem 5.1.6 and Lemma 5.2.1, we finally have a theorem for asymptotic convergence of $U_1^{(0,t)}$ as $t \to \infty$ in the dhLV system (5.34).

Theorem 5.2.2. *Let us assume that* $\{f_s^{(t)}\}_{s=0}^{(M+1)m-1}$ *is the* (m, M)*-step Fibonacci sequence. Furthermore, let* $C_k^{(s,t)}$ *with* $s = 0, 1, \ldots, M$ *and* $k = 1, 2, \ldots, m$ *be nonzero constants. Then,* $U_1^{(0,t)} = f_M^{(t)}/f_0^{(t)}$ *converges to* $c_m = \lim_{k\to\infty} G_{k+M}/G_k$*, which is a real solution to the* mM*-degree algebraic equation* (5.31)*, as* $t \to \infty$*.*

As an example, let us consider the case where $m = 2$, $M = 3$ and $f_0^{(0)} = 1$, $f_1^{(0)} = 2$, $f_2^{(0)} = 3$, $f_3^{(0)} = 1$, $f_4^{(0)} = 6$, $f_5^{(0)} = 7$, $f_6^{(0)} = 2$, $f_7^{(0)} = 8$. It is easy to check that $\{f_s^{(t)}\}_{s=0}^{7}$ is the $(2, 3)$-step Fibonacci sequence. The Casorati determinants $C_1^{(0,0)}$, $C_1^{(1,0)}$, $C_1^{(2,0)}$, $C_1^{(3,0)}$, $C_2^{(0,0)}$, $C_2^{(1,0)}$, $C_2^{(2,0)}$, $C_2^{(3,0)}$ are nonzero constants such that $C_1^{(0,0)} = 1$, $C_1^{(1,0)} = 2$, $C_1^{(2,0)} = 3$, $C_1^{(3,0)} = 1$, $C_2^{(0,0)} = 4$, $C_2^{(1,0)} = -4$, $C_2^{(2,0)} = -1$, $C_2^{(3,0)} = -4$. Furthermore, from considering Lemma 4.3.5 with $s = 0$ and Lemma 5.1.3, we get $U_1^{(0,0)} = 1$, $U_2^{(0,0)} = 1$, $U_3^{(0,0)} = -1/6$, $U_4^{(0,0)} = -1/10$, $U_5^{(0,0)} = -2/3$ as an initial setting in the dhLV system (5.34) with $s = 0$. A demonstration was carried out with the scientific computing software Maple 15 on our computer with OS: Windows 7 Professional, CPU: Intel Core i5-2540M 2.60GHz and Memory: 8GB. Figuare 5.1 shows the values of $U_1^{(0,0)}$, $U_1^{(0,1)}$, $\ldots$, $U_1^{(0,30)}$ denoted by the symbol $\times$. Figuare 5.1 suggests $U_1^{(0,0)}, U_1^{(0,1)}, \ldots, U_1^{(0,30)}$ are given without overflow by repeating the evolution from t to $t+1$ in the dhLV system (5.34) and $U_1^{(0,t)}$ converges to some positive constant c_2 as $t \to \infty$. It is verified that $c_2 \approx 1.6180339887498951$ is the golden ratio and coincides with one of the real solution to $x^2 - x - 1 = 0$.

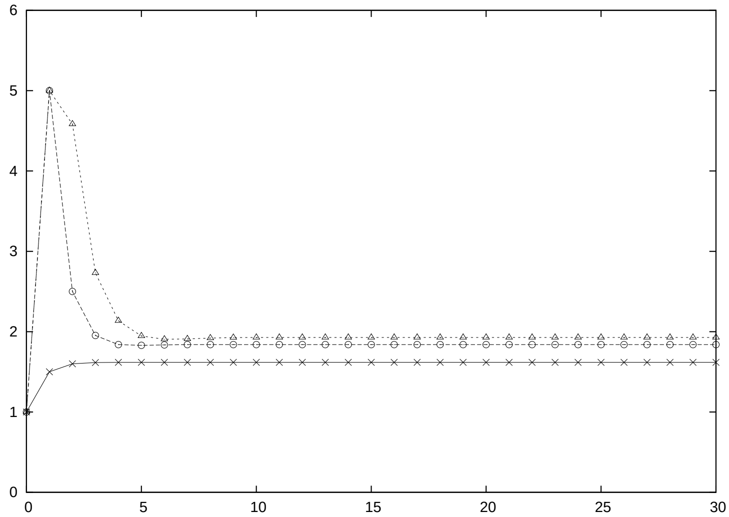

Figure 5.1: A graph of the disrete-time t (x-axis) and the value of $U_1^{(0,t)}$ (y-axis) in the dhLV system (5.34) with $M = 3$ and $m = 2, 3, 4$. The symbols Cross $\times$, Circle $\circ$ and Triangle $\triangle$ are in the cases $m = 2, 3$ and 4, respectively.

Similarly, in the cases where $\{f_s^{(0)}\}_{s=0}^{11}$ is the $(3, 3)$-step Fibonacci sequence and $\{f_s^{(0)}\}_{s=0}^{14}$ is the $(4, 3)$-step one, it is observed in Figuare 5.1 that $U_1^{(0,t)} \to c_3 \approx 1.8371367696097378$ and $U_1^{(0,t)} \to c_4 \approx 1.9275619754829241$ as $t \to \infty$, respectively. It is also realized that c_3 and c_4 are equal to one of positive solutions to $x^3 - x^2 - x - 1 = 0$ and $x^4 - x^3 - x^3 - x^2 - x - 1 = 0$, respectively.

Chapter 6

Concluding remarks

Each chapter of the thesis is summarized as follows.

In Chapter 2 [A1,A3,A4], as a preliminary chapter, we have examined the properties of the extended Hankel determinants associated with the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$ with respect to two types of the discrete-time variables s and t. We next have presented asymptotic expansions of the extended Hankel determinants $H_k^{(s,t)}$, in terms of the moment sequence $\{f_s^{(t)}\}_{s,t=0}^{\infty}$, as the discrete-time variables s and t go to infinity. Finally, we have associated a formal power series $F_s(z) = \sum_{t=0}^{\infty} f_s^{(t)} z^t$ with the Casorati determinants $C_k^{(s,t)}$ and gave asymptotic expansions of the Casorati determinants as $t \to \infty$.

In Chapter 3 [A3], the ordering indices in the moment sequence has been corresponded to the discrete-time s in the dhToda equation. We then have derived the determinant solution to the dhToda equation using the extended Hankel determinants $H_k^{(s,t)}$. We next have clarified a Lax pair of the dhToda equation by examining the extended Hadamard polynomials associated with the extended Hankel determinants $H_k^{(s,t)}$. Eigenvectors of band matrices have been simultaneously shown to be expressed by using the extended Hadamard polynomials. Furthermore, by observing the initial settings of the dhToda equation, we have clarified the general solution to the dhToda equation which generates similarity transformations of band matrices. Finally, with the help of the asymptotic expansions of the extended Hankel determinants $H_k^{(s,t)}$, we have understood the convergence of the general solution to eigenvalues of band matrices as the discrete-time variable s goes to infinity. An explicit expression of the general solution is vital for proving a convergence theorem that is not limited to the case where the band matrices are totally nonnegative but applied to band matrices which can be decomposed into a product of lower and upper triangular factors. The present framework will surely contribute to asymptotic analysis of other discrete integrable systems related to similarity transformations of matrices.

In Chapter 4 [A4], by observing the extended Hadamard polynomials based on the extended Hankel determinants, we have expressed the solution to the non-autonomous dhToda equation using the extended Hankel determinants. We thus clarified the eigenvalue problem of Hessenberg matrices associated with the non-autonomous dhToda equation. We also have given the determinant solution to the dhLV system in terms of the extended Hankel determinants. Finally, we have showed the asymptotic convergence to eigenvalues in Hessenberg matrices associate with the non-autonomous dhToda equation and dhLV system as the discrete-time variable t goes to infinity. It is worth noting that, through this chapter, the non-autonomous dhToda and dhLV variables are not restricted to be all positive. The Hessenberg matrices thus differ from those restricted to be totally nonnegative in Sumikura et al. [33]. The discrete-time variable t and an auxiliary discrete-time variable s enable us to get a deeper understanding of the non-autonomous dhToda equation and dhLV system. The auxiliary discrete-time variable s plays a key role in completing asymptotic analysis of the autonomous dhToda equation and the dhLV system.

In Chapter 5 [A2], we have clarified that the extended Fibonacci and the extended golden numbers lie behind the dhLV system. We first have showed in Theorems 3.6 that if entries of Casorati determinants appearing in the determinant solution to the predator-prey dhLV system are the (m, M)-step Fibonacci sequence at the initial discrete time $t = 0$, then they are so at any discrete time t. We next have proved that a variable related to the dhLV variable converges to the extended golden ratio as $t \to \infty$ if entries of the Casorati determinants are the (m, M)-step Fibonacci numbers at $t = 0$. In studies of integrable systems, the semi-infinite dynamical systems are often considered. The semi-infinite dhLV system can be regarded as a dynamical system in which a kind of species is assumed to be not finite. Theorems similar to Theorems 5.1.6 and 5.2.2 are easily proved in such an artificial dhLV system. This is because the evolution from t to $t + 1$ is given for every entry of the Casorati determinants. Some other properties in the Casorati determinant solution play key roles for deriving Theorems 5.1.6 and 5.2.2 in the finite realistic dhLV system.

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List of the Author's Papers Related to the Thesis

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[A1] Masato Shinjo, Masashi Iwasaki, Akiko Fukuda, Emiko Ishiwata, Yusaku Yamamoto, Yoshimasa Nakamura, An asymptotic analysis for an integrable variant of the Lotka-Volterra prey-predator model via a determinant expansion technique, *Cogent Mathematics* **2** (2015) 1046538 DOI:10.1080/23311835.2015.1046538. (Chapter 2)

[A2] Masato Shinjo, Kanae Akaiwa, Masashi Iwasaki, Yoshimasa Nakamura, An extended Fibonacci sequence associated with the discrete hungry Lotka-Volterra system, *Int. J. Biomathematics* **10** (2017) 1750043 DOI:10.1142/S1793524517500437. (Chapter 5)
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[A4] Masato Shinjo, Yoshimasa Nakamura, Masashi Iwasaki, Koichi Kondo, Asymptotic analysis of non-autonomous discrete hungry integrable systems, submitted. (Chapter 4)