Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces

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Abstract

Let $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of degree 0 on a compact connected Riemann surface. Once we fix a flat metric $h_{\det(E)}$ on the determinant of E, we have the harmonic metrics h_t $(t > 0)$ for the stable Higgs bundles $(E, \overline{\partial}_E, t\theta)$ such that $\det(h_t) = h_{\det(E)}$. We study the behaviour of h_t when t goes to ∞. First, we show that the Hitchin equation is asymptotically decoupled under the assumption that the Higgs field is generically regular semisimple. We apply it to the study of the so called Hitchin WKB-problem. Second, we study the convergence of the sequence $(E, \overline{\partial}_E, \theta, h_t)$ in the case $\operatorname{rank} E = 2$. We introduce a rule to determine the parabolic weights of a "limiting configuration", and we show the convergence of the sequence to the limiting configuration in an appropriate sense. The results can be appropriately generalized in the context of Higgs bundles with a Hermitian-Einstein metric on curves.

Keywords: harmonic bundle, asymptotic behaviour, asymptotic decoupling, Hitchin WKB-problem, limiting configuration, Hermitian-Einstein metric
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1 Introduction

Let X be a compact connected Riemann surface. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle of rank r on X with $\deg(E) = 0$. Let h be a Hermitian metric of E. We have the Chern connection ∇_h associated to $(E, \overline{\partial}_E, h)$. Let $R(h)$ denote the curvature of ∇_h. Let $\theta^\dagger_h$ denote the adjoint of θ with respect to h. Recall the celebrated Hitchin equation [6]:

$$R(h) + \big[\theta, \theta^\dagger_h\big] = 0$$

If the Hitchin equation is satisfied, h is called a harmonic metric of $(E, \overline{\partial}_E, \theta)$, and $(E, \overline{\partial}_E, \theta, h)$ is called a harmonic bundle.

Remark 1.1 *The Hitchin equation makes sense for a Higgs bundle with a Hermitian metric on any complex curve. In this introduction, X is assumed to be compact to simplify the explanation.*

If $\deg(E)$ is not 0, we take a Hermitian metric $h_{\det(E)}$ of the determinant line bundle $\det(E)$, and we consider the Hermitian-Einstein condition $R(h)^\perp + [\theta, \theta^\dagger_h] = 0$, where $R(h)^\perp$ is the trace-free part of $R(h)$. The condition is also called the Hitchin equation. In this introduction, we assume $\deg(E) = 0$ for simplicity. ∎

If $\operatorname{rank} E = 1$, we always have $[\theta, \theta^\dagger_h] = 0$. Hence, the Hitchin equation is reduced to $R(h) = 0$, i.e., the metric h is flat with respect to the Chern connection. By the classical harmonic theory, we can always find such a harmonic metric in the rank one case, which is unique up to the multiplication of positive constants.

In the higher rank case, we fix a harmonic metric $h_{\det E}$ of $(\det(E), \overline{\partial}_{\det E}, \operatorname{tr}\theta)$. According to Hitchin [6] and Simpson [19], if the Higgs bundle $(E, \overline{\partial}_E, \theta)$ is stable, we have a unique harmonic metric h of $(E, \overline{\partial}_E, \theta)$ such that $\det(h) = h_{\det E}$.

For any non-zero complex number t, the Higgs bundle $(E, \overline{\partial}_E, t\theta)$ is also stable. We obtain a family of harmonic metrics h_t $(t \in \mathbb{C}^*)$ satisfying $\det(h_t) = h_{\det E}$. It is easy to observe that $h_{t_1} = h_{t_2}$ if $|t_1| = |t_2|$. So, it is enough to consider the case where t are positive numbers.

Simpson studied the behaviour of $(E, \overline{\partial}_E, t\theta, h_t)$ when $t \to 0$. (See [19, 20, 21, 22], for example.) He discovered the convergence to a polarized variation of Hodge structure, and he gave various applications of this interesting phenomena.

More recently, there has been a growing interest to the behaviour of $(E, \overline{\partial}_E, t\theta, h_t)$ when $t \to \infty$. In [7], Katzarkov, Noll, Pandit and Simpson proposed "Hitchin WKB-problem" on the behaviour of the family of harmonic metrics h_t and the monodromy of the associated flat connections $\nabla_h + \theta + \theta_h^\dagger$, in relation with their magnificent theory of harmonic maps to buildings. In [11, 12], Mazzeo, Swoboda, Weiss and Witt studied the rank 2 case under the assumption that the zeroes of $\det(\theta) - (\operatorname{tr}\theta)^2/4$ are simple, i.e., the spectral curve of the Higgs field is smooth irreducible and simply ramified over X, motivated by the study on the structure of the end of the moduli spaces of Higgs bundles. They introduced the concept of "limiting configuration", and they proved the convergence to the limiting configuration under the assumption, inspired by the work of Gaiotto, Moore and Neitzke [4, 5]. In [3], Collier and Li closely studied the issue for some Toda-like harmonic bundles in a rather explicit way, and they resolved Hitchin WKB-problem in these cases for some kind of non-critical paths.

In this paper, we shall give two results on the asymptotic behaviour of the harmonic bundles $(E, \overline{\partial}_E, t\theta, h_t)$ $(t \to \infty)$. One is the asymptotic decoupling, and the other is the convergence to the limiting configuration for harmonic bundles of rank two.

1.1 Asymptotic decoupling

The Hitchin equation is much simplified if $R(h) = [\theta, \theta_h^\dagger] = 0$ holds. The equation $R(h) = 0$ implies that (E, ∇_h, h) is a unitary flat bundle. The additional condition $[\theta, \theta_h^\dagger] = 0$ implies that, at least locally, we have a flat decomposition $(E, \nabla, h) = \bigoplus_{i=1}^r (E_i, \nabla_i, h_i)$ into flat line bundles such that $\theta = \bigoplus \phi_i \cdot \mathrm{id}_{E_i}$, where ϕ_i are holomorphic one forms. In [11], this kind of simplification seems to be called "decoupling" of the Hitchin equation.

Our first purpose is to show that if t is sufficiently large, the Hitchin equation for $(E, \overline{\partial}_E, t\theta)$ is almost decoupled in some sense.

1.1.1 Generically regular semisimple Higgs bundles

To state the claim more precisely, we introduce a condition for Higgs bundles. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on X. We have the associated coherent sheaf $M_{E,\theta}$ on the cotangent bundle T^*X. The support $\Sigma(E, \theta)$ is called the spectral curve of the Higgs bundle. The number of the points of $\rho(P) := T_P^*X \cap \Sigma(E, \theta)$ are finite for any $P \in X$. We say that the Higgs bundle $(E, \overline{\partial}_E, \theta)$ is generically regular semisimple if the following holds:

- We have a discrete subset $D \subset X$ such that $\rho(P) = \operatorname{rank} E$ for any $P \in X \setminus D$.

Let $D(E, \theta)$ denote the set of the points $P \in X$ such that $\rho(P) < \operatorname{rank} E$, which we call the discriminant of the Higgs bundle.

Suppose that $(E, \overline{\partial}_E, \theta, h)$ is generically regular semisimple. Then, the following holds for any point $P \in X \setminus D(E, \theta)$ with a small neighbourhood U_P of P:

- We have holomorphic 1-forms $\phi_{P,1}, \ldots, \phi_{P,r}$ on U_P and a decomposition of the Higgs bundle

$$(E, \overline{\partial}_E, \theta)_{|U_P} = \bigoplus_{i=1}^r (E_{P,i}, \overline{\partial}_{E_{P,i}}, \phi_{P,i}\, \mathrm{id}_{E_{P,i}}),$$

  where we assume that $\operatorname{rank} E_i = 1$ $(i = 1, \ldots, r)$, and that $\phi_{P,i} - \phi_{P,j}$ $(i \neq j)$ have no zero.

1.1.2 Asymptotic decoupling

Let $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of degree 0 on X. Suppose that it is generically regular semisimple. We take any Kähler metric g_X of X. For any local section s of $\mathrm{End}(E) \otimes \Omega^{p,q}$, we have the function $|s|_{h_t, g_X} : X \longrightarrow \mathbb{R}$, where $|s|_{h_t, g_X}(P)$ $(P \in X)$ are the norm of $s_{|P}$ with respect to h_t and g_X. We have the asymptotic decoupling in the following sense.

Theorem 1.2 (Theorem 2.9) *We take any neighbourhood N of the discriminant $D(E,\theta)$. Then, there exist positive constants C_0 and ϵ_0 such that the following holds on $X\setminus N$:*

$$\big|R(h_t)\big|_{h_t,g_X}=|t|^2\big|\big[\theta,\theta^\dagger_{h_t}\big]\big|_{h_t,g_X}\leq C_0\exp\big(-\epsilon_0 t\big).$$

The constants C_0 and ϵ_0 may depend only on (X,g_X), N and $\Sigma(E,\theta)$.

We also have the family of flat connections $\mathbb{D}^1_{h_t}:=\nabla_{h_t}+t\theta+(t\theta)^\dagger_{h_t}$, which are correctly associated to the harmonic bundles $(E,\overline{\partial}_E,t\theta,h_t)$. Let us describe that we have nice approximations of these connections.

Let P be any point of $X\setminus D(E,\theta)$. We take a small neighbourhood U_P of P in $X\setminus D(E,\theta)$. We have a decomposition of the Higgs bundle $(E,\overline{\partial}_E,\theta)_{|U_P}=\bigoplus_{i=1}^r(E_{P,i},\overline{\partial}_{E_{P,i}},\theta_{P,i})$, where $\operatorname{rank}E_{P,i}=1$. Let $h_{t,E_{P,i}}$ be the restriction of h_t to $E_{P,i}$. By taking the direct sum, we obtain a Hermitian metric $h_{t,P,0}:=\bigoplus_{i=1}^r h_{t,E_{P,i}}$ of $E_{|U_P}$. Note that Theorem 1.2 implies the following.

Lemma 1.3 *We have positive constants $C'_{P,0}$ and $\epsilon'_{P,0}$ such that the following estimate holds for any local sections u_i and u_j $(i\neq j)$ of $E_{P,i}$ and $E_{P,j}$:*

$$\big|h_t(u_i,u_j)\big|\leq C'_{P,0}\exp(-\epsilon'_{P,0}t)|u_i|_{h_t}\,|u_j|_{h_t}$$

In particular, we have a constant $K_P>1$ such that $K_P^{-1}h_{t,P,0}\leq h_{t|U_P}\leq K_P h_{t,P,0}$ for any $t>1$.

By varying $P\in X\setminus D(E,\theta)$ and by gluing $h_{t,P,0}$, we obtain a family of Hermitian metrics $h_{t,0}$ $(t>1)$ on $E_{|X\setminus D(E,\theta)}$. We have the Chern connection $\nabla_{t,0}$ of $(E_{|X\setminus D(E,\theta)},h_{t,0})$. Let $(t\theta)^\dagger_{h_{t,0}}$ denote the adjoint of $t\theta_{|X\setminus D(E,\theta)}$ with respect to $h_{t,0}$. We set $\mathbb{D}^1_{h_{t,0}}:=\nabla_{t,0}+t\theta+(t\theta)^\dagger_{h_{t,0}}$.

Theorem 1.4 (Proposition 2.12, Corollary 2.13) *Take any neighbourhood N of $D(E,\theta)$. Then, we have a constant $K>1$ such that $K^{-1}h_{t,0|X\setminus N}\leq h_{t|X\setminus N}\leq Kh_{t,0|X\setminus N}$ for any $t>1$. We also have positive constants C_1 and ϵ_1 such that the following holds on $X\setminus N$:*

$$\big|R(h_{t,0})\big|_{h_t,g_X}\leq C_1\exp(-\epsilon_1 t),\qquad \big|\mathbb{D}^1_{h_t}-\mathbb{D}^1_{h_{t,0}}\big|_{h_t,g_X}\leq C_1\exp(-\epsilon_1 t)$$

The constants C_1 and ϵ_1 may depend only on (X,g_X), N and $\Sigma(E,\theta)$.

We note that we can obtain these estimates in an elementary way which is standard in the study of the asymptotic behaviour of harmonic bundles around the singularity, pioneered by Simpson [20], and pursued further by the author [13, 16]. We also emphasize that we can obtain these kinds of estimates without the assumption that harmonic bundles are given on a compact Riemann surface. Indeed, we shall study harmonic bundles given on discs in §2, which is clearly enough for the above estimates on relatively compact regions.

Finally, we remark that the estimates can be generalized in the case $\deg(E)\neq 0$, i.e., in the context of Higgs bundles with Hermitian-Einstein metrics on curves. (See §2.5.)

1.1.3 Hitchin WKB-problem

Together with a rather standard argument of singular perturbations, we can apply Theorem 1.4 to the Hitchin WKB-problem in [7].

We recall a notation in [7]. Let V be an r-dimensional complex vector space. For Hermitian metrics h_1,h_2, we can take a base $e_1,\ldots,e_r$ of V which is orthogonal with respect to both h_i $(i=1,2)$. We have the real numbers κ_j $(j=1,\ldots,r)$ determined by $\kappa_j:=\log|e_j|_{h_2}-\log|e_j|_{h_1}$. We impose $\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_r$. Then, we set

$$\vec{d}(h_1,h_2):=\big(\kappa_1,\ldots,\kappa_r\big)\in\mathbb{R}^r.$$

Let us return to the study on the family of harmonic bundles $(E,\overline{\partial}_E,t\theta,h_t)$ $(t>0)$ for a stable Higgs bundle $(E,\overline{\partial}_E,\theta)$ of rank r with $\deg(E)=0$ on a compact Riemann surface X, which is generically regular semisimple. We take a universal covering $\pi:Y\longrightarrow X\setminus D(E,\theta)$. Then, we have the decomposition of the Higgs bundle $\pi^*(E,\overline{\partial}_E,\theta)=\bigoplus_{i=1}^r(E_i,\overline{\partial}_{E_i},\phi_i\,\mathrm{id}_{E_i})$, where ϕ_i are holomorphic 1-forms. We have $\operatorname{rank}E_i=1$, and $\phi_i-\phi_j$ $(i\neq j)$ have no zeroes.

Let $[0,1]$ denote the closed interval $\{0 \leq s \leq 1\}$. Let $\gamma : [0,1] \longrightarrow Y$ be a C^∞-path. We have the expressions $\gamma^*(\phi_i) = a_i\, ds$ where a_i are C^∞-functions on $[0,1]$. The path γ is called non-critical if $\operatorname{Re} a_i(s) \neq \operatorname{Re} a_j(s)$ $(i \neq j)$ for any s. In that case, we may assume $\operatorname{Re} a_i(s) < \operatorname{Re} a_j(s)$ $(i < j)$. We set

$$\alpha_i := -\int_0^1 \operatorname{Re}(a_i) ds.$$

We have the families of Hermitian metrics $h_{t,\gamma(\kappa)}$ $(t > 0)$ on the fibers $E_{\gamma(\kappa)}$ $(\kappa = 0, 1)$, induced by the harmonic metrics h_t. Let $\Pi_{\gamma,t} : E_{\gamma(0)} \longrightarrow E_{\gamma(1)}$ denote the parallel transport of the flat connection $\mathbb{D}^1_{h_t}$ along γ. Let $\Pi^*_{\gamma,t} h_{t,\gamma(1)}$ denote the family of Hermitian metrics on $E_{\gamma(0)}$ induced by $h_{t,\gamma(1)}$ and $\Pi_{\gamma,t}$.

Theorem 1.5 (Theorem 2.17) *If γ is non-critical, there exist positive constants C_2 and ϵ_2 such that the following holds:*

$$\Bigl|\frac{1}{t}\vec{d}\bigl(h_{t,\gamma(0)}, \Pi^*_{\gamma,t} h_{t,\gamma(1)}\bigr) - (2\alpha_1, \ldots, 2\alpha_r)\Bigr| \leq C_2 \exp(-\epsilon_2 t)$$

The constants C_2 and ϵ_2 may depend only on X, $\phi_1, \ldots, \phi_r$ and γ.

The theorem was conjectured in [7], and a different version of the problem called the Riemann-Hilbert WKB-problem was studied in detail. Some cases were verified in [3]. (See [3, 7] for the precise statements.) We emphasize that we can obtain this kind of estimate for more general families of harmonic bundles given on complex curves which are not necessarily compact.

1.2 Convergence to the limiting configuration

1.2.1 Limiting configuration

Let $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of rank r with $\deg(E) = 0$ on a compact Riemann surface X, which is generically regular semisimple. We have the family of harmonic bundles $(E, \overline{\partial}_E, t\theta, h_t)$ $(t > 0)$. Let ∇_t denote the Chern connection of $(E, \overline{\partial}_E, h_t)$. By Theorem 1.2, we can take a sub-sequence $t_i \to \infty$ such that the sequence of vector bundles with a Hermitian metric and a unitary connection $(E, \nabla_{t_i}, h_{t_i})_{|X \setminus D(E,\theta)}$ converges in some sense to a vector bundle with a Hermitian metric and a unitary flat connection $(E_\infty, \nabla_\infty, h_\infty)$ on $X \setminus D(E, \theta)$. It is interesting to determine the flat connection ∇_∞. Following [11, 12], such a limit (or the associated parabolic bundle) is called a limiting configuration of the Higgs bundle $(E, \overline{\partial}_E, \theta)$ in this paper. Note that it is not clear, in general, whether ∇_∞ is independent of the choice of a sub-sequence. In this paper, we shall study the case $\operatorname{rank} E = 2$ by assuming that θ is generically regular semisimple, but without assuming that the zeroes of $\det(\theta) - (\operatorname{tr}\theta)^2/4$ are simple. See §1.2.6 for a remark on the case where the Higgs bundle is not generically regular semisimple.

Remark 1.6 *The higher rank case has not yet been studied in general. Under some assumptions on the spectral curves, it looks possible to generalize the method in* [11, 12] *and our method in this paper. See also* [3] *for some Toda-like cases.* ∎

1.2.2 The case where the spectral curve is smooth irreducible and simply ramified over X

In [11, 12], Mazzeo, Swoboda, Weiss, and Witt studied the case under the assumption that $\operatorname{rank} E = 2$ and that the zeroes of $\det(\theta) - (\operatorname{tr}\theta)^2/4$ are simple, i.e., the orders of the zeroes of the quadratic differential $\det(\theta) - (\operatorname{tr}\theta)^2/4$ are at most 1, inspired by the work of Gaiotto, Moore and Neitzke [5]. (We remark that in [11, 12], the condition $\deg(E) = 0$ is not imposed.) Let us describe the limiting configuration in this case. It is enough to consider the case $\operatorname{tr}(\theta) = 0$. Under the assumption, the spectral curve $\widetilde{X} := \Sigma(E, \theta)$ is smooth and connected. The natural projection $\pi : \widetilde{X} \longrightarrow X$ is the ramified covering of degree 2. The ramification index is at most 2. The discriminant $D(E, \theta)$ is exactly the set of the points on which π is ramified. Let $\iota : \widetilde{X} \longrightarrow T^*X$ denote the inclusion. We have the line bundle $\widetilde{L}$ on $\widetilde{X}$ such that $M_{E,\theta} \simeq \iota_* \widetilde{L}$. We have $E \simeq \pi_* \widetilde{L}$.

We have the unique non-trivial involution $\rho : \widetilde{X} \longrightarrow \widetilde{X}$ over X, i.e., $\pi \circ \rho = \pi$, $\rho \circ \rho = \mathrm{id}_{\widetilde{X}}$ and $\rho \neq \mathrm{id}_{\widetilde{X}}$. We have the line bundle $\rho^* \widetilde{L}$ on $\widetilde{X}$. We have a natural inclusion of $\mathcal{O}_{\widetilde{X}}$-modules $\pi^* E \longrightarrow \widetilde{L} \oplus \rho^* \widetilde{L}$, and the cokernel is isomorphic to the structure sheaf of $\widetilde{D}(E, \theta) := \pi^{-1} D(E, \theta)$. We have $\widetilde{L} \otimes \rho^* \widetilde{L} \simeq \pi^* \det(E) \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{D}(E, \theta))$.

Because $\deg(\widetilde{L}) - \bigl|\widetilde{D}(E, \theta)\bigr|/2 = 0$, we have a Hermitian metric $h^{\lim}_{\widetilde{L}}$ of $\widetilde{L}_{|\widetilde{X} \setminus \widetilde{D}(E,\theta)}$ with the following property.

- The Chern connection of $(\widetilde{L}_{|\widetilde{X}\setminus\widetilde{D}(E,\theta)}, h^{\lim}_{\widetilde{L}})$ is flat.
- Let $\widetilde{P} \in \widetilde{D}(E,\theta)$. Let $e_{\widetilde{P}}$ be a local frame of $\widetilde{L}$ around $\widetilde{P}$. Let $(\widetilde{U}_{\widetilde{P}}, w)$ be a holomorphic coordinate system around $\widetilde{P}$ with $w(\widetilde{P}) = 0$. Then, $\Bigl|\log\bigl(|w| h^{\lim}_{\widetilde{L}}(e_{\widetilde{P}}, e_{\widetilde{P}})\bigr)\Bigr|$ is bounded on $\widetilde{U}_{\widetilde{P}} \setminus \widetilde{P}$.
- We have $h^{\lim}_{\widetilde{L}} \otimes \rho^* h^{\lim}_{\widetilde{L}} = \pi^* h_{\det(E)}$ on $\widetilde{X} \setminus \widetilde{D}(E,\theta)$.

Such $h^{\lim}_{\widetilde{L}}$ is uniquely determined. We have the induced Hermitian metric $h^{\lim}_{E,\theta}$ of $E_{|X\setminus D(E,\theta)} = \pi_*(\widetilde{L})_{|X\setminus D(E,\theta)}$. We have the Chern connection $\nabla^{\lim}_{E,\theta}$ of $(E_{|X\setminus D(E,\theta)}, h^{\lim}_{E,\theta})$. This is the limiting configuration of $(E, \overline{\partial}_E, \theta)$ in this case. Interestingly, in [11, 12], it is proved that the family $(E, \overline{\partial}_E, h_t, \theta)_{|X\setminus D(E,\theta)}$ $(t > 0)$ is convergent to $(E_{|X\setminus D(E,\theta)}, \overline{\partial}_E, h^{\lim}_{E,\theta}, \theta)$ in an appropriate sense.

1.2.3 Limiting configuration in the general case

It is natural to study the case where θ is generically regular semisimple, but the zeroes of $\det(\theta) - (\operatorname{tr}\theta)^2/4$ are not necessarily simple. We may assume $\operatorname{tr}\theta = 0$. It is enough to consider the case where the spectral curve of the Higgs bundle $(E, \overline{\partial}_E, \theta)$ is reducible to the two components, i.e., we have holomorphic 1-forms $\omega \neq 0$ such that $\Sigma(E,\theta) = \operatorname{Im}(\omega) \cup \operatorname{Im}(-\omega)$. Indeed, if the spectral curve is irreducible, we have only to consider the pull back of the Higgs bundle by a ramified covering map of degree 2 given by the normalization of the spectral curve. (See §1.2.5. See §4.3.2 and §5.1.3 for more details.) Let $Z(\omega)$ denote the zero set of ω, which is equal to the discriminant $D(E,\theta)$.

We have two line bundles L_i $(i = 1, 2)$ with an inclusion $\iota_E : E \longrightarrow L_1 \oplus L_2$ with the following property:

- Set $\theta_{L_1\oplus L_2} := \omega \operatorname{id}_{L_1} \oplus (-\omega) \operatorname{id}_{L_2}$. Then, we have $\theta_{L_1\oplus L_2} \circ \iota_E = \iota_E \circ \theta$.
- The restriction $\iota_{E|X\setminus Z(\omega)}$ is an isomorphism.
- The induced morphisms $E \longrightarrow L_i$ are surjective.

Set $d_i := \deg(L_i)$. We may assume that $d_1 \leq d_2$. For each $P \in Z(\omega)$, let m_P denote the order of the zero of ω at P. Namely, for a holomorphic coordinate system (U_P, z) around P with $z(P) = 0$, we have $\omega_{|U_P} = g_P \cdot z^{m_P} dz$ with $g_P(P) \neq 0$. The support of the cokernel $(L_1 \oplus L_2)/E$ is contained in $Z(\omega)$. For each $P \in Z(\omega)$, let ℓ_P denote the length of the stalk of $(L_1 \oplus L_2)/E$ at P. Note that we have $L_1 \otimes L_2 = \det(E) \otimes \mathcal{O}\bigl(\sum_{P\in Z(\omega)} \ell_P P\bigr)$. In particular, we have

$$d_1 + d_2 - \sum_{P\in Z(\omega)} \ell_P = \deg(E) = 0.$$

Let L_1' denote the kernel of $E \longrightarrow L_2$. Similarly, let L_2' denote the kernel of $E \longrightarrow L_1$. Proper subbundles of E preserved by θ are only L_1' and L_2'. Hence, the stability condition for (E,θ) is equivalent to $\deg(L_i') < 0$ $(i = 1, 2)$. It is equivalent to $d_i = \deg(L_i) > 0$ $(i = 1, 2)$.

To give the limiting configuration of $(E, \overline{\partial}_E, \theta)$, we would like to give "parabolic weights" of L_i $(i = 1, 2)$ at $Z(\omega)$. In §1.2.2, all the parabolic weights are $1/2$. In the general case, it turns out that the parabolic weights are given by the following rule.

For each $P \in Z(\omega)$, we consider the piecewise linear function $\chi_P : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\leq 0}$ given by

$$\chi_P(a) := \begin{cases} (m_P + 1)a - \ell_P/2 & (a \leq \ell_P/2(m_P + 1)) \\ 0 & (a > \ell_P/2(m_P + 1)) \end{cases}$$

We set $\chi_{E,\theta}(a) := \sum_{P\in Z(\omega)} \chi_P(a)$. Then, we have a unique $0 \leq a_{E,\theta} < \max_{P\in Z(\omega)}\bigl\{\ell_P/2(m_P + 1)\bigr\}$ such that $d_1 + \chi_{E,\theta}(a_{E,\theta}) = 0$.

We determine the parabolic weights of L_1 at $P \in Z(\omega)$ as $-\chi_P(a_{E,\theta})$, and the parabolic weights of L_2 at $P \in Z(\omega)$ as $\chi_P(a_{E,\theta}) + \ell_P$. We have associated singular Hermitian metrics on L_i, i.e., we consider Hermitian metrics $h^{\lim}_{L_i}$ $(i = 1, 2)$ on $L_{i|X\setminus Z(\omega)}$ satisfying the following conditions.

- The Chern connections of $(L_{i|X\setminus Z(\omega)}, h^{\lim}_{L_i})$ are flat.

- Let $P \in Z(\omega)$. Let $v_{P,i}$ be local frames of L_i around P. Let (U_P, z) be a holomorphic coordinate neighbourhood around P with $z(P) = 0$. Then, the functions

$$\log\bigl(|z|^{-2\chi_P(a_{E,\theta})} h^{\lim}_{L_1}(v_{P,1}, v_{P,1})\bigr), \quad \log\bigl(|z|^{2\chi_P(a_{E,\theta})+2\ell_P} h^{\lim}_{L_2}(v_{P,2}, v_{P,2})\bigr)$$

  are bounded on $U_P \setminus P$.

- We have $h^{\lim}_{L_1} \otimes h^{\lim}_{L_2} = h_{\det(E)}$.

We obtain the Hermitian metric $h^{\lim}_{E,\theta} = h^{\lim}_{L_1} \oplus h^{\lim}_{L_2}$ of $E_{|X\setminus Z(\omega)}$, and the Chern connection $\nabla^{\lim}_{E,\theta}$.

Remark 1.7 *We still have the ambiguity for $h^{\lim}_{L_i}$. Namely, if $h^{\lim}_{L_i}$ $(i = 1, 2)$ satisfy the above conditions, $\alpha h^{\lim}_{L_1}$ and $\alpha^{-1} h^{\lim}_{L_2}$ also satisfy the above conditions for any $\alpha > 0$. In §1.2.2, we have an additional condition $\rho^* h^{\lim}_{L_1} = h^{\lim}_{L_2}$ under the natural isomorphism $\rho^* L_1 \simeq L_2$, with which the metrics are uniquely determined. But, in general, it seems that we do not have such an extra condition.*

The metric $h^{\lim}_{E,\theta}$ is also characterized as a harmonic metric of the polystable Higgs bundle $(E, \overline{\partial}_E, \theta)_{|X\setminus Z(\omega)}$ adapted to the parabolic structure given as above.

Although we have the ambiguity of the metrics, the connection $\nabla^{\lim}_{E,\theta}$ is uniquely determined. ▮

Remark 1.8 *Let us consider the case where $m_P = \ell_P = 1$ for any $P \in Z(\omega)$, and $d_1 = d_2 = |Z(\omega)|/2$. Then, we have $a_{E,\theta} = 0$, and hence $-\chi_P(a_{E,\theta}) = \chi_P(a_{E,\theta}) + \ell_P = 1/2$. These are the parabolic weights appeared in §1.2.2.* ▮

Remark 1.9 *It might be instructive to mention that if we have* $\deg(L_1) = \deg(L_2)$ *then we have $a_{E,\theta} = 0$, and $-\chi_P(a_{E,\theta}) = \chi_P(a_{E,\theta}) + \ell_P = \ell_P/2$.* ▮

1.2.4 Convergence to the limiting configuration

Suppose that $(E, \overline{\partial}_E, \theta)$ is stable of degree 0 with $\operatorname{rank} E = 2$ and that θ is generically regular semisimple. For simplicity, we assume $\operatorname{tr}\theta = 0$. Let h_t $(t > 0)$ denote the harmonic metric of the Higgs bundle $(E, \overline{\partial}_E, t\theta)$ with $\det(h_t) = h_{\det(E)}$.

For any $\alpha > 0$, let Ψ_α be the automorphism of $L_1 \oplus L_2$ given by $\Psi_\alpha = \alpha \operatorname{id}_{L_1} \oplus \alpha^{-1} \operatorname{id}_{L_2}$. Let $\Psi^*_\alpha h_t$ be the metric of $E_{|X\setminus D(E,\theta)}$ given by $h_t(\Psi_\alpha s_1, \Psi_\alpha s_2)$ for local sections s_j $(j = 1, 2)$ of $E_{|X\setminus D(E,\theta)}$. We take any point $Q \in X \setminus D(E, \theta)$ and a frame v_Q of $L_{1|Q}$, and we put

$$\gamma(t, Q) := \left(\frac{h^{\lim}_{L_1}(v_Q, v_Q)}{h_t(v_Q, v_Q)}\right)^{1/2}.$$

The following theorem is our second main result in this paper.

Theorem 1.10 (The degree 0 case of Theorem 5.1) *When t goes to ∞, the sequence $\Psi^*_{\gamma(t,Q)} h_t$ converges to $h^{\lim}_{E,\theta}$ in the C^∞-sense on any compact subset in $X \setminus Z(\omega)$. In particular, the sequence of the connections ∇_t converges to $\nabla^{\lim}_{E,\theta}$. (See §1.2.3 for $h^{\lim}_{E,\theta}$ and $\nabla^{\lim}_{E,\theta}$.)*

For the proof of Theorem 5.1, we need to study the global property of $(E, \overline{\partial}_E, \theta)$, in contrast that we can obtain the estimates in §1.1.2 locally at any point of $X \setminus D(E, \theta)$. A key is the construction of a family of Hermitian metrics h^0_t of E such that the L^p-norms of $R(h^0_t) + \bigl[t\theta, (t\theta)^\dagger_{h^0_t}\bigr]$ are bounded, for which we naturally encounter the above parabolic weights for the limiting configuration. We have the family of the self-adjoint endomorphisms k_t of (E, h^0_t) such that $h_t(u_1, u_2) = h^0_t(k_t u_1, u_2)$ for any local sections u_i $(i = 1, 2)$. By applying a variant of the arguments used in [15] with the tools given in [19], we shall observe that the sequence $\kappa_i k_{t_i}$ converges in some sense for an appropriate sequence $\kappa_i > 0$. Then, the claim of the theorem follows.

We note that for the construction of the family of metrics h^0_t, we use the general theory of wild harmonic bundles. We need a family of harmonic metrics given on a neighbourhood of the discriminant locus, for which we apply the Kobayashi-Hitchin correspondence for wild harmonic bundles on curves given in [2] (see [20] for the tame case). On the basis of the general results on the asymptotic behaviour of wild harmonic bundles studied in [20] and [16], we can deduce rather detailed properties of the family of the harmonic metrics as in Proposition 3.17 and Proposition 3.19.

At this moment, it is not clear to the author whether we could directly use the argument in [11] to prove Theorem 1.10 or its variant in our setting. But, we should note that we do not study the order of the convergence in Theorem 1.10. In contrast, when all the zeroes of $\det\theta-(\operatorname{tr}\theta)^2/4$ are simple, the method in [11] is strong enough to give the order of the convergence.

Remark 1.11 *Let $(E,\overline{\partial}_E,\theta)$ be a stable Higgs bundle of rank 2 on X such that θ is generically regular semisimple, but that $\deg(E)$ is not necessarily 0. We fix a Hermitian metric $h_{\det(E)}$ of $\det(E)$. For each $t>0$, we have the Hermitian-Einstein metric h_t^{HE} of $(E,\overline{\partial}_E,\theta)$ such that $\det(h_t^{HE})=h_{\det E}$, according to [6] and [19]. Here, the Hermitian-Einstein condition means the trace-free part of $R(h_t^{HE})+[t\theta,(t\theta)^\dagger_{h_t^{HE}}]$ is 0. We can study the behaviour of h_t^{HE} $(t\to\infty)$ by using Theorem 1.10. (See Theorem 5.1.)*

Suppose that $\deg(E)=2m$ for an integer m. We take a line bundle L with $\deg(L)=-m$. We can easily reduce the study on the behaviour of h_t^{HE} to the study on the behaviour of harmonic metrics for $(E,\overline{\partial}_E,\theta)\otimes L$. Suppose that $\deg(E)$ is odd. We take any (unramified) covering map $\varphi:X'\longrightarrow X$ of degree 2. Then, the degree of $\varphi^*(E,\overline{\partial}_E,\theta)$ is even. Hence, the study can be reduced to the degree 0 case. (See also §5.2.) ∎

1.2.5 Symmetric case and irreducible case

Let us explain that we can obtain a stronger result if the Higgs bundle is equipped with an extra symmetry. Suppose that X is equipped with a non-trivial involution ρ, i.e., $\rho:X\longrightarrow X$ is a holomorphic automorphism such that $\rho\circ\rho=\mathrm{id}_X$ and $\rho\neq\mathrm{id}_X$. Let $(E,\overline{\partial}_E,\theta)$ be a stable Higgs bundle of degree 0 on X with $\operatorname{tr}\theta=0$ such that the spectral curve is reducible, i.e., $\Sigma(E,\theta)=\mathrm{Im}(\omega)\cup\mathrm{Im}(-\omega)$ for a holomorphic one form $\omega\neq0$ on X. We impose the following conditions.

- We have $\rho^*\omega=-\omega$.
- The Higgs bundle $(E,\overline{\partial}_E,\theta)$ is equivariant with respect to the action of $\{1,\rho\}$ on X.

We have the induced isomorphism $\rho^*\det(E)\simeq\det(E)$. We naturally have $\rho^*h_{\det(E)}=h_{\det(E)}$.

Let L_i $(i=1,2)$ be as in §1.2.3. We naturally have the isomorphisms $\rho^*L_1\simeq L_2$ and $\rho^*L_2\simeq L_1$, which are compatible with the isomorphism $\rho^*E\simeq E$. Because $\deg(L_1)=\deg(L_2)$, we have $a_{E,\theta}=0$ and $-\chi_P(a_{E,\theta})=\chi_P(a_{E,\theta})+\ell_P=\ell_P/2$ for any $P\in Z(\omega)$. We can uniquely determine the metrics $h^{\lim}_{L_j}$ by imposing the extra condition $\rho^*h^{\lim}_{L_1}=h^{\lim}_{L_2}$.

Theorem 1.12 (The degree 0 case of Theorem 5.3) *For $t>0$, let h_t be the harmonic metrics of the Higgs bundles $(E,\overline{\partial}_E,t\theta)$ such that $\det(h_t)=h_{\det(E)}$. Then, the sequence h_t is convergent to $h^{\lim}_{E,\theta}=h^{\lim}_{L_1}\oplus h^{\lim}_{L_2}$ on any compact subset in $X\setminus Z(\omega)$.*

Let us remark that we can apply Theorem 1.12 to the case where the spectral curve is irreducible and the Higgs field is generically regular semisimple. Let X' be a compact connected Riemann surface. Let $(E',\overline{\partial}_{E'},\theta')$ be a stable Higgs bundle of rank 2 with $\deg(E')=0$ on X' such that (i) the spectral curve $\Sigma(E',\theta')$ is irreducible, (ii) $\operatorname{tr}\theta'=0$, (iii) θ' is generically regular semisimple. We take the normalization $\kappa:X\longrightarrow\Sigma(E',\theta')$. Let $p:X\longrightarrow X'$ be the morphism obtained as the composite of κ and the projection $\Sigma(E',\theta')\longrightarrow X'$. We set $(E,\overline{\partial}_E,\theta)=p^*(E',\overline{\partial}_{E'},\theta')$. We have the involution on $\Sigma(E',\theta')$ induced by the multiplication of -1 on the cotangent bundle T^*X'. It induces an involution ρ on X. We fix a Hermitian metric $h_{\det(E')}$ of $\det(E')$.

If $(E,\overline{\partial}_E,\theta)$ is polystable, we can easily observe that the harmonic metrics h'_t for the Higgs bundles $(E',\overline{\partial}_{E'},t\theta')$ are independent of t. (See §4.3.2.) We formally set $h^{\lim}_{E',\theta'}:=h'_t$.

If $(E,\overline{\partial}_E,\theta)$ is stable, the above symmetry conditions for $(E,\overline{\partial}_E,\theta)$ and ρ are satisfied. The pull back $h_{\det(E)}=p^*h_{\det(E')}$ satisfies the condition $\rho^*h_{\det(E)}=h_{\det(E)}$. We have the metrics $h^{\lim}_{L_i}$ and $h^{\lim}_{E,\theta}=h^{\lim}_{L_1}\oplus h^{\lim}_{L_2}$ as above. Because we have $\rho^*h^{\lim}_{E,\theta}=h^{\lim}_{E,\theta}$, we have a unique Hermitian metric $h^{\lim}_{E',\theta'}$ on $X'\setminus D(E',\theta')$ such that $p^*h^{\lim}_{E',\theta'}=h^{\lim}_{E,\theta}$. We also have a characterization of $h^{\lim}_{E',\theta'}$ as a harmonic metric for the stable filtered

Higgs bundles on $(X', D(E',\theta'))$ induced by the limiting configuration of $(E,\overline{\partial}_E,\theta)$. (See §4.3.2.) We have the following direct corollary.

Corollary 1.13 (The degree 0 case of Corollary 5.2) *For $t>0$, let h'_t be the harmonic metrics of the Higgs bundles $(E',\overline{\partial}_{E'},t\theta')$ such that $\det(h'_t)=h_{\det(E')}$. Then, the sequence h'_t is convergent to $h^{\lim}_{E',\theta'}$ on any compact subset in $X'\setminus D(E',\theta')$.* ∎

1.2.6 Complement

Let $(E,\overline{\partial}_E,\theta,h)$ be a Higgs bundle of rank 2. Suppose that θ is nilpotent, i.e., $\Sigma(E,\theta)$ is the 0-section in T^*X. Then, we can easily observe that $\lim_{t\to\infty}(E,\overline{\partial}_E,t\theta,h_t)$ is convergent to a polarized complex variation of Hodge structure. We sketch it. We have a polystable Higgs bundle $(E_\infty,\overline{\partial}_\infty,\theta_\infty)$ obtained as the limit $\lim_{t\to\infty}(E,\overline{\partial}_E,t\theta)$ in the coarse moduli space of semistable Higgs bundles of degree 0 on X. Note that the norms $\big|R(\nabla_{h_t})\big|_{h_t,g_X}=\big|t\theta\big|^2_{h_t,g_X}$ $(t>0)$ are uniformly bounded, where g_X is a Kähler metric of X. (See Proposition 2.1, for example.) Hence, we obtain a harmonic metric h_∞ of $(E_\infty,\overline{\partial}_\infty,\theta_\infty)$ as the limit of a convergent subsequence h_{t_i} $(t_i\to\infty)$. If $(E_\infty,\overline{\partial}_\infty,\theta_\infty)$ is not stable, θ_∞ is trivial, $(E_\infty,\overline{\partial}_\infty)$ is a direct sum of line bundles, and h_∞ is a flat metric. If $(E_\infty,\overline{\partial}_\infty,\theta_\infty)$ is stable, then as in the case of the limit for $t\to 0$ studied by Simpson, we can observe that $(E_\infty,\overline{\partial}_\infty,\theta_\infty)$ is a Hodge bundle, and h_∞ is equivariant with respect to the natural grading S^1-action. Hence, $(E_\infty,\overline{\partial}_\infty,\theta_\infty,h_\infty)$ comes from a polarized complex variation of Hodge structure.

In view of the moduli theoretic picture [6, 22], we can see it as follows. The energies of the family of $(E,\overline{\partial}_E,t\theta,h_t)$ $(t>0)$ are bounded. So, the family is relatively compact in the moduli space of harmonic bundles. Hence, when t goes to ∞, $(E,\overline{\partial}_E,\theta,h_t)$ goes to a fixed point in the moduli space induced by the natural $\mathbb{C}^*$-action on the moduli space of Higgs bundles. It means that $(E,\overline{\partial}_E,\theta,h_t)$ is convergent to a polarized complex variation of Hodge structure.

If $(E,\overline{\partial}_E,\theta)$ is not generically regular semisimple but $\operatorname{rank}E=2$, then the study of $\lim_{t\to\infty}(E,\overline{\partial}_E,t\theta,h_t)$ is easily reduced to the above case.

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2 Asymptotic decoupling

2.1 Simpson's main estimate for harmonic bundles on discs

Let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle on a complex curve X. We have the spectral curve $\Sigma(E,\theta)$ in the cotangent bundle of X. We have an estimate of the norm of θ, depending only on $\Sigma(E,\theta)$ (Proposition 2.10). If $\Sigma(E,\theta)$ is decomposed into a disjoint union $\Sigma_1\sqcup\Sigma_2$, then we have the corresponding decomposition of the Higgs bundle $(E,\overline{\partial}_E,\theta)=(E_1,\overline{\partial}_{E_1},\theta_1)\oplus(E_2,\overline{\partial}_{E_2},\theta_2)$. The Hermitian product of sections of E_1 and E_2 given by h should be small. We have such estimates depending only on $\Sigma(E,\theta)$ (Corollary 2.6). Because we use the arguments essentially given in [20], such estimates are called Simpson's main estimate. It is enough to consider the case where X is a disc.

2.1.1 Estimate of the sup norm in terms of the eigenvalues

For $R>0$, we set $\Delta(R):=\bigl\{z\in\mathbb{C}\,\big|\,|z|<R\bigr\}$. We consider a harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ of rank r on $\Delta(R)$. We have the description $\theta=f\,dz$, where f is a holomorphic endomorphism of E. Fix $M>0$, and suppose the following.

- For any $P\in\Delta(R)$, the eigenvalues γ of $f_{|P}$ satisfy $|\gamma|<M$.

We recall the following proposition, for which a proof was given in [14], for example. We include an outline of the proof for the convenience of the readers.

Proposition 2.1 *Fix $0<R_1<R$. Then, we have $C_1,C_2>0$ depending only on r,R_1,R such that*

$$|f|_h\leq C_1M+C_2$$

holds on $\Delta(R_1)$.

Proof Let $f_h^{\dagger}$ denote the adjoint of f with respect to h. As in [20], we have the following inequality on $\Delta(R)$ (see also Lemma 2.4 below):

$$-\partial_z\partial_{\overline{z}}\log|f|_h^2\leq-\frac{\bigl|[f,f_h^{\dagger}]\bigr|_h^2}{|f|_h^2}$$

For any $P\in\Delta(R)$, we set $g(P):=\sum_{i=1}^{r}|\alpha_i|^2$, where $\alpha_1,\ldots,\alpha_r$ are the eigenvalues of $f_{|P}$. There exists a constant $C_3>0$ depending only on r such that

$$\bigl|[f,f_h^{\dagger}]_{|P}\bigr|_h\geq C_3\bigl(|f_{|P}|_h^2-g(P)\bigr).$$

We obtain

$$-\partial_z\partial_{\overline{z}}\log|f|_h^2(P)\leq-C_3^2\frac{\bigl||f_{|P}|_h^2-g(P)\bigr|^2}{|f_{|P}|_h^2}$$

Note that if $|f_{|P}|_h^2\geq 2g(P)$ for some P, then we have

$$-\frac{\bigl||f_{|P}|_h^2-g(P)\bigr|^2}{|f_{|P}|_h^2}\leq-\frac{1}{4}|f_{|P}|_h^2.$$

For any positive number B, we have

$$-\partial_z\partial_{\overline{z}}\log\frac{B}{(R^2-|z|^2)^2}=-\frac{2R^2}{B}\frac{B}{(R^2-|z|^2)^2}.$$

Take B satisfying $B\geq 8R^2/C_3^2$ and $B\geq 2R^4rM^2$. Then, we use an idea in the proof of a lemma of Ahlfors [1]. Set $\mathcal{Z}:=\bigl\{P\in\Delta(R)\,\big|\,|f_{|P}|_h^2>B(R^2-|z(P)|^2)^{-2}\bigr\}$. Suppose that $\mathcal{Z}\neq\emptyset$, and we shall derive a contradiction. We have $|f_{|P}|_h^2>BR^{-4}\geq 2rM^2>2g(P)$ for any $P\in\mathcal{Z}$. Hence, the following inequality holds on $\mathcal{Z}$:

$$-\partial_z\partial_{\overline{z}}\Bigl(\log|f|_h^2-\log\bigl(B(R^2-|z|^2)^{-2}\bigr)\Bigr)\leq-\frac{C_3^2}{4}\Bigl(|f|_h^2-B(R^2-|z|^2)^{-2}\Bigr)\leq 0$$

Note that $\mathcal{Z}$ is relatively compact in $\Delta(R)$. So, we have $|f|_h^2=B(R^2-|z|^2)^{-2}$ on the boundary of $\mathcal{Z}$, and hence we obtain $|f|_h^2\leq B(R^2-|z|^2)^{-2}$ on $\mathcal{Z}$. But, it contradicts with the choice of $\mathcal{Z}$. So, we obtain $\mathcal{Z}=\emptyset$. Namely, we have $|f|_h^2\leq B(R^2-|z|^2)^{-2}$ on $\Delta(R)$. Then, we obtain the claim of the proposition. ∎

2.1.2 Asymptotic orthogonality

We continue to use the notation in §2.1.1. Suppose that we have a finite subset $S\subset\mathbb{C}$ and a decomposition $(E,\theta)=\bigoplus_{\alpha\in S}(E_\alpha,\theta_\alpha)$. We have the expression $\theta_\alpha=f_\alpha dz$ for each α, where f_α is a holomorphic endomorphism of E_α.

Assumption 2.2 *Fix $C_{10}>1$, and we impose the following conditions.*

- *We set $d:=\min\bigl\{|\alpha-\beta|\,\big|\,\alpha,\beta\in S,\alpha\neq\beta\bigr\}$. Then, we have $d\geq 1$ and $M\leq C_{10}d$. Here, M is the constant in the beginning of* §2.1.1.
- *For any $P\in\Delta(R)$, the eigenvalues γ of $f_{\alpha|P}$ satisfy $|\gamma-\alpha|\leq d/100$.*

Let π_α be the projection of E onto E_α with respect to the above decomposition. Let π'_α denote the orthogonal projection of E onto E_α. We set $\rho_\alpha:=\pi_\alpha-\pi'_\alpha$. The following proposition is a variant of the estimates given in [20] and [13, 16] for harmonic bundles given on punctured discs or the products of punctured discs, but it is more useful for the purpose in this paper.

Proposition 2.3 *Let R_1 be as in Proposition* 2.1*. Fix $0<R_2<R_1$. There exist positive constants ϵ_0 and C_{11} depending only on R, R_1, R_2, r and C_{10}, such that $|\rho_\alpha|_h\leq C_{11}\exp(-\epsilon_0 d)$ on $\Delta(R_2)$.*

Proof We recall a general inequality for holomorphic sections of $\mathrm{End}(E)$ which commutes with θ, for the convenience of the readers.

Lemma 2.4 *Let s be any holomorphic section of* $\mathrm{End}(E)$ *such that $[\theta,s]=0$. Then, we have the following inequality:*

$$-\partial_z\partial_{\overline{z}}\log|s|_h^2\leq-\frac{\bigl|[f_h^\dagger,s]\bigr|_h^2}{|s|_h^2}\tag{1}$$

Proof We have the equality $-\partial_z\partial_{\overline{z}}|s|^2_h=-|\partial_{z,h}s|_h^2-h\bigl(s,\partial_{\overline{z}}\partial_{z,h}s\bigr)$. Hence, we obtain the following:

$$\begin{aligned}-\partial_z\partial_{\overline{z}}\log|s|_h^2&=-\frac{\partial_z\partial_{\overline{z}}|s|_h^2}{|s|_h^2}+\frac{\partial_z|s|_h^2}{|s|_h^2}\frac{\partial_{\overline{z}}|s|_h^2}{|s|_h^2}=-\frac{h(s,\partial_{\overline{z}}\partial_{z,h}s)}{|s|_h^2}-\frac{\bigl|\partial_{z,h}s\bigr|_h^2}{|s|_h^2}+\frac{h(\partial_{z,h}s,s)}{|s|_h^2}\frac{h(z,\partial_{z,h}s)}{|s|_h^2}\\&\leq-\frac{h(s,\partial_{\overline{z}}\partial_{z,h}s)}{|s|_h^2}=-\frac{h(s,(\partial_{\overline{z}}\partial_{z,h}-\partial_{z,h}\partial_{\overline{z}})s)}{|s|_h^2}.\end{aligned}\tag{2}$$

By the Hitchin equation, we have $R(h)+[\theta,\theta_h^\dagger]=R(h)+[f,f_h^\dagger]\,dz\,d\overline{z}=0$, i.e., $R(h)=[f,f_h^\dagger]d\overline{z}\,dz$. We obtain the following from (2):

$$-\partial_z\partial_{\overline{z}}\log|s|_h^2\leq-\frac{h\bigl(s,[[f,f_h^\dagger],s]\bigr)}{|s|_h^2}$$

Because $[f,s]=0$, we obtain

$$-\partial_z\partial_{\overline{z}}\log|s|_h^2\leq-\frac{h\bigl(s,[f,[f_h^\dagger,s]]\bigr)}{|s|_h^2}=-\frac{h\bigl([f_h^\dagger,s],[f_h^\dagger,s]\bigr)}{|s|_h^2}=-\frac{\bigl|[f_h^\dagger,s]\bigr|_h^2}{|s|_h^2}$$

Thus, we obtain Lemma 2.4. ∎

Let $r_\alpha:=\operatorname{rank}E_\alpha$. Because $\partial_{\overline{z}}\partial_z\log r_\alpha=0$, we obtain the following inequality on $\Delta(R)$ from (1):

$$-\partial_z\partial_{\overline{z}}\log\bigl(|\pi_\alpha|_h^2/r_\alpha\bigr)\leq-\frac{\bigl|[f_h^\dagger,\pi_\alpha]\bigr|_h^2}{|\pi_\alpha|_h^2}$$

According to Proposition 2.1 and Lemma 2.7 below, there exists a positive constant C_{12} depending only on R, R_1, r and C_{10} such that we have $\bigl|\pi_\alpha\bigr|_h\leq C_{12}$ and $|\rho_\alpha|_h\leq C_{12}$ on $\Delta(R_1)$. We set $k_\alpha:=\log(|\pi_\alpha|_h^2/r_\alpha)$. Note that $|\pi_\alpha|_h^2 = |\pi'_\alpha|_h^2 + |\rho_\alpha|_h^2$ and $|\pi'_\alpha|_h^2 = r_\alpha$. Hence, we have $k_\alpha = \log(1+|\rho_\alpha|^2/r_\alpha)$. There exists a positive constant C_{13} depending only on R, R_1, r and C_{10} such that $C_{13}^{-1}|\rho_\alpha|_h^2 \leq k_\alpha \leq C_{13}|\rho_\alpha|_h^2$ on $\Delta(R_1)$.

According to Lemma 2.8 below, there exists a constant $\epsilon_1 > 0$ depending only on R, R_1, r, C_{10} such that we have $\big|[f_h^\dagger, \pi_\alpha]\big|_h \geq \epsilon_1 d|\rho_\alpha|_h$ on $\Delta(R_1)$. Hence, we have a constant $\epsilon_2 > 0$ depending only on R, R_1, r, C_{10} such that the following holds on $\Delta(R_1)$:

$$-\partial_z\partial_{\overline{z}} k_\alpha \leq -\epsilon_2 d^2 k_\alpha$$

For any positive number $\epsilon_3 > 0$, we have the following on $\Delta(R_1)$:

$$-\partial_z\partial_{\overline{z}} \exp\bigl(\epsilon_3 d|z|^2\bigr) = -(d\epsilon_3 + d^2|z|^2\epsilon_3^2)\exp(\epsilon_3 d|z|^2) \geq -(d\epsilon_3 + d^2\epsilon_3^2 R_1^2)\exp\bigl(\epsilon_3 d|z|^2\bigr)$$

Take $\epsilon_3 > 0$ such that $\epsilon_3 \leq (\epsilon_2 R_1^{-1})/2$ and $\epsilon_3 \leq d\epsilon_2/2$. Then, we have

$$-\partial_z\partial_{\overline{z}} \exp\bigl(\epsilon_3 d|z|^2\bigr) \geq -\epsilon_2 d^2 \exp\bigl(\epsilon_3 d|z|^2\bigr).$$

We take $C_{14} > 0$ depending only on R, R_1, r, C_{10} such that $k_\alpha(P) < C_{14}$ for $|z(P)| = R_1$. We set

$$\mathcal{Z} := \Bigl\{P \in \Delta(R_1) \Bigm| k_\alpha(P) > C_{14}\exp\bigl(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2\bigr)(P)\Bigr\}.$$

Suppose that $\mathcal{Z}$ is non-empty. By the choice of C_{14}, $\mathcal{Z}$ is relatively compact in $\Delta(R_1)$. So, we have $k_\alpha = C_{14}\exp\bigl(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2\bigr)$ on the boundary of $\mathcal{Z}$. We also have the following inequality on $\mathcal{Z}$:

$$-\partial_z\partial_{\overline{z}}\Bigl(k_\alpha - C_{14}\exp\bigl(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2\bigr)\Bigr) \leq -\epsilon_2 d^2\Bigl(k_\alpha - C_{14}\exp\bigl(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2\bigr)\Bigr) \leq 0$$

We obtain $k_\alpha \leq C_{14}\exp\bigl(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2\bigr)$ on $\mathcal{Z}$, which contradicts with the construction of $\mathcal{Z}$. Hence, we obtain that $\mathcal{Z} = \emptyset$, i.e., $k_\alpha \leq C_{14}\exp\bigl(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2\bigr) = C_{14}\exp\bigl(-\epsilon_3(R_1^2 - |z|^2)d\bigr)$ holds on $\Delta(R_1)$. We obtain the following on $\Delta(R_2)$:

$$k_\alpha \leq C_{14}\exp\bigl(-\epsilon_3(R_1^2 - R_2^2)d\bigr)$$

Then, by setting $C_{11} := C_{14} > 0$ and $\epsilon_0 := \epsilon_3(R_1^2 - R_2^2) > 0$, we obtain the claim of Proposition 2.3. ∎

For any endomorphism g of E, let $g_h^\dagger$ denote the adjoint of g with respect to h. We also denote it by $g^\dagger$ if there is no risk of confusion.

Corollary 2.5 *There exist positive constants C_{20} and ϵ_{20}, depending only on R, R_1, R_2, r and C_{10} such that $\big|[f, (\pi_\alpha)_h^\dagger]\big|_h = \big|[f_h^\dagger, \pi_\alpha]\big|_h \leq C_{20}\exp(-\epsilon_{20} d)$ on $\Delta(R_2)$.*

Proof We have $|\rho_\alpha|_h = |\pi_\alpha - \pi'_\alpha|_h \leq C_{11}\exp(-\epsilon_0 d)$. We also have $\big|(\rho_\alpha)_h^\dagger\big|_h = \big|(\pi_\alpha)_h^\dagger - \pi'_\alpha\big|_h \leq C_{11}\exp(-\epsilon_0 d)$. We obtain

$$\big|[f, \pi_\alpha^\dagger]\big|_h = \big|[f, \pi_\alpha^\dagger - \pi_\alpha]\big|_h \leq \big|[f, \rho_\alpha^\dagger]\big|_h + \big|[f, \rho_\alpha]\big|_h \leq 2(C_1 M + C_2)C_{11}\exp(-\epsilon_0 d).$$

Then, we obtain the desired estimate. ∎

Corollary 2.6 *There exist positive constants C_{21} and ϵ_{21} depending only on R, R_1, R_2, r and C_{10}, such that the following holds:*

- *Take any $\alpha, \beta \in S$ with $\alpha \neq \beta$. Let s_α and s_β be local sections of E_α and E_β. Then, we have*

$$\big|h(s_\alpha, s_\beta)\big| \leq C_{21}\exp(-\epsilon_{21} d)\cdot|s_\alpha|_h\cdot|s_\beta|_h \tag{3}$$

Proof We have $h(s_\alpha, s_\beta) = h(\pi_\alpha s_\alpha, s_\beta) = h(s_\alpha, \pi_\alpha^\dagger s_\beta) = h(s_\alpha, (\pi_\alpha^\dagger - \pi_\alpha)s_\beta)$. We have $\pi_\alpha^\dagger - \pi_\alpha = \rho_\alpha^\dagger - \rho_\alpha$, where $\rho_\alpha^\dagger$ denote the adjoint of ρ_α with respect to h. Then, we obtain (3) from Proposition 2.3. ∎

2.1.3 Appendix: Preliminary from linear algebra

Fix positive constants G_i $(i=1,2)$. Fix a positive integer $r>0$. Let U be any r-dimensional $\mathbb{C}$-vector space with a Hermitian metric h. Let f be an endomorphism on U. Suppose that we have a finite subset $S\subset\mathbb{C}$ and a decomposition $(U,f)=\bigoplus_{\alpha\in S}(U_\alpha,f_\alpha)$, which is not necessarily orthogonal. We impose the following conditions.

- We set $d:=\min\bigl\{|\alpha-\beta|\,\bigm|\,\alpha,\beta\in S,\alpha\neq\beta\bigr\}$. Then, we have $d\geq 1$ and $|f|_h\leq G_1d+G_2$.
- The eigenvalues γ of f_α satisfy $|\gamma-\alpha|\leq d/100$.

Let π_α be the projection with respect to the decomposition $U=\bigoplus U_\alpha$. Let π'_α denote the orthogonal projection of U onto U_α. We set $\rho_\alpha:=\pi_\alpha-\pi'_\alpha$. To clarify the argument, we recall the following lemma from [20] which was used in [13, 16].

Lemma 2.7 *There exists a positive constant B_1 depending only on r and G_i $(i=1,2)$ such that*

$$\bigl|\pi_\alpha\bigr|_h\leq B_1,\quad |\rho_\alpha|_h\leq B_1.$$

Proof Because ρ_α and π'_α are orthogonal, we have $|\rho_\alpha|_h\leq|\pi_\alpha|_h$. Hence, it is enough to obtain the estimate for π_α. Let id_U denote the identity on U. We have

$$\pi_\alpha=\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_\alpha}\bigl(\zeta\,\mathrm{id}_U-f\bigr)^{-1}d\zeta.$$

Here, γ_α denotes the loop $\gamma_\alpha(\theta)=\alpha+de^{\sqrt{-1}\theta}/10$ $(0\leq\theta\leq 2\pi)$. There exist positive constants B_i $(i=2,3)$ depending only on r and G_i $(i=1,2)$ such that

$$\bigl|(\gamma_\alpha(\theta)\,\mathrm{id}_U-f)^{-1}\bigr|_h\leq d^{-1}B_2\Bigl(\bigl(G_1+d^{-1}G_2\bigr)^r+1\Bigr)\leq d^{-1}B_3.$$

Thus, we obtain the claim of the lemma. ∎

Let $f_h^\dagger$ denote the adjoint of f with respect to h. To clarify our argument, we recall the following lemma from [20] which was used in [13, 16].

Lemma 2.8 *We have $\delta>0$ depending only on r and G_i $(i=1,2)$ such that $\bigl|[f_h^\dagger,\pi_\alpha]\bigr|_h\geq\delta\cdot d\cdot|\rho_\alpha|_h$.*

Proof We take a numbering of the elements of S, i.e., $S=\{\alpha_1,\ldots,\alpha_m\}$. We impose $\alpha_1=\alpha$. We set $F_j(U):=\bigoplus_{i\leq j}U_{\alpha_i}$ and $F_{<j}(U):=\bigoplus_{i<j}U_{\alpha_i}$. Let U'_j be the orthogonal complement of $F_{<j}(U)$ in $F_j(U)$. We have the orthogonal decomposition $F_j(U)=\bigoplus_{i\leq j}U'_i$. Because $f(F_j)\subset F_j$, we have the decomposition $f=\sum_{i\leq j}f_{ij}$, where $f_{ij}:U'_j\longrightarrow U'_i$. As the adjoint, we have $f_h^\dagger=\sum_{i\leq j}(f_{ij})_h^\dagger$, where $(f_{ij})_h^\dagger:U'_i\longrightarrow U'_j$. We set $f_{ij}^\dagger:=(f_{ji})_h^\dagger$. Then, we have $f_h^\dagger=\sum_{i\geq j}f_{ij}^\dagger$, where $f_{ij}^\dagger:U'_j\longrightarrow U'_i$.

For each i, we take an orthonormal base $e_1^{(i)},\ldots,e_{r_i}^{(i)}$ of U'_i for which $f_{ii}^\dagger$ is represented by a lower triangular matrix A_i. Let $\Gamma_i\in\mathrm{End}(U'_i)$ be determined by $\Gamma_i(e_k^{(i)})=\alpha_k^{(i)}e_k^{(i)}$, where $\alpha_k^{(i)}$ denote the (k,k)-entry of A_i. Then, $f_{ii}^\dagger-\Gamma_i$ is nilpotent. We put $\Gamma=\sum\Gamma_i\in\mathrm{End}(U)$. Then, $f_h^\dagger-\Gamma$ is nilpotent, and we have $|f_h^\dagger-\Gamma|_h\leq G_1d+G_2$.

Let Π denote the orthogonal projection of $\mathrm{End}(E)=\bigoplus\mathrm{Hom}(U'_j,U'_i)$ onto $\bigoplus_{j>i}\mathrm{Hom}(U'_j,U'_i)$. Let F_1 be the endomorphism on $\bigoplus_{j>i}\mathrm{Hom}(U'_j,U'_i)$ given by $F_1(B):=\Pi([f_h^\dagger,B])$. Let F_2 be the endomorphism on $\bigoplus_{j>i}\mathrm{Hom}(U'_j,U'_i)$ given by $F_2(B):=[\Gamma,B]=\Pi([\Gamma,B])$.

Let us observe that F_1-F_2 is nilpotent. We have the nilpotent endomorphism $\widetilde{F}$ on $\mathrm{End}(E)$ given by $\widetilde{F}(B)=[f_h^\dagger-\Gamma,B]$, which preserves $\bigoplus_{j\leq i}\mathrm{Hom}(U'_j,U'_i)$. Then, F_1-F_2 is equal to the endomorphism on $\bigoplus_{j>i}\mathrm{Hom}(U'_j,U'_i)\simeq\mathrm{End}(E)/\bigoplus_{j\leq i}\mathrm{Hom}(U'_j,U'_i)$ induced by $\widetilde{F}$, and hence F_1-F_2 is nilpotent.

There exists a constant $B_{11}>0$ depending only on r such that $|F_1-F_2|_h\leq B_{11}(G_1d+G_2)$. Moreover, F_2 is invertible, and there exists a constant $B_{12}>0$ depending only on r such that $|F_2^{-1}|_h\leq B_{12}d^{-1}$. Hence, we obtain that $|F_1^{-1}|_h\leq B_{13}d^{-1}(1+(G_1+G_2/d)^r)$ for a constant $B_{13}>0$ depending only on r.

Note that we have $\Pi([f_h^\dagger, \pi'_\alpha]) = 0$. Hence, we have the following:

$$\big|[f_h^\dagger, \pi_\alpha]\big|_h \geq \Big|\Pi([f_h^\dagger, \pi_\alpha])\Big|_h = \Big|\Pi([f_h^\dagger, \rho_\alpha])\Big|_h = \big|F_1(\rho_\alpha)\big|_h \geq B_{13}^{-1} d\big(1 + (G_1 + G_2 d^{-1})^r\big)^{-1} |\rho_\alpha|_h$$

Thus, we obtain the claim of the lemma. ∎

2.2 Asymptotic decoupling of harmonic bundles on discs

We continue to use the setting in §2.1. We further impose the following condition:

- $\operatorname{rank} E_\alpha = 1$ for each $\alpha \in S$.

In other words, we assume to have holomorphic functions g_α ($\alpha \in S$) on $\Delta(R)$ such that $\theta = \bigoplus g_\alpha \, \mathrm{id}_{E_\alpha} \, dz$. (Recall the generically regular semisimple condition in §1.1.1.) By the condition, we have $|g_\alpha(P) - \alpha| \leq d/100$. In this setting, we explain that Simpson's main estimate implies the asymptotic decoupling of the Hitchin equation.

Let $g_{\mathbb{C}}$ denote the Euclidean metric $dz\, d\overline{z}$ of $\mathbb{C}$. For any section s of $\operatorname{End}(E) \otimes \Omega^{p,q}$ on $\Delta(R')$ ($R' > 0$), let $|s|_{h,g_{\mathbb{C}}}$ denote the function on $\Delta(R')$ by taking the norm of s at each $P \in \Delta(R')$ with respect to h and $g_{\mathbb{C}}$.

2.2.1 Decay of the curvatures

Let $R(h)$ denote the curvature of the Chern connection of $(E, \overline{\partial}_E, h)$. We obtain the following "asymptotic decoupling" of the Hitchin equation $R(h) + [\theta, \theta_h^\dagger] = 0$.

Theorem 2.9 *There exist positive constants C_{30} and ϵ_{30}, depending only on R, R_1, R_2, r and C_{10} such that $\big|R(h)\big|_{h,g_{\mathbb{C}}} = \big|[\theta, \theta_h^\dagger]\big|_{h,g_{\mathbb{C}}} \leq C_{30} \exp(-\epsilon_{30} d)$ on $\Delta(R_2)$.*

Proof It is enough to obtain the estimate for $[\theta, \theta_h^\dagger]$. We have the decomposition $f_h^\dagger = \sum_{\alpha,\beta} \pi_\alpha \circ f_h^\dagger \circ \pi_\beta$. By Corollary 2.5, if $\alpha \neq \beta$, we have $C_{31} > 0$, depending only on R, R_1, R_2, r and C_{10}, such that

$$\big|\pi_\alpha \circ f_h^\dagger \circ \pi_\beta\big|_h = \big|[\pi_\alpha, f_h^\dagger] \circ \pi_\beta\big|_h \leq C_{31} \exp(-\epsilon_{20} d).$$

For $\alpha \neq \beta$, we have $[f_\alpha, \pi_\beta \circ f_h^\dagger \circ \pi_\beta] = 0$. Because $\operatorname{rank} E_\alpha = 1$, we also have $\big[f_\alpha, \pi_\alpha \circ f_h^\dagger \circ \pi_\alpha\big] = 0$. Then, we obtain the estimate for $[\theta, \theta_h^\dagger]$. ∎

2.2.2 The connections and the projections

Let ∂_E denote the $(1,0)$-part of the Chern connection associated to h and $\overline{\partial}_E$. According to Proposition 2.3, the decomposition $E = \bigoplus E_\alpha$ is almost orthogonal. Let us see that such an almost orthogonality holds at the level of the first derivative in the sense that $\partial_E \pi_\alpha$ is very small.

Proposition 2.10 *Take $0 < R_3 < R_2$. There exist positive constants ϵ_{40} and C_{40} depending only on R, R_1, R_2, R_3, r and C_{10} such that $\big|\partial_E \pi_\alpha\big|_{h,g_{\mathbb{C}}} = \big|\overline{\partial}_E \pi_\alpha^\dagger\big|_{h,g_{\mathbb{C}}} \leq C_{40} \exp(-\epsilon_{40} d)$ on $\Delta(R_3)$.*

Proof It is enough to obtain the estimate for $\partial_E \pi_\alpha$. In the following, the constants may depend only on R, R_1, R_2, r and C_{10}. We have $C_{41} > 0$ such that the following holds on $\Delta(R_2)$:

$$\big|\overline{\partial}_E \partial_E \pi_\alpha\big|_{h,g_{\mathbb{C}}} = \big|[R(h), \pi_\alpha]\big|_{h,g_{\mathbb{C}}} \leq C_{41} \exp(-\epsilon_{30} d).$$

Because $\partial_E \pi_\alpha^\dagger = 0$, we have the following on $\Delta(R_2)$:

$$\big|\overline{\partial}_E \partial_E (\pi_\alpha - \pi_\alpha^\dagger)\big|_{h,g_{\mathbb{C}}} \leq C_{41} \exp(-\epsilon_{30} d)$$

We already have $\big|\pi_\alpha - \pi_\alpha^\dagger\big|_h \leq C_{42} \exp(-\epsilon_0 d)$ for a constant $C_{42} > 0$. We may assume $\epsilon_{30} < \epsilon_0$. Hence, we have a constant $C_{43} > 0$ such that $\big\|\pi_\alpha - \pi_\alpha^\dagger\big\|_{h,g_{\mathbb{C}},L_2^p} \leq C_{43} \exp(-\epsilon_{30} d)$ for a large $p > 1$. Here, $\|\cdot\|_{h,g_{\mathbb{C}},L_2^p}$ denote the L_2^p-norm with respect to h and $g_{\mathbb{C}}$ on $\Delta(R_2)$. Then, we obtain the claim of the lemma. ∎

Recall that π'_α denote the orthogonal projection of E onto E_α. Let us see that it is almost holomorphic.

Proposition 2.11 *We have a positive constant C_{50}, depending only on R, R_1, R_2, R_3, r and C_{10} such that the following holds on $\Delta(R_3)$:*

$$\big|\overline{\partial}_E\pi'_\alpha\big|_{h,g_{\mathbb{C}}}=\big|\partial_E\pi'_\alpha\big|_{h,g_{\mathbb{C}}}\leq C_{50}\exp(-\epsilon_{40}d).$$

Proof It is enough to prove the estimate for $\overline{\partial}_E\pi'_\alpha$. Let $E=E_\alpha\oplus E_\alpha^\perp$ denote the orthogonal decomposition. We may naturally regard $\rho_\alpha:=\pi_\alpha-\pi'_\alpha$ as a morphism $E_\alpha^\perp\longrightarrow E_\alpha$. We may also regard $\rho_\alpha^\dagger:=\pi_\alpha^\dagger-\pi'_\alpha$ as a morphism $E_\alpha\longrightarrow E_\alpha^\perp$.

We have the induced holomorphic structure on $E_\alpha^\perp\simeq E/E_\alpha$. Let $\overline{\partial}_E^{(0)}$ be the holomorphic structure on $E_\alpha\oplus E_\alpha^\perp$ obtained as the direct sum. We have $\overline{\partial}_E=\overline{\partial}_E^{(0)}+\kappa$, where κ is a section of $\mathrm{Hom}(E_\alpha^\perp,E_\alpha)\otimes\Omega^{0,1}$. We have

$$\overline{\partial}_E(\pi_\alpha-\pi_\alpha^\dagger)=\overline{\partial}_E\rho_\alpha-\overline{\partial}_E\rho_\alpha^\dagger=\overline{\partial}_E^{(0)}\rho_\alpha-\overline{\partial}_E^{(0)}\rho_\alpha^\dagger-\kappa\circ\rho_\alpha^\dagger+\rho_\alpha^\dagger\circ\kappa \tag{4}$$

Note that $\overline{\partial}_E^{(0)}(\rho_\alpha)$, $\overline{\partial}_E^{(0)}(\rho_\alpha^\dagger)$, $\kappa\circ\rho_\alpha^\dagger$ and $\rho_\alpha^\dagger\circ\kappa$ are sections of $\mathrm{Hom}(E_\alpha^\perp,E_\alpha)\otimes\Omega^{0,1}$, $\mathrm{Hom}(E_\alpha,E_\alpha^\perp)\otimes\Omega^{0,1}$, $\mathrm{Hom}(E_\alpha,E_\alpha)\otimes\Omega^{0,1}$ and $\mathrm{Hom}(E_\alpha^\perp,E_\alpha^\perp)\otimes\Omega^{0,1}$, respectively. Note that the bundles are orthogonal with respect to h and $g_{\mathbb{C}}$. In general, for any orthogonal decomposition of bundles $\mathcal{V}=\bigoplus\mathcal{V}_i$ and for any section $s=\sum s_i$ of $\mathcal{V}$, the norms of s_i are smaller than the norm of s. Hence, we obtain the following from (4):

$$\big|\overline{\partial}_E^{(0)}\rho_\alpha\big|_{h,g_{\mathbb{C}}}\leq\big|\overline{\partial}_E(\pi_\alpha-\pi_\alpha^\dagger)\big|_{h,g_{\mathbb{C}}}=\big|\overline{\partial}_E(\pi_\alpha^\dagger)\big|_{h,g_{\mathbb{C}}} \tag{5}$$

We also have $\overline{\partial}_E\pi'_\alpha=-\overline{\partial}_E\rho_\alpha=-\overline{\partial}_E^{(0)}\rho_\alpha$. Hence, we obtain $\big|\overline{\partial}_E\pi'_\alpha\big|_{h,g_{\mathbb{C}}}\leq\big|\overline{\partial}_E(\pi_\alpha^\dagger)\big|_{h,g_{\mathbb{C}}}$ from (5). Then, the claim of Proposition 2.11 follows from the estimate in Proposition 2.10. ∎

2.2.3 The decay of the curvatures on the line bundles

Let $\overline{\partial}_\alpha$ denote the holomorphic structure of E_α. Let h_α be the restriction of h to E_α. Let ∂_α denote the $(1,0)$-part of the Chern connection of $(E_\alpha,\overline{\partial}_\alpha,h_\alpha)$. Let $R(h_\alpha)$ denote the curvature of the connection $\overline{\partial}_\alpha+\partial_\alpha$. We have $\partial_\alpha s=\pi'_\alpha\circ\partial_E s$ and $\overline{\partial}_\alpha s=\overline{\partial}_E s$ for any section s of E_α.

Proposition 2.12 *We have a positive constant C_{60} depending only on R, R_1, R_2, R_3, r and C_{10} such that $\big|R(h_\alpha)\big|_{h_\alpha,g_{\mathbb{C}}}\leq C_{60}\exp(-\epsilon_{40}d)$ on $\Delta(R_3)$.*

Proof In the following, the constants may depend only on R, R_1, R_2, R_3, r and C_{10}. Let s be any section of E_α. We have

$$\overline{\partial}_E\circ\pi'_\alpha\big(\partial_E s\big)=\overline{\partial}_E\circ\pi'_\alpha\circ\partial_E\big(\pi_\alpha s\big)=\overline{\partial}_E\circ\pi_\alpha(\partial_E s)+\overline{\partial}_E\circ\pi'_\alpha\circ(\partial_E\pi_\alpha)(s)=\pi_\alpha(\overline{\partial}_E\partial_E s)+\overline{\partial}_E\big(\pi'_\alpha(\partial_E\pi_\alpha)s\big).$$

We also have

$$\pi'_\alpha\circ\partial_E\circ\overline{\partial}_E s=\pi'_\alpha\circ\partial_E(\pi_\alpha\overline{\partial}_E s)=\pi_\alpha\partial_E\overline{\partial}_E s+\pi'_\alpha\partial_E(\pi_\alpha)\overline{\partial}_E s.$$

Hence, it is enough to obtain an estimate of $\overline{\partial}_E(\pi'_\alpha\circ\partial_E\pi_\alpha)=\overline{\partial}_E(\pi'_\alpha)\circ\partial_E(\pi_\alpha)+\pi'_\alpha\circ\overline{\partial}_E\partial_E\pi_\alpha$. By Proposition 2.10 and Proposition 2.11, we have $C_{61}>0$ such that $\big|\overline{\partial}_E(\pi'_\alpha)\circ\partial_E(\pi_\alpha)\big|_{h,g_{\mathbb{C}}}\leq C_{61}\exp(-\epsilon_{40}d)$. We also have $\big|\pi'_\alpha\circ\overline{\partial}_E\partial_E\pi_\alpha\big|_{h,g_{\mathbb{C}}}=\big|\pi'_\alpha\circ[R(h),\pi_\alpha]\big|_{h,g_{\mathbb{C}}}\leq C_{62}\exp(-\epsilon_{40}d)$ for a constant $C_{62}>0$. Then, the claim of the lemma follows. ∎

2.2.4 Approximation of the flat connections

We consider the flat connection $\mathbb{D}^1:=\overline{\partial}_E+\partial_E+\theta+\theta^\dagger$ on E. We also have the connection $\mathbb{D}^1_0$ on $E=\bigoplus_{\alpha\in S}E_\alpha$ given by

$$\mathbb{D}^1_0=\bigoplus_{\alpha\in S}\Big(\big(\overline{\partial}_\alpha+\partial_\alpha\big)+\big(g_\alpha dz+\overline{g_\alpha}d\overline{z}\big)\,\mathrm{id}_{E_\alpha}\Big).$$

Here, g_α are holomorphic functions such that $\theta=\bigoplus g_\alpha\,\mathrm{id}_{E_\alpha}\,dz$, as in the beginning of §2.2.

Corollary 2.13 *There exists a positive constant C_{70}, depending only on R, R_1, R_2, R_3, r and C_{10} such that $\big|\mathbb{D}^1-\mathbb{D}^1_0\big|_{h,g_{\mathbb{C}}}\leq C_{70}\exp(-\epsilon_{40}d)$ on $\Delta(R_3)$.*

Proof It is enough to obtain an estimate of $\partial_E-\bigoplus_{\alpha\in S}\partial_\alpha$. We have a constant $C_{71}>0$, depending only on R, R_1, R_2, R_3, r and C_{10} such that $\big|\pi'_\alpha\circ(\partial_E\pi_\alpha)\big|_{h,g_{\mathbb{C}}}\leq C_{71}\exp(-\epsilon_{40}d)$. Hence, for any section s of E_α, we have

$$\big|\pi_\alpha\circ\partial_Es-\partial_\alpha s\big|_{h,g_{\mathbb{C}}}=\big|\pi'_\alpha\circ\pi_\alpha\circ\partial_Es-\pi'_\alpha\partial_E(\pi_\alpha s)\big|_{h,g_{\mathbb{C}}}=\big|\pi'_\alpha\circ\partial_E\pi_\alpha(s)\big|_{h,g_{\mathbb{C}}}\leq C_{71}\exp(-\epsilon_{40}d)|s|_h.$$

Then, the claim of the lemma follows. ∎

2.2.5 Higher derivatives

We take a numbering $\{\alpha_1,\ldots,\alpha_r\}$ on S. For each i, we can take a holomorphic frame u_i of E_{α_i} such that $|u_{i|0}|_h=1$ and $|\partial_{\alpha_i}u_i|_{h,g_{\mathbb{C}}}\leq C_{80}\exp(-\epsilon_{80}d)$ on $\Delta(R_3)$ for some positive constants C_{80} and ϵ_{80}, depending only on R, R_1, R_2, R_3, r and C_{10}. Let us sketch how to obtain such sections in an elementary way. For a real coordinate $z=x+\sqrt{-1}y$, we can take a section s_i of E_{α_i} on $\Delta(R_2)$ such that $\big(\overline{\partial}_{\alpha_i}+\partial_{\alpha_i}\big)s_i=s_i\cdot\nu_idx=s_i\cdot\nu_i(dz+d\overline{z})/2$ such that $\nu_i(x,0)=0$. The curvature form is given by $\partial_y\nu_i\,dy\,dx$. By Proposition 2.12, we have $|\nu_i|\leq C'_{80}\exp(-\epsilon'_{80}d)$. Take R'_2 such that $R_3<R'_2<R_2$. We can take a function ρ_i on $\Delta(R'_2)$ satisfying $\partial_{\overline{z}}\rho_i=\nu_i/2$, $|\rho_i|\leq C''_{80}\exp(-\epsilon''_{80}d)$ and $\|\partial_z\rho_i\|_{L^p(\Delta(R'_2))}\leq C''_{80,p}\exp(-\epsilon''_{80})$ for any $p\geq1$. We set $v_i:=s_ie^{-\rho_i}$ on $\Delta(R'_2)$. Then, we have $\overline{\partial}_{\alpha_i}v_i=0$ and $\partial_{\alpha_i}v_i=v_i\cdot\kappa_idz$, where $\|\kappa_i\|_{L^p(\Delta(R'_2))}\leq C^{(3)}_{80,p}\exp(-\epsilon^{(3)}_{80}d)$. Because $\partial_{\overline{z}}\kappa_id\,\overline{z}\,dz$ is the curvature of E_α, we have $\big|\partial_{\overline{z}}\kappa_i\big|\leq C^{(4)}_{80}\exp(-\epsilon^{(4)}_{80}d)$. Hence, we obtain $|\kappa_i|\leq C^{(5)}_{80}\exp(-\epsilon^{(5)}_{80}d)$ on $\Delta(R_3)$. By adjusting the norm of v_i at the origin, we obtain the desired section u_i. Note that we have $\big|\log|u_i|_h\big|\leq C_{81}\exp(-\epsilon_{81}d)$ for some positive constants C_{81} and ϵ_{81} depending only on R, R_1, R_2, R_3, r and C_{10}.

We obtain a frame $\boldsymbol{u}=(u_1,\ldots,u_r)$ of E on $\Delta(R_3)$. Let H be the r-square Hermitian matrix valued function given by $H_{ij}=h(u_i,u_j)$. Let Θ be the holomorphic r-square matrix valued function such that $\Theta_{ii}=g_{\alpha_i}$ and $\Theta_{ij}=0$ $(i\neq j)$. We have $\theta\boldsymbol{u}=\boldsymbol{u}\cdot\Theta dz$. Let $\Theta^\dagger$ be the r-square matrix valued function given by $\Theta^\dagger=\overline{H}^{-1}\,{}^t\overline{\Theta}\overline{H}$. We have $\theta^\dagger_h\boldsymbol{u}=\boldsymbol{u}\Theta^\dagger\,d\overline{z}$.

Because $H_{ii}=|u_i|^2_h$, we have $\big|\log H_{ii}\big|\leq 2C_{81}\exp(-\epsilon_{81}d)$, as remarked above. We also have positive constants C_{82} and ϵ_{82} depending only on R, R_1, R_2, R_3, r and C_{10} such that the following holds:

$$|H_{ij}|\leq C_{82}\exp(-\epsilon_{82}d),\qquad(i\neq j)\tag{6}$$

$$\big|\partial_zH_{ij}\big|=\big|\partial_{\overline{z}}H_{ij}\big|\leq C_{82}\exp(-\epsilon_{82}d)\tag{7}$$

$$\big|\partial_{\overline{z}}\partial_zH_{ij}\big|\leq C_{82}\exp(-\epsilon_{82}d)\tag{8}$$

Indeed, (6) follows from Corollary 2.6. We have $\partial h(u_i,u_j)=h\big((\partial_E\pi_{\alpha_i})u_i,u_j\big)+h\big(\pi_{\alpha_i}(\partial_Eu_i),u_j\big)$. As in the proof of Corollary 2.13, we have $\big|\pi_{\alpha_i}\partial_Eu_i-\partial_{\alpha_i}u_i\big|_{h,g_{\mathbb{C}}}\leq C'_{82}\exp(-\epsilon'_{82}d)|u_i|_h$. Then, we obtain (7) from Proposition 2.10 and our choice of u_i. We obtain (8) from the estimate for the curvature $\overline{\partial}\big(\overline{H}^{-1}\partial\overline{H}\big)$ and (7).

Lemma 2.14 *Take any $\boldsymbol{\ell}=(\ell_1,\ell_2)\in\big(\mathbb{Z}_{\geq0}\times\mathbb{Z}_{\geq0}\big)\setminus\{(0,0)\}$ and $R_4<R_3$. We also take any $p>1$. Then, we have positive constants $C_{83,\boldsymbol{\ell},p}$ and $\epsilon_{83,\boldsymbol{\ell},p}$ depending only on R, R_1, R_2, R_3, R_4, r, C_{10}, $\boldsymbol{\ell}$ and p, such that the following holds:*

$$\big\|\partial_z^{\ell_1}\partial_{\overline{z}}^{\ell_2}H_{|\Delta(R_4)}\big\|_{g_{\mathbb{C}},L^p}\leq C_{83,\boldsymbol{\ell},p}\exp\big(-\epsilon_{83,\boldsymbol{\ell},p}d\big).$$

Proof The proof is given in a standard inductive argument using the Hitchin equation and the elliptic regularity. Because Θ_{ii} in the Hitchin equation can be large, we give a rather detailed argument.

Let us consider the case $\boldsymbol{\ell}=(2,0)$. By (7) and (8), we have positive constants C_{84} and ϵ_{84} such that $\|\partial_zH_{|\Delta(R_4)}\|_{L^p_1}\leq C_{84}\exp(-\epsilon_{84}d)$. So, we obtain $\|\partial_z\partial_zH_{|\Delta(R_4)}\|_{L^p}\leq C_{85}\exp(-\epsilon_{85}d)$ for some $C_{85}>0$ and $\epsilon_{85}>0$. Similarly, we can obtain the estimate for $\|\partial_{\overline{z}}\partial_{\overline{z}}H_{\Delta(R_4)}\|_{L^p}$.

Suppose the claim has already been proved for any R_4 and p if $\ell_1+\ell_2<k$. We consider $\boldsymbol{\ell}=(\ell_1,\ell_2)$ satisfying $\ell_1+\ell_2=k$ and $\ell_i>0$. The Hitchin equation is described as follows:

$$\overline{H}^{-1}\partial_{\overline{z}}\partial_z\overline{H}-\overline{H}^{-1}\partial_{\overline{z}}\overline{H}\cdot\overline{H}^{-1}\partial_z\overline{H}-\big[\Theta,\overline{H}^{-1}({}^t\overline{\Theta})\overline{H}\big]=0 \tag{9}$$

For each ℓ, we have $C_{86,\ell}$ depending on C_{10} and ℓ, such that the following holds:

$$\Big|\partial_z^{\ell}\Theta_{ii}\Big|=\Big|\partial_{\overline{z}}^{\ell}\overline{\Theta}_{ii}\Big|\leq C_{86,\ell}d$$

Applying $\partial_z^{\ell_1-1}\partial_{\overline{z}}^{\ell_2-1}$ to (9), we obtain the following:

$$\overline{H}^{-1}\partial_z^{\ell_1}\partial_{\overline{z}}^{\ell_2}\overline{H}-\big[\partial_z^{\ell_1-1}\Theta,\overline{H}^{-1}\overline{{}^t(\partial_z^{\ell_2-1}\Theta)}\overline{H}\big]+G=0$$

Here, G is expressed as a linear combination of terms which contains derivatives of $\overline{H}$. Hence, we have $\|G_{|\Delta(R_4)}\|_{L^p}\leq C_{87}\exp(-\epsilon_{87}d)$. We also have $\Big\|\big[\partial_z^{\ell_1-1}\Theta,\overline{H}^{-1}\overline{{}^t(\partial_z^{\ell_2-1}\Theta)}\overline{H}\big]_{|\Delta(R_4)}\Big\|_{L^p}\leq C_{88}\exp(-\epsilon_{88}d)$, by (6) and (9). Hence, we obtain the desired estimate if $\ell_1+\ell_2=k$ and $\ell_i\neq 0$. We can deal with the cases $\boldsymbol{\ell}=(k,0),(0,k)$ by using the elliptic regularity. Then, by an inductive argument, we can obtain the claim of Lemma 2.14. ∎

Corollary 2.15 *Take any $\boldsymbol{\ell}=(\ell_1,\ell_2)\in\big(\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\big)\setminus\{(0,0)\}$ and $R_4<R_3$. Then, we have positive constants $C_{90,\boldsymbol{\ell}}$ and $\epsilon_{90,\boldsymbol{\ell}}$ depending only on R, R_1, R_2, R_3, R_4, r, C_{10} and $\boldsymbol{\ell}$ such that the following holds:*

$$\sup_{\Delta(R_4)}\big|\partial_z^{\ell_1}\partial_{\overline{z}}^{\ell_2}H\big|\leq C_{90,\boldsymbol{\ell}}\exp\big(-\epsilon_{90,\boldsymbol{\ell}}d\big).$$

∎

2.3 Hitchin WKB-problem

2.3.1 Preliminary

Let V be an r-dimensional complex vector space. For Hermitian metrics h_j $(j=1,2)$, we can take a base $e_1,\ldots,e_r$ of V which is orthogonal with respect to both h_1 and h_2. We have the real numbers α_i determined by $\kappa_i:=\log|e_i|_{h_2}-\log|e_i|_{h_1}$. We impose $\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_r$. Then, we set

$$\vec{d}(h_1,h_2):=\big(\kappa_1,\ldots,\kappa_r\big)\in\mathbb{R}^r.$$

Let V_j $(j=1,2)$ be r-dimensional complex vector spaces with Hermitian metrics h_j. Let $f:V_1\longrightarrow V_2$ be a linear isomorphism. We define the Hermitian metric f^*h_2 on V_1 by $f^*h_2(u,v)=h_2(f(u),f(v))$. We recall the following lemma from [7], which can be easily proved.

Lemma 2.16 *We have orthonormal frames $\boldsymbol{q}(s)=(q_1(s),\ldots,q_r(s))$ $(s=0,1)$ on V_s such that $f(q_j(0))=e^{\beta_j}q_j(1)$ $(j=1,\ldots,r)$, where β_j are real numbers satisfying $\beta_1\geq\beta_2\geq\cdots\geq\beta_r$. In this case, we have*

$$\vec{d}(h_1,f^*h_2)=(\beta_1,\ldots,\beta_r). \tag{10}$$

∎

We set $\|f\|_{\mathrm{op}}:=\sup\Big\{\big|f(u)\big|_{h_2}\,\Big|\,u\in V_1,|u|_{h_1}=1\Big\}$. By Lemma 2.16, we have $\beta_1=\log\|f\|_{\mathrm{op}}$. We also have

$$\sum_{j=1}^{k}\beta_j=\log\Big\|\bigwedge^k f\Big\|_{\mathrm{op}},$$

where $\bigwedge^k f:\bigwedge^k V_1\longrightarrow\bigwedge^k V_2$ are the induced morphisms. Hence, we have the following formula, as noted in [7]:

$$\beta_k=\log\Big\|\bigwedge^k f\Big\|_{\mathrm{op}}-\log\Big\|\bigwedge^{k-1} f\Big\|_{\mathrm{op}}. \tag{11}$$

2.3.2 Hitchin WKB-problem

Let X be any complex curve. Let ϕ_i $(i = 1, \ldots, r)$ be holomorphic 1-forms on X. Let $[0,1] := \{s \in \mathbb{R} \,|\, 0 \leq s \leq 1\}$ be the closed interval. Let $\gamma : [0,1] \longrightarrow X$ be a C^1-map. Suppose that it is a non-critical path in the sense of [7], i.e., the following holds:

- At any point $s \in [0,1]$, we have $\gamma^* \operatorname{Re}(\phi_i)_s \neq \gamma^* \operatorname{Re}(\phi_j)_s$ $(i \neq j)$.

We have the expression $\gamma^*(\phi_i)_s = a_i(s)\, ds$ for some C^∞-functions $a_i : [0,1] \longrightarrow \mathbb{C}$. We may assume $\operatorname{Re} a_i(s) < \operatorname{Re} a_j(s)$ $(i < j)$ for any s. We set

$$\alpha_i := -\int_0^1 \operatorname{Re}(a_i)\, ds.$$

We have $\alpha_1 > \alpha_2 > \cdots > \alpha_r$.

Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle of rank r on X. We suppose the following.

- We have the decomposition $(E, \overline{\partial}_E, \theta) = \bigoplus_{i=1}^r (E_i, \overline{\partial}_{E_i}, t\phi_i \operatorname{id}_{E_i})$ for some $t > 0$, where $\operatorname{rank} E_i = 1$.

We have the associated flat connection $\mathbb{D}^1 = \overline{\partial}_E + \partial_E + \theta + \theta^\dagger$ on E. Let $\Pi_\gamma : E_{|\gamma(0)} \longrightarrow E_{|\gamma(1)}$ be the parallel transport of $\mathbb{D}^1$ along γ. We have the metrics $h_{\gamma(0)}$ on the fibers $E_{|\gamma(\kappa)}$ $(\kappa = 0, 1)$ induced by h. We obtain the metric $\Pi_\gamma^* h_{\gamma(1)}$ on $E_{|\gamma(0)}$ induced by $h_{\gamma(1)}$ and Π_γ.

The following theorem was conjectured in [7], and some cases were verified in [3].

Theorem 2.17 *There exist positive constants t_0, ϵ_0 and C_0, which may depend only on X, $\phi_1, \ldots, \phi_r$ and γ, such that the following holds if $t \geq t_0$:*

$$\Big| \frac{1}{t} \vec{d}(h_{\gamma(0)}, \Pi_\gamma^* h_{\gamma(1)}) - (2\alpha_1, \ldots, 2\alpha_r) \Big| \leq C_0 \exp(-\epsilon_0 t)$$

Proof In the following, the constants C_i and ϵ_i may depend only on X, $\phi_1, \ldots, \phi_r$ and γ, unless otherwise specified.

We can take finite points $s_0 = 0 < s_1 < \cdots < s_{N-1} < s_N = 1$ and coordinate neighbourhoods (U_k, z_k) around $\gamma(s_k)$ such that the following holds:

- We can take $R_k > 0$ $(k = 0, \ldots, N)$ such that U_k contains the disc $\Delta_k(R_k) := \{|z_k| < R_k\}$ and that $\Delta_k(R_k/2) := \{|z_k| < R_k/2\}$ $(k = 0, \ldots, N)$ give a covering of $\gamma([0,1])$.
- We have the expressions $\phi_{i|U_k} = f_{k,i} dz_k$. Set $d_k := \min\big\{ |f_{k,i}(0) - f_{k,j}(0)| \,\big|\, i \neq j \big\}$. Then, we have $|f_{k,i}(z_k) - f_{k,i}(0)| \leq d_k/100$ on $\Delta_k(R_k)$.

If t_0 is sufficiently large, we have $t_0 d_k \geq 1$ for any k. Then, Assumption 2.2 is satisfied for the harmonic bundles $(E, \overline{\partial}_E, \theta, h)_{|\Delta_k(R_k)}$ if $t \geq t_0$. So, we can apply the results in §2.1–2.2 to each $(E, \overline{\partial}_E, \theta, h)_{|\Delta_k(R_k)}$.

Let h_{E_i} be the restriction of h to E_i. Let ∇_i be the Chern connection of $(E_i, \overline{\partial}_{E_i}, h_{E_i})$, and we have the following connection on E:

$$\mathbb{D}_0^1 = \bigoplus_{i=1}^r \big(\nabla_i + t(\phi_i + \overline{\phi_i}) \operatorname{id}_{E_i} \big)$$

We fix a Kähler metric g_X of X. By Corollary 2.13, we have the following estimate on the union of $\Delta_k(R_k/2)$ with respect to h and g_X for some positive constants C_1 and ϵ_1:

$$\Big| \mathbb{D}^1 - \mathbb{D}_0^1 \Big|_{h, g_X} \leq C_1 \exp(-\epsilon_1 t)$$

We have the vector bundle $\gamma^* E = \bigoplus_{i=1}^r \gamma^* E_i$ with the metric $\gamma^* h$ and the connections $\gamma^* \mathbb{D}^1$ and $\gamma^* \mathbb{D}_0^1$. We take any orthonormal frames u_i of $\gamma^* E_i$ such that $\gamma^* \nabla_i u_i = 0$. They give a frame $\boldsymbol{u} = (u_1, \ldots, u_r)$ of $\gamma^* E$. We have the following estimates for some positive constants C_2 and ϵ_2:

$$\Big| \gamma^* h(u_i, u_j) \Big| \leq C_2 \exp(-\epsilon_2 t), \quad (i \neq j).$$

The connection $\gamma^*\mathbb{D}^1_0$ is represented by the diagonal matrix $A\,ds$ with respect to the frame $\boldsymbol{u}$, where the (i,i)-entry of A is $2t\,\mathrm{Re}\,a_i(s)$. Hence, we have

$$\gamma^*\mathbb{D}^1\boldsymbol{u} = \boldsymbol{u}\big(A(s) + B_0(s) + B_1(s)\big)\,ds$$

where $B_m(s)$ $(m = 0,1)$ are r-square matrix valued C^∞-functions such that (i) $B_0(s)_{ij} = 0$ $(i \neq j)$ and $B_1(s)_{ij} = 0$ $(i = j)$, (ii) there exist $\epsilon_3 > 0$ and $C_3 > 0$ such that

$$\big|B_m(s)\big| \leq C_3 \exp(-\epsilon_3 t) \quad (m = 0,1).$$

We may assume that $C_3 \exp(-\epsilon_3 t)$ is sufficiently small for any $t \geq t_0$ if t_0 is sufficiently large. Then, applying Corollary 2.19 in §2.4 below, we have a C^1-function $G : [0,1] \longrightarrow M_r(\mathbb{C})$ and a C^0-function $H : [0,1] \longrightarrow M_r(\mathbb{C})$ such that the following holds for some positive constants C_4 and ϵ_4:

- The C^1-norm of G is dominated by $C_4 \exp(-\epsilon_4 t)$.
- $H(s)$ are diagonal, and the C^0-norm of H is dominated by $C_4 \exp(-\epsilon_4 t)$.
- The connection $\gamma^*\mathbb{D}^1$ is represented by $(A + B_0 + H)\,ds$ with respect to the frame $\boldsymbol{v} := \boldsymbol{u}(I + G)$, where I denotes the r-square identity matrix.

The frame $\boldsymbol{v}\exp\Big(-\int_0^s (A + B_0 + H)d\tau\Big)$ of γ^*E is flat with respect to $\gamma^*\mathbb{D}^1$. The parallel transport Π_γ of $\gamma^*\mathbb{D}^1$ from $E_{\gamma(0)}$ to $E_{\gamma(1)}$ is represented by the following matrix with respect to $\boldsymbol{u}(0)$ and $\boldsymbol{u}(1)$:

$$\big(I + G(1)\big)\exp\Big(-\int_0^1 (A + B_0 + H)d\tau\Big)\big(I + G(0)\big)^{-1}$$

Here, I denotes the r-square identity matrix.

Let $\boldsymbol{p}(s) = (p_1(s),\ldots,p_r(s))$ $(s = 0,1)$ be the orthonormal frames of $E_{|\gamma(s)}$, induced by the frames $\boldsymbol{u}(s)$ and the Gram-Schmidt process. We have $\boldsymbol{p}(s) = \boldsymbol{u}(s)\cdot(I + K(s))$, where we have positive constants C_5 and ϵ_5 such that $|K(s)| \leq C_5 \exp(-\epsilon_5 t)$. Let $L(s)$ $(s = 0,1)$ be determined by $L(s) = (I + K(s))(I + G(s)) - I$. Then, Π_γ is represented by the following matrix with respect to the orthonormal bases $\boldsymbol{p}(0)$ and $\boldsymbol{p}(1)$:

$$Z_\gamma := \big(I + L(1)\big)\exp\Big(-\int_0^1 (A + B_0 + H)d\tau\Big)\big(I + L(0)\big)^{-1}$$

For any r-square matrix Y, we set

$$\|Y\|_{\mathrm{op}} := \sup\Big\{|Y\boldsymbol{v}|\;\Big|\;\boldsymbol{v} \in \mathbb{C}^r,\ |\boldsymbol{v}| = 1\Big\}.$$

We clearly have $\|Y_1Y_2\|_{\mathrm{op}} \leq \|Y_1\|_{\mathrm{op}}\|Y_2\|_{\mathrm{op}}$. So, we have positive constants C_6 and ϵ_6, such that the following holds:

$$\log\|Z_\gamma\|_{\mathrm{op}} \leq \log\Big\|\exp\Big(-\int_0^1 (A + B_0 + H)d\tau\Big)\Big\|_{\mathrm{op}} + \rho_1, \qquad |\rho_1| \leq C_6 \exp(-\epsilon_6 t)$$

We also have positive constants C_7 and ϵ_7, such that the following holds:

$$\log\Big\|\exp\Big(-\int_0^1 (A + B_0 + H)d\tau\Big)\Big\|_{\mathrm{op}} \leq 2t\alpha_1 + \rho_2, \qquad |\rho_2| \leq C_7 \exp(-\epsilon_7 t)$$

We also have positive constants C_8 and ϵ_8, such that the following holds:

$$\big|\Pi_\gamma p_1(0)\big|_{h_{\gamma(1)}} = \exp(2t\alpha_1)\cdot(1 + \rho_3), \qquad |\rho_3| \leq C_8 \exp(-\epsilon_8 t)$$

Hence, we have positive constants C_9 and ϵ_9, such that the following holds:

$$\log\|Z_\gamma\|_{\mathrm{op}} \geq 2t\alpha_1 + \rho_4, \qquad |\rho_4| \leq C_9 \exp(-\epsilon_9 t)$$

Therefore, we have positive constants C_{10} and ϵ_{10}, such that the following holds:

$$\Big|\log\|Z_\gamma\|_{\rm op}-2t\alpha_1\Big|\leq C_{10}\exp(-\epsilon_{10}t)$$

By applying the argument to $\bigwedge^k Z_\gamma$, we obtain positive constants C_{11} and ϵ_{11}, such that the following holds for any k:

$$\Big|\log\|\bigwedge^k Z_\gamma\|_{\rm op}-2t\sum_{j=1}^k\alpha_j\Big|\leq C_{11}\exp(-\epsilon_{11}t) \tag{12}$$

By using (11), we can deduce the claim of the theorem from (12). ∎

2.4 Appendix: A singular perturbation theory

We explain a singular perturbation theory which is available in our situation, and which seems slightly different from those in [3] and [7]. Note that we applied Corollary 2.19 in the proof of Theorem 2.17 to find a family of small gauge transforms with which the family of connections $\gamma^*\boldsymbol{D}$ on $[0,1]$ are transformed to the connections whose connection matrices are diagonal.

2.4.1 Preliminary

We set $[0,1]:=\big\{s\in\mathbb{R}\,\big|\,0\leq s\leq 1\big\}$. For any non-negative integer ℓ, let $C^\ell([0,1])$ denote the space of $\mathbb{C}$-valued C^ℓ-functions on $[0,1]$. We set $\|f\|_0:=\sup_{s\in[0,1]}|f(s)|$ for any $f\in C^0([0,1])$.

Let $M_r(\mathbb{C})$ denote the space of r-square matrices. Let $M_r(\mathbb{C})_0$ denote the space of the r-square diagonal matrices, i.e., $M_r(\mathbb{C})_0=\big\{(a_{ij})\in M_r(\mathbb{C})\,\big|\,a_{ij}=0\ (i\neq j)\big\}$. Let $M_r(\mathbb{C})_1$ denote the set of the off-diagonal matrices, i.e., $M_r(\mathbb{C})_1:=\big\{(a_{ij})\in M_r(\mathbb{C})\,\big|\,a_{ij}=0\ (i=j)\big\}$. We have $M_r(\mathbb{C})=M_r(\mathbb{C})_0\oplus M_r(\mathbb{C})_1$.

For any non-negative integer ℓ, let $C^\ell([0,1],M_r(\mathbb{C}))$ denote the space of C^ℓ-maps $X:[0,1]\longrightarrow M_r(\mathbb{C})$. Similarly, let $C^\ell([0,1],M_r(\mathbb{C})_\kappa)$ $(\kappa=0,1)$ be the space of C^ℓ-maps $X:[0,1]\longrightarrow M_r(\mathbb{C})_\kappa$. We set $\|X\|_0:=\sup_{i,j}\|X_{ij}\|_0$ for $X\in C^0([0,1],M_r(\mathbb{C}))$.

2.4.2 Statement

Fix $C_0>0$. We consider $a_j,b_j\in C^0([0,1])$ $(j=1,\ldots,r)$ satisfying the following conditions:

- $\operatorname{Re}a_1(s)<\operatorname{Re}a_2(s)<\cdots<\operatorname{Re}a_r(s)$ for any s.
- $\big|b_j(s)\big|\leq C_0$ for any s.

For any $t\geq 0$, we put $\alpha_j^t(s):=ta_j(s)+b_j(s)$. Let A^t denote the r-square matrix whose (j,j)-entries are α_j^t.

Proposition 2.18 *There exist constants $C_1>0$ and $\epsilon_1>0$, depending only on C_0, such that the following holds:*

- *For any $t\geq 0$ and any $B\in C^0([0,1],M_r(\mathbb{C})_1)$ satisfying $\|B\|_0\leq\epsilon_1$, we can take $G^t\in C^1\big([0,1],M_r(\mathbb{C})_1\big)$ and $H^t\in C^0\big([0,1],M_r(\mathbb{C})_0\big)$ such that (i) $\|G^t\|_0+\|\partial_sG^t+[A^t,G^t]\|_0+\|H^t\|_0\leq C_1\|B\|_0$, (ii) we have*

$$A^t+B=(I+G^t)^{-1}(A^t+H^t)(I+G^t)+(I+G^t)^{-1}\partial_sG^t. \tag{13}$$

Here, I denotes the r-square identity matrix.

We shall prove the proposition in §2.4.3–2.4.4. Indeed, we shall give a more refined claim (see Corollary 2.23).

We give a reformulation of Proposition 2.18. Recall that when we have a vector bundle V with a connection ∇ and a frame $\boldsymbol{w}=(w_1,\ldots,w_m)$, we have the matrix-valued 1-form $A=(A_{ij})$ determined by $\nabla w_j=\sum A_{ij}w_i$, and we describe the relation by $\nabla\boldsymbol{w}=\boldsymbol{w}A$.

Corollary 2.19 *Let E be a C^1-vector bundle on $[0,1]$ with a frame $\boldsymbol{v}=(v_1,\ldots,v_r)$. Let $B\in C^0([0,1],M_r(\mathbb{C})_1)$ satisfying $\|B\|_0\leq\epsilon_1$. Take $t\geq 0$. Let ∇^t be the connection on E given as follows:*

$$\nabla^t\boldsymbol{v}=\boldsymbol{v}\big(A^t(s)+B(s)\big)ds.$$

Then, we can take $G^t\in C^1([0,1],M_r(\mathbb{C})_1)$ and $H^t\in C^0([0,1],M_r(\mathbb{C})_0)$ such that (i) $\|G^t\|_0+\|\partial_sG^t+[A^t,G^t]\|_0+\|H^t\|_0\leq C_1\|B\|_0$, (ii) for the frame $\boldsymbol{u}^t=\boldsymbol{v}\cdot(I+G^t)^{-1}$, we have $\nabla^t\boldsymbol{u}^t=\boldsymbol{u}^t\cdot(A^t+H^t)ds$. Here, ϵ_1 and C_1 are constants as in Proposition 2.18. ∎

Remark 2.20 *In Proposition 2.18, the off-diagonal part B is required to be sufficiently small. It is satisfied in the Hitchin WKB-problem if t is sufficiently large, as observed in Corollary 2.13. However, it is not satisfied in the Riemann-Hilbert WKB-problem [7] even if t is large, in general. But, we may still apply Proposition 2.18 after dividing the path to shorter paths* [1] *as did in [7]. It also seems possible to apply Proposition 2.18 to the WKB-problem for family of λ-connections without going to small paths.* ∎

2.4.3 Some linear maps

Let $C^1([0,1],M_r(\mathbb{C})_1)_\partial$ denote the subspace of $C^1([0,1],M_r(\mathbb{C})_1)$ which consists of the functions $X=(X_{ij}):[0,1]\longrightarrow M_r(\mathbb{C})$ such that $X_{ij}(1)=0$ $(i<j)$ and $X_{ij}(0)=0$ $(i>j)$. We have the linear map $D_0^t: C^1([0,1],M_r(\mathbb{C})_1)_\partial\longrightarrow C^0([0,1],M_r(\mathbb{C})_1)$ given by

$$D_0^t(X):=\partial_sX+[A^t,X],\quad\text{i.e., } D_0^t(X)_{ij}=\partial_sX_{ij}+(\alpha_i^t-\alpha_j^t)X_{ij}.$$

For $i\neq j$, we put

$$F_{i,j}^t(s):=\int_0^s\big(\alpha_i^t(\tau)-\alpha_j^t(\tau)\big)d\tau.$$

We have the map $I_0^t: C^0\big([0,1],M_r(\mathbb{C})_1\big)\longrightarrow C^1\big([0,1],M_r(\mathbb{C})_1\big)_\partial$ given as follows:

$$I_0^t(X)_{i,j}:=\begin{cases}\int_1^s\exp(-F_{i,j}^t(s)+F_{i,j}^t(\tau))X_{i,j}(\tau)d\tau & (i<j)\\[2ex] \int_0^s\exp(-F_{i,j}^t(s)+F_{i,j}^t(\tau))X_{i,j}(\tau)d\tau & (i>j)\end{cases}$$

Together with the identity map on $C^0([0,1],M_r(\mathbb{C})_0)$, we also obtain the following maps:

$$I_0^t: C^0([0,1],M_r(\mathbb{C})_0)\oplus C^0([0,1],M_r(\mathbb{C})_1)\longrightarrow C^0([0,1],M_r(\mathbb{C})_0)\oplus C^1([0,1],M_r(\mathbb{C})_1)_\partial$$
$$D_0^t: C^0([0,1],M_r(\mathbb{C})_0)\oplus C^1([0,1],M_r(\mathbb{C})_1)_\partial\longrightarrow C^0([0,1],M_r(\mathbb{C})_0)\oplus C^0([0,1],M_r(\mathbb{C})_1)$$

Then, I_0^t and D_0^t are mutually inverse.

Lemma 2.21 *There exists a constant $K_1>0$, depending only on C_0, such that the following holds for any $t\geq 0$ and for any $(Z,W)\in C^0([0,1],M_r(\mathbb{C})_0)\oplus C^0([0,1],M_r(\mathbb{C})_1)$:*

$$\big\|I_0^t(Z,W)\big\|_0\leq K_1\big\|(Z,W)\big\|_0.$$

Proof The estimate for (i,i)-entries are obvious by the construction. Let us consider the estimate for (i,j)-entries $(i\neq j)$. We set $Q_{i,j}^t(s):=t\int_0^s\big(a_i(\tau)-a_j(\tau)\big)\,d\tau$ and $R_{i,j}(s):=\int_0^s\big(b_i(\tau)-b_j(\tau)\big)d\tau$. We have $F_{i,j}^t=Q_{i,j}^t+R_{i,j}$. Because $\mathrm{Re}(a_k)<\mathrm{Re}(a_\ell)$ $(k<\ell)$, we have the following for $s_1\leq s_2$:

$$\mathrm{Re}\,Q_{i,j}^t(s_1)-\mathrm{Re}\,Q_{i,j}^t(s_2)\geq 0\ (i<j),\quad \mathrm{Re}\,Q_{i,j}^t(s_1)-\mathrm{Re}\,Q_{i,j}^t(s_2)\leq 0\ (i>j).$$

Hence, we have a constant K_1', depending only on C_0, such that $\Big|\exp\big(-F_{i,j}^t(s)+F_{i,j}^t(\tau)\big)\Big|\leq K_1'$ holds in the cases (i) $s\leq\tau$ and $i<j$, (ii) $s\geq\tau$ and $i>j$. Then, we obtain the following in the case $i<j$:

$$\big|I_0^t(X)_{i,j}\big|\leq\int_s^1\Big|\exp\big(-F_{i,j}^t(s)+F_{i,j}^t(\tau)\big)X_{i,j}(\tau)\Big|\,d\tau\leq K_1'\int_s^1|X_{i,j}(\tau)|d\tau\leq K_1'\|X_{i,j}\|_0$$

Hence, we obtain $\|I_0^t(X)_{i,j}\|_0\leq K_1'\|X_{i,j}\|_0$ in the case $i<j$. Similarly, we obtain $\|I_0^t(X)_{i,j}\|_0\leq K_1'\|X_{i,j}\|_0$ in the case $i>j$. ∎

[1] This was remarked by C. Simpson.

2.4.4 Proof of Proposition 2.18

We take a small number $\epsilon > 0$. We set

$$\mathcal{H}_\epsilon := \big\{ H \in C^0([0,1], M_r(\mathbb{C})_0) \,\big|\, \|H\|_0 \leq \epsilon \big\}, \qquad \mathcal{G}^t_\epsilon := \big\{ G \in C^1([0,1], M_r(\mathbb{C})_1)_\partial \,\big|\, \|D^t_0 G\|_0 \leq \epsilon \big\}.$$

Note that $\|G\|_0 \leq K_1 \|D^t_0 G\|_0$ for any $G \in \mathcal{G}^t_\epsilon$.

Let $I \in M_r(\mathbb{C})$ denote the identity matrix. If $\epsilon K_1 < 1/2$, then $(I+G)(s)$ are invertible for any $G \in \mathcal{G}^t_\epsilon$. So, we have the maps $J^t : \mathcal{H}_\epsilon \times \mathcal{G}^t_\epsilon \longrightarrow C^0([0,1], M_r(\mathbb{C})_0) \oplus C^0([0,1], M_r(\mathbb{C})_1)$ given by

$$J^t(H,G) := (I+G)^{-1}(A^t + H)(I+G) + (I+G)^{-1}\partial_s G - A^t.$$

Let $T_{(H,G)}J^t$ denote the derivative at (H,G). We have $T_{(0,0)}J^t(X,Y) = X + D^t_0 Y$. More generally, we have

$$\begin{aligned} T_{(H,G)}J^t(X,Y) = (I+G)^{-1}X(I+G) + (I+G)^{-1}(A^t+H)Y - (I+G)^{-1}Y(I+G)^{-1}(A^t+H)(I+G) \\ - (I+G)^{-1}Y(I+G)^{-1}\partial_s G + (I+G)^{-1}\partial_s Y \end{aligned} \tag{14}$$

We regard $T_{(H,G)}J^t$ as maps $C^0([0,1], M_r(\mathbb{C})_0) \oplus C^1([0,1], M_r(\mathbb{C})_1)_\partial \longrightarrow C^0([0,1], M_r(\mathbb{C})_0) \oplus C^0([0,1], M_r(\mathbb{C})_1)$. We obtain the following family of endomorphisms on $C^0([0,1], M_r(\mathbb{C})_0) \oplus C^0([0,1], M_r(\mathbb{C})_1)$:

$$T_{(H,G)}J^t \circ \big(T_{(0,0)}J^t\big)^{-1}, \qquad ((H,G) \in \mathcal{H}_\epsilon \times \mathcal{G}^t_\epsilon).$$

Lemma 2.22 *There exists a constant $C_2 > 0$, depending only on C_0, such that the following holds for any $t \geq 0$ and any $(Z,W) \in C^0([0,1], M_r(\mathbb{C})_0) \oplus C^0([0,1], M_r(\mathbb{C})_1)$:*

$$\Big\| T_{(H,G)}J^t \circ \big(T_{(0,0)}J^t\big)^{-1}(Z,W) - (Z,W) \Big\|_0 \leq C_2 \big(\|H\|_0 + \|D^t_0 G\|_0\big)\big(\|Z\|_0 + \|W\|_0\big)$$

Proof We consider

$$(I+G)^{-1}(A^t)Y - (I+G)^{-1}Y(I+G)^{-1}(A^t)(I+G) - (I+G)^{-1}Y(I+G)^{-1}\partial_s G + (I+G)^{-1}\partial_s Y - D^t_0 Y. \tag{15}$$

We have $(I+G)^{-1}\partial_s Y = (I+G)^{-1}D^t_0 Y - (I+G)^{-1}[A^t, Y]$. Hence, (15) is rewritten as

$$\begin{aligned} (I+G)^{-1}(A^t)Y - (I+G)^{-1}Y(I+G)^{-1}(A^t)(I+G) - (I+G)^{-1}Y(I+G)^{-1}\partial_s G \\ - (I+G)^{-1}[A^t, Y] + \big((I+G)^{-1} - I\big)D^t_0 Y \end{aligned} \tag{16}$$

It is equal to the following:

$$\begin{aligned} (I+G)^{-1}Y(A^t) - (I+G)^{-1}Y(I+G)^{-1}(A^t)(I+G) - (I+G)^{-1}Y(I+G)^{-1}\partial_s G \\ + \big((I+G)^{-1} - I\big)D^t_0 Y \end{aligned} \tag{17}$$

We have the following:

$$\begin{aligned} &- (I+G)^{-1}Y(I+G)^{-1}(A^t)(I+G) - (I+G)^{-1}Y(I+G)^{-1}\partial_s G \\ &\qquad = -(I+G)^{-1}Y(I+G)^{-1}(A^t) - (I+G)^{-1}Y(I+G)^{-1}\big(\partial_s G + (A^t)G\big) \\ &\qquad = -(I+G)^{-1}Y(I+G)^{-1}(A^t) - (I+G)^{-1}Y(I+G)^{-1}G(A^t) - (I+G)^{-1}Y(I+G)^{-1}D^t_0 G \end{aligned} \tag{18}$$

Hence, (17) is equal to the following:

$$\begin{aligned} &(I+G)^{-1}Y(A^t) - (I+G)^{-1}Y(I+G)^{-1}(A^t) - (I+G)^{-1}Y(I+G)^{-1}G(A^t) \\ &\qquad - (I+G)^{-1}Y(I+G)^{-1}D^t_0 G + \big((I+G)^{-1} - I\big)D^t_0 Y \\ &\qquad = -(I+G)^{-1}Y(I+G)^{-1}D^t_0 G + \big((I+G)^{-1} - I\big)D^t_0 Y \end{aligned} \tag{19}$$

In all, we obtain the following:

$$T_{(H,G)}J^t \circ \big(T_{(0,0)}J^t\big)^{-1}(Z,W) - (Z,W) = -(I+G)^{-1}I_0^t(W)(I+G)^{-1}D_0^tG - \big((I+G)^{-1}-I\big)W \\ + (I+G)^{-1}Z(I+G) - Z + \big[(I+G)^{-1}H(I+G), (I+G)^{-1}I_0^t(W)\big] \quad (20)$$

Because $\|G\|_0 \leq K_1\|D_0^tG\|_0$, we have $\|(I+G)^{-1}-I\|_0 \leq C_3\|D_0^tG\|_0$ and $\|(I+G)^{-1}X(I+G)-X\|_0 \leq C_4(\|D_0^tG\|_0\|X\|_0)$ for positive constants C_i $(i=3,4)$ depending only on C_0. We also have the following for a positive constant C_5 depending only on C_0:

$$\Big\|\big[(I+G)^{-1}H(I+G), (I+G)^{-1}Y\big]\Big\|_0 \leq C_5\big(\|D_0^tG\|_0 + \|H\|_0\big)\|D_0^tY\|_0$$

Hence, we obtain the claim of the lemma. ∎

Corollary 2.23 *There exist positive constants $\epsilon_{10}>0$ and $C_{10}>0$, depending only on C_0, with the following property:*

- *For any $t\geq 0$ and any $B\in C^0([0,1],M_r(\mathbb{C})_1)$ such that $\|B\|_0\leq\epsilon_{10}$, we have a unique $(H^t,G^t)\in\mathcal{H}_\epsilon\times\mathcal{G}_\epsilon^t$ such that $J^t(H^t,G^t)=B$ and that $\|H^t\|_0+\|D_0^tG^t\|_0\leq C_{10}\|B\|_0$.*

Proof We set $\overline{\mathcal{G}}_\epsilon := \big\{\overline{G}\in C^0([0,1],M_r(\mathbb{C})_1)\,\big|\,\|\overline{G}\|_0\leq\epsilon\big\}\subset C^0([0,1],M_r(\mathbb{C})_1)$. We have the bijections $I_0^t:\overline{\mathcal{G}}_\epsilon\simeq\mathcal{G}_\epsilon^t$ for any $t\geq 0$. We consider the maps

$$\mathcal{F}^t := J^t\circ(T_{(0,0)}J^t)^{-1}:\mathcal{H}_\epsilon\times\overline{\mathcal{G}}_\epsilon \longrightarrow C^0([0,1],M_r(\mathbb{C})_0)\oplus C^0([0,1],M_r(\mathbb{C})_1).$$

Let $T\mathcal{F}^t: C^0([0,1],M_r(\mathbb{C}))\longrightarrow C^0([0,1],M_r(\mathbb{C}))$ denote the derivative of $\mathcal{F}^t$. According to Lemma 2.22, we have a positive constant C_3 such that the operator norms of $T_{(H,\overline{G})}\mathcal{F}^t-\mathrm{id}$ are dominated by $C_3\big(\|H\|_0+\|\overline{G}\|_0\big)$ for any $(H,\overline{G})\in\mathcal{H}_\epsilon\times\overline{\mathcal{G}}_\epsilon$. The constant C_3 may depend only on C_0, and the estimate is uniform for t. By the inverse function theorem (see [8], for instance), there exist positive constants ϵ_{10} and C_{10}, depending only on C_0, with the following property:

- For any $t\geq 0$ and any $B\in C^0([0,1],M_r(\mathbb{C})_1)$ such that $\|B\|_0\leq\epsilon_{10}$, we have a unique $(H^t,\overline{G}^t)\in\mathcal{H}_\epsilon\times\overline{\mathcal{G}}_\epsilon$ such that $\mathcal{F}^t(H^t,\overline{G}^t)=B$ and that $\|H^t\|_0+\|\overline{G}^t\|_0\leq C_{10}\|B\|_0$.

By setting $G^t:=(T_{(0,0)}J^t)^{-1}\overline{G}^t$, we obtain the claim of Corollary 2.23. We also finish the proof of Proposition 2.18. ∎

2.5 Appendix: The case of Hermitian-Einstein metrics

For $R>0$, we set $\Delta(R):=\big\{z\in\mathbb{C}\,\big|\,|z|<R\big\}$. Let $(E,\overline{\partial}_E,\theta)$ be a Higgs bundle of rank r on $\Delta(R)$. We fix a Hermitian metric $h_{\det(E)}$ of $\det(E)$. Let h be a Hermitian-Einstein metric of $(E,\overline{\partial}_E,\theta)$ such that $\det(h)=h_{\det(E)}$, i.e., $R(h)^\perp+[\theta,\theta_h^\dagger]=0$, where $R(h)^\perp$ denotes the trace-free part of the curvature $R(h)$. We have an obvious generalization of the results in §2.1 and §2.2, which we state explicitly in this subsection.

We have a C^∞-function ν such that $\overline{\partial}\partial\nu=R(h_{\det(E)})/r$ such that the L_2^p-norm of ν is dominated by $C_{0,p}\|R(h_{\det(E)})\|_\infty$ $(p>1)$, where $\|R(h_{\det(E)})\|_\infty$ denotes the sup norm of $R(h_{\det(E)})$ with respect to the Euclidean metric, and $C_{0,p}$ denotes the constant depending on p and the radius R. Then, the metric $\widetilde{h}=he^{-\nu}$ is a harmonic metric for the Higgs bundle $(E,\overline{\partial}_E,\theta)$. Note that h and $\widetilde{h}$ induce the same metrics on the vector bundles $\mathrm{End}(E)\otimes\Omega^{p,q}$.

We have the description $\theta=f\,dz$. Let M be the constant as in §2.1.1. We obtain the following from Proposition 2.1.

Corollary 2.24 *Fix $0<R_1<R$. Then, we have $C_1,C_2>0$ depending only on r,R_1,R such that $|f|_h\leq C_1M+C_2$ holds on $\Delta(R_1)$.* ∎

We impose the conditions as in Assumption 2.2. We obtain the following from Proposition 2.3.

Corollary 2.25 *Let R_1 be as in Proposition 2.1. Fix $0 < R_2 < R_1$. There exist positive constants ϵ_0 and C_{11} depending only on R, R_1, R_2, r and C_{10}, such that $|\rho_\alpha|_h \leq C_{11}\exp(-\epsilon_0 d)$ on $\Delta(R_2)$.* ∎

We impose the condition that $\operatorname{rank} E_\alpha = 1$ as in §2.2. We have $R(\widetilde{h}) = R(h)^\perp$ and $\theta_h^\dagger = \theta_{\widetilde{h}}^\dagger$. Hence, we obtain the following from Theorem 2.9.

Corollary 2.26 *There exist positive constants C_{30} and ϵ_{30}, depending only on R, R_1, R_2, r and C_{10} such that $\big|R(h)^\perp\big|_{h,g_{\mathbb{C}}} = \big|[\theta, \theta_h^\dagger]\big|_{h,g_{\mathbb{C}}} \leq C_{30}\exp(-\epsilon_{30}d)$ on $\Delta(R_2)$.* ∎

Because h and $\widetilde{h}$ give the same connection on $\operatorname{End}(E)$ as the Chern connections, we obtain the following from Proposition 2.10 and Proposition 2.11.

Corollary 2.27 *Take $0 < R_3 < R_2$. There exist positive constants ϵ_{40} and C_{40} depending only on R, R_1, R_2, R_3, r and C_{10} such that $\big|\partial_{E,h}\pi_\alpha\big|_{h,g_{\mathbb{C}}} = \big|\overline{\partial}_E\pi_\alpha^\dagger\big|_{h,g_{\mathbb{C}}} \leq C_{40}\exp(-\epsilon_{40}d)$ and $\big|\overline{\partial}_E\pi'_\alpha\big|_{h,g_{\mathbb{C}}} = \big|\partial_{E,h}\pi'_\alpha\big|_{h,g_{\mathbb{C}}} \leq C_{40}\exp(-\epsilon_{40}d)$ on $\Delta(R_3)$.* ∎

Let h_α (resp. $h_{0,\alpha}$) denote the restriction of h (resp. $\widetilde{h}$) to L_α. Because $h_\alpha = \widetilde{h}_\alpha e^{\nu/r}$, we obtain the following from Proposition 2.12.

Corollary 2.28 *We have a positive constant C_{60} depending only on R, R_1, R_2, R_3, r and C_{10} such that $\big|R(h_\alpha) - R(h_{\det(E)})/r\big|_{h_\alpha,g_{\mathbb{C}}} \leq C_{60}\exp(-\epsilon_{40}d)$ on $\Delta(R_3)$.* ∎

We have the projectively flat connection $\mathbb{D}^1 := \overline{\partial}_E + \partial_{E,h} + \theta + \theta_h^\dagger$ induced by h. We also have the flat connection $\widetilde{\mathbb{D}}^1 := \overline{\partial}_E + \partial_{E,\widetilde{h}} + \theta + \theta_{\widetilde{h}}^\dagger$. They are related as $\mathbb{D}^1 = \widetilde{\mathbb{D}}^1 + \partial\nu \cdot \mathrm{id}_E$. We have the Chern connections $\overline{\partial}_\alpha + \partial_{\alpha,h}$ on E_α induced by h_α. Similarly, we have the Chern connections $\overline{\partial}_\alpha + \partial_{\alpha,\widetilde{h}}$ on E_α induced by $\widetilde{h}_\alpha$. They are related as $\partial_{\alpha,h} = \partial_{\alpha,\widetilde{h}} + \partial\nu \cdot \mathrm{id}_{E_\alpha}$. Hence, we obtain the following from Corollary 2.13.

Corollary 2.29 *We set $\mathbb{D}^1_0 := \bigoplus\big(\overline{\partial}_\alpha + \partial_{\alpha,h} + (g_\alpha dz + \overline{g_\alpha}d\overline{z})\cdot \mathrm{id}_{E_\alpha}\big)$ as in §2.2.4. Then, there exists a positive constant C_{70}, depending only on R, R_1, R_2, R_3, r and C_{10} such that $\big|\mathbb{D}^1 - \mathbb{D}^1_0\big|_{h,g_{\mathbb{C}}} \leq C_{70}\exp(-\epsilon_{40}d)$ holds on $\Delta(R_3)$.* ∎

We also have an obvious generalization of the estimates of the higher derivatives in §2.2.5, which we omit to describe.

3 Local models

3.1 Review on unramifiedly good filtered Higgs bundles

3.1.1 Filtered bundles on curves

Let X be a complex curve with a discrete subset D. We recall the concept of filtered bundles on (X, D) [20]. Let $\mathbb{R}^D$ denote the set of maps $D \longrightarrow \mathbb{R}$. Elements of $\mathbb{R}^D$ are denoted by $\boldsymbol{a} = (a_P \,|\, P \in D)$. Let $\mathcal{O}_X(*D)$ be the sheaf of meromorphic functions on X whose poles are contained in D. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X(*D)$-module of rank r. A filtered bundle over $\mathcal{E}$ is a family of coherent $\mathcal{O}_X$-submodules $\mathcal{P}_*\mathcal{E} = \big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \subset \mathcal{E} \,\big|\, \boldsymbol{a} \in \mathbb{R}^D\big)$ with the following property.

- $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}$ are lattices of $\mathcal{E}$, i.e., $\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D) = \mathcal{E}$.
- The stalk of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}$ at $P \in D$ depends only on $a_P \in \mathbb{R}$. We denote it by $\mathcal{P}^P_{a_P}(\mathcal{E}_P)$.
- We have $\mathcal{P}^P_a(\mathcal{E}_P) \subset \mathcal{P}^P_b(\mathcal{E}_P)$ for $a \leq b$. For any $a \in \mathbb{R}$, there exists $\epsilon > 0$ such that $\mathcal{P}^P_a(\mathcal{E}_P) = \mathcal{P}^P_{a+\epsilon}(\mathcal{E}_P)$.
- For $n \in \mathbb{Z}$, we have $\mathcal{P}^P_a(\mathcal{E}_P) \otimes_{\mathcal{O}_{X,P}} \mathcal{O}_X(nP)_P = \mathcal{P}^P_{a+n}(\mathcal{E}_P)$.

Such $\mathcal{P}_*\mathcal{E}$ is called a filtered bundle over $\mathcal{E}$ on (X, D). For any $\mathcal{O}_X(*D)$-submodule $\mathcal{G} \subset \mathcal{E}$, we have the induced filtered bundle $\mathcal{P}_*\mathcal{G}$ over $\mathcal{G}$ given by $\mathcal{P}_{\boldsymbol{a}}\mathcal{G} = \mathcal{P}_{\boldsymbol{a}}\mathcal{E} \cap \mathcal{G}$ in $\mathcal{E}$.

Parabolic structure Let $\mathcal{P}_*\mathcal{E}$ be a filtered bundle on (X,D). Let $\mathbf{0}\in\mathbb{R}^D$ denote the element such that the P-th component of $\mathbf{0}$ are 0 for any $P\in D$. We have the locally free $\mathcal{O}_X$-module $\mathcal{P}_{\mathbf{0}}\mathcal{E}$ on X. For each $Q\in X$, let $\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|Q}$ denote the fiber of the vector bundle $\mathcal{P}_{\mathbf{0}}\mathcal{E}$ over Q. For $P\in D$ and for $-1<a\leq 0$, we set

$$F^P_a(\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P}):=\mathrm{Im}\Big(\mathcal{P}^P_a(\mathcal{E}_P)\longrightarrow\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P}\Big).$$

In this way, we obtain a filtration F^P of $\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P}$ indexed by $\{-1<a\leq 0\}$. We have $F^P_0(\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P})=\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P}$. Note that we have the natural identification $\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P}\simeq\mathcal{P}^P_0(\mathcal{E}_P)/\mathcal{P}^P_{-1}(\mathcal{E}_P)$ because $\mathcal{P}^P_{a+n}(\mathcal{E})=\mathcal{P}^P_a(\mathcal{E})\otimes_{\mathcal{O}_{X,P}}\mathcal{O}_{X,P}(nP)$, and hence we have $F^P_{-1}(\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P})=0$. For any $-1<a\leq 0$, there exists $\epsilon>0$ such that $F^P_a(\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P})=F^P_{a+\epsilon}\mathcal{P}_{\mathbf{0}}\mathcal{E}_{|P}$. Such a family of filtrations $(F^P\,|\,P\in D)$ is called a parabolic structure on the vector bundle $\mathcal{P}_{\mathbf{0}}\mathcal{E}$ along D. We can easily observe that filtered bundles on (X,D) are equivalent to vector bundles on X with a parabolic structure along D.

3.1.2 Unramifiedly good filtered Higgs bundles

Let X and D be as in §3.1.1. Let $\mathcal{P}_*\mathcal{E}$ be a filtered bundle on (X,D). Let θ be a Higgs field of $\mathcal{E}$, i.e., θ is a holomorphic section of $\mathrm{End}(\mathcal{E})\otimes\Omega^1_X$. The filtered bundle with a Higgs field $(\mathcal{P}_*\mathcal{E},\theta)$ is called unramifiedly good at P, if the following holds.

- We have a finite subset $\mathcal{I}(P)\subset\mathcal{O}_X(*D)_P$ and a decomposition of the stalk $\mathcal{P}^P_a\mathcal{E}_P=\bigoplus_{\mathfrak{a}\in\mathcal{I}(P)}\mathcal{P}^P_a\mathcal{E}_{P,\mathfrak{a}}$ such that
$$(\theta-d\mathfrak{a}\,\mathrm{id})\mathcal{P}^P_a\mathcal{E}_{P,\mathfrak{a}}\subset\mathcal{P}^P_a\mathcal{E}_{P,\mathfrak{a}}\otimes\Omega^1_X(\log D)_P.$$
Here, $d\mathfrak{a}$ denotes the exterior derivative of $\mathfrak{a}\in\mathcal{I}(P)\subset\mathcal{O}_X(*D)_P$. We have $\mathcal{P}^P_a\mathcal{E}_{P,\mathfrak{a}}\subset\mathcal{P}^P_b\mathcal{E}_{P,\mathfrak{a}}$ for $a\leq b$. The induced map $\mathcal{I}(P)\longrightarrow\mathcal{O}_X(*D)_P/\mathcal{O}_{X,P}$ is assumed to be injective.

If $(\mathcal{P}_*\mathcal{E},\theta)$ is unramifiedly good at any $P\in D$, it is called an unramifiedly good filtered Higgs bundle.

Remark 3.1 *More generally, a filtered bundle with a Higgs field $(\mathcal{P}_*\mathcal{E},\theta)$ is called good at P if we have a neighbourhood X_P of P, a Galois covering map $X'_P\longrightarrow X_P$ ramified along P, and an unramifiedly good filtered Higgs bundle $(\mathcal{P}^P_*\mathcal{E}',\theta')$ on $(X'_P,\pi^{-1}(P))$ such that $(\mathcal{P}_*\mathcal{E},\theta)_{|X_P}$ is the descent of $(\mathcal{P}^P_*\mathcal{E}',\theta')$. In this paper, we consider only unramified ones.* ∎

3.1.3 Unramified wild harmonic bundles and the associated unramifiedly good filtered Higgs bundles

Let X and D be as in §3.1.1. Let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle on $X\setminus D$. It is called wild and unramified over (X,D) if the following holds for any $P\in D$.

- Let (U_P,z) be a holomorphic coordinate neighbourhood of P in X with $z(P)=0$. Then, we have a finite subset $\mathcal{I}(P)\subset z^{-1}\mathbb{C}[z^{-1}]$ and a decomposition
$$(E,\overline{\partial}_E,\theta)_{|U_P\setminus\{P\}}=\bigoplus_{\mathfrak{a}\in\mathcal{I}(P)}(E_{\mathfrak{a}},\overline{\partial}_{E_{\mathfrak{a}}},\theta_{\mathfrak{a}}),$$
and the coefficients $a_{\mathfrak{a},j}(z)$ of the characteristic polynomials $\det(t\,\mathrm{id}-g_{\mathfrak{a}})=\sum a_{\mathfrak{a},j}(z)t^j$ are holomorphic at $z=0$, where $g_{\mathfrak{a}}\in\mathrm{End}(E_{\mathfrak{a}})$ are determined by $\theta_{\mathfrak{a}}-d\mathfrak{a}\,\mathrm{id}_{E_{\mathfrak{a}}}=g_{\mathfrak{a}}\,dz/z$.

A unramified wild harmonic bundle is called tame if $\mathcal{I}(P)=\{0\}$ in $\mathcal{O}_X(*D)_P/\mathcal{O}_{X,P}$ at each $P\in D$.

The tame case of the following proposition was established in [20], and generalized to the wild case in [16].

Proposition 3.2 *Let $(E,\overline{\partial}_E,\theta,h)$ be an unramified wild harmonic bundle on (X,D). Then, we have an unramifiedly good filtered Higgs bundle $(\mathcal{P}^h_*E,\theta)$ on (X,D) with the following property.*

- *We have $(\mathcal{P}^h_{\boldsymbol{a}}E,\theta)_{|X\setminus D}=(E,\theta)$ for any $\boldsymbol{a}\in\mathbb{R}^D$.*
- *Let U be any open subset of X. Let f be a holomorphic section of E on $U\setminus D$. Then, f is a section of $\mathcal{P}^h_{\boldsymbol{a}}E$ on U if and only if $|f|_h=O(|z_P|^{-a_P-\epsilon})$ for any $\epsilon>0$ around each $P\in U\cap D$, where z_P denotes a holomorphic coordinate around P with $z_P(P)=0$.* ∎

In this situation, h is called adapted to the filtered bundle $\mathcal{P}^h_*E$.

3.1.4 Filtered Higgs bundles on compact Riemann surfaces

Suppose that X is a compact connected Riemann surface with a finite subset D. Let $(\mathcal{P}_*\mathcal{E},\theta)$ be an unramifiedly good filtered Higgs bundle on (X,D). Let $\mathcal{G}\subset\mathcal{E}$ be any locally free $\mathcal{O}_X(*D)$-submodule. We set $\mathcal{P}_{\boldsymbol{a}}\mathcal{G}:=\mathcal{P}_{\boldsymbol{a}}\mathcal{E}\cap\mathcal{G}$ in $\mathcal{E}$ for any $\boldsymbol{a}\in\mathbb{R}^D$, and then we obtain a filtered bundle $\mathcal{P}_*\mathcal{G}$ over $\mathcal{G}$. The degree of $\mathcal{P}_*\mathcal{G}$ is defined as follows [9, 10] for any $\boldsymbol{b}\in\mathbb{R}^D$:

$$\deg(\mathcal{P}_*\mathcal{G})=\deg(\mathcal{P}_{\boldsymbol{b}}\mathcal{G})-\sum_{P\in D}\sum_{b_P-1<a\leq b_P}a\cdot\dim_{\mathbb{C}}\bigl(\mathcal{P}^P_a(\mathcal{G}_P)/\mathcal{P}^P_{<a}(\mathcal{G}_P)\bigr)\tag{21}$$

Here, we set $\mathcal{P}^P_{<a}(\mathcal{G}_P)=\bigcup_{c<a}\mathcal{P}^P_c(\mathcal{G}_P)$, and we regard $\mathcal{P}^P_a(\mathcal{G}_P)/\mathcal{P}^P_{<a}(\mathcal{G}_P)$ as finite dimensional $\mathbb{C}$-vector spaces in a natural way. It is easy to see that the right hand side of (21) is independent of the choice of $\boldsymbol{b}$. The unramifiedly good filtered Higgs bundle $(\mathcal{P}_*\mathcal{E},\theta)$ is called stable (resp. semistable) if the following inequality holds for any locally free $\mathcal{O}_X(*D)$-submodule $\mathcal{G}\subset\mathcal{E}$ with (i) $\theta\mathcal{G}\subset\mathcal{G}\otimes\Omega^1_X$, (ii) $0<\operatorname{rank}\mathcal{G}<\operatorname{rank}\mathcal{E}$:

$$\frac{\deg(\mathcal{P}_*\mathcal{G})}{\operatorname{rank}\mathcal{G}}<\frac{\deg(\mathcal{P}_*\mathcal{E})}{\operatorname{rank}\mathcal{E}}\qquad\left(\text{resp. }\frac{\deg(\mathcal{P}_*\mathcal{G})}{\operatorname{rank}\mathcal{G}}\leq\frac{\deg(\mathcal{P}_*\mathcal{E})}{\operatorname{rank}\mathcal{E}}\right)$$

The unramifiedly good filtered Higgs bundle $(\mathcal{P}_*\mathcal{E},\theta)$ on (X,D) is called polystable if it is the direct sum of stable ones $(\mathcal{P}_*\mathcal{E}_i,\theta_i)$ $(i=1,\dots,m)$ with $\deg(\mathcal{P}_*\mathcal{E}_i)=\deg(\mathcal{P}_*\mathcal{E})$. In the following proposition, the tame case was proved by Simpson [20], and see [16] for the wild case, for example.

Proposition 3.3 *Let $(E,\overline{\partial}_E,\theta,h)$ be an unramifiedly good wild harmonic bundle on (X,D). Then, the associated filtered bundle $(\mathcal{P}^h_*E,\theta)$ on (X,D) is poly-stable with* $\deg(\mathcal{P}^h_*E)=0$. ∎

In the following proposition, the tame case was proved by Simpson [20], and the wild case was proved by Biquard and Boalch [2].

Proposition 3.4 *Let $(\mathcal{P}_*\mathcal{E},\theta)$ be an unramifiedly good filtered Higgs bundle on (X,D) with $\deg(\mathcal{P}_*\mathcal{E})=0$. If $(\mathcal{P}_*\mathcal{E},\theta)$ is stable, we have a harmonic metric h of $(E,\theta)=(\mathcal{E},\theta)_{|X\setminus D}$ with an isomorphism $(\mathcal{P}^h_*E,\theta)\simeq(\mathcal{P}_*\mathcal{E},\theta)$. Such a metric h is unique up to the multiplication of positive constants.* ∎

3.1.5 Filtered bundles of rank one on curves

Let X be any complex curve with a discrete subset D. Let L be a line bundle on X. Suppose that a tuple of real numbers $\boldsymbol{b}=(b_P\,|\,P\in D)$ is attached. Then, we have the filtered bundle $\mathcal{P}^{\boldsymbol{b}}_*L$ over the meromorphic bundle $L(*D)$ given as follows. For $\boldsymbol{a}=(a_P\,|\,P\in D)\in\mathbb{R}^D$, we have the integers $n(a_P)$ $(P\in D)$ such that $a_P-1<n(a_P)+b_P\leq a_P$, and we set

$$\mathcal{P}^{\boldsymbol{b}}_{\boldsymbol{a}}L:=L\Bigl(\sum_{P\in D}n(a_P)P\Bigr).$$

If X is compact, we have $\deg(\mathcal{P}^{\boldsymbol{b}}_*L)=\deg(L)-\sum_{P\in D}b_P$.

3.1.6 Comparison of filtered bundles

Let X be a compact connected Riemann surface with a finite subset D. Let $\mathcal{P}_*\mathcal{E}$ be a filtered bundle over a locally free $\mathcal{O}_X(*D)$-module $\mathcal{E}$ on (X,D). Fix a point $P\in D$. Set $D_1:=D\setminus P$. For any $b\in\mathbb{R}$, let $\mathcal{P}^P_b\mathcal{E}$ denote the $\mathcal{O}_X(*D_1)$-module such that $\mathcal{P}^P_b\mathcal{E}(*P)=\mathcal{E}$ and the stalk of $\mathcal{P}^P_b\mathcal{E}$ at P is $\mathcal{P}^P_b(\mathcal{E}_P)$. For any $b\in\mathbb{R}$ and $\boldsymbol{c}\in\mathbb{R}^{D_1}$, we set $\mathcal{P}_{\boldsymbol{c}}(\mathcal{P}^P_b\mathcal{E}):=\mathcal{P}_{(b,\boldsymbol{c})}\mathcal{E}$. For any $b\in\mathbb{R}$, we have the filtered bundle $\mathcal{P}_*(\mathcal{P}^P_b\mathcal{E})=\bigl(\mathcal{P}_{\boldsymbol{c}}\mathcal{P}^P_b\mathcal{E}\,\big|\,\boldsymbol{c}\in\mathbb{R}^{D_1}\bigr)$ over $\mathcal{P}^P_b\mathcal{E}$ on (X,D_1).

We have the set $\bigl\{a\,\big|\,0<a\leq1,\ \mathcal{P}^P_a(\mathcal{E}_P)/\mathcal{P}^P_{<a}(\mathcal{E}_P)\neq0\bigr\}=\{a_1,\dots,a_\ell\}$. We assume $a_i<a_{i+1}$. The following observation is given in [9].

Lemma 3.5 *We have the following equality:*

$$\int_0^1 \deg\bigl(\mathcal{P}_*(\mathcal{P}^P_b\mathcal{E})\bigr)\,db = \deg\bigl(\mathcal{P}_*(\mathcal{P}^P_0\mathcal{E})\bigr) - \sum_{i=1}^{\ell}(a_i-1)\dim\bigl(\mathcal{P}^P_{a_i}(\mathcal{E}_P)/\mathcal{P}^P_{<a_i}(\mathcal{E}_P)\bigr) \tag{22}$$

Here, we regard $\deg\bigl(\mathcal{P}_*(\mathcal{P}^P_b\mathcal{E})\bigr)$ *as a measurable function in variable* b*. In particular, we have* $\deg(\mathcal{P}_*\mathcal{E}) = \int_0^1 \deg\bigl(\mathcal{P}_*(\mathcal{P}^P_b\mathcal{E})\bigr)\,db$*.*

Proof We obtain (22) by a direct computation. Because $\mathcal{P}^P_{a_i}(\mathcal{E}_P)/\mathcal{P}^P_{<a_i}(\mathcal{E}_P) \simeq \mathcal{P}^P_{a_i-1}(\mathcal{E}_P)/\mathcal{P}^P_{<a_i-1}(\mathcal{E}_P)$, we can observe the right hand side of (22) is equal to $\deg(\mathcal{P}_*\mathcal{E})$. ∎

We recall a lemma for the comparison of filtered bundles for the convenience of the readers.

Lemma 3.6 *Let* $\mathcal{E}$ *be a locally free* $\mathcal{O}_X(*D)$*-module. Let* $\mathcal{P}^i_*\mathcal{E}$ $(i=1,2)$ *be filtered bundles over* $\mathcal{E}$ *such that* $\mathcal{P}^1_{\boldsymbol{a}}\mathcal{E} \subset \mathcal{P}^2_{\boldsymbol{a}}\mathcal{E}$ *for any* $\boldsymbol{a} \in \mathbb{R}^D$*. If* $\deg(\mathcal{P}^1_*\mathcal{E}) = \deg(\mathcal{P}^2_*\mathcal{E})$*, then we have* $\mathcal{P}^1_{\boldsymbol{a}}\mathcal{E} = \mathcal{P}^2_{\boldsymbol{a}}\mathcal{E}$ *for any* $\boldsymbol{a} \in \mathbb{R}^D$*.*

Proof We use an induction on $|D|$. If $D = \emptyset$, the claim is clear. Take $P \in D$, and set $D_1 := D \setminus \{P\}$. We obtain the filtered bundles $\mathcal{P}^i_*\mathcal{P}^P_b\mathcal{E}$ $(i=1,2)$ for any $b \in \mathbb{R}$ as above. By using Lemma 3.5, we can easily deduce $\deg\bigl(\mathcal{P}^1_*(\mathcal{P}^P_b\mathcal{E})\bigr) = \deg\bigl(\mathcal{P}^2_*(\mathcal{P}^P_b\mathcal{E})\bigr)$ for any $b \in \mathbb{R}$ from the equality $\deg(\mathcal{P}^1_*\mathcal{E}) = \deg(\mathcal{P}^2_*\mathcal{E})$. By the hypothesis of the induction, we obtain $\mathcal{P}^1_{\boldsymbol{c}}(\mathcal{P}^P_b\mathcal{E}) = \mathcal{P}^2_{\boldsymbol{c}}(\mathcal{P}^P_b\mathcal{E})$ for any $\boldsymbol{c}$. Then, the claim of the lemma follows. ∎

3.2 Filtered Higgs bundles of rank 2 on $(\mathbb{P}^1, \infty)$

Let ζ be the standard coordinate on $\mathbb{C} \subset \mathbb{P}^1$. We set $\widetilde{V} = \mathcal{O}_{\mathbb{P}^1}(*\infty)\widetilde{v}_1 \oplus \mathcal{O}_{\mathbb{P}^1}(*\infty)\widetilde{v}_2$. Take a non-zero complex number α and a positive integer m. We have the Higgs field $\widetilde{\theta}$ on $\widetilde{V}$ given by

$$\widetilde{\theta}(\widetilde{v}_1, \widetilde{v}_2) = (\widetilde{v}_1, \widetilde{v}_2)\begin{pmatrix} \alpha\zeta^m d\zeta & 0 \\ 0 & -\alpha\zeta^m d\zeta \end{pmatrix}.$$

Fix $0 \leq \ell \leq m$. Let $\widetilde{E}_\ell \subset \widetilde{V}$ be the $\mathcal{O}_{\mathbb{P}^1}(*\infty)$-submodule generated by $\widetilde{e}_1 = \widetilde{v}_1 + \widetilde{v}_2$ and $\widetilde{e}_2 = \zeta^\ell\widetilde{v}_2$. Because $\widetilde{\theta}(\widetilde{E}_\ell) \subset \widetilde{E}_\ell \otimes \Omega^1_{\mathbb{P}^1}$, we obtain the meromorphic Higgs bundle $(\widetilde{E}_\ell, \widetilde{\theta})$. We have

$$\widetilde{\theta}(\widetilde{e}_1, \widetilde{e}_2) = (\widetilde{e}_1, \widetilde{e}_2)\begin{pmatrix} \alpha\zeta^m d\zeta & 0 \\ -2\alpha\zeta^{m-\ell}d\zeta & -\alpha\zeta^m d\zeta \end{pmatrix}.$$

Take $c \in \mathbb{R}$, and set $c_1 := c$ and $c_2 := -c-\ell$. We have the parabolic Higgs bundle $(\mathcal{P}^c_*\widetilde{E}_\ell, \widetilde{\theta})$ on $(\mathbb{P}^1, \infty)$ with $\deg(\mathcal{P}^c_*\widetilde{E}_\ell) = 0$ given as follows

$$\mathcal{P}^c_b(\widetilde{E}_\ell)_{|\mathbb{P}^1\setminus\{\infty\}} = \Bigl(\mathcal{O}_{\mathbb{P}^1}([b-c_1]\infty)\widetilde{v}_1 \oplus \mathcal{O}_{\mathbb{P}^1}([b-c_2]\infty)\widetilde{v}_2\Bigr)_{|\mathbb{P}^1\setminus\{\infty\}}$$

Here, $[x] := \max\{n \in \mathbb{Z} \,|\, n \leq x\}$. We shall impose that $c_1 \geq c_2$, i.e., $c \geq -\ell/2$.

3.2.1 Stability condition

We set $L_i := \mathcal{O}_{\mathbb{P}^1}(*\infty)\widetilde{v}_i \subset \widetilde{V}$ $(i=1,2)$. We have $\widetilde{E}_\ell \cap L_i = \mathcal{O}_{\mathbb{P}^1}(*\infty)\cdot\zeta^\ell\widetilde{v}_i$. We have the induced parabolic bundle $\mathcal{P}^c_*(\widetilde{E}_\ell \cap L_i)$. We have $\deg\bigl(\mathcal{P}^c_*(\widetilde{E}_\ell \cap L_i)\bigr) = -\ell - c_i$. Hence, $(\mathcal{P}^c_*\widetilde{E}_\ell, \widetilde{\theta})$ is stable if and only if $-\ell - c_i < 0$ $(i=1,2)$, i.e., $c_i < 0$ $(i=1,2)$. We also have that $(\mathcal{P}^c_*\widetilde{E}_\ell, \widetilde{\theta})$ is semistable if and only if $-\ell - c_1 = 0$ or $-\ell - c_2 = 0$, i.e., $c_1 = 0, -\ell$. Under the assumption $c_1 \geq c_2$, the semistability is equivalent to $c_1 = 0$.

3.2.2 The determinant bundles

We have $\det(\widetilde{E}_\ell) = \mathcal{O}_{\mathbb{P}^1}(*\infty)\cdot\widetilde{e}_1\wedge\widetilde{e}_2 = \mathcal{O}_{\mathbb{P}^1}(*\infty)\cdot\zeta^\ell\widetilde{v}_1\wedge\widetilde{v}_2$. The induced filtered bundle $\det(\mathcal{P}^c_*\widetilde{E}_\ell, \widetilde{\theta})$ over $\det(\widetilde{E}_\ell)$ is equal to $\mathcal{P}^0_*\bigl(\mathcal{O}_{\mathbb{P}^1}\cdot\widetilde{e}_1\wedge\widetilde{e}_2\bigr)$, where $\mathcal{P}^0_b\bigl(\mathcal{O}_{\mathbb{P}^1}\cdot\widetilde{e}_1\wedge\widetilde{e}_2\bigr) = \mathcal{O}_{\mathbb{P}^1}([b])\cdot\widetilde{e}_1\wedge\widetilde{e}_2$ for $[b] := \max\{n \in \mathbb{Z} \,|\, n \leq b\}$. In particular, they are independent of c. We fix the Hermitian metric $h_{\det(\widetilde{E}_\ell)}$ of $\det(\widetilde{E}_\ell)$ given by $h_{\det(\widetilde{E}_\ell)}\bigl(\widetilde{e}_1\wedge\widetilde{e}_2, \widetilde{e}_1\wedge\widetilde{e}_2\bigr) = 1$.

3.2.3 Harmonic metrics in the case $\ell = 0$

If $\ell = 0$, $(\mathcal{P}^c_* \widetilde{E}_0, \widetilde{\theta})$ cannot be stable. And, $(\mathcal{P}^c_* \widetilde{E}_0, \widetilde{\theta})$ is semistable if and only if $c = 0$. Indeed, it is polystable in that case, i.e., we have the decomposition $(\mathcal{P}^0_* \widetilde{E}_0, \widetilde{\theta}) = (\mathcal{P}^0_* L_1, \widetilde{\theta}_1) \oplus (\mathcal{P}^0_* L_2, \widetilde{\theta}_2)$, where $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ are the multiplications of $\alpha\zeta^m d\zeta$ and $-\alpha\zeta^m d\zeta$, respectively. We have the harmonic metrics h_{L_i} for $(\mathcal{P}^0_* L_i, \widetilde{\theta}_i)$. We impose $h_{L_1} \otimes h_{L_2} = h_{\det(\widetilde{E}_0)}$. Then, we have a harmonic metric $h_{\widetilde{E}_0} = h_{L_1} \oplus h_{L_2}$ for $(\mathcal{P}^0_* \widetilde{E}_0, \widetilde{\theta})$. Note that we have an ambiguity given by automorphisms of $(\mathcal{P}^0_* \widetilde{E}_0, \widetilde{\theta})$, i.e., for any $\alpha > 0$, $\alpha h_{L_1} \oplus \alpha^{-1} h_{L_2}$ is also a harmonic metric for $(\mathcal{P}^0_* \widetilde{E}_0, \widetilde{\theta})$. We also note that $h_{L_i}(\widetilde{v}_i, \widetilde{v}_i)$ are constants.

3.2.4 Harmonic metrics in the case $\ell > 0$ and their homogeneous property

Suppose that $\ell > 0$. According to Proposition 3.4, if the unramifiedly good filtered Higgs bundle $(\mathcal{P}^c_* \widetilde{E}_\ell, \widetilde{\theta})$ is stable, i.e., $-\ell < c < 0$, we have the harmonic metric $h_{c,\ell}$ of $(\widetilde{E}_\ell, \widetilde{\theta})_{|\mathbb{C}}$ adapted to the filtered Higgs bundle $(\mathcal{P}^c_* \widetilde{E}_\ell, \widetilde{\theta})$ such that $\det(h_{c,\ell}) = h_{\det \widetilde{E}_\ell}$.

For any $\tau \in \mathbb{C}^*$, let $\varphi_\tau : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be given by $\varphi_\tau(\zeta) = \tau^2\zeta$. We have the isomorphism $\Phi_\tau : \varphi_\tau^* \widetilde{E}_\ell \simeq \widetilde{E}_\ell$ given by $\tau^\ell \varphi_\tau^* \widetilde{e}_1 \longleftrightarrow \widetilde{e}_1$ and $\tau^{-\ell} \varphi_\tau^* \widetilde{e}_2 \longleftrightarrow \widetilde{e}_2$. Under the isomorphism, we have $\tau^\ell \varphi_\tau^* \widetilde{v}_i \longleftrightarrow \widetilde{v}_i$ $(i = 1, 2)$. Hence, Φ_τ induces an isomorphism of the filtered bundles $\varphi_\tau^* \mathcal{P}_* \widetilde{E}_\ell \simeq \mathcal{P}_* \widetilde{E}_\ell$. Under the isomorphism, we have $\varphi_\tau^* \widetilde{\theta} = \tau^{2(m+1)} \widetilde{\theta}$. Hence, the Hermitian metric $\varphi_\tau^* h_{c,\ell}$ gives a harmonic metric of $(\mathcal{P}^c_* \widetilde{E}_\ell, \tau^{2(m+1)} \widetilde{\theta})$. Note that $\det \varphi^* h_{c,\ell} = h_{\det \widetilde{E}_\ell}$.

Proposition 3.7 *If $|\tau| = 1$, we have $\varphi_\tau^* h_{c,\ell} = h_{c,\ell}$.*

Proof If $|\tau| = 1$, both the Hermitian metrics $\varphi_\tau^* h_{c,\ell}$ and $h_{c,\ell}$ are harmonic metrics of $(\widetilde{E}_\ell, \widetilde{\theta})$. They are adapted to the filtered bundle $\mathcal{P}^c_* \widetilde{E}_\ell$, and they satisfy $\det(h_{c,\ell}) = \det(\varphi_\tau^* h_{c,\ell}) = h_{\det \widetilde{E}_\ell}$. Hence, we have $h_{c,\ell} = \varphi_\tau^* h_{c,\ell}$ by the uniqueness. ∎

Corollary 3.8 *The functions $h_{c,\ell}(\widetilde{e}_i, \widetilde{e}_i)$ $(i = 1, 2)$ depend only on $|\zeta|$. The function $\zeta^{-\ell} h_{c,\ell}(\widetilde{e}_1, \widetilde{e}_2)$ depend only on $|\zeta|$. The functions $h_{c,\ell}(\widetilde{v}_i, \widetilde{v}_j)$ $(i, j \in \{1, 2\})$ depend only on $|\zeta|$.* ∎

Remark 3.9 *We clearly have the homogeneous property of the harmonic metrics such as Corollary 3.8 even in the case $\ell = 0$.* ∎

3.2.5 The norms of $\widetilde{v}_j$

By the norm estimate of wild harmonic bundles [16], we have constants $C_{i,c} > 0$ $(i = 1, 2)$ depending on c such that $C_{1,c}|\zeta|^{c_j} \leq \bigl|\widetilde{v}_j\bigr|_{h_{c,\ell}} \leq C_{2,c}|\zeta|^{c_j}$, where $c_1 = c$ and $c_2 = -c - \ell$. Let us refine it in our situation.

Proposition 3.10 *There exist positive constants b_c, C_3 and ϵ_3 such that the following holds on $\{|\zeta| > 1\}$:*

$$\Bigl|\log |\widetilde{v}_1|_{h_{c,\ell}} - \log\bigl(b_c |\zeta|^{c_1}\bigr)\Bigr| \leq C_3 \exp(-\epsilon_3 |\zeta|^{m+1}) \tag{23}$$

$$\Bigl|\log |\widetilde{v}_2|_{h_{c,\ell}} - \log\bigl(b_c^{-1} |\zeta|^{c_2}\bigr)\Bigr| \leq C_3 \exp(-\epsilon_3 |\zeta|^{m+1}) \tag{24}$$

$$\Bigl|\zeta\partial_\zeta \log |\widetilde{v}_i|_{h_{c,\ell}} - c_i/2\Bigr| \leq C_3 \exp\bigl(-\epsilon_3 |\zeta|^{m+1}\bigr) \quad (i = 1, 2) \tag{25}$$

Here, b_c may depend on c , but C_3 and ϵ_3 are independent of c.

Proof Let $\eta := \zeta^{-1}$ be the coordinate around ∞. Set $U_\infty = \{|\eta| \leq 1\} \subset \mathbb{P}^1$.

Lemma 3.11 *There exist $C_4 > 0$ and $\epsilon_4 > 0$ which are independent of c, such that the following holds on $U_\infty \setminus \{\infty\}$:*

$$\Bigl|\partial_\eta \partial_{\overline{\eta}} \bigl(\log |\widetilde{v}_j|_{h_{c,\ell}}\bigr)\Bigr| \leq C_4 \exp\bigl(-\epsilon_4 |\eta|^{-m-1}\bigr)$$

Proof Let $h_{L_j,c,\ell}$ denote the restriction of $h_{c,\ell}$ to L_j. Let $R(h_{L_j,c,\ell})$ denote the curvature of the Chern connection of $(L_j, h_{L_j,c,\ell})$. Let $g_{\mathbb{C}}$ be the Euclidean metric $d\zeta\, d\overline{\zeta}$. By Proposition 2.12, we have positive constants C_5 and ϵ_5 which are independent of c, such that the following holds:

$$\left|R(h_{L_j,c,\ell})\right|_{h_{L_j,c,\ell},g_{\mathbb{C}}} \leq C_5 \exp\bigl(-\epsilon_5|\zeta|^{m+1}\bigr)$$

Because $\overline{\partial}\partial \log\bigl|\widetilde{v}_j\bigr|_{h_{c,\ell}} = R(h_{L_j,c,\ell})$, we obtain the claim of the lemma. ∎

We set $Y_{\widetilde{\eta}} := \bigl\{\widetilde{\eta} \in \mathbb{C} \,\big|\, |\widetilde{\eta}| < 1\bigr\}$. For any $\kappa < 1$, we have the isomorphism $\Psi_\kappa : Y_{\widetilde{\eta}} \longrightarrow \{|\eta| < \kappa\}$ given by $\Psi_\kappa(\widetilde{\eta}) = \kappa\widetilde{\eta}$. We have the following on $Y_{\widetilde{\eta}} \setminus \{0\}$:

$$\left|\partial_{\widetilde{\eta}}\partial_{\overline{\widetilde{\eta}}}\Psi_\kappa^*(\log|\widetilde{v}_1|_{h_{c,\ell}})\right| \leq C_4\kappa^2 \exp(-\epsilon_4\kappa^{-m-1}|\widetilde{\eta}|^{-m-1})$$

Take a large $p > 1$. For each c and κ, we can take an $\mathbb{R}$-valued L_2^p-function $G_{\kappa,c}$ on a neighbourhood of the closure of $Y_{\widetilde{\eta}}$ such that (i) $G_{\kappa,c}$ is a function of $|\eta|$, (ii) there exist positive constants C_6 and ϵ_6 such that the L_2^p-norm of $G_{\kappa,c}$ on $Y_{\widetilde{\eta}}$ is dominated by $C_6 \exp(-\epsilon_6\kappa^{-m-1})$, (iii) $G_{\kappa,c}(0) = 0$, (iv) the following holds on $Y_{\widetilde{\eta}}$:

$$\partial_{\widetilde{\eta}}\partial_{\overline{\widetilde{\eta}}}\bigl(\Psi_\kappa^*(\log|\widetilde{v}_1|_{h_{c,\ell}}) - G_{\kappa,c}\bigr) = 0$$

Because $\Psi_\kappa^*(\log|\widetilde{v}_1|_{h_{c,\ell}}) - G_{\kappa,c} - \log|\widetilde{\eta}|^{-c_1}$ is bounded, we have a holomorphic function $g_{\kappa,c}$ on $Y_{\widetilde{\eta}}$ such that $\Psi_\kappa^*(\log|\widetilde{v}_1|_{h_{c,\ell}}) - G_{\kappa,c} - \log|\widetilde{\eta}|^{-c_1} = \mathrm{Re}(g_{\kappa,c})$. Because $\Psi_\kappa^*(\log|\widetilde{v}_1|_{h_{c,\ell}}) - G_{\kappa,c} - \log|\widetilde{\eta}|^{-c_1}$ depends only on $|\eta|$, we obtain that $g_{\kappa,c}$ is constant. We also obtain

$$\widetilde{\eta}\partial_{\widetilde{\eta}}\Psi_\kappa^*(\log|\widetilde{v}_1|_{h_{c,\ell}}) + c_1/2 - \widetilde{\eta}_{\partial_{\widetilde{\eta}}}G_{\kappa,c} = 0.$$

Hence, on $\{|\eta| < \kappa\}$, the function $\log|\widetilde{v}_1|_{h_{c,\ell}} - (\Psi_\kappa^{-1})^*G_{\kappa,c} - \log|\eta|^{-c_1}$ is a constant. We have positive constants C_7 and ϵ_7 such that the following holds on $\{\kappa/2 < |\eta| < \kappa\}$:

$$\left|(\Psi_\kappa^{-1})^*G_{\kappa,c}\right| \leq C_7 \exp\bigl(-\epsilon_7|\eta|^{-m-1}\bigr)$$

$$\left|\eta\partial_\eta(\Psi_\kappa^{-1})^*G_{\kappa,c}\right| \leq C_7 \exp\bigl(-\epsilon_7|\eta|^{-m-1}\bigr)$$

We also obtain that the function $F := \log|\widetilde{v}_1|_{h_{c,\ell}} - \log|\eta|^{-c_1}$ gives a continuous function on $\{|\eta| < 1\}$. We set $b_c := \exp(F(0))$. Then, we obtain (23), and (25) for $i = 1$. By using $\bigl|\widetilde{v}_1 \wedge \widetilde{v}_2\bigr|_{h_{c,\ell}} = |\zeta|^{-\ell}$ and the estimate (26) below, we obtain the estimate (24). We also obtain (25) for $i = 2$ with a similar argument. ∎

3.2.6 Asymptotic orthogonality and some complements

As studied in [16], we have the following estimate on $\{|\zeta| \geq 1\}$, which also follows from Corollary 2.6:

$$\bigl|h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2)\bigr| \leq K \exp(-\delta|\zeta|^{m+1}) \cdot \bigl|\widetilde{v}_1\bigr|_{h_{c,\ell}} \cdot \bigl|\widetilde{v}_2\bigr|_{h_{c,\ell}} \leq K' \exp(-\delta|\zeta|^{m+1}) \tag{26}$$

Here, K, K' and δ are positive constants which are independent of c.

Let $\overline{\partial}_{c,\ell}$ denote the holomorphic structure of $\widetilde{E}_\ell$. Let $\partial_{c,\ell}$ denote the $(1,0)$-part of the Chern connection of $(\widetilde{E}_\ell, h_{c,\ell})$. Let $g_{\mathbb{C}}$ denote the Euclidean metric of $\mathbb{C}_\zeta$.

Lemma 3.12 *We have positive constants K_1 which are independent of c, such that $\bigl|\zeta\partial_{c,\ell}\widetilde{v}_i\bigr|_{h_{c,\ell},g_{\mathbb{C}}} \leq K_1 \cdot |\widetilde{v}_i|_{h_{c,\ell}}$ on $\{|\zeta| \geq 1\}$.*

Proof In this proof, the constants K_i and δ_i are independent of c. To simplify the description, we denote $h_{c,\ell}$ by h. By the asymptotic orthogonality (26), we have a positive constant K_2 such that the following holds on $\{|\zeta| \geq 1\}$:

$$\bigl|\partial_{c,\ell}\widetilde{v}_1\bigr|_{h,g_{\mathbb{C}}} \leq K_2\Bigl(|\widetilde{v}_1|_h^{-1} \cdot \bigl|h(\partial_{c,\ell}\widetilde{v}_1, \widetilde{v}_1)\bigr|_{g_{\mathbb{C}}} + |\widetilde{v}_2|_h^{-1} \cdot \bigl|h(\partial_{c,\ell}\widetilde{v}_1, \widetilde{v}_2)\bigr|_{g_{\mathbb{C}}}\Bigr)$$

By Proposition 3.10, we have a constant $K_3 > 0$ such that the following holds on $\{|\zeta| \geq 1\}$:

$$|\widetilde{v}_1|_h^{-1} \big|h(\partial_{c,\ell}\widetilde{v}_1, \widetilde{v}_1)\big|_{g_{\mathbb{C}}} = |\widetilde{v}_1|_h \cdot \Big|\partial \log |\widetilde{v}_1|_h^2\Big|_{g_{\mathbb{C}}} \leq K_3 |\widetilde{v}_1|_h \cdot |\zeta|^{-1}$$

Let π_1 denote the projection of $\widetilde{V} = L_1 \oplus L_2$ onto L_1. We also have the following on $\{|\zeta| \geq 1\}$:

$$|\widetilde{v}_2|_h^{-1} \cdot \big|h(\partial_{c,\ell}\widetilde{v}_1, \widetilde{v}_2)\big|_{g_{\mathbb{C}}} \leq |\widetilde{v}_2|_h^{-1} \cdot \big|h\big((\partial_{c,\ell}\pi_1) \cdot \widetilde{v}_1, \widetilde{v}_2\big)\big| + |\widetilde{v}_2|_h^{-1} \cdot \big|h\big(\pi_1(\partial_{c,\ell}\widetilde{v}_1), \widetilde{v}_2\big)\big| \tag{27}$$

By Proposition 2.10, we have positive constants K_4 and δ_4 such that

$$\big|\partial_{c,\ell}\pi_1\big|_{h,g_{\mathbb{C}}} \leq K_4 \exp(-\delta_4 |\zeta|^{m+1}). \tag{28}$$

By (26) and Proposition 2.3, we have positive constants K_5 and δ_5 such that

$$\big|h\big(\pi_1(\partial_{c,\ell}\widetilde{v}_1), \widetilde{v}_2\big)\big| \leq K_5 \exp(-\delta_5 |\zeta|^{m+1}) \big|\partial_{c,\ell}\widetilde{v}_1\big|_{h,g_{\mathbb{C}}} \cdot \big|\widetilde{v}_2\big|_h. \tag{29}$$

Hence, we obtain the following on $\{|\zeta| \geq 1\}$:

$$\big|\partial_{c,\ell}\widetilde{v}_1\big|_{h,g_{\mathbb{C}}} \leq K_2\big(K_3|\zeta|^{-1} + K_4 \exp(-\delta_4|\zeta|^{m+1})\big)|\widetilde{v}_1|_h + K_2 K_5 \exp(-\delta_5 |\zeta|^{m+1}) \big|\partial_{c,\ell}\widetilde{v}_1\big|_{h,g_{\mathbb{C}}}$$

Hence, we obtain the desired estimate for $\partial_{c,\ell}\widetilde{v}_1$. Similarly, we obtain the estimate for $\partial_{c,\ell}\widetilde{v}_2$. ∎

Lemma 3.13 *We have positive constants K_6 and δ_6 which are independent of c, such that the following holds:*

$$\Big|\partial_\zeta h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2)\Big| \leq K_6 \exp(-\delta_6 |\zeta|^{m+1}), \qquad \Big|\partial_{\overline{\zeta}} h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2)\Big| \leq K_6 \exp(-\delta_6 |\zeta|^{m+1}),$$

$$\Big|\partial_\zeta \partial_{\overline{\zeta}} h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2)\Big| \leq K_6 \exp(-\delta_6 |\zeta|^{m+1}).$$

Proof In this proof, the constants K_i and δ_i are independent of c. We obtain the estimate for $\partial_\zeta h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2)$ from (27–29) and Lemma 3.12. We obtain the estimate for $\partial_{\overline{\zeta}} h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2)$ in a similar way. Let $R(h_{c,\ell})$ denote the curvature of $(\widetilde{E}_\ell, h_{c,\ell})$. We have the following:

$$\partial\overline{\partial} h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2) = h_{c,\ell}(\partial_{c,\ell}\widetilde{v}_1, \partial_{c,\ell}\widetilde{v}_2) + h_{c,\ell}(\widetilde{v}_1, R(h_{c,\ell})\widetilde{v}_2)$$

We have positive constants K_7 and δ_7 such that $|R(h_{c,\ell})|_{h_{c,\ell},g_{\mathbb{C}}} \leq K_7 \exp(-\delta_n |\zeta|^{m+1})$. Hence, we have

$$\big|h_{c,\ell}(\widetilde{v}_1, R(h_{c,\ell})\widetilde{v}_2)\big|_{g_{\mathbb{C}}} \leq K_8 \exp(-\delta_8 |\zeta|^{m+1})$$

for positive constants K_8 and δ_8. Let π_i denote the projection of $\widetilde{V} = L_1 \oplus L_2$ to L_i. We have the following:

$$\begin{aligned} h_{c,\ell}(\partial_{c,\ell}\widetilde{v}_1, \partial_{c,\ell}\widetilde{v}_2) &= h_{c,\ell}\big(\pi_1(\partial_{c,\ell}\widetilde{v}_1), \pi_2(\partial_{c,\ell}\widetilde{v}_2)\big) + h_{c,\ell}\big((\partial_{c,\ell}\pi_1)\widetilde{v}_1, \pi_2(\partial_{c,\ell}\widetilde{v}_2)\big) \\ &\quad + h_{c,\ell}\big(\pi_1(\partial_{c,\ell}\widetilde{v}_1), (\partial_{c,\ell}\pi_2)\widetilde{v}_2\big) + h_{c,\ell}\big((\partial_{c,\ell}\pi_1)\widetilde{v}_1, (\partial_{c,\ell}\pi_2)\widetilde{v}_2\big) \end{aligned} \tag{30}$$

Hence, as in the case of the estimate for $h_{c,\ell}(\partial_{c,\ell}\widetilde{v}_1, \widetilde{v}_2)$, we have positive constants K_9 and δ_9 such that $\big|h_{c,\ell}(\partial_{c,\ell}\widetilde{v}_1, \partial_{c,\ell}\widetilde{v}_2)\big|_{g_{\mathbb{C}}} \leq K_9 \exp(-\delta_9 |\zeta|^{m+1})$. Thus, we obtain the desired estimate for $\partial_\zeta \partial_{\overline{\zeta}} h_{c,\ell}(\widetilde{v}_1, \widetilde{v}_2)$. ∎

3.2.7 Convergence of some sequences

Set $t := \tau^{2(m+1)}$ for $\tau \in \mathbb{C}^*$. We use the notation in §3.2.4. We have the family of the harmonic metrics $h_{t,c,\ell} := \varphi_\tau^* h_{c,\ell}$ for $(\mathcal{P}_*^c \widetilde{E}_\ell, t\widetilde{\theta})$ satisfying $\det(h_{t,c,\ell}) = h_{\det(E)}$.

Under the isomorphism $\Phi_\tau : \varphi_\tau^* \widetilde{E}_\ell \simeq \widetilde{E}_\ell$, we have $\tau^\ell \varphi_\tau^* \widetilde{v}_i \longleftrightarrow \widetilde{v}_i$ $(i = 1, 2)$. Take any $T > 0$. We have positive constants C and ϵ, which are independent of c and T, such that the following holds on $\{|\zeta| \geq T\}$ for any t satisfying $|t| > T^{-1}$:

$$\Big|\log\big|\widetilde{v}_1\big|_{h_{t,c,\ell}} - \log\big(b_c |t|^{(\ell+2c)/2(m+1)} |\zeta|^c\big)\Big| \leq C \exp\big(-\epsilon |\zeta|^{m+1} |t|\big)$$

$$\Big|\log\big|\widetilde{v}_2\big|_{h_{t,c,\ell}} - \log\big(b_c^{-1}|t|^{-(\ell+2c)/2(m+1)}|\zeta|^{-c-\ell}\big)\Big| \leq C\exp\big(-\epsilon|\zeta|^{m+1}|t|\big)$$

$$\Big|h_{t,c,\ell}(\widetilde{v}_1,\widetilde{v}_2)\Big| \leq C\exp\big(-\epsilon|\zeta|^{m+1}|t|\big)\cdot\big|\widetilde{v}_1\big|_{h_{t,c,\ell}}\cdot\big|\widetilde{v}_2\big|_{h_{t,c,\ell}}$$

$$\big|R(h_{t,c,\ell})\big|_{h_{t,c,\ell},g_{\mathbb{C}}} \leq C\exp\big(-\epsilon|\zeta|^{m+1}|t|\big)$$

Here, $g_{\mathbb{C}}$ denote the standard Euclidean metric on $\mathbb{C}$.

Let $h^{\lim}_{c,\ell}$ be the Hermitian metric of $\widetilde{V}_{|\mathbb{C}^*}$ given by

$$h^{\lim}_{c,\ell}(\widetilde{v}_1,\widetilde{v}_1)=|\zeta|^{2c},\quad h^{\lim}_{c,\ell}(\widetilde{v}_2,\widetilde{v}_2)=|\zeta|^{-2c-2\ell},\quad h^{\lim}_{c,\ell}(\widetilde{v}_1,\widetilde{v}_2)=0.$$

For any $\gamma>0$, the automorphism Ψ_γ on $\widetilde{V}_{|\mathbb{C}^*}$ is given by $\Psi_\gamma=\gamma\,\mathrm{id}_{L_1}\oplus\gamma^{-1}\,\mathrm{id}_{L_2}$. We define $\Psi_\gamma^*h_{t,c,\ell}(u_1,u_2):=h_{t,c,\ell}(\Psi_\gamma u_1,\Psi_\gamma u_2)$. The following is clear.

Proposition 3.14 *Set $\gamma(t):=b_c^{-1}t^{-(\ell+2c)/2(m+1)}$. Then, we have the convergence $\lim_{|t|\to\infty}\Psi^*_{\gamma(t)}h_{t,c,\ell}=h^{\lim}_{c,\ell}$ on $\mathbb{C}^*$. For any fixed $T>0$, we have a constant $\delta_T>0$ depending on T, such that the order of the convergence is $\exp(-\delta_T|t|)$ on $\{|\zeta|\geq T\}$.* ∎

3.3 Family of harmonic metrics in the case $\ell>0$

We continue to use the notation in §3.2. We study the dependence of the harmonic metrics $h_{c,\ell}$ on c in the case $\ell>0$.

3.3.1 Continuity of b_c with respect to the parabolic weights

Proposition 3.15 *b_c is continuous with respect to c.*

Proof Fix $-\ell<c^0<0$. It is enough to study the continuity at c^0. We give a preliminary. Take a neighbourhood U of c^0. We have a family of Hermitian metrics $h^0_{c,\ell}$ $(c\in U)$ of $\widetilde{E}_\ell$ satisfying the following conditions:

- We have $h^0_{c^0,\ell}=h_{c^0,\ell}$ on $\mathbb{C}$.
- We have $h^0_{c,\ell}(\widetilde{v}_1,\widetilde{v}_2)=h_{c^0,\ell}(\widetilde{v}_1,\widetilde{v}_2)$, $h^0_{c,\ell}(\widetilde{v}_1,\widetilde{v}_1)=h_{c^0,\ell}(\widetilde{v}_1,\widetilde{v}_1)|\zeta|^{c-c^0}$ and $h^0_{c,\ell}(\widetilde{v}_2,\widetilde{v}_2)=h_{c^0,\ell}(\widetilde{v}_2,\widetilde{v}_2)|\zeta|^{c^0-c}$ on $\{|\zeta|\geq 1\}$.
- $\big|\widetilde{v}_1\wedge\widetilde{v}_2\big|_{h^0_{c,\ell}}=|\zeta|^\ell$.
- We have $\lim_{c\to c^0}h^0_{c,\ell}=h^0_{c^0,\ell}$ in the C^∞-sense on any compact subset in $\mathbb{C}$.

Note that the conditions are compatible on $\big\{|\zeta|\geq 1\big\}$. We fix a C^∞-Kähler metric $g_{\mathbb{P}^1}$ of $\mathbb{P}^1$.

Lemma 3.16 *We have a constant $C>0$ such that the following holds on $\mathbb{P}^1\setminus\{\infty\}$ for any $c\in U$:*

$$\big|R(h^0_{c,\ell})\big|_{h^0_{c,\ell},g_{\mathbb{P}^1}}\leq C,\quad \big|[\widetilde{\theta},\widetilde{\theta}^\dagger_{h^0_{c,\ell}}]\big|_{h^0_{c,\ell},g_{\mathbb{P}^1}}\leq C$$

Proof In the proof, C_i and ϵ_i are positive constants, which are independent of c. It is enough to consider the issue on $\big\{|\zeta|\geq 1\big\}$. The estimate for $[\widetilde{\theta},\widetilde{\theta}^\dagger_{h^0_{c,\ell}}]$ follows from (26) and the construction of $h^0_{c,\ell}$. Let us study the estimate for $R(h^0_{c,\ell})$.

Let H be the $M_2(\mathbb{C})$-valued function on $\{|\zeta|\geq 1\}$ given by $H_{ij}=h_{c^0,\ell}(\widetilde{v}_i,\widetilde{v}_j)$. By Lemma 3.13, we have positive constants C_1 and ϵ_1 such that the following holds:

$$\big|\partial H_{12}\big|_{g_{\mathbb{P}^1}}\leq C_1\exp(-\epsilon_1|\eta|^{-m-1}),\qquad \big|\partial H_{21}\big|_{g_{\mathbb{P}^1}}\leq C_1\exp(-\epsilon_1|\eta|^{-m-1}),$$

Set $c^0_1=c^0$ and $c^0_2=-c^0-\ell$. By Lemma 3.12, we have a positive constant C_2 such that the following holds:

$$\big|\partial H_{ii}\big|_{g_{\mathbb{P}^1}}\leq C_2|\eta|^{-2c^0_i-1}\ \ (i=1,2)$$

Let Γ_c be the $M_2(\mathbb{C})$-valued function given as $\Gamma_{c,11}=|\zeta|^{c-c_0}$, $\Gamma_{c,22}=|\zeta|^{c_0-c}$, and $\Gamma_{c,ij}=0$ $(i\neq j)$. Then, $R(h^0_{c,\ell})$ on $\{|\zeta|\geq 1\}$ is represented by the following matrix with respect to the frame $(\widetilde{v}_1,\widetilde{v}_2)$:

$$\overline{\partial}\big((\Gamma_c\overline{H}\Gamma_c)^{-1}\partial(\Gamma_c\overline{H}\Gamma_c)\big)=\overline{\partial}(\Gamma_c^{-1}\overline{H}^{-1})\cdot\Gamma_c^{-1}\partial\Gamma\cdot\overline{H}\Gamma_c-\Gamma_c^{-1}\overline{H}^{-1}(\Gamma_c^{-1}\partial\Gamma_c)\overline{\partial}(\overline{H}\Gamma)\\+\overline{\partial}\Gamma_c^{-1}(\overline{H}^{-1}\partial\overline{H})\Gamma_c-\Gamma_c^{-1}\overline{H}^{-1}\partial\overline{H}\cdot\overline{\partial}\Gamma_c+\Gamma_c^{-1}\overline{\partial}(\overline{H}^{-1}\partial\overline{H})\cdot\Gamma_c \quad (31)$$

Because $\big|R(h_{c^0,\ell})\big|_{h_{c^0,\ell},g_{\mathbb{P}^1}}\leq C_3\exp(-\epsilon_3|\eta|^{-m-1})$, we have

$$\big|\overline{\partial}(\overline{H}^{-1}\partial\overline{H})\big|_{g_{\mathbb{P}^1}}\leq C_4\exp(-\epsilon_4|\eta|^{-m-1}),$$

and hence $\big|\Gamma_c^{-1}\overline{\partial}(\overline{H}^{-1}\partial\overline{H})\cdot\Gamma_c\big|_{g_{\mathbb{P}^1}}\leq C_5\exp(-\epsilon_5|\eta|^{-m-1})$. We also have

$$\overline{\partial}(\Gamma_c^{-1}\overline{H}^{-1})\cdot\Gamma_c^{-1}\partial\Gamma_c\cdot\overline{H}\Gamma_c-\Gamma_c^{-1}\overline{H}^{-1}(\Gamma_c^{-1}\partial\Gamma_c)\overline{\partial}(\overline{H}\Gamma_c)=-\big[(\overline{H}\Gamma_c)^{-1}\overline{\partial}(\overline{H}\Gamma_c),(\overline{H}\Gamma_c)^{-1}(\Gamma_c^{-1}\partial\Gamma_c)\overline{H}\Gamma_c\big] \quad (32)$$

Because the off-diagonal part of $\overline{H}\Gamma_c$ and $\overline{\partial}(\overline{H}\Gamma_c)$ are dominated by $C_6\exp(-\epsilon_6|\eta|^{-m-1})$, the term (32) is dominated by $C_7\exp(-\epsilon_7|\eta|^{-m-1})\,d\zeta\,d\overline{\zeta}$. Similarly,

$$\overline{\partial}\Gamma_c^{-1}(\overline{H}^{-1}\partial\overline{H})\Gamma_c-\Gamma_c^{-1}\overline{H}^{-1}\partial\overline{H}\cdot\overline{\partial}\Gamma_c=-\big[\Gamma_c^{-1}\overline{\partial}\Gamma_c,\Gamma_c^{-1}(\overline{H}^{-1}\partial\overline{H})\Gamma_c\big]$$

is dominated by $C_8\exp(-\epsilon_8|\eta|^{-m-1})\,d\zeta\,d\overline{\zeta}$. Then, the claim of the lemma follows. ∎

We have the self-adjoint endomorphisms k_c of $(\widetilde{E}_\ell,h^0_{c,\ell})$ determined by $h_{c,\ell}(u_1,u_2)=h^0_{c,\ell}(k_cu_1,u_2)$. Note that k_c are bounded with respect to $h^0_{c,\ell}$, although the estimate might depend on c at this stage. We also remark that $\operatorname{Tr}k_c(P)\geq 1$ at any $P\in\mathbb{C}$. The claim of Proposition 3.15 is deduced from the following proposition.

Proposition 3.17 *We have the convergence* $k_c\longrightarrow\mathrm{id}$ $(c\to c^0)$ *uniformly on* $\mathbb{C}$.

Proof Take a large $p>1$. Let $\|k_c\|_{h^0_{c,\ell},g_{\mathbb{P}^1},L^p}$ be the L^p-norm of k_c with respect to $h^0_{c,\ell}$ and $g_{\mathbb{P}^1}$. We set $s_c:=k_c/\|k_c\|_{h^0_{c,\ell},g_{\mathbb{P}^1},L^p}$. According to [19, Lemma 3.1], we have the following inequality on $\mathbb{C}$:

$$\sqrt{-1}\Lambda_{g_{\mathbb{P}^1}}\overline{\partial}\partial\operatorname{Tr}(s_c)\leq\Big|\Lambda_{g_{\mathbb{P}^1}}\operatorname{Tr}\Big(\big(R(h^0_{c,\ell})+[\widetilde{\theta},\widetilde{\theta}^\dagger_{h^0_{c,\ell}}]\big)s_c\Big)\Big| \quad (33)$$

We recall the following general lemma, which is a variant of [20, Lemma 2.2].

Lemma 3.18 *Let φ and g be bounded $\mathbb{R}$-valued C^∞-functions on a punctured disc $\{x\in\mathbb{C}\,|\,0<|x|<1\}$. Suppose that $-\partial_x\partial_{\overline{x}}\varphi\leq g$ holds on $\{x\in\mathbb{C}\,|\,0<|x|<1\}$. Then, the inequality holds on $\{x\in\mathbb{C}\,|\,|x|<1\}$ in the sense of distributions.*

Proof We give only an outline of the proof. We take a C^∞-function $\rho:\mathbb{R}\longrightarrow\mathbb{R}_{\geq 0}$ satisfying $\rho(t)=1$ $(t\leq 1)$ and $\rho(t)=0$ $(t\geq 2)$. For any $N>0$, we set $\chi_N(x)=\rho\big(-N^{-1}\log|x|\big)$. Note that $\partial\chi_N$, $\overline{\partial}\chi_N$ and $\partial\overline{\partial}\chi_N$ are bounded with respect to the Poincaré metric $(\log|x|^2)^{-1}|x|^{-2}dx\,d\overline{x}$, which are uniformly for N.

Let f be any $\mathbb{R}_{\geq 0}$-valued test function on $\{|x|<1\}$. The claim of the lemma is the following inequality:

$$\int_{|x|<1}-\varphi\partial_x\partial_{\overline{x}}f\,|dx\,d\overline{x}|\leq\int_{|x|<1}g\cdot f\,|dx\,d\overline{x}|$$

By the assumption, we have $\int_{|x|<1}-\varphi\partial_x\partial_{\overline{x}}(\chi_Nf)\,|dx\,d\overline{x}|\leq\int_{|x|<1}g\cdot\chi_Nf\,|dx\,d\overline{x}|$. It is enough to prove that $\lim_{N\to\infty}\int_{|x|<1}\varphi\,\big(\partial\overline{\partial}(\chi_Nf)-\chi_N\partial\overline{\partial}(f)\big)=0$. It follows from the uniform boundedness of $\partial\chi_N$, $\overline{\partial}\chi_N$ and $\partial\overline{\partial}\chi_N$ with respect to the Poincaré metric. ∎

In particular, the inequality (33) holds on $\mathbb{P}^1$. The right hand side of (33) is uniformly bounded in L^p. We can find L^p_2-functions M_c on $\mathbb{P}^1$ and constants $C_i>0$ $(i=10,11)$ such that the following holds for any $c\in U$:

$$\Lambda_{g_{\mathbb{P}^1}}\overline{\partial}\partial\big(\operatorname{Tr}(s_c)-M_c\big)\leq C_{10},\qquad\sup|M_c|\leq C_{11}$$

By [19, Proposition 2.1], we can take constants $C_i > 0$ ($i = 12, 13$) such that the following holds for any $c \in U$:

$$\sup_{\mathbb{P}^1} \bigl(\mathrm{Tr}(s_c) - M_c\bigr) \leq C_{12} \int_{\mathbb{P}^1} \bigl|\mathrm{Tr}(s_c) - M_c\bigr| \, \mathrm{dvol}_{\mathbb{P}^1} \leq C_{13}$$

Hence, we can take a constant $C_{14} > 0$ such that the following holds for any $c \in U$:

$$\sup_{\mathbb{P}^1} |s_c| \leq C_{14} \tag{34}$$

Again, according to [19, Proposition 3.1], we have

$$\sqrt{-1}\Lambda_{g_{\mathbb{P}^1}} \overline{\partial}\partial \, \mathrm{Tr}(s_c) = \sqrt{-1}\Lambda_{g_{\mathbb{P}^1}} \, \mathrm{Tr}\Bigl(s_c\bigl(R(h^0_{c,\ell} + [\widetilde{\theta}, \widetilde{\theta}^{\dagger}_{h^0_{c,\ell}}])\bigr)\Bigr) - \bigl|s_c^{-1/2}(\overline{\partial} + \widetilde{\theta})s_c\bigr|^2_{h^0_{c,\ell}, g_{\mathbb{P}^1}}.$$

Hence, we obtain the boundedness of $\bigl\|s_c^{-1/2}(\overline{\partial} + \widetilde{\theta})s_c\bigr\|_{h^0_{c,\ell}, g_{\mathbb{P}^1}, L^2}$ ($c \in U$). Moreover, we have

$$\bigl\|s_c^{-1/2}(\overline{\partial} + \widetilde{\theta})s_c\bigr\|_{h^0_{c,\ell}, g_{\mathbb{P}^1}, L^2} \to 0 \quad (c \to c^0).$$

Because $s_c^{1/2}$ is uniformly bounded with respect to $h^0_{c,\ell}$, we obtain the boundedness of $\bigl\|(\overline{\partial} + \widetilde{\theta})s_c\bigr\|_{h^0_{c,\ell}, g_{\mathbb{P}^1}, L^2}$ ($c \in U$). We also have $\bigl\|(\overline{\partial} + \widetilde{\theta})s_c\bigr\|_{h^0_{c,\ell}, g_{\mathbb{P}^1}, L^2} \to 0$ ($c \to c^0$). In particular, we obtain the uniform boundedness of s_c in L^2_1 with respect to $h^0_{c,\ell}$.

We take any subsequence s_{c^i} which is weakly convergent in L^2_1 locally on $\mathbb{P}^1 \setminus \{\infty\}$. Let s_∞ denote the limit. The sequence s_{c^i} converges to s_∞ almost everywhere. By the uniform boundedness (34), we have the boundedness of s_∞. It also implies $\|s_\infty\|_{L^p} = \lim \|s_{c^i}\|_{L^p} = 1$. In particular, $s_\infty \neq 0$. By the construction, we have $(\overline{\partial} + \widetilde{\theta})s_\infty = 0$. Hence, s_∞ gives a non-zero endomorphism of the stable filtered Higgs bundle $(\mathcal{P}^{c^0}_* \widetilde{E}_\ell, \widetilde{\theta})$. It implies that s_∞ is the multiplication of a non-zero complex number. In particular, $\det(s_\infty) \neq 0$. It implies that $\bigl\|k_{c^i}\bigr\|_{L^p, h^0_{c,\ell}, g_{\mathbb{P}^1}}$ are bounded. We obtain that $\bigl\|k_c\bigr\|_{L^p, h^0_{c,\ell}, g_{\mathbb{P}^1}}$ ($c \in U$) are bounded.

It implies that the sequence k_c ($c \in U$) are bounded in L^2_1. We also have $C_{15} > 0$ such that $\sup_{\mathbb{P}^1} |k_c| < C_{15}$ ($c \in U$). We take any subsequence k_{c^i} which is weakly convergent in L^2_1 locally on $\mathbb{P}^1 \setminus \{\infty\}$. Then, the limit k_∞ is the multiplication of a non-zero positive number. Because $\det(k_c) = 1$ for any c, we obtain that $k_\infty = \mathrm{id}$. It implies that k_c ($c \in U$) is weakly convergent to id in L^2_1 locally on $\mathbb{P}^1 \setminus \{\infty\}$.

By using [19, Proposition 3.1], we obtain $\sqrt{-1}\Lambda_{g_{\mathbb{P}^1}} \overline{\partial}\partial\bigl(\mathrm{Tr}(k_c) - 2\bigr) \leq C_{16}$ ($c \in U$) for a constant $C_{16} > 0$. By [19, Proposition 2.1], we obtain $\sup_{\mathbb{P}^1}\bigl(\mathrm{Tr}(k_c) - 2\bigr) \leq C_{17} \int_{\mathbb{P}^1} \bigl|\mathrm{Tr}(k_c) - 2\bigr| \, \mathrm{dvol}_{\mathbb{P}^1}$ ($c \in U$) for a constant $C_{17} > 0$. Note that we always have $\mathrm{Tr}(k_c) - 2 \geq 0$, and $\mathrm{Tr}(k_c) - 2 = 0$ implies that $k_c = \mathrm{id}$. Then, we obtain the uniform convergence $\mathrm{Tr}(k_c) - 2 \to 0$ ($c \to c_0$). It implies the uniform convergence $k_c \to \mathrm{id}$ ($c \to c_0$). Thus, the claims of Proposition 3.17 and Proposition 3.15 are proved. ∎

3.3.2 Behaviour of b_c when $c \to 0$

Recall that we have the following description on $\{|\zeta| > 1\}$:

$$\log\bigl|\widetilde{v}_1\bigr|_{h_{c,\ell}} = c \log|\zeta| + \log b_c + \rho_{1,c}, \qquad \log\bigl|\widetilde{v}_2\bigr|_{h_{c,\ell}} = -(\ell + c)\log|\zeta| - \log b_c + \rho_{2,c}.$$

According to Proposition 3.10, we have $|\rho_{i,c}| \leq C_{30} \exp(-\epsilon_{30}|\zeta|^{m+1})$ ($i = 1, 2$) for positive constants C_{30} and ϵ_{30} which are independent of c.

Proposition 3.19 *When $c \to 0$, we have $b_c \to \infty$. We also have the uniform convergences $\rho_{i,c} \longrightarrow 0$ ($i = 1, 2$).*

Proof First, let us study the convergence of $\rho_{i,c}$. We begin with a preliminary. We set $Y_0 := \mathbb{C}_w \times \mathbb{P}^1$ and $D_0 := \mathbb{C}_w \times \{\infty\}$. Let $p : Y_0 \longrightarrow \mathbb{P}^1$ be the projection. We have the locally free $\mathcal{O}_{Y_0}(*D_0)$-module $p^*(L_1 \oplus L_2)$. Let $\overline{v}_i$ be the pull back of $\widetilde{v}_i$. We have the $\mathcal{O}_Y(*D_Y)$-submodule $\overline{E}_\ell \subset p^*(L_1 \oplus L_2)$ generated by $\overline{e}_1 = \overline{v}_1 + w \cdot \overline{v}_2$ and $\overline{e}_2 = \zeta^\ell \overline{v}_2$.

The restriction of $\overline{E}_\ell$ to $\{w\}\times\mathbb{P}^1$ is denoted by $\overline{E}_{\ell,w}$. For each w, we may naturally regard $\overline{E}_{\ell,w}$ as a subsheaf of $\widetilde{V}=L_1\oplus L_2$. We have $\widetilde{\theta}(\overline{E}_{\ell,w})\subset\overline{E}_{\ell,w}\otimes\Omega^1_{\mathbb{P}^1}$. So, we have the family of Higgs bundles $(\overline{E}_{\ell,w},\widetilde{\theta})$. The restrictions of $\overline{v}_i$ and $\overline{e}_i$ to $\{w\}\times\mathbb{P}^1$ are denoted by $\overline{v}_{i,w}$ and $\overline{e}_{i,w}$, respectively. We have

$$\widetilde{\theta}(\overline{v}_{1,w},\overline{v}_{2,w})=(\overline{v}_{1,w},\overline{v}_{2,w})\begin{pmatrix}\alpha\zeta^m d\zeta & 0\\ 0 & -\alpha\zeta^m d\zeta\end{pmatrix},$$

$$\widetilde{\theta}(\overline{e}_{1,w},\overline{e}_{2,w})=(\overline{e}_{1,w},\overline{e}_{2,w})\begin{pmatrix}\alpha\zeta^m d\zeta & 0\\ -2w\alpha\zeta^{m-\ell}d\zeta & -\alpha\zeta^m d\zeta\end{pmatrix}.$$

We have a natural isomorphism $\overline{E}_{\ell,0}\simeq L_1\oplus\zeta^\ell L_2$. For $w\neq 0$, we pick $w^{1/2}$, then we have the isomorphism $\Phi_w:(\overline{E}_{\ell,w},\widetilde{\theta})\simeq(\widetilde{E}_\ell,\widetilde{\theta})$ given by $w^{-1/2}\overline{e}_{1,w}\longleftrightarrow e_1$ and $w^{1/2}\overline{e}_{2,w}\longleftrightarrow e_2$, i.e., $w^{-1/2}\overline{v}_{1,w}\longleftrightarrow v_1$ and $w^{1/2}\overline{v}_{2,w}\longleftrightarrow v_2$.

We take a C^∞-metric $\overline{h}$ of $\overline{E}_{\ell|\mathbb{C}_w\times\mathbb{C}_\zeta}$ satisfying the following conditions:

- On $\mathbb{C}_w\times\{|\zeta|\geq 1\}$, we have $\overline{h}(\overline{v}_1,\overline{v}_1)=|\zeta|^{-2|w|}$, $\overline{h}(\overline{v}_2,\overline{v}_2)=|\zeta|^{-2(\ell-|w|)}$, and $\overline{h}(\overline{v}_1,\overline{v}_2)=0$.
- On $\{w=0\}\times\mathbb{C}_\zeta$, we have $\overline{h}(\overline{e}_{1,0},\overline{e}_{1,0})=1$, $\overline{h}(\overline{e}_{2,0},\overline{e}_{2,0})=1$, and $\overline{h}(\overline{e}_{1,0},\overline{e}_{2,0})=0$.

The restriction of $\overline{h}$ to $\{w\}\times\mathbb{P}^1$ is denoted by $\overline{h}_w$. We take a small $\delta>0$, and consider $U_w:=\{|w|\leq\delta\}$. We have the following uniform boundedness on $\mathbb{C}_\zeta$:

$$\Big|R(\overline{h}_w)+\big[\widetilde{\theta},\widetilde{\theta}^\dagger_{\overline{h}_w}\big]\Big|_{\overline{h}_w,g_{\mathbb{P}^1}}\leq C_{31}\qquad(w\in U_w).$$

Moreover, we have the uniform convergence $\lim_{w\to 0}\Big(R(\overline{h}_w)+\big[\widetilde{\theta},\widetilde{\theta}^\dagger_{\overline{h}_w}\big]\Big)=0$.

For any c satisfying $-\delta<c<0$, we have the self-adjoint endomorphism k_c of $(\overline{E}_{\ell,-c},\overline{h}_{-c})$ determined by $\Phi^*_{-c}h_{c,\ell}(u_1,u_2)=\overline{h}_{-c}(k_cu_1,u_2)$. Take a large $p>1$. Let $\|k_c\|_{\overline{h}_{-c},g_{\mathbb{P}^1},L^p}$ be the L^p-norm of k_c with respect to $\overline{h}_{-c}$ and $g_{\mathbb{P}^1}$. We set $s_c:=k_c/\|k_c\|_{\overline{h}_{-c},g_{\mathbb{P}^1},L^p}$.

Suppose that $|\rho_{1,c}|+|\rho_{2,c}|$ is not uniformly convergent to 0 when $c\to 0$, and we shall deduce a contradiction. Under the assumption, we have a positive number $\delta>0$ and a subsequence $c^j\to 0$ such that

$$\sup_{|\zeta|>1}\big(|\rho_{1,c^j}|+|\rho_{2,c^j}|\big)\geq\delta. \tag{35}$$

By the argument in the proof of Proposition 3.17, we can assume that s_{c^j} weakly converges to a non-zero endomorphism s_∞ of $\overline{E}_{\ell,0}$ in L_1^2 locally on $\mathbb{P}^1\setminus\{\infty\}$, such that (i) $(\overline{\partial}+\widetilde{\theta})s_\infty=0$, (ii) s_∞ is bounded with respect to $\overline{h}_0$. So, it gives an endomorphism of the poly-stable parabolic Higgs bundle $(\mathcal{P}^{\overline{h}_0}_*\overline{E}_0,\widetilde{\theta})$. We obtain that $s_\infty=\alpha_1\operatorname{id}_{L_1}\oplus\alpha_2\operatorname{id}_{\zeta^\ell L_2}$. Here, α_i are non-negative real numbers, and $(\alpha_1,\alpha_2)\neq(0,0)$. Suppose that $\alpha_1\neq 0$. We have the following uniform convergence on any compact subset in $\mathbb{C}$.

$$\lim_{j\to\infty}\Big(h_{c^j,\ell}(\widetilde{v}_1,\widetilde{v}_1)|\zeta|^{-2c^j}\|k_{c^j}\|^{-1}_{\overline{h}_{-c^j},g_{\mathbb{P}^1},L^p}\alpha_1^{-1}\Big)=1$$

It implies that ρ_{1,c^j} is convergent to 0 on any compact subset in $\{|\zeta|\geq 1\}$. Together with the uniform estimate $\big|\rho_{1,c^j}\big|\leq C_{30}\exp(-\epsilon_{30}|\zeta|^{m+1})$, we obtain the uniform convergence $\lim_{j\to\infty}\sup_{|\zeta|>1}|\rho_{1,c^j}|=0$. We have $\big|\widetilde{v}_1\wedge\widetilde{v}_2\big|_{h_{c,\ell}}=|\zeta|^{-\ell}$. By (26), we have the following estimate:

$$\Big|\big|\widetilde{v}_1\big|_{h_{c,\ell}}\cdot\big|\widetilde{v}_2\big|_{h_{c,\ell}}-|\zeta|^{-\ell}\Big|\leq C_{32}\exp(-\epsilon_{32}|\zeta|^{m+1})$$

Here, the constants C_{32} and ϵ_{32} are independent of c. Then, we can deduce the uniform convergence $\rho_{2,c^j}\to 0$. Hence, we obtain $\lim_{j\to\infty}\sup_{|\zeta|>1}|\rho_{i,c^j}|=0$ $(i=1,2)$ in the case $\alpha_1\neq 0$. Similarly, we can deduce the uniform convergences $\lim_{j\to\infty}\sup_{|\zeta|>1}|\rho_{i,c^j}|=0$ $(i=1,2)$ in the case $\alpha_2\neq 0$. But, it contradicts with (35). Thus, we can conclude the uniform convergences $\lim_{c\to 0}\rho_{i,c}=0$ $(i=1,2)$.

Let us study the divergence of b_c. By the convergence of $\rho_{i,c}$, we have the following on $\{|\zeta| \geq 1\}$:

$$\lim_{c\to 0}\Big(b_c^{-1}|\widetilde{v}_1|_{h_{c,\ell}}\Big) = 1, \qquad \lim_{c\to 0}\Big(b_c|\widetilde{v}_2|_{h_{c,\ell}}\Big) = |\zeta|^{-\ell}. \tag{36}$$

Suppose that there exists a subsequence $c^i \to 0$ such that b_{c^i} are bounded, and we shall deduce a contradiction. We may assume the convergence $b_{c^i} \to \check{b}$. We shall give a detailed argument in the case $\check{b} = 0$. Later, we shall sketch the argument for the simpler case $\check{b} \neq 0$.

We set $Y_1 := \mathbb{C}_x \times \mathbb{P}^1$ and $D_1 := \mathbb{C}_x \times \{\infty\}$. Let $p_1 : Y_1 \longrightarrow \mathbb{P}^1$ be the projection. We have the locally free $\mathcal{O}_{Y_1}(*D_1)$-module $p_1^*(L_1 \oplus L_2)$. The pull back of v_i are denoted by $\widehat{v}_i$. Let $\widehat{E}_\ell \subset p_1^*(L_1 \oplus L_2)$ generated by $\widehat{f}_1 = x\widehat{v}_1 + \widehat{v}_2$ and $\widehat{f}_2 = \zeta^\ell \widehat{v}_1$. The restriction of $\widehat{E}_\ell$ to $\{x\} \times \mathbb{P}^1$ is denoted by $\widehat{E}_{\ell,x}$. The restriction of $\widehat{v}_i$ and $\widehat{f}_i$ to $\{x\} \times \mathbb{P}^1$ are denoted by $\widehat{v}_{i,x}$ and $\widehat{f}_{i,x}$. We have

$$\widetilde{\theta}(\widehat{v}_{1,x}, \widehat{v}_{2,x}) = (\widehat{v}_{1,x}, \widehat{v}_{2,x}) \begin{pmatrix} \alpha\zeta^m d\zeta & 0 \\ 0 & -\alpha\zeta^m d\zeta \end{pmatrix}$$

$$\widetilde{\theta}(\widehat{f}_{1,x}, \widehat{f}_{2,x}) = (\widehat{f}_{1,x}, \widehat{f}_{2,x}) \begin{pmatrix} \alpha\zeta^m d\zeta & 0 \\ 2x\alpha\zeta^{m-\ell} d\zeta & -\alpha\zeta^m d\zeta \end{pmatrix}$$

We have $\widehat{E}_{\ell,0} \simeq \zeta^\ell L_1 \oplus L_2$. For $x \neq 0$, taking $x^{1/2}$, we have the isomorphism $\Psi_x : (\widehat{E}_{\ell,x}, \widetilde{\theta}) \simeq (\widetilde{E}_\ell, \widetilde{\theta})$ given by the correspondences $x^{-1/2}\widehat{f}_{1,x} \longleftrightarrow v_1 + v_2$ and $x^{1/2}\widehat{f}_{2,x} \longleftrightarrow \zeta^\ell v_1$, i.e., $x^{1/2}\widehat{v}_1 \longleftrightarrow v_1$ and $x^{-1/2}\widehat{v}_2 \longleftrightarrow v_2$.

We have the isomorphisms of holomorphic vector bundles $\Upsilon_{x_i} : \widehat{E}_{\ell,0} \simeq \widehat{E}_{\ell,x_i}$ given by $\Upsilon_{x_i}(\widehat{f}_{j,0}) = \widehat{f}_{j,x}$. We shall implicitly identify the vector bundles $\widehat{E}_{\ell,0}$ and $\widehat{E}_{\ell,x_i}$ by Υ_{x_i} in the following argument.

Let $x_i := b_{c^i}^2$. By the assumption, we have $\lim_{i\to\infty} x_i = 0$. We obtain the harmonic metrics $\Psi_{x_i}^* h_{c^i,\ell}$ on $(\widehat{E}_{\ell,x_i}, \theta)$. We take a family of C^∞-Hermitian metrics $h^0_{x_i}$ of $\widehat{E}_{\ell,x_i|\mathbb{C}_\zeta}$ satisfying the following conditions.

- $h^0_0(\widehat{v}_{1,0}, \widehat{v}_{1,0}) = 1$, $h^0_0(\widehat{v}_{2,0}, \widehat{v}_{2,0}) = |\zeta|^{-2\ell}$, and $h^0_0(\widehat{v}_{1,0}, \widehat{v}_{2,0}) = 0$.
- $\Upsilon_{x_i}^* h^0_{x_i} \to h^0_0$ in the C^∞-sense on any compact subset in $\mathbb{C}_\zeta$.
- We have $h^0_{x_i}(\widehat{v}_{1,x}, \widehat{v}_{1,x}) = |\zeta|^{2c^i}$, $h^0_{x_i}(\widehat{v}_{2,x}, \widehat{v}_{2,x}) = |\zeta|^{-2c^i-2\ell}$ and $h^0_{x_i}(\widehat{v}_{1,x}, \widehat{v}_{2,x}) = 0$ on $\{|\zeta| > 1\}$.

Let k_i be the self-adjoint endomorphism of $(\widehat{E}_{\ell,x_i}, h^0_{x_i})$ determined by $\Psi_{x_i}^* h_{c^i,\ell} = h^0_{x_i} \cdot k_i$. By (36), we have the convergence of $\Upsilon_{x_i}^* k_i$ to the identity on any compact subset in $\{|\zeta| > 1\}$. We have the convergence of $\Upsilon_{x_i}^* k_i^{-1}$ to the identity on any compact subset in $\{|\zeta| > 1\}$.

Let us study the convergence of $\Upsilon_{x_i}^* k_i$ and $\Upsilon_{x_i}^* k_i^{-1}$ on $\{|\zeta| < 2\}$. We have the following uniform estimate:

$$\big|R(h_{c^i,\ell})\big|_{h_{c^i,\ell}, g_{\mathbb{P}^1}} \leq C_{42}.$$

Hence, we have a constant $C_{43} > 0$ which is independent of c^i such that $-\partial_\zeta\partial_{\overline{\zeta}} \log|\widehat{f}_1|_{h_{c^i,\ell}} \leq C_{43}$ on $\{|\zeta| < 2\}$. Namely, we have

$$-\partial_\zeta\partial_{\overline{\zeta}}\Big(\log|\widehat{f}_1|_{h_{c^i,\ell}} - C_{43}|\zeta|^2\Big) \leq 0$$

We have already known the uniform boundedness of $|\widehat{f}_1|_{h_{c^i,\ell}}$ on $\{1 < |\zeta| < 2\}$. Then, we obtain the uniform boundedness of $|\widehat{f}_1|_{h_{c^i,\ell}}$ on $\{|\zeta| < 2\}$. Similarly, we obtain the uniform boundedness of $|\widehat{f}_2|_{h_{c^i,\ell}}$ on $\{|\zeta| < 2\}$. Hence, k_i and k_i^{-1} are uniformly bounded on $\{|\zeta| < 2\}$.

Then, we may assume that the sequence $\Upsilon_{x_i}^* k_i$ is weakly convergent to k_∞ in L_2^p on any compact subset in $\mathbb{C}_\zeta$. We have the convergence of $\Upsilon_{x_i}^* k_i^{-1}$ to k_∞^{-1}. We obtain a harmonic metric $h_\infty = h^0_0 \cdot k_\infty$. By the construction, we have $\mathcal{P}^0_*\widehat{E}_{\ell,0} \subset \mathcal{P}^{h_\infty}_*\widehat{E}_{\ell,0}$. Because $\deg(\mathcal{P}^0_*\widehat{E}_{\ell,0}) = \deg(\mathcal{P}^{h_\infty}_*\widehat{E}_{\ell,0}) = 0$, we obtain $\mathcal{P}^0_*\widehat{E}_{\ell,0} = \mathcal{P}^{h_\infty}_*\widehat{E}_{\ell,0}$ by Lemma 3.6. But, $(\mathcal{P}^0_*\widehat{E}_{\ell,0}, \widetilde{\theta})$ is not poly-stable because $\mathcal{P}^0_0 L_2 \simeq \mathcal{O}_{\mathbb{P}^1}(\ell)$ and $\deg \mathcal{P}^0_0 L_2 = \ell > 0$. Hence, we have deduced a contradiction from the assumption $b_{c^i} \to 0$.

Let us give a sketch of the argument in the case $\lim b_{c^i} = \check{b} \neq 0$. Under the assumption, the sequence $h_{c^i,\ell}$ is convergent on $\{|\zeta| \geq 1\}$. As in the previous case, by taking a subsequence, we may assume that the sequence $h_{c^i,\ell}$ is weakly convergent in L_2^p on any compact subset in $\mathbb{C}_\zeta$, and the limit h_∞ is a harmonic metric of $(\widetilde{E}_\ell, \theta)$ which is adapted to the filtered bundle $\mathcal{P}^0_*\widetilde{E}_\ell$. But, $(\mathcal{P}^0_*\widetilde{E}_\ell, \theta)$ is not poly-stable. Hence, we obtain a contradiction even in the case $\check{b} \neq 0$. ∎

3.3.3 Convergence of some sequences

Let t_i be a sequence of positive numbers such that $t_i \to \infty$. According to Proposition 3.15 and Proposition 3.19, we can take a sequence of negative numbers c_i such that $c_i \to 0$ and $b_{c_i} t_i^{c_i/(m+1)} = 1$. We set $\tau_i := t_i^{1/2(m+1)}$.

We use the notation in §3.2.4. We have the isomorphisms $\Phi_{\tau_i} : \varphi_{\tau_i}^* \widetilde{E}_\ell \simeq \widetilde{E}_\ell$. Let h_i denote the Hermitian metric of $\widetilde{E}_\ell$ induced by Φ_{τ_i} and $\varphi_{\tau_i}^* h_{c_i,\ell}$. Take any $T > 0$. We have the following estimates on $\{|\zeta| \geq T\}$ for any i such that $t_i^{1/(m+1)} T > 1$, where C and ϵ are positive constants independent of i and T:

$$\Big|\log\big|\widetilde{v}_1\big|_{h_i} - \log\big(|\zeta|^{c_i} t_i^{\ell/2(m+1)}\big)\Big| \leq C \exp(-\epsilon |\zeta|^{m+1} t_i)$$

$$\Big|\log\big|\widetilde{v}_2\big|_{h_i} - \log\big(|\zeta|^{-\ell-c_i} t_i^{-\ell/2(m+1)}\big)\Big| \leq C \exp(-\epsilon |\zeta|^{m+1} t_i)$$

$$\big|h_i(\widetilde{v}_1, \widetilde{v}_2)\big| \leq C \exp(-\epsilon |\zeta|^{m+1} t_i) \cdot |\widetilde{v}_1|_{h_i} \cdot |\widetilde{v}_2|_{h_i}$$

Let $h^{\lim}$ be the Hermitian metric of $\widetilde{V}_{|\mathbb{C}^*}$ given by

$$h^{\lim}(\widetilde{v}_1, \widetilde{v}_1) = 1, \quad h^{\lim}(\widetilde{v}_2, \widetilde{v}_2) = |\zeta|^{-2\ell}, \quad h^{\lim}(\widetilde{v}_1, \widetilde{v}_2) = 0.$$

For any $\gamma > 0$, the automorphism Ψ_γ on $\widetilde{V}_{|\mathbb{C}^*}$ is given by $\Psi_\gamma = \gamma\, \mathrm{id}_{L_1} \oplus \gamma^{-1}\, \mathrm{id}_{L_2}$. We define $\Psi_\gamma^* h_i(u_1, u_2) := h_i(\Psi_\gamma u_1, \Psi_\gamma u_2)$. The following is clear.

Proposition 3.20 *Set $\gamma_i := t_i^{-\ell/2(m+1)}$. Then, we have the convergence $\lim_{i\to\infty} \Psi_{\gamma_i}^* h_i = h^{\lim}$ on $\mathbb{C}^*$.* ∎

3.4 Complement

We use the notation in §3.2. Take $0 < \kappa < 1$. We take a C^∞-function $\rho : \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$ satisfying $\rho(\zeta) = 1$ ($|\zeta| \leq 1/2$) and $\rho(\zeta) = 0$ ($|\zeta| \geq 1$). We set

$$u_1 := -(\kappa\rho(\zeta) + |\zeta|^{2\ell})^{-1} \overline{\zeta}^\ell e_2 + e_1, \qquad u_2 := e_2.$$

They give a C^∞-frame of $\widetilde{E}_\ell$. Note that we have $u_1 = v_1$ and $u_2 = \zeta^\ell v_2$ on $\{|\zeta| \geq 1\}$.

We take a large integer L. Let h_κ be the Hermitian metric of $\widetilde{E}_\ell$ determined by the following conditions:

$$|u_2|_{h_\kappa} = \kappa^L, \quad |u_1|_{h_\kappa} = \kappa^{-L}, \quad h_\kappa(u_1, u_2) = 0$$

Let ∇_κ be the Chern connection of $(\widetilde{E}_\ell, h_\kappa)$, and let $R(h_\kappa)$ denote the curvature of ∇_κ. Let $\widetilde{\theta}_\kappa^\dagger$ denote the adjoint of $\widetilde{\theta}$ with respect to h_κ. Let $\widetilde{\theta}_\kappa^\dagger$ denote the adjoint of $\widetilde{\theta}$ with respect to h_κ.

Lemma 3.21 *On $\big\{|\zeta| \leq 1\big\}$, we have $\big|R(h_\kappa)\big|_{h_\kappa, g_\mathbb{C}} \leq C_{50}\kappa^L$ and $\big|[\widetilde{\theta}, \widetilde{\theta}_\kappa^\dagger]\big|_{h_\kappa, g_\mathbb{C}} \leq C_{50}\kappa^L$ for a constant $C_{50} > 0$. On $\big\{|\zeta| \geq 1\big\}$, we have $R(h_\kappa) = [\widetilde{\theta}, \widetilde{\theta}_\kappa^\dagger] = 0$.*

Proof In the proof, $O(\kappa^L)$ denotes functions which are dominated by $C_{51}\kappa^L$ for a constant $C_{51} > 0$. The equalities on $\{|\zeta| \geq 1\}$ are clear. Let us argue the estimates on $\big\{|\zeta| \leq 1\big\}$. We have $\overline{\partial}\big(\kappa^{-L} u_2\big) = 0$, and

$$\overline{\partial}(\kappa^L u_1) = -\frac{\overline{\partial}(\overline{\zeta}^\ell \rho) \cdot \kappa^{2L+1}}{(\kappa\rho(\zeta) + |\zeta|^{2\ell})^2} (\kappa^{-L} u_2).$$

We have

$$\nabla_\kappa\big(\kappa^L u_1,\kappa^{-L}u_2\big)=\big(\kappa^L u_1,\kappa^{-L}u_2\big)A,\qquad A=\begin{pmatrix} 0 & \frac{\partial(\zeta^\ell\rho(\zeta))\kappa^{2L+1}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2}\\ \frac{-\overline{\partial}(\overline{\zeta}^\ell\rho(\zeta))\kappa^{2L+1}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2} & 0\end{pmatrix}$$

It is easy to obtain the following estimate on $\{|\zeta|\leq 1\}$:

$$\frac{\overline{\partial}(\overline{\zeta}^\ell\rho(\zeta))\kappa^{2L+1}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2}=O(\kappa^L)d\overline{\zeta},\qquad \frac{\partial(\zeta^\ell\rho(\zeta))\kappa^{2L+1}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2}=O(\kappa^L)d\zeta$$

We also have the following on $\{|\zeta|\leq 1\}$:

$$d\Big(\frac{\overline{\partial}(\overline{\zeta}^\ell\rho(\zeta))\kappa^{2L+1}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2}\Big)=O(\kappa^L)\,d\zeta\,d\overline{\zeta}$$

Hence, we have the following on $\{|\zeta|\leq 1\}$:

$$dA+A\wedge A=O(\kappa^L)\,d\zeta\,d\overline{\zeta}$$

It implies the estimate for $R(h_\kappa)$ on $\{|\zeta|\leq 1\}$.

The Higgs field $\widetilde{\theta}$ is represented as follows:

$$\widetilde{\theta}(\kappa^L u_1,\kappa^{-L}u_2)=(\kappa^L u_1,\kappa^{-L}u_2)\begin{pmatrix}\alpha\zeta^m d\zeta & 0\\ \frac{-2\kappa^{2L+1}\alpha\rho(\zeta)\zeta^{m-\ell}}{\kappa\rho(\zeta)+|\zeta|^{2\ell}}d\zeta & -\alpha\zeta^m d\zeta\end{pmatrix}$$

The adjoint $\widetilde{\theta}^\dagger_\kappa$ is represented as follows:

$$\widetilde{\theta}^\dagger_\kappa(\kappa^L u_1,\kappa^{-L}u_2)=(\kappa^L u_1,\kappa^{-L}u_2)\begin{pmatrix}\overline{\alpha}\overline{\zeta}^m d\overline{\zeta} & \frac{-2\kappa^{2L+1}\overline{\alpha}\rho(\zeta)\overline{\zeta}^{m-\ell}}{\kappa\rho(\zeta)+|\zeta|^{2\ell}}d\overline{\zeta}\\ 0 & -\overline{\alpha}\overline{\zeta}^m d\overline{\zeta}\end{pmatrix}$$

Hence, $[\widetilde{\theta},\widetilde{\theta}^\dagger_\kappa]$ is represented as follows:

$$\big[\widetilde{\theta},\widetilde{\theta}^\dagger_\kappa\big](\kappa^L u_1,\kappa^{-L}u_2)=(\kappa^L u_1,\kappa^{-L}u_2)A_2$$

$$A_2:=\begin{pmatrix}4\kappa^{4L+2}|\alpha|^2\rho(\zeta)^2\frac{|\zeta|^{2(m-\ell)}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2}d\overline{\zeta}d\zeta & -4\kappa^{2L+1}|\alpha|^2\rho(\zeta)\frac{|\zeta|^{2m}\zeta^{-\ell}}{\kappa\rho(\zeta)+|\zeta|^{2\ell}}d\overline{\zeta}\,d\zeta\\ 4\kappa^{2L+1}|\alpha|^2\rho(\zeta)\frac{|\zeta|^{2m}\zeta^{-\ell}}{\kappa\rho(\zeta)+|\zeta|^{2\ell}}d\overline{\zeta}\,d\zeta & -4\kappa^{4L+2}|\alpha|^2\rho(\zeta)^2\frac{|\zeta|^{2(m-\ell)}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2}d\overline{\zeta}d\zeta\end{pmatrix}$$

In particular, we have the following on $\{|\zeta|\leq 1\}$:

$$A_2=O\Big(\frac{\kappa^{4L+2}\rho(\zeta)|\zeta|^{2(m-\ell)}}{(\kappa\rho(\zeta)+|\zeta|^{2\ell})^2}+\frac{\kappa^{2L+1}\rho(\zeta)|\zeta|^{2m-\ell}}{\kappa\rho(\zeta)+|\zeta|^{2\ell}}\Big)d\zeta\,d\overline{\zeta}=O(\kappa^L)d\zeta\,d\overline{\zeta}$$

Thus, we are done. ∎

3.4.1 Convergence of some sequences

Suppose that we are give a number $a>\ell/2(m+1)$. Set $\nu:=a-\ell/2(m+1)>0$. Let t_i be a sequence of positive numbers such that $t_i\to\infty$. We set $\kappa_i:=t_i^{-\nu/L}$, for which we have $\kappa_i\to 0$. We have the Hermitian metric h_{κ_i} of $\widetilde{E}_\ell$ as above.

We use the notation in §3.2.4. We set $\tau_i:=t_i^{1/2(m+1)}$. We have the isomorphisms $\Phi_{\tau_i}:\varphi_{\tau_i}^*\widetilde{E}_\ell\simeq\widetilde{E}_\ell$. Let h_i be the Hermitian metric of $\widetilde{E}_\ell$ induced by $\varphi_{\tau_i}^*h_{\kappa_i}$ and Φ_{τ_i}.

On $\{|\tau| \geq \kappa_i\}$, we have the following:

$$|\widetilde{v}_1|_{h_i} = t_i^a, \quad |\widetilde{v}_2|_{h_i} = t_i^{-a}|\zeta|^{-\ell}, \quad h_i(\widetilde{v}_1, \widetilde{v}_2) = 0$$

Let $h^{\lim}$ be the Hermitian metric of $\widetilde{V}_{|\mathbb{C}^*}$ given by

$$h^{\lim}(\widetilde{v}_1, \widetilde{v}_1) = 1, \quad h^{\lim}(\widetilde{v}_2, \widetilde{v}_2) = |\zeta|^{-2\ell}, \quad h^{\lim}(\widetilde{v}_1, \widetilde{v}_2) = 0.$$

For any $\gamma > 0$, the automorphism Ψ_γ on $\widetilde{V}_{|\mathbb{C}^*}$ is given by $\Psi_\gamma = \gamma \operatorname{id}_{L_1} \oplus \gamma^{-1} \operatorname{id}_{L_2}$. We define $\Psi_\gamma^* h_i(u_1, u_2) := h_i(\Psi_\gamma u_1, \Psi_\gamma u_2)$. The following is clear.

Proposition 3.22 *Set $\gamma_i := t_i^{-a}$. Then, we have the convergence $\lim_{i\to\infty} \Psi_{\gamma_i}^* h_i = h^{\lim}$ on $\mathbb{C}^*$.* ∎

4 Limiting configurations

4.1 A description of generically regular semisimple Higgs bundles of rank 2

Let X be a connected complex curve. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on X of rank 2 which is generically regular semisimple. For simplicity, we assume that $\operatorname{tr}(\theta) = 0$. Let us consider the case that the spectral curve $\Sigma(E, \theta)$ is reducible, i.e., we have a holomorphic 1-form $\omega \neq 0$ on X such that $\Sigma(E, \theta)$ is the union of the images of ω and $-\omega$. Such ω is determined up to the multiplication of ± 1.

Remark 4.1 *Suppose that $\operatorname{tr}(\theta) \neq 0$. Then, we set $\theta' := \theta - (\operatorname{tr}\theta/2) \operatorname{id}_E$. Then, the Higgs bundle $(E, \overline{\partial}_E, \theta')$ satisfies $\operatorname{tr}(\theta') = 0$. Hence, it is enough to study Higgs bundles such that the trace of the Higgs field is 0.*

Suppose that $\Sigma(E, \theta)$ is irreducible. We take a normalization $\varphi : \widetilde{X} \longrightarrow \Sigma(E, \theta)$. We have the induced morphism $\varphi_1 : \widetilde{X} \longrightarrow X$. The Higgs bundle $\varphi_1^*(E, \overline{\partial}_E, \theta)$ satisfies the above condition. If X is compact and $(E, \overline{\partial}_E, \theta)$ is stable, then $\varphi_1^*(E, \overline{\partial}_E, \theta)$ is stable or poly-stable. So, the study on the irreducible case can be reduced to the reducible case. (See also §4.3.2.) ∎

Let $Z(\omega)$ denote the zero set of ω. We have the decomposition of the $\mathcal{O}_X\bigl(*Z(\omega)\bigr)$-module

$$E \otimes \mathcal{O}_X\bigl(*Z(\omega)\bigr) = L'_\omega \oplus L'_{-\omega}$$

corresponding to the decomposition of the spectral curve, i.e., $\theta = \omega \cdot \operatorname{id}_{L'_\omega} \oplus (-\omega) \cdot \operatorname{id}_{L'_{-\omega}}$. Let L_ω (resp. $L_{-\omega}$) denote the $\mathcal{O}_X$-module obtained as the image of E by the induced morphism $E \longrightarrow L'_\omega$ (resp. $E \longrightarrow L_{-\omega}$). We can regard E as an $\mathcal{O}_X$-submodule of $L_\omega \oplus L_{-\omega}$.

We obtain the $\mathcal{O}_X$-module $\det(L_\omega \oplus L_{-\omega}) \big/ \det(E)$ whose supports are contained in $Z(\omega)$. For each $P \in Z(\omega)$, let ℓ_P denote the length of the stalk of $\det(L_\omega \oplus L_{-\omega}) / \det(E)$ at P. Because we have the exact sequence

$$0 \longrightarrow L_\omega \cap E \longrightarrow E \longrightarrow L_{-\omega} \longrightarrow 0,$$

we have $L_\omega \cap E = L_\omega\bigl(-\sum_{P \in Z(\omega)} \ell_P \cdot P\bigr)$ in $L_\omega \oplus L_{-\omega}$. Similarly, we have $L_{-\omega} \cap E = L_{-\omega}\bigl(-\sum_{P \in Z(\omega)} \ell_P \cdot P\bigr)$ in $L_\omega \oplus L_{-\omega}$.

Local description Let (U_P, z) be a small holomorphic coordinate neighbourhood around P satisfying $z(P) = 0$. Let s_P be any frame of $\det(E)_{|U_P}$.

Lemma 4.2 *We have local frames $v_\pm$ of $L_{\pm\omega}$ around P such that (i) $e_1 = v_+ + v_-$ and $e_2 = z^{\ell_P} v_-$ is a frame of $E_{|U_P}$, (ii) $e_1 \wedge e_2 = s_P$.*

Proof If $\ell_P = 0$, we have $E = L_\omega \oplus L_{-\omega}$ on U_P, and hence the claim is clear. We shall consider the case $\ell_P > 0$. We omit to distinguish the restriction to U_P. We take a frame e'_2 of $z^\ell L_{-\omega}$. We take a section e'_1 of E such that e'_1 and e'_2 give a frame of E. We have the unique decomposition $e'_1 = v'_+ + v'_-$, where $v'_\pm$ are sections of $L_{\pm\omega}$. By the construction, $L_{\pm\omega}$ is generated by the images of e'_1 and e'_2. Because $\ell_P > 0$, we can observe that the image of $v'_\pm$ generates $L_{\pm\omega}$, i.e., $v'_\pm$ are frames of $L_{\pm\omega}$. We may assume that $e'_2 = z^{\ell_P} v'_-$.

Let g be the holomorphic function determined by $e_1' \wedge e_2' = g \cdot s$. Because $g(P) \neq 0$, we can take a holomorphic function g_1 such that $g_1^2 = g$. We set $e_i := g_1^{-1} \cdot e_i'$ and $v_\pm := g_1^{-1} v_\pm'$. Then, they satisfy the desired condition. ∎

Let m_P denote the order of zero of ω at P, i.e., we have $\omega = z^{m_P} g_P(z)\, dz$ on U_P for a holomorphic function $g_P(z)$ with $g_P(0) \neq 0$. For the frame (e_1, e_2), the Higgs field $\theta_{|U_P}$ is described as

$$\theta(e_1, e_2) = (e_1, e_2) \begin{pmatrix} \omega & 0 \\ -2z^{-\ell_P}\omega & -\omega \end{pmatrix}.$$

In particular, we have $\ell_P \leq m_P$.

4.2 The limiting configurations of stable Higgs bundles

Let X be a compact connected Riemann surface. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle of rank 2 on X such that (i) $(E, \overline{\partial}_E, \theta)$ is stable, (ii) $(E, \overline{\partial}_E, \theta)$ is generically regular semisimple, (iii) $\mathrm{tr}(\theta) = 0$. We assume that the spectral curve of θ is reducible. We obtain a holomorphic 1-form $\omega \neq 0$ and the line bundles $L_{\pm\omega}$ as in §4.1. We impose that $\deg(L_\omega) \leq \deg(L_{-\omega})$. We set $L_1 := L_\omega$ and $L_2 := L_{-\omega}$.

We set $d_i := \deg(L_i)$ $(i = 1, 2)$. We have

$$d_1 + d_2 - \sum_{P \in Z(\omega)} \ell_P = \deg(E).$$

The stability condition for (E, θ) is equivalent to the inequalities $\deg(L_j \cap E) = d_j - \sum_{P \in Z(\omega)} \ell_P < \deg(E)/2$ $(j = 1, 2)$, i.e.,

$$d_j - \deg(E)/2 > 0 \quad (j = 1, 2).$$

The local scaling factor a_P of $(E, \overline{\partial}, \theta)$ at $P \in Z(\omega)$ is defined as follows:

$$a_P := \frac{\ell_P}{2(m_P + 1)}.$$

We have the functions $\chi_P : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\leq 0}$ $(P \in Z(\omega))$ given as follows:

$$\chi_P(a) := \begin{cases} (m_P + 1)(a - a_P) & (0 \leq a \leq a_P) \\ 0 & (a \geq a_P) \end{cases}$$

We set $\chi_{E,\theta}(a) := \sum_{P \in Z(\omega)} \chi_i(a)$.

Lemma 4.3 *We have the unique number $a_{E,\theta}$ satisfying the conditions*

$$d_1 - \frac{\deg(E)}{2} + \chi_{E,\theta}(a_{E,\theta}) = 0, \quad 0 \leq a_{E,\theta} < \max\bigl\{a_P \bigm| P \in Z(\omega)\bigr\}.$$

Proof The function $\chi_{E,\theta}$ is strictly increasing for $0 \leq a \leq a_1 := \max\{a_P \,|\, P \in Z(\omega)\}$, and we have $\chi_{E,\theta}(a) = 0$ for $a \geq a_1$. Hence, we have $d_1 - \frac{\deg(E)}{2} + \chi_{E,\theta}(a_1) = d_1 - \frac{\deg(E)}{2} > 0$. We have $\chi_{E,\theta}(0) = -\sum_{P \in Z(\omega)} \ell_P/2$. Because of $d_1 + d_2 - \deg(E) - \sum_{P \in Z(\omega)} \ell_P = 0$ and $d_1 \leq d_2$, we have $d_1 - \frac{\deg(E)}{2} + \chi_{E,\theta}(0) \leq 0$. Hence, the claim of the lemma follows. ∎

As in §3.1.5, the numbers $-\chi_P(a_{E,\theta})$ $(P \in Z(\omega))$ give a parabolic structure on the line bundle L_1. For any $\boldsymbol{c} = (c_P \,|\, P \in Z(\omega)) \in \mathbb{R}^{Z(\omega)}$, let $\boldsymbol{n}(\boldsymbol{c}) = (n_P(\boldsymbol{c}) \,|\, P \in Z(\omega)) \in \mathbb{Z}^{Z(\omega)}$ be determined by the condition

$$c_P - 1 < n_P(\boldsymbol{c}) - \chi_P(a_{E,\theta}) \leq c_P.$$

Then, we set $\mathcal{P}^{\lim}_{\boldsymbol{c}}(L_1) = L_1\Bigl(\sum_{P \in Z(\omega)} n_P(\boldsymbol{c}) P\Bigr)$. We obtain a filtered bundle $\mathcal{P}^{\lim}_*(L_1) = \bigl(\mathcal{P}^{\lim}_{\boldsymbol{c}}(L_1) \bigm| \boldsymbol{c} \in \mathbb{R}^{Z(\omega)}\bigr)$ over the meromorphic line bundle $L_1\bigl(*Z(\omega)\bigr)$. The parabolic degree of $\mathcal{P}^{\lim}_*(L_1)$ is

$$d_1 - \sum_{P \in Z(\omega)} (-\chi_P(a_{E,\theta})) = d_1 + \chi_{E,\theta}(a_{E,\theta}) = \frac{\deg(E)}{2}.$$

Similarly, the numbers $\chi_P(a_{E,\theta})+\ell_P$ ($P \in Z(\omega)$) determine a filtered bundle over $L_2(*Z(\omega))$. The parabolic degree of $\mathcal{P}^{\lim}_*(L_2)$ is

$$d_2 - \sum_{P \in Z(\omega)} (\chi_P(a_{E,\theta}) + \ell_P) = d_1 - \frac{\deg(E)}{2} + d_2 - \sum_{P \in Z(\omega)} \ell_P = \frac{\deg(E)}{2}.$$

The direct sum $\mathcal{P}^{\lim}_* L_1 \oplus \mathcal{P}^{\lim}_* L_2$ is called the limiting configuration of $(E, \overline{\partial}_E, \theta)$.

4.2.1 Hermitian metrics of the limiting configuration

We fix a Hermitian metric $h_{\det(E)}$ on the determinant line bundle $\det(E)$. We have the 2-form $R(h_{\det(E)})$ obtained as the curvature of the Chern connection of $(\det(E), h_{\det(E)})$.

Lemma 4.4 *We have Hermitian metrics $h^{\lim}_{L_j}$ of the holomorphic line bundles $L_{j|X\setminus Z(\omega)}$ satisfying the following conditions:*

- *The curvature of the Chern connection $\nabla^{\lim}_j$ of $(L_{j|X\setminus Z(\omega)}, h^{\lim}_{L_j})$ is equal to $R(h_{\det(E)})/2$.*
- *For each $P \in Z(\omega)$, let (U_P, z_P) be any holomorphic coordinate neighbourhood of P with $z_P(P) = 0$. Then, $|z_P|^{-2\chi_P(a_{E,\theta})} h^{\lim}_{L_1|U_P\setminus\{P\}}$ and $|z_P|^{2\chi_P(a_{E,\theta})+2\ell_P} h^{\lim}_{L_2|U_P\setminus\{P\}}$ induce Hermitian metrics of C^∞-class on $L_{1|U_P}$ and $L_{2|U_P}$, respectively.*
- *Under the isomorphism $\det(E)_{|X\setminus Z(\omega)} \simeq (L_1 \otimes L_2)_{|X\setminus Z(\omega)}$, we have $h^{\lim}_{L_1} \otimes h^{\lim}_{L_2} = h_{\det(E)}$ on $X \setminus Z(\omega)$.*

Proof Because this is standard, we give only a sketch of the proof. We can take Hermitian metrics h_{L_j} ($j = 1, 2$) of $L_{j|X\setminus Z(\omega)}$ such that $|z_P|^{-2\chi_P(a_{E,\theta})} h_{L_1|U_P\setminus\{P\}}$ and $|z_P|^{2\chi_P(a_{E,\theta})+2\ell_P} h_{L_2|U_P\setminus\{P\}}$ induce Hermitian metrics of C^∞-class on $L_{1|U_P}$ and $L_{2|U_P}$, respectively. Let $R(h_{L_j})$ denote the curvature form of $(L_{j|X\setminus Z(\omega)}, h_{L_j})$. They naturally induce 2-forms of C^∞-class on X, which are also denoted by $R(h_{L_j})$. Because $\frac{\sqrt{-1}}{2\pi}\int R(h_{L_j})$ is equal to the parabolic degree of $\mathcal{P}^{\lim}_* L_j$, we have $\int \Big(R(h_{L_j}) - R(h_{\det(E)})/2\Big) = 0$. We have C^∞-functions ρ_j ($j = 1, 2$) on X such that $\overline{\partial}\partial\rho_j = R(h_{L_j}) - R(h_{\det(E)})/2$. We set $h^{(1)}_{L_j} := e^{-\rho_j} h_{L_j}$. Then, we have $R(h^{(1)}_{L_j}) = R(h_{\det(E)})/2$. Because $\det(E) = L_1 \otimes L_2 \otimes \mathcal{O}_X\big(-\sum \ell_P P\big)$, the tensor product $h^{(1)}_{L_1} \otimes h^{(2)}_{L_2}$ induces a C^∞-Hermitian metric of $\det(E)$. By comparison of the curvature, we have $h^{(1)}_{L_1} \otimes h^{(2)}_{L_2} = \alpha \cdot h_{\det(E)}$ for a positive constant α. Hence, we obtain Hermitian metrics $h^{\lim}_{L_j}$ with the desired property by adjusting $h^{(1)}_{L_j}$, for example by setting $h^{\lim}_{L_1} = \alpha^{-1} h^{(1)}_{L_1}$ and $h^{\lim}_{L_2} = h^{(1)}_{L_2}$. ∎

Such a metric $h^{\lim}_{E,\theta} := h^{\lim}_{L_1} \oplus h^{\lim}_{L_2}$ is also called the limiting configuration of $(E, \overline{\partial}_E, \theta)$. Note that we have the ambiguity of the actions of automorphisms $\alpha\, \mathrm{id}_{L_1} \oplus \alpha^{-1}\, \mathrm{id}_{L_2}$ ($\alpha > 0$), i.e., the pair of $\alpha \cdot h^{\lim}_{L_1}$ and $\alpha^{-1} \cdot h^{\lim}_{L_2}$ satisfies the conditions in Lemma 4.4. But, the induced Chern connection $\nabla^{\lim}_{E,\theta} := \nabla^{\lim}_1 \oplus \nabla^{\lim}_2$ on $E_{|X\setminus Z(\omega)}$ is well defined. Note that $\nabla^{\lim}_{E,\theta}$ is projectively flat, whose curvature is given by the multiplication of $R(h_{\det(E)})/2$.

Remark 4.5 *The metric $h^{\lim}_{E,\theta}$ is also characterized as a Hermitian-Einstein metric for the Higgs bundle $\Big((L_1, t\omega) \oplus (L_2, -t\omega)\Big)_{|X\setminus Z(\omega)}$ adapted to the filtered bundle $\mathcal{P}^{\lim}_* L_1 \oplus \mathcal{P}^{\lim}_* L_2$ for any $t \neq 0$ such that $\det(h^{\lim}_{E,\theta}) = h_{\det(E)}$. Because $(\mathcal{P}^{\lim}_* L_1, t\omega) \oplus (\mathcal{P}^{\lim}_* L_2, -t\omega)$ is polystable, we have the ambiguity of the metric $h^{\lim}_{E,\theta}$ by the automorphisms $\alpha\, \mathrm{id}_{L_1} \oplus \alpha^{-1}\, \mathrm{id}_{L_2}$, as usual.* ∎

4.3 The limiting configuration in complementary cases

Let X be a compact connected Riemann surface. Let $(E, \overline{\partial}_E, \theta)$ be any Higgs bundle of rank 2 on X such that (i) $(E, \overline{\partial}_E, \theta)$ is generically regular semisimple, (ii) $\mathrm{tr}(\theta) = 0$. We give limiting configurations in some complementary cases.

4.3.1 Polystable Higgs bundles

Suppose that $(E,\overline{\partial}_E,\theta)$ is polystable. Then, we have the decomposition $(E,\theta)=(L_\omega,\omega)\oplus(L_{-\omega},-\omega)$. In particular, the spectral curve is reducible, and we have $\ell_P=0$ in the description in §4.1. In this case, we set (E,θ) as the limiting configuration.

We take a Hermitian metric $h_{\det(E)}$ on the line bundle $\det(E)$. We have Hermitian metrics $h^{\lim}_{L_{\pm\omega}}$ on $L_{\pm\omega}$ such that $R(h^{\lim}_{L_{\pm\omega}})=R(h_{\det(E)})/2$. We set $h^{\lim}_E:=h^{\lim}_{L_\omega}\oplus h^{\lim}_{L_{-\omega}}$ such that $h^{\lim}_{L_\omega}\otimes h^{\lim}_{L_{-\omega}}=h_{\det(E)}$. The metric $h^{\lim}_{E,\theta}$ is also called limiting configuration. It is characterized as a Hermitian-Einstein metric on the Higgs bundle $(E,\overline{\partial}_E,\theta)$ such that $\det(h^{\lim}_{E,\theta})=h_{\det(E)}$. We have the ambiguity of the metric $h^{\lim}_{E,\theta}$ caused by automorphisms of (E,θ).

The limiting configuration can also be given as a filtered bundle on $(X,Z(\omega))$ as in the case of stable Higgs bundles §4.2. We consider the trivial parabolic structures for $L_{\pm\omega}$ at $P\in Z(\omega)$. Namely, for any $\boldsymbol{c}=(c_P\,|\,P\in Z(\omega))\in\mathbb{R}^{Z(\omega)}$, let $\boldsymbol{n}(\boldsymbol{c})=(n_P(\boldsymbol{c}))\in\mathbb{Z}^{Z(\omega)}$ be determined by the condition $c_P-1<n_P(\boldsymbol{c})\le c_P$. Then, we set $\mathcal{P}^{\lim}_{\boldsymbol{c}}(L_{\pm\omega})=L_{\pm\omega}\Bigl(\sum_{P\in Z(\omega)}n_P(\boldsymbol{c})P\Bigr)$. The parabolic degrees of $\mathcal{P}^{\lim}_*(L_{\pm\omega})$ is $\deg(E)/2$. The filtered bundle $\mathcal{P}^{\lim}_*L_\omega\oplus\mathcal{P}^{\lim}_*L_{-\omega}$ is called the limiting configuration of $(E,\overline{\partial}_E,\theta)$.

Remark 4.6 *The metric $h^{\lim}_{E,\theta}$ can be characterized as a Hermitian-Einstein metric for $(E,\theta)_{|X\setminus Z(\omega)}$ adapted to the filtered bundle such that $\det(h^{\lim}_{E,\theta})=h_{\det(E)}$.* ∎

4.3.2 The case where the spectral curve is irreducible

Suppose that the spectral curve $\Sigma(E,\theta)$ is irreducible. It implies that the Higgs bundle $(E,\overline{\partial}_E,\theta)$ is stable. We take a normalization $\widetilde{X}\longrightarrow\Sigma(E,\theta)$. We have the induced morphism $p:\widetilde{X}\longrightarrow X$, which is a ramified covering of degree 2. We have the involution of $\Sigma(E,\theta)$ induced by the multiplication of -1 on the cotangent bundle T^*X. It induces an involution ρ of $\widetilde{X}$ over X. We can regard X as the quotient space of $\widetilde{X}$ by the action of the group $\{1,\rho\}$.

We set $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta}):=p^*(E,\overline{\partial}_E,\theta)$. The spectral curve of $(\widetilde{E},\widetilde{\theta})$ is reducible, i.e., it is the union $\mathrm{Im}(\widetilde{\omega})\cup\mathrm{Im}(-\widetilde{\omega})$ for a holomorphic one form $\widetilde{\omega}$ on $\widetilde{X}$. We have a natural isomorphism $\rho^*(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta})\simeq(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta})$. By the construction, we have $\rho^*\widetilde{\omega}=-\widetilde{\omega}$. We have $Z(\widetilde{\omega})=p^{-1}D(E,\theta)$. We have the line bundles $\widetilde{L}_{\widetilde{\omega}}$ and $\widetilde{L}_{-\widetilde{\omega}}$ on $\widetilde{X}$ with an inclusion $\widetilde{E}\longrightarrow\widetilde{L}_{\widetilde{\omega}}\oplus\widetilde{L}_{-\widetilde{\omega}}$ as in §4.1. Because $\rho^*\widetilde{\omega}=-\widetilde{\omega}$, we have natural isomorphisms $\rho^*\widetilde{L}_{\pm\widetilde{\omega}}\simeq\widetilde{L}_{\mp\widetilde{\omega}}$ such that the following is commutative:

$$\begin{array}{ccc}\rho^*\widetilde{E} & \longrightarrow & \rho^*\widetilde{L}_{\widetilde{\omega}}\oplus\rho^*\widetilde{L}_{-\widetilde{\omega}}\\ \simeq\downarrow & & \simeq\downarrow\\ \widetilde{E} & \longrightarrow & \widetilde{L}_{-\widetilde{\omega}}\oplus\widetilde{L}_{\widetilde{\omega}}\end{array}$$

Note that we have already known that $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta})$ is poly-stable. Indeed, because $(E,\overline{\partial}_E,\theta)$ is stable, we have a Hermitian-Einstein metric h^{HE} for $(E,\overline{\partial}_E,\theta)$. The pull back p^*h^{HE} is a Hermitian-Einstein metric for $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta})$, which implies the poly-stability of the Higgs bundle.

Stable case When $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta})$ is stable, we obtain the limiting configuration for $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta})$ as a filtered bundle on $(\widetilde{X},Z(\widetilde{\omega}))$ by the procedure in §4.2. Namely, we obtain the filtered line bundles $\mathcal{P}^{\lim}_*\widetilde{L}_{\pm\widetilde{\omega}}$ as in §4.2. In this case, we have $\deg(\widetilde{L}_{\widetilde{\omega}})=\deg(\widetilde{L}_{-\widetilde{\omega}})$ which implies that $a_{\widetilde{E},\widetilde{\theta}}=0$ and that the parabolic weights are $-\chi_P(a_{\widetilde{E},\widetilde{\theta}})=\chi_P(a_{\widetilde{E},\widetilde{\theta}})+\ell_P=\ell_P/2$. Because $\ell_P=\ell_{\rho(P)}$, the isomorphisms $\rho^*\widetilde{L}_{\pm\widetilde{\omega}}\simeq\widetilde{L}_{\mp\widetilde{\omega}}$ induce the isomorphisms of filtered line bundles $\rho^*\mathcal{P}^{\lim}_*\widetilde{L}_{\pm\widetilde{\omega}}\simeq\mathcal{P}^{\lim}_*\widetilde{L}_{\mp\widetilde{\omega}}$. We have the Hermitian metrics $h^{\lim}_{\widetilde{L}_{\pm\widetilde{\omega}}}$ of $\widetilde{L}_{\pm\widetilde{\omega}|\widetilde{X}\setminus Z(\widetilde{\omega})}$ as in Lemma 4.4. We may also impose the condition $\rho^*h^{\lim}_{\widetilde{L}_{\widetilde{\omega}}}=h^{\lim}_{\widetilde{L}_{-\widetilde{\omega}}}$ with which the metrics $h^{\lim}_{\widetilde{L}_{\pm\widetilde{\omega}}}$ are uniquely determined.

We set $h^{\lim}_{\widetilde{E},\widetilde{\theta}}:=h^{\lim}_{\widetilde{L}_{\widetilde{\omega}}}\oplus h^{\lim}_{\widetilde{L}_{-\widetilde{\omega}}}$ on $\widetilde{E}_{|\widetilde{X}\setminus Z(\widetilde{\omega})}$. Because $\rho^*h^{\lim}_{\widetilde{E},\widetilde{\theta}}=h^{\lim}_{\widetilde{E},\widetilde{\theta}}$, we have the Hermitian metric $h^{\lim}_{E,\theta}$ of $E_{|X\setminus D(E,\theta)}$ such that $p^*h^{\lim}_{E,\theta}=h^{\lim}_{\widetilde{E},\widetilde{\theta}}$. The metric $h^{\lim}_{E,\theta}$ and the associated Chern connection $\nabla^{\lim}_{E,\theta}$ of $E_{|X\setminus D(E,\theta)}$ are uniquely determined.

Remark 4.7 *The metric $h^{\lim}_{E,\theta}$ is also characterized as follows. Because the filtered bundle $\mathcal{P}^{\lim}_*\widetilde{L}_\omega \oplus \mathcal{P}^{\lim}_*\widetilde{L}_{-\omega}$ is equivariant with respect to the action of $\{1,\rho\}$, we have the filtered bundle $\mathcal{P}^{\lim}_*E$ on $(X, D(E,\theta))$ obtained as the descent of $\mathcal{P}^{\lim}_*\widetilde{L}_\omega \oplus \mathcal{P}^{\lim}_*\widetilde{L}_{-\omega}$. (See §4.3.3 below for the descent of filtered bundles in this situation.) We can easily observe that the filtered Higgs bundles $(\mathcal{P}^{\lim}_*E, t\theta)$ $(t \neq 0)$ are stable, and that the parabolic degree is $\deg(E)$. The metric $h^{\lim}_{E,\theta}$ is a unique Hermitian-Einstein metric for the Higgs bundle $(E, \overline{\partial}_E, t\theta)_{|X\setminus D(E,\theta)}$ for any t, such that $\det(h^{\lim}_{E,\theta}) = h_{\det(E)}$ adapted to the filtered bundle $\mathcal{P}^{\lim}_*E$.* ∎

Polystable case We have $\widetilde{E} = \widetilde{L}_{\widetilde{\omega}} \oplus \widetilde{L}_{-\widetilde{\omega}}$. As in §4.3.1, we take Hermitian metrics $h^{\lim}_{\widetilde{L}_{\pm\widetilde{\omega}}}$ of $\widetilde{L}_{\pm\widetilde{\omega}}$ satisfying $R(h^{\lim}_{\widetilde{L}_{\pm\widetilde{\omega}}}) = p^*R(h_{\det(E)})/2$ and $h^{\lim}_{\widetilde{L}_{\widetilde{\omega}}} \otimes h^{\lim}_{\widetilde{L}_{-\widetilde{\omega}}} = p^*h_{\det(E)}$, and we set $h^{\lim}_{\widetilde{E},\widetilde{\theta}} := h^{\lim}_{\widetilde{L}_{\widetilde{\omega}}} \oplus h^{\lim}_{\widetilde{L}_{-\widetilde{\omega}}}$. We also impose the condition $\rho^*h^{\lim}_{\widetilde{L}_{\widetilde{\omega}}} = h^{\lim}_{\widetilde{L}_{-\widetilde{\omega}}}$, with which the metrics $h^{\lim}_{\widetilde{L}_{\pm\widetilde{\omega}}}$ and $h^{\lim}_{\widetilde{E},\widetilde{\theta}}$ are uniquely determined. Because $\rho^*h^{\lim}_{\widetilde{E},\widetilde{\theta}}$, we have a unique Hermitian metric $h^{\lim}_{E,\theta}$ of E such that $p^*h^{\lim}_{E,\theta} = h^{\lim}_{\widetilde{E},\widetilde{\theta}}$.

The metric $h^{\lim}_{E,\theta}$ is also characterized as follows.

Lemma 4.8 *For any $t \neq 0$, $h^{\lim}_{E,\theta}$ is a unique Hermitian-Einstein metric for the stable Higgs bundle $(E, \overline{\partial}_E, t\theta)$ such that $\det(h^{\lim}_{E,\theta}) = h_{\det(E)}$. In particular, the Hermitian-Einstein metrics for the Higgs bundles $(E, \overline{\partial}_E, t\theta)$ are independent of t.*

Proof The claim is clear by the construction of $h^{\lim}_{E,\theta}$. ∎

Remark 4.9 *We have the filtered bundle $\mathcal{P}^{\lim}_*\widetilde{E}$ as in §4.3.1. Because it is equivariant with respect to ρ, we obtain a filtered bundle $\mathcal{P}^{\lim}_*E$ obtained as the descent of $\mathcal{P}^{\lim}_*\widetilde{E}$. The metric $h^{\lim}_{E,\theta}$ is also characterized as a unique Hermitian metric for the stable Higgs bundle $(\mathcal{P}^{\lim}_*E, t\theta)$ $(t \neq 0)$.* ∎

4.3.3 Descent (Appendix)

Let $\widetilde{X}, X, p, \rho$, $D(E,\theta)$ and $Z(\widetilde{\omega})$ be as in §4.3.2. Let $\widetilde{V}$ be a locally free $\mathcal{O}_{\widetilde{X}}(*Z(\widetilde{\omega}))$-module of finite rank which is equivariant with respect to $\{1,\rho\}$, i.e., we are given an isomorphism $\Phi : \rho^*\widetilde{V} \simeq \widetilde{V}$ such that $\Phi \circ \rho^*\Phi = \mathrm{id}$. Let $\mathcal{P}_*\widetilde{V}$ be a filtered bundle over $\widetilde{V}$, which is equivariant with respect to $\{1,\rho\}$, i.e., $\rho^*\mathcal{P}_{\boldsymbol{a}}\widetilde{V} = \mathcal{P}_{\boldsymbol{a}}\widetilde{V}$ for any $\boldsymbol{a} \in \mathbb{R}^{Z(\widetilde{\omega})}$ under the above isomorphism. In this case, the decent of $\mathcal{P}_*\widetilde{V}$ is given as follows. By the equivariance of $\widetilde{V}$, we have a locally free $\mathcal{O}_X(*D(E,\theta))$-module V with an isomorphism $p^*V \simeq \widetilde{V}$. It is also described as follows. We have the locally free $\mathcal{O}_X(*D(E,\theta))$-module $p_*\widetilde{V}$. It is equivariant with respect to $\{1,\rho\}$, where the action of $\{1,\rho\}$ on X is trivial. Then, V is the invariant part of $p_*\widetilde{V}$ with respect to the action. For any $P \in D(E,\theta)$, let $q(P)$ denote the number of the set $p^{-1}(P)$, which are 1 or 2. Let $\boldsymbol{a} = (a_P \,|\, P \in D(E,\theta)) \in \mathbb{R}^{D(E,\theta)}$. For $Q \in p^{-1}(P)$, we set $\widetilde{a}_Q := q(P) \cdot a_P$. We obtain the locally free $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{P}_{\widetilde{\boldsymbol{a}}}(\widetilde{V})$. It is equivariant with respect to $\{1,\rho\}$. We obtain the locally free $\mathcal{O}_X$-module $p_*\mathcal{P}_{\widetilde{\boldsymbol{a}}}\widetilde{V}$. It is equivariant with respect to $\{1,\rho\}$. The invariant part is denoted by $\mathcal{P}_{\boldsymbol{a}}V$. Thus, we obtain a filtered bundle $\mathcal{P}_*V$ over V, which is the decent of $\mathcal{P}_*\widetilde{V}$.

5 Convergence to the limiting configurations

5.1 Statements

5.1.1 General case

Let $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of rank 2 on a compact connected Riemann surface X, such that (i) $(E, \overline{\partial}_E, \theta)$ is generically regular semisimple, (ii) the spectral curve is reducible, (iii) $\mathrm{tr}(\theta) = 0$. We fix a Hermitian metric $h_{\det(E)}$ of $\det(E)$. We use the notation in §4.2. We have the limiting configuration $\mathcal{P}^{\lim}_*L_1 \oplus \mathcal{P}^{\lim}_*L_2$. We take Hermitian metrics $h^{\lim}_{L_j}$ for the parabolic line bundle $\mathcal{P}^{\lim}_*L_j$ satisfying the condition in Lemma 4.4. We set $h^{\lim}_{E,\theta} := h^{\lim}_{L_1} \oplus h^{\lim}_{L_2}$. We have the associated Chern connection $\nabla^{\lim}_{E,\theta}$ of $E_{|X\setminus Z(\omega)}$, which is projectively flat.

For any $t > 0$, the Higgs bundle $(E, \overline{\partial}_E, t\theta)$ is also stable. We have the Hermitian-Einstein metrics h_t of the Higgs bundles $(E, \overline{\partial}_E, t\theta)$, i.e., $R(h_t) + [t\theta, t\theta^\dagger_{h_t}]$ is equal to the multiplication of $R(h_{\det(E)})/2$ according to Hitchin [6] and Simpson [19]. We impose that $\det(h_t) = h_{\det(E)}$. We have the Chern connection ∇_{h_t} of $(E, \overline{\partial}_E, h_t)$.

For any $\gamma > 0$, let Ψ_γ denote the automorphism of $L_1 \oplus L_2$ given by $\Psi_\gamma = \gamma \operatorname{id}_{L_1} \oplus \gamma^{-1} \operatorname{id}_{L_2}$. We define the metric $\Psi_\gamma^* h_t$ of $E_{|X\setminus Z(\omega)}$ by $\Psi_\gamma^* h_t(u_1, u_2) = h_t(\Psi_\gamma u_1, \Psi_\gamma u_2)$ for local sections u_i of $E_{|X\setminus Z(\omega)}$.

Take any $Q \in X \setminus Z(\omega)$. Let v_Q be any frame of $L_{1|Q}$. We set

$$\gamma(t,Q) := \left(\frac{h^{\lim}_{L_1}(v_Q, v_Q)}{h_t(v_Q, v_Q)} \right)^{1/2}$$

We shall prove the following theorem in §5.2–§5.4.

Theorem 5.1 *When t goes to ∞, the sequence $\Psi^*_{\gamma(t,Q)} h_t$ converges to $h^{\lim}_{E,\theta}$ in the C^∞-sense on any compact subset in $X \setminus Z(\omega)$.*

In particular, we obtain the following convergence of unitary connections ∇_{h_t}.

Corollary 5.2 *The sequence ∇_{h_t} $(t \to \infty)$ converges to $\nabla^{\lim}_{E,\theta}$ on any compact subset in $X \setminus Z(\omega)$.*

Proof If $\deg(E) = 0$ and $R(h_{\det(E)}) = 0$, according to the estimates in §2.2, it is enough to prove that the sequence of the Chern connections of $(L_{j|X\setminus Z(\omega)}, h_{t|L_j})$ converges to the Chern connection of $(L_{j|X\setminus Z(\omega)}, h^{\lim}_{L_j})$. It follows from Theorem 5.1.

Let us consider the case where $\deg(E)$ is even. We have a holomorphic line bundle L_0 on X with an isomorphism $\det(E) \simeq L_0 \otimes L_0$. We have the Hermitian metric h_{L_0} of L_0 such that $h_{\det(E)} = h_{L_0} \otimes h_{L_0}$ under the isomorphism. Then, $h_t \otimes h_{L_0}^{-1}$ on $E \otimes L_0^{-1}$ is a harmonic metric of $(E \otimes L_0^{-1}, \overline{\partial}_{E\otimes L_0^{-1}}, \theta, h_t \otimes h_{L_0}^{-1})$. We have the convergence of $\nabla_{h_t \otimes h_{L_0}^{-1}}$ to the Chern connection of $\big((L_1 \otimes L_0^{-1})_{|X\setminus Z(\omega)}, h^{\lim}_{L_1} \otimes h^{-1}_{L_0}\big) \oplus \big(L_2 \otimes L_0^{-1})_{|X\setminus Z(\omega)}, h^{\lim}_{L_2} \otimes h^{-1}_{L_0}\big)\big)$ by the consideration in the case $\deg(E) = 0$ and $h_{\det(E)} = 0$. Hence, we obtain the convergence of ∇_{h_t} to $\nabla^{\lim}_{E,\theta}$ in the case where $\deg(E)$ is even.

Let us consider the case where $\deg(E)$ is odd. We take a covering map $p : \widetilde{X} \longrightarrow X$ of degree 2 such that $\widetilde{X}$ is connected. Note that $p^*(E, \overline{\partial}_E, \theta)$ is stable because $\deg(p^* L_i) - \deg(p^* E)/2 > 0$. Because $\deg(p^* E)$ is even, we have the convergence of $p^* \nabla_{h_t}$ to $p^* \nabla^{\lim}_{E,\theta}$. Hence, we obtain the convergence of ∇_{h_t} to $\nabla^{\lim}_{E,\theta}$. ∎

5.1.2 Symmetric case

We can deduce a stronger result if X and $(E, \overline{\partial}_E, \theta)$ are equipped with an extra symmetry. Suppose that X is equipped with a holomorphic non-trivial involution ρ, i.e., ρ is an automorphism of X such that $\rho \circ \rho = \operatorname{id}_X$ and $\rho \neq \operatorname{id}_X$. Let $(E, \overline{\partial}_E, \theta)$ be as in §5.1.1. We impose the following additional conditions.

- $(E, \overline{\partial}_E, \theta)$ is equivariant with respect to the action of $\{\operatorname{id}_X, \rho\}$. Namely, we have an isomorphism $\upsilon_\rho : \rho^*(E, \overline{\partial}_E, \theta) \simeq (E, \overline{\partial}_E, \theta)$ such that $\rho^* \upsilon_\rho \circ \upsilon_\rho = \operatorname{id}$.
- We have $\rho^* \omega = -\omega$.

We impose the condition $\rho^* h_{\det(E)} = h_{\det(E)}$ to the metric $h_{\det(E)}$ under the induced isomorphism $\rho^* \det(E) \simeq \det(E)$.

The conditions imply that we have natural isomorphisms $\rho^* L_1 \simeq L_2$ and $\rho^* L_2 \simeq L_1$ which are compatible with $\rho^* E \simeq E$. Because $\deg(L_1) = \deg(L_2)$, we have $a_{E,\theta} = 0$ and $-\chi_P(a_{E,\theta}) = \chi_P(a_{E,\theta}) + \ell_P = \ell_P/2$ for any $P \in Z(\omega)$. We also have $\ell_P = \ell_{\rho(P)}$. So, we have natural isomorphisms $\rho^* \mathcal{P}^{\lim}_* L_1 \simeq \mathcal{P}^{\lim}_* L_2$ and $\rho^* \mathcal{P}^{\lim}_* L_2 \simeq \mathcal{P}^{\lim}_* L_1$ compatible with the isomorphism $\rho^* E \simeq E$. We can impose the additional condition $\rho^* h^{\lim}_{L_1} = h^{\lim}_{L_2}$ to the conditions in Lemma 4.4, with which the metrics $h^{\lim}_{L_i}$ are uniquely determined. We shall prove the following theorem in §5.5.

Theorem 5.3 *Suppose the symmetric property of $(E, \overline{\partial}_E, \theta)$ as above. For any $t > 0$, let h_t be the Hermitian-Einstein metric for $(E, \overline{\partial}_E, t\theta)$ satisfying $\det(h_t) = h_{\det(E)}$. Then, when t goes to ∞, the sequence h_t is convergent to $h^{\lim}_{E,\theta}$ in the C^∞-sense on any compact subset in $X \setminus Z(\omega)$.*

5.1.3 The case where the spectral curve is irreducible

As a complement, we explain how to deduce the results in the case where the spectral curve is irreducible, from Theorem 5.3.

Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle of rank 2 such that (i) $(E, \overline{\partial}_E, \theta)$ is generically regular semisimple, (ii) the spectral curve $\Sigma(E, \theta)$ is irreducible, (iii) $\mathrm{tr}(\theta) = 0$. Note that the Higgs bundle is stable. We fix a Hermitian metric $h_{\det(E)}$ on $\det(E)$. We have the Hermitian-Einstein metrics h_t of $(E, \overline{\partial}_E, t\theta)$ such that $\det(h_t) = h_{\det(E)}$. Recall that we have constructed a Hermitian metric $h^{\lim}_{E,\theta}$ of $E_{|X\setminus D(E,\theta)}$ in §4.3.2.

Corollary 5.4 *When t goes to ∞, the sequence h_t converges to $h^{\lim}_{E,\theta}$ in the C^∞-sense on any compact subset in $X \setminus D(E, \theta)$.*

Proof We use the notation in §4.3.2. If the Higgs bundle $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$ is polystable, then the metric h_t are independent of t and equal to $h^{\lim}_{E,\theta}$ as remarked in Lemma 4.8. Let us consider the case where $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$ is stable. Then, by Theorem 5.3, p^*h_t is convergent to $h^{\lim}_{\widetilde{E},\widetilde{\theta}} = p^*h^{\lim}_{E,\theta}$ in the C^∞-sense on any compact subset in $\widetilde{X} \setminus Z(\widetilde{\omega})$. Hence, we obtain the convergence of h_t to $h^{\lim}_{E,\theta}$. ∎

5.2 A reduction for the proof of Theorem 5.1

Let us observe that for the proof of Theorem 5.1 it is enough to consider the case where $\deg(E) = 0$ and $R(h_{\det(E)}) = 0$. The argument already appeared in the proof of Corollary 5.2.

Lemma 5.5 *Suppose that we have already proved the claim of Theorem* 5.1 *in the case where* $\deg(E) = 0$ *and* $R(h_{\det(E)}) = 0$. *Then, we obtain the claim of Theorem* 5.1 *in the general case.*

Proof Let us consider the case where $\deg(E)$ is even. We have a holomorphic line bundle L_0 with an isomorphism $L_0^{\otimes 2} \simeq \det(E)$. We have a Hermitian metric h_{L_0} on L_0 such that $h_{L_0} \otimes h_{L_0} = h_{\det(E)}$. By the assumption and the construction of the limiting configuration, we obtain that the sequence $\Psi^*_{\gamma(t,Q)}(h_t \otimes h^{-1}_{L_0})$ converges to $h^{\lim}_{E,\theta} \otimes h^{-1}_{L_0}$ in the C^∞-sense on any compact subset in $X \setminus Z(\omega)$. Hence, we obtain the convergence of $\Psi^*_{\gamma(t,Q)}(h_t)$ to $h^{\lim}_{E,\theta}$.

Let us consider the case where $\deg(E)$ is odd. We take a covering $p : \widetilde{X} \longrightarrow X$ of degree 2 such that $\widetilde{X}$ is connected. Note that $p^*(E, \overline{\partial}_E, \theta)$ is stable because $\deg(p^*L_i) - \deg(p^*E)/2 > 0$. We take $\widetilde{Q}$ such that $p(\widetilde{Q}) = Q$. Because $\deg(p^*E)$ is even, we obtain that the sequence $\Psi^*_{\gamma(t,\widetilde{Q})}(p^*h_t)$ is convergent to $p^*h^{\lim}_{E,\theta}$ in the C^∞-sense on any compact subset in $\widetilde{X} \setminus p^{-1}D(E, \theta)$. Hence, we obtain the convergence of $\Psi^*_{\gamma(t,Q)}h_t$ to $h^{\lim}_{E,\theta}$. Thus, the proof of Lemma 5.5 is finished. ∎

It remains to prove Theorem 5.1 in the case where $\deg(E) = 0$ and $R(h_{\det(E)}) = 0$, which will be established in §5.3–5.4.

5.3 Construction of approximate solutions

Let X and $(E, \overline{\partial}_E, \theta)$ be the Higgs bundle as in §5.1.1. We impose $\deg(E) = 0$. We fix a flat metric $h_{\det(E)}$ on $\det(E)$.

5.3.1 Rescaling around the zeroes

Let $P \in Z(\omega)$. We take a holomorphic coordinate system (U_P, z) such that the eigenvalues of θ are $\pm d(z^{m_P+1}) = \pm(m_P + 1)z^{m_P}dz$. Such z is determined up to the multiplication of a $(m_P + 1)$-th square root of 1. We take frames v_i of $L_{i|U_P}$ $(i = 1, 2)$ such that (i) $e_1 = v_1 + v_2$ and $e_2 = z^{\ell_P}v_2$ give a frame of $E_{|U_P}$, (ii) $|e_1 \wedge e_2|_{h_{\det(E)}} = 1$.

We use the Higgs bundle $(\widetilde{E}_\ell, \widetilde{\theta})$ in §3 by setting $\alpha = m_P + 1$. Let $\varphi_t : U_P \longrightarrow \mathbb{C}$ be given by $\varphi_t(z) = t^{1/(m_P+1)}z = \zeta$. We have

$$\varphi_t^*\widetilde{\theta}(\varphi_t^*\widetilde{e}_1, \varphi_t^*\widetilde{e}_2) = (\varphi_t^*\widetilde{e}_1, \varphi_t^*\widetilde{e}_2)\begin{pmatrix} t\alpha z^{m_P}dz & 0 \\ -2\alpha t \cdot t^{-\ell_P/(m_P+1)}z^{m_P-\ell_P}dz & -t\alpha z^{m_P}dz \end{pmatrix}.$$

Hence, we have the isomorphism $\varphi_t^*(\widetilde{E}_\ell, \widetilde{\theta}) \simeq (E, t\theta)_{|U_P}$ given by the following correspondence:

$$t^{\ell_P/2(m_P+1)}\varphi_t^*\widetilde{e}_1 \longleftrightarrow e_1, \qquad t^{-\ell_P/2(m_P+1)}\varphi_t^*\widetilde{e}_2 \longleftrightarrow e_2$$

Moreover, we have $t^{\ell_P/2(m_P+1)}\varphi_t^*\widetilde{v}_j \longleftrightarrow v_j$.

5.3.2 Local constructions around the zeroes

In the following, for a given positive function ν, let $O(\nu)$ denote a function f such that $|f| \leq C\nu$, where C is a positive constant independent of t. Let ϵ denote small positive numbers which are independent of t.

Let $P \in Z(\omega)$. Suppose that $a_{E,\theta} - \ell_P/2(m_P+1) < 0$. We have the harmonic metric $h_{\chi_P(a_{E,\theta}),\ell_P}$ of $(\widetilde{E}_\ell, \widetilde{\theta})$ as in §3.3. We obtain harmonic metrics $h^0_{t,P} := \varphi_t^* h_{\chi_P(a_{E,\theta}),\ell_P}$ of $(E, \overline{\partial}_E, t\theta)_{|U_P}$. By construction, we obtain the following from Proposition 3.10:

Lemma 5.6 *Take $R_{1,P} > 0$ such that $\{|z| \leq R_{1,P}\} \subset U_P$. Take $0 < R_{2,P} < R_{1,P}$. On $\{R_{2,P} \leq |z| < R_{1,P}\} \subset U_P$, we have*

$$\big|v_1\big|_{h^0_{t,P}} = t^{a_{E,\theta}}|z|^{\chi_P(a_{E,\theta})} \cdot b_{\chi_P(a_{E,\theta})}\Big(1 + O\big(\exp(-\epsilon|z|^{m_P+1}t)\big)\Big)$$

$$\big|v_2\big|_{h^0_{t,P}} = t^{-a_{E,\theta}}|z|^{-\ell_P-\chi_P(a_{E,\theta})}b^{-1}_{\chi_P(a_{E,\theta})}\Big(1 + O\big(\exp(-\epsilon|z|^{m_P+1}t)\big)\Big)$$

$$h^0_{t,P}\big(v_1, v_2\big) = O\big(\exp(-\epsilon|z|^{m_P+1}t)\big)$$

∎

We also have the following lemma.

Lemma 5.7 *On $\big\{R_{2,P} \leq |z| \leq R_{1,P}\big\}$, we have the following:*

$$\partial \log|v_1|^2_{h^0_{t,P}} = \chi_P(a_{E,\theta})dz/z + O\big(\exp(-\epsilon t)\big)\,dz = O(1)\,dz$$

$$\partial \log|v_2|^2_{h^0_{t,P}} = -(\ell_P + \chi_P(a_{E,\theta}))dz/z + O\big(\exp(-\epsilon t)\big)\,dz = O(1)\,dz$$

$$\overline{\partial}\partial \log|v_j|^2_{h^0_{t,P}} = O\big(\exp(-\epsilon t)\big)\,dz\,d\overline{z} \quad (j = 1, 2)$$

$$\partial h^0_t(v_1, v_2) = O\big(\exp(-\epsilon t)\big)\,dz, \qquad \overline{\partial}h^0_t(v_1, v_2) = O\big(\exp(-\epsilon t)\big)\,d\overline{z},$$

$$\partial\overline{\partial}h^0_t(v_1, v_2) = O\big(\exp(-\epsilon t)\big)\,dz\,d\overline{z}$$

Proof According to Proposition 3.10, we have $z\partial_z \log|v_1|^2_{h^0_{t,P}} - \chi_P(a_{E,\theta}) = O\big(\exp(-\epsilon|z|^{m+1}t)\big)$. Hence, we obtain the estimate for $\partial \log|v_1|^2_{h^0_{t,P}}$ on the domain. We obtain the estimate for $\partial \log|v_2|^2_{h^0_{t,P}}$ in a similar way. We obtain the estimate for $\overline{\partial}\partial \log|v_j|^2_{h^0_{t,P}}$ from Lemma 3.11. We obtain the estimate for $\partial h^0_t(v_1, v_2)$ and $\partial\overline{\partial}h^0_t(v_1, v_2)$ from Lemma 3.13. ∎

Suppose that $a_{E,\theta} - \ell_P/2(m_P+1) = 0$. If $\ell_P = a_{E,\theta} = 0$, we set $h_{t_i,P} = \varphi^*_{t_i}h_{\widetilde{E}_0}$, where $h_{\widetilde{E}_0}$ is the harmonic metric given in §3.2.3. Suppose $\ell_P > 0$. According to Proposition 3.15 and Proposition 3.19, for a given sequence $t_i \to \infty$, we can take a sequence of negative numbers $c_i \to 0$ such that

$$\frac{\log b_{c_i}}{-c_i} = \frac{\log t_i}{m_P + 1}, \quad \text{i.e.,} \quad b_{c_i}t^{c_i/(m_P+1)} = 1$$

We obtain the sequence of harmonic metrics $h_{t_i,P} := \varphi^*_{t_i}h_{c_i,\ell_P}$ of $(E, \overline{\partial}_E, t_i\theta)_{|U_P}$, where h_{c_i,ℓ_P} are given as in §3.3. By Proposition 3.10, we have the following.

Lemma 5.8 *Take $0 < R_{2,P} < R_{1,P}$ as in Lemma 5.6. On $\{R_{2,P} \leq |z| \leq R_{1,P}\}$, we have the following estimates:*

$$\big|v_1\big|_{h^0_{t_i,P}} = t_i^{a_{E,\theta}}|z|^{c_i}\Big(1 + O\big(\exp(-\epsilon|z|^{m_P+1}t_i)\big)\Big)$$

$$\big|v_2\big|_{h^0_{t_i,P}} = t_i^{-a_{E,\theta}}|z|^{-c_i-\ell_P}\Big(1 + O\big(\exp(-\epsilon|z|^{m_P+1}t_i)\big)\Big)$$

$$h^0_{t_i,P}\big(v_1, v_2\big) = O\big(\exp(-\epsilon|z|^{m_P+1}t_i)\big)$$

∎

As in the case of Lemma 5.7, we have the following.

Lemma 5.9 *Take $0 < R_{2,P} < R_{1,P}$. On $\big\{R_{2,P} \leq |z| \leq R_{1,P}\big\}$, we have the following:*

$$\partial \log |v_1|^2_{h^0_{t_i,P}} = c_i dz/z + O\big(\exp(-\epsilon t_i)\big)\, dz = O(1)\, dz$$

$$\partial \log |v_2|^2_{h^0_{t_i,P}} = -(c_i + \ell_P) dz/z + O\big(\exp(-\epsilon t_i)\big)\, dz = O(1)\, dz$$

$$\overline{\partial}\partial \log |v_j|^2_{h^0_{t_i,P}} = O\big(\exp(-\epsilon t_i)\big)\, dz\, d\overline{z}$$

$$\partial h^0_{t_i}(v_1, v_2) = O\big(\exp(-\epsilon t_i)\big)\, dz, \qquad \overline{\partial} h^0_{t_i}(v_1, v_2) = O\big(\exp(-\epsilon t_i)\big)\, d\overline{z}$$

$$\partial\overline{\partial} h^0_{t_i}(v_1, v_2) = O\big(\exp(-\epsilon t_i)\big)\, dz\, d\overline{z}$$

∎

Suppose that $j_P := a_{E,\theta} - \ell_P/2(m_P + 1) > 0$. We use the notation in §3.4. For a given sequence $t_i \to \infty$, we set $\kappa_i := t_i^{-j_P/L}$. We have the Hermitian metrics h_{κ_i} of $\widetilde{E}_\ell$ as in §3.4. We obtain the Hermitian metrics $h^0_{t_i,P} := \varphi^*_{t_i} h_{\kappa_i}$ of $E_{|U_P}$. By construction, we have the following on $\{t_i^{-1/(m_P+1)} \leq |z| < 1\}$:

$$\big|v_1\big|_{h^0_{t_i,P}} = t_i^{a_{E,\theta}}, \quad \big|v_2\big|_{h^0_{t_i,P}} = t_i^{-a_{E,\theta}}|z|^{-\ell_P}, \quad h^0_{t_i,P}(v_1, v_2) = 0$$

Lemma 5.10 *Take $0 < \epsilon_P << (m_P + 1)j_P$. We have the following estimate:*

$$R(h^0_{t_i,P}) = O\Big(t_i^{-j_P+\epsilon_P/(m_P+1)}\Big)|z|^{\epsilon_P-2} dz\, d\overline{z}$$

$$\big[t_i\theta, (t_i\theta)^\dagger_{h^0_{t_i,P}}\big] = O\Big(t_i^{-j_P+\epsilon_P/(m_P+1)}\Big)|z|^{\epsilon_P-2} dz\, d\overline{z}$$

Proof Because we have $R(h^0_{t_i,P}) = [t\theta, (t\theta)^\dagger_{h^0_{t_i,P}}] = 0$ on $\{|z| \geq t^{-1/(m_P+1)}\}$, it is enough to argue the estimates on $|z| < t_i^{-1/(m_P+1)}$. We have the following:

$$R(h^0_{t_i,P}) = O\Big(t_i^{-j_P} t_i^{2/(m_P+1)}\Big) dz\, d\overline{z}, \quad \big[t_i\theta, (t_i\theta)^\dagger_{h^0_{t_i,P}}\big] = O\Big(t_i^{-j_P} t_i^{2/(m_P+1)}\Big) dz\, d\overline{z}$$

Both of them are dominated by

$$O\Big(t_i^{-j_P+2/(m_P+1)}\Big)|z|^{2-\epsilon_P}\big(|z|^{\epsilon_P-2} dz d\overline{z}\big) = O\Big(t_i^{-j_P+\epsilon/(m_P+1)}\Big)|z|^{\epsilon_P-2} dz\, d\overline{z}$$

Thus, we obtain the claim of the lemma. ∎

5.3.3 Global construction

We take a Kähler metric g_X of X. Let t_i be any sequence of positive numbers going to ∞. We set $\beta_i := t_i^{a_{E,\theta}}$. We shall construct a family of Hermitian metrics $h^0_{t_i}$ of $(E, \overline{\partial}_E, \theta)$ with the following property:

- There exists $p > 1$ such that the L^p-norms of $R(h^0_{t_i}) + \big[t_i\theta, (t_i\theta)^\dagger_{h^0_{t_i}}\big]$ with respect to g_X and $h^0_{t_i}$ are bounded.
- There exists $C > 0$ such that $C^{-1}h^0_{t_i} \leq h^0_{t_i,P} \leq Ch^0_{t_i}$ on the neighbourhood U_P for each $P \in Z(\omega)$, and that $C^{-1}h^0_{t_i} \leq \Psi^*_{\beta_i} h^{\lim}_{E,\theta} \leq Ch^0_{t_i}$ on $X \setminus \bigcup_{P\in Z(\omega)} U_P$.

Let $P \in Z(\omega)$. We take $0 < R_{2,P} < R_{1,P}$ such that $\{|z| \leq R_{1,P}\} \subset U_P$. We take a function $\rho_P : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ such that $\rho_P(s) = 1$ $(s \leq R_{2,P})$ and $\rho_P(s) = 0$ $(s \geq R_{1,P})$. On $\{R_{2,P} \leq |z| \leq R_{1,P}\} \subset U_P$, we define $h^0_{t_i}$ by the following conditions:

$$\log h^0_{t_i}(v_j, v_j) = \rho_P(|z|) \log h^0_{t_i,P}(v_j, v_j) + (1 - \rho_P(|z|)) \log h^{\lim}_{E,\theta}(\Psi_{\beta_i} v_j, \Psi_{\beta_i} v_j) \quad (j = 1, 2)$$

$$h^0_{t_i}(v_1, v_2) = \rho_P(|z|) h^0_{t_i,P}(v_1, v_2)$$

Note that $\log h^0_{t_i,P}(v_j, v_j) - \log h^{\lim}_{E,\theta}(\Psi_{\beta_i} v_j, \Psi_{\beta_i} v_j)$ are uniformly bounded on $\{R_{2,P} \leq |z| \leq R_{1,P}\}$. On $\{|z| \leq R_{2,P}\}$, we set $h^0_{t_i} := h^0_{t_i,P}$. On $X \setminus \bigcup_{P\in Z(\omega)}\{|z| \leq R_{1,P}\}$, we set $h^0_{t_i} := \Psi^*_{\beta_i} h^{\lim}_{E,\theta}$. Then, we can check that the family of the Hermitian metrics $h^0_{t_i}$ has the desired property by using the estimates in §5.3.2.

The following lemma is clear by the construction and Proposition 3.14, Proposition 3.20 and Proposition 3.22

Lemma 5.11 *The sequence of Hermitian metrics $\Psi^*_{\beta_i^{-1}} h^0_{t_i|X\setminus Z(\omega)}$ is convergent in the C^∞-sense on any compact subset in $X \setminus Z(\omega)$. For the limit $\widetilde{h}^0_\infty$, the decomposition $L_1 \oplus L_2$ is orthogonal. There exists $M_1 > 0$ such that $M_1^{-1} h^{\lim}_{E,\theta} \leq \widetilde{h}^0_\infty \leq M_1 h^{\lim}_{E,\theta}$. In particular, there exists $M_2 > 0$ with the following property.*

- *For any neighbourhood N of $Z(\omega)$, there exists $i_0(N)$ such that $M_2^{-1} h^{\lim}_{E,\theta} \leq \Psi^*_{\beta_i^{-1}} h^0_{t_i} \leq M_2 h^{\lim}_{E,\theta}$ on $X \setminus N$ for any $i \geq i_0(N)$.* ∎

Let ρ_i be the self-adjoint endomorphisms of $\big(E_{|X\setminus Z(\omega)}, h^{\lim}_{E,\theta}\big)$ determined by $\Psi^*_{\beta_i^{-1}} h^0_{t_i}(u, v) = h^{\lim}_{E,\theta}(\rho_i u, v)$ for any local sections u and v. We also have the following.

Lemma 5.12 *The sequence ρ_i are convergent in the C^∞-sense with respect to $h^{\lim}_{E,\theta}$ on any compact subset in $X \setminus Z(\omega)$. The limit ρ_∞ preserves the decomposition $L_1 \oplus L_2$. We have the boundedness of ρ_∞ and ρ_∞^{-1} with respect to $h^{\lim}_{E,\theta}$.* ∎

5.4 Proof of Theorem 5.1

We continue to use the notation in §5.3.

5.4.1 Boundedness of a modified sequence

Let $t_i \to \infty$ be any sequence. It is enough to prove that we can take a subsequence t'_i such that the sequence $\Psi^*_{\gamma(t'_i,Q)} h_{t'_i}$ converges to $h^{\lim}_{E,\theta}$.

Let Δ_X be the Laplacian with respect to the Kähler metric g_X of X. We construct the family of Hermitian metrics of $h^0_{t_i}$ on E as in §5.3. Let k_i be the self adjoint endomorphism of $(E, h^0_{t_i})$ determined by $h_{t_i} = h^0_{t_i} k_i$, i.e., $h_{t_i}(u, v) = h^0_{t_i}(k_i u, v)$ for local sections u and v. According to [19, Proposition 3.1], we have the following on X:

$$\Delta_X \operatorname{Tr}(k_i) \leq \Big|\Lambda_{g_X} \operatorname{Tr}\Big(k_i \cdot \big(R(h^0_{t_i}) + [t_i\theta, (t_i\theta)^\dagger_{h^0_{t_i}}]\big)\Big)\Big|$$

Let $p>1$ be as in §5.3.3, i.e., the L^p-norms of $R(h^0_{t_i})+\big[t_i\theta,(t_i\theta)^\dagger_{h^0_{t_i}}\big]$ with respect to g_X and $h^0_{t_i}$ are bounded. We take $q>1$ such that $p^{-1}+q^{-1}<1$. Set $r:=(p^{-1}+q^{-1})^{-1}$. Let $\nu_i:=\|k_i\|_{L^q,h^0_{t_i},g_X}$ be the L^q-norm of k_i with respect to $h^0_{t_i}$ and g_X. Set $s_i:=\nu_i^{-1}k_i$. We have

$$\Delta_X \operatorname{Tr}(s_i)\leq \Big|\Lambda_{g_X}\operatorname{Tr}\Big(s_i\cdot\big(R(h^0_{t_i})+[t_i\theta,(t_i\theta)^\dagger_{h^0_{t_i}}]\big)\Big)\Big| \tag{37}$$

The L^r-norms of the right hand side in (37) are bounded for i. So we have a constant $C_1>0$ and L^r_2-functions G_i such that

$$\Delta_X\big(\operatorname{Tr}(s_i)-G_i\big)\leq C_1,\quad \|G_i\|_{L^r_2}\leq C_1.$$

Hence, we have $C_2>0$ such that $\sup_X|s_i|_{h^0_{t_i}}\leq C_2$ holds for any i. Again, according to [19, Proposition 3.1], we have the following:

$$\Delta_X\operatorname{Tr}(s_i)=\sqrt{-1}\Lambda_X\operatorname{Tr}\Big(s_i\cdot\big(R(h^0_{t_i})+[t_i\theta,(t_i\theta)^\dagger_{h^0_{t_i}}]\big)\Big)-\big|s_i^{-1/2}(\overline{\partial}+t_i\theta)s_i\big|^2_{h^0_{t_i},g_X}$$

We have a constant $C_3>0$ such that $\big|(\overline{\partial}_E+t_i\theta)s_i\big|^2_{h^0_{t_i},g_X}\leq C_3\big|s_i^{-1/2}(\overline{\partial}_E+t_i\theta)s_i\big|^2_{h^0_{t_i},g_X}$. Hence, we obtain the following for a constant $C_4>0$:

$$\int_X\big|\overline{\partial}_E s_i\big|^2_{h^0_{t_i},g_X}+\int_X\big|t_i[\theta,s_i]\big|^2_{h^0_{t_i},g_X}\leq C_4$$

5.4.2 Weak convergence of a subsequence

We consider the sequences of metrics $\overline{h}_{t_i}:=\Psi^*_{\beta_i^{-1}}h_{t_i}$ and $\overline{h}^0_{t_i}:=\Psi^*_{\beta_i^{-1}}h^0_{t_i}$ on $E_{|X\setminus Z(\omega)}$. Let $\overline{k}_i$ be the self-adjoint endomorphism of $\big(E_{|X\setminus Z(\omega)},\overline{h}^0_{t_i}\big)$ determined by $\overline{h}_{t_i}=\overline{h}^0_{t_i}\overline{k}_i$. We have $\|\overline{k}_i\|_{L^q,\overline{h}^0_{t_i},g_X}=\nu_i$. Set $\overline{s}_i:=\nu_i^{-1}\overline{k}_i$. We have

$$\sup_X|\overline{s}_i|_{\overline{h}^0_{t_i}}\leq C_2. \tag{38}$$

We also have

$$\int_X\big|\overline{\partial}_E\overline{s}_i\big|^2_{\overline{h}^0_{t_i},g_X}+\int_X\big|t_i[\theta,\overline{s}_i]\big|^2_{\overline{h}^0_{t_i},g_X}\leq C_4 \tag{39}$$

By Lemma 5.11 and (39), we may assume that the sequence $\overline{s}_i$ is weakly convergent in L^2_1 on any compact subset in $X\setminus Z(\omega)$ with respect to g_X and $h^{\lim}_{E,\theta}$. Let $\overline{s}_\infty$ denote the weak limit.

Lemma 5.13 *$\overline{s}_\infty$ is bounded with respect to $h^{\lim}_{E,\theta}$. We have $\overline{s}_\infty\neq 0$.*

Proof By Lemma 5.12 and (38), there exists $M_3>0$ with the following property.

- For any neighbourhood N of $Z(\omega)$, there exists $i_3(N)$ such that $|\overline{s}_i|_{h^{\lim}_{E,\theta}}\leq M_3$ on $X\setminus N$ for any $i\geq i_3(N)$.

Hence, we have the boundedness $\big|\overline{s}_\infty\big|_{h^{\lim}_{E,\theta}}\leq M_3$.

Take a small $\delta>0$. By (38), we have a small neighbourhood N_1 of $Z(\omega)$ such that

$$\int_{X\setminus N_1}\big|\overline{s}_i\big|^q_{\overline{h}^0_{t_i}}\,\mathrm{dvol}_{g_X}\geq 1-\delta>0$$

Hence, we obtain $\int_{X\setminus N_1}\big|\overline{s}_\infty\big|^q_{h^{\lim}_{E,\theta}}\,\mathrm{dvol}_{g_X}>0$. In particular, we have $\overline{s}_\infty\neq 0$. ∎

Lemma 5.14 *We have $[\overline{s}_\infty,\theta]=0$. In particular, $\overline{s}_\infty$ preserves the decomposition $L_1\oplus L_2$.*

Proof We have $\lim_{i\to\infty}\int_X\big|[\overline{s}_i,\theta]\big|^2_{\overline{h}^0_{t_i},g_X}=0$ from (38), which implies the claim of the lemma. ∎

5.4.3 Modification

Let ρ_i be as in Lemma 5.12. We set $\overline{s}_i^1 := \rho_i \circ \overline{s}_i$. It is self-adjoint with respect to $h^{\lim}_{E,\theta}$, and we have $\nu_i^{-1}\Psi^*_{\beta_i^{-1}} h_{t_i} = h^{\lim}_{E,\theta} \cdot \overline{s}_i^1$. The sequence $\overline{s}_i^1$ is weakly convergent in L_1^2 on any compact subset in $X \setminus Z(\omega)$. Let $\overline{s}_\infty^1$ denote the weak limit. We have $\overline{s}_\infty^1 = \rho_\infty \circ \overline{s}_\infty$. We obtain the following from Lemma 5.12, Lemma 5.13 and Lemma 5.14.

Lemma 5.15 *$\overline{s}_\infty^1$ is bounded with respect to $h^{\lim}_{E,\theta}$. We have $\overline{s}_\infty^1 \neq 0$ and $[\overline{s}_\infty^1, \theta] = 0$.* ∎

Lemma 5.16 *We have $\overline{\partial}_E \overline{s}_\infty^1 = 0$.*

Proof Applying [19, Proposition 3.1] to $h^{\lim}_\infty$ and $\nu_i^{-1}\Psi^*_{\beta_i^{-1}} h_{t_i}$, we obtain the following on $X \setminus Z(\omega)$:

$$\left|(\overline{s}_i^1)^{-1/2}\overline{\partial}_E \overline{s}_i^1\right|^2_{h^{\lim}_{E,\theta},g_X} \leq \left|(\overline{s}_i^1)^{-1/2}(\overline{\partial}+t_i\theta)\overline{s}_i^1\right|^2_{h^{\lim}_{E,\theta},g_X} = -\sqrt{-1}\Lambda_{g_X}\overline{\partial}\partial \operatorname{Tr}(\overline{s}_i^1)$$

We take a C^∞-function $\mu : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ such that $\mu(s) = 1$ $(s \leq 1)$ and $\mu(s) = 0$ $(s \geq 2)$. For any sufficiently large real number T, let $\chi_T : X \longrightarrow \mathbb{R}_{\geq 0}$ be the C^∞-function such that (i) $\chi_T \equiv 1$ on $X \setminus \bigcup_{P\in Z(\omega)} U_P$, (ii) $\chi_T(z) = \mu\bigl(-T^{-1}\log|z|\bigr)$ on the coordinate neighbourhoods (U_P, z) for $P \in Z(\omega)$.

We have a constant $M_4 > 0$ with the following property.

- For any neighbourhood N of $Z(\omega)$, there exists $i_4(N)$ such that $|\overline{s}_i^1|_{h^{\lim}_{E,\theta}} \leq M_4$ on $X \setminus N$ for any $i \geq i_4(N)$.

Then, we have a constant C_{10} such that for any fixed $T > 0$ the following holds for a large i:

$$\int \chi_T\left|\overline{\partial}_E\overline{s}_i^1\right|^2_{h^{\lim}_{E,\theta},g_X}\operatorname{dvol}_{g_X} \leq C_{10}\int \chi_T\left|(\overline{s}_i^1)^{-1/2}\overline{\partial}_E\overline{s}_i^1\right|^2_{h^{\lim}_{E,\theta},g_X}\operatorname{dvol}_{g_X}$$
$$\leq C_{10}\left|\int \chi_T\overline{\partial}\cdot\partial\operatorname{Tr}(\overline{s}_i^1)\right| = C_{10}\left|\int\bigl(\overline{\partial}\partial\chi_T\bigr)\cdot\operatorname{Tr}(\overline{s}_i^1)\right| \quad (40)$$

By taking the limit for $i \to \infty$, we obtain

$$\int \chi_T\left|\overline{\partial}_E\overline{s}_\infty^1\right|^2_{h^{\lim}_{E,\theta},g_X}\operatorname{dvol}_{g_X} \leq C_{10}\left|\int\bigl(\overline{\partial}\partial\chi_T\bigr)\cdot\operatorname{Tr}(\overline{s}_\infty^1)\right|$$

Note that we have already known the boundedness of $\operatorname{Tr}(\overline{s}_\infty^1)$. We also have the uniform boundedness of $\overline{\partial}\partial\chi_T$ with respect to the Poincaré like metric on $X \setminus Z(\omega)$. Hence, by taking the limit for $T \to \infty$, we obtain

$$\int\left|\overline{\partial}_E\overline{s}_\infty^1\right|^2_{h^{\lim}_{E,\theta},g_X}\operatorname{dvol}_{g_X} \leq 0$$

Hence, we obtain $\overline{\partial}_E\overline{s}_\infty^1 = 0$. ∎

By Lemma 5.15 and Lemma 5.16, we have $\overline{s}_\infty^1 = \alpha_1 \operatorname{id}_{L_1} \oplus \alpha_2 \operatorname{id}_{L_2}$ for non-negative real numbers α_i $(i = 1, 2)$. We have $(\alpha_1, \alpha_2) \neq (0, 0)$.

5.4.4 End of the proof of Theorem 5.1

Suppose that $\alpha_1 \neq 0$. Let u_j be local frames of L_j $(j = 1, 2)$ on a relatively compact open subset in $X \setminus Z(\omega)$. We set $\gamma_i := \beta_i^{-1}\nu_i^{-1/2}\alpha_1^{-1/2}$. Then, we have the following:

$$\lim_{i\to\infty}\left|u_1\right|_{h_{t_i}}\gamma_i = \left|u_1\right|_{h^{\lim}_{L_1}} \quad (41)$$

By the asymptotic orthogonality in §2, we have the following on $X \setminus N$, where N is any neighbourhood of $Z(\omega)$:

$$\left|u_1 \wedge u_2\right|^2_{\det(E)} = |u_1|^2_{h_{t_i}}|u_2|^2_{h_{t_i}} - |h_{t_i}(u_1, u_2)|^2 = |u_1|^2_{h_{t_i}}|u_2|^2_{h_{t_i}}\cdot\Bigl(1 + O\bigl(\exp(-\epsilon t_i)\bigr)\Bigr)$$

We also have $\big|u_1\wedge u_2\big|_{\det(E)}=\big|u_1\big|_{h^{\lim}_{E,\theta}}\cdot\big|u_2\big|_{h^{\lim}_{E,\theta}}$. Hence, we obtain the following from (41):

$$\lim_{i\to\infty}\big|u_2\big|_{h_{t_i}}\gamma_i^{-1}=\big|u_2\big|_{h^{\lim}_{L_2}} \tag{42}$$

We obtain the following for $j=1,2$ from (41) and (42):

$$\lim_{i\to\infty}\Psi^*_{\gamma_i}h_{t_i}(u_j,u_j)=h^{\lim}_{E,\theta}(u_j,u_j).$$

We also have the following on $X\setminus N$, where N is any neighbourhood of $Z(\omega)$:

$$\Psi^*_{\gamma_i}h_{t_i}(u_1,u_2)=h_{t_i}(u_1,u_2)=O(\exp(-\epsilon t_i))\cdot|u_1|_{h_{t_i}}\cdot|u_2|_{h_{t_i}}=O(\exp(-\epsilon t_i))\cdot|u_1|_{\Psi^*_{\gamma_i}h_{t_i}}\cdot|u_2|_{\Psi^*_{\gamma_i}h_{t_i}}$$

So, we obtain $\lim_{i\to\infty}\Psi^*_{\gamma_i}h_{t_i}(u_1,u_2)=0=h^{\lim}_{E,\theta}(u_1,u_2)$. Hence, we have the convergence of $\Psi^*_{\gamma_i}h_{t_i}$ to $h^{\lim}_{E,\theta}$ in C^0 on any compact subset in $X\setminus Z(\omega)$.

By the construction of the sequence $\gamma(t_i,Q)$, we have $\lim_{i\to\infty}\gamma(t_i,Q)\cdot\gamma_i^{-1}=1$. Hence, we have the convergence of $\Psi^*_{\gamma(t_i,Q)}h_{t_i}$ to $h^{\lim}_{E,\theta}$ in C^0 on any compact subset in $X\setminus Z(\omega)$. We can obtain the convergence of the higher derivative from Corollary 2.15. Thus, we are done in the case $\alpha_1\neq 0$.

We can argue the case $\alpha_2\neq 0$ in a similar way. Thus, the proof of Theorem 5.1 is finished. ∎

5.5 Proof of Theorem 5.3

5.5.1 Preliminary

Let $(E,\overline{\partial}_E,\theta)$ be any stable Higgs bundle of rank 2 on X such that (i) $(E,\overline{\partial}_E,\theta)$ is generically regular semisimple, (ii) the spectral curve $\Sigma(E,\theta)$ is reducible, (iii) $\mathrm{tr}(\theta)=0$. We have holomorphic line bundles L_i $(i=1,2)$ with an inclusion $E\longrightarrow L_1\oplus L_2$ as in §4.2. We assume that $\deg(L_1)=\deg(L_2)$. Then, we have $a_{E,\theta}=0$ and $-\chi_P(a_{E,\theta})=\chi_P(a_{E,\theta})+\ell_P=\ell_P/2$.

We fix a Hermitian metric $h_{\det(E)}$ of $\det(E)$. For any $t>0$, we have Hermitian-Einstein metrics h_t for $(E,\overline{\partial}_E,t\theta)$ such that $\det(h_t)=h_{\det(E)}$. We have the metric $h^{\lim}_{E,\theta}$ as in §5.1.1.

We take any sequence $t_i\to\infty$. Let $k_i^{(2)}$ be the self-adjoint endomorphisms of $(E_{|X\setminus Z(\omega)},h^{\lim}_{E,\theta})$ determined by $h_{t_i}=h^{\lim}_{E,\theta}\cdot k_i^{(2)}$.

Lemma 5.17 *After going to a subsequence $\{i(p)\}\subset\{i\}$ there exists a sequence of positive numbers $\nu_{i(p)}$ such that the sequence $\nu_{i(p)}^{-1}k^{(2)}_{i(p)}$ weakly converges to a morphism $\alpha_1\cdot\mathrm{id}_{L_1}\oplus\alpha_2\cdot\mathrm{id}_{L_2}$ in L_1^2 locally on $X\setminus Z(\omega)$ for non-negative real numbers α_j $(j=1,2)$ with $(\alpha_1,\alpha_2)\neq(0,0)$.*

Proof Let us consider the case where $\deg(E)$ is even. We take a holomorphic line bundle L_0 with an isomorphism $L_0\otimes L_0\simeq\det(E)$. We have a Hermitian metric h_{L_0} such that $h_{L_0}\otimes h_{L_0}=h_{\det(E)}$. We have the harmonic metrics $h_{E\otimes L_0^{-1},t_i}$ for $(E\otimes L_0^{-1},\overline{\partial}_{E\otimes L_0^{-1}},t_i\theta)$. We have $h_{E\otimes L_0^{-1},t_i}=h_{t_i}\otimes h_{L_0}^{-1}$ and $h^{\lim}_{E\otimes L_0^{-1},\theta}=h^{\lim}_{E,\theta}\otimes h^{-1}_{L_0}$. Hence, $k_i^{(2)}$ is the self-adjoint endomorphisms of $((E\otimes L_0^{-1})_{|X\setminus Z(\omega)},h^{\lim}_{E\otimes L_0^{-1},\theta})$ determined by $h_{E\otimes L_0^{-1},t_i}=h^{\lim}_{E\otimes L_0^{-1},\theta}\cdot k_i^{(2)}$. Note that $a_{E\otimes L_0^{-1},\theta}=0$ and $\beta_i=t_i^{a_{E\otimes L_0^{-1},\theta}}=1$ in §5.3.3 and §5.4. Then, the claim of the lemma for $k_i^{(2)}$ has been already observed in §5.4.2–§5.4.3. We can easily reduce the case where $\deg(E)$ is odd to the case where $\deg(E)$ is even, by taking the pull back by a covering $\widetilde{X}\longrightarrow X$ of degree 2. ∎

5.5.2 Proof of Theorem 5.3

We take any sequence $t_i\to\infty$. It is enough to prove that we can take a subsequence t_i' such that $h_{t_i'}$ converges to $h^{\lim}_{E,\theta}$ on any compact subsets in $X\setminus Z(\omega)$.

Let $k_i^{(2)}$ be the self-adjoint endomorphisms of $(E_{|X\setminus Z(\omega)},h^{\lim}_{E,\theta})$ determined by $h_{t_i}=h^{\lim}_{E,\theta}\cdot k_i^{(2)}$, as in §5.5.1. Because $\rho^*h^{\lim}_{E,\theta}=h^{\lim}_{E,\theta}$ and $\rho^*h_{t_i}=h_{t_i}$, we have $\rho^*k_i^{(2)}=k_i^{(2)}$. As remarked in Lemma 5.17, by going to a subsequence, we may assume to have a sequence of positive numbers ν_i such that the sequence $\nu_i^{-1}k_i^{(2)}$ is weakly convergent to $\alpha_1 \,\mathrm{id}_{L_1} \oplus \alpha_2 \,\mathrm{id}_{L_2}$ in L_1^2 locally on $X \setminus Z(\omega)$, where α_i are non-negative numbers such that $(\alpha_1, \alpha_2) \neq (0,0)$. Because $\rho^*\bigl(\nu_i^{-1} k_i^{(2)}\bigr) = \nu_i^{-1} k_i^{(2)}$, $\rho^*(L_1) = L_2$ and $\rho^*(L_2) = L_1$, we have $\alpha_1 = \alpha_2$. In particular, $\alpha_1 \cdot \alpha_2 \neq 0$. Because $\det(k_i^{(2)}) = 1$, the sequence ν_i^{-2} converges to $\alpha_1 \cdot \alpha_2$. In particular, the sequences ν_i and ν_i^{-1} are bounded.

We take a subsequence $h_{t_{i(p)}}$ for which the sequence $k_{i(p)}^{(2)}$ is convergent to $\beta \,\mathrm{id}_{L_1} \oplus \beta \,\mathrm{id}_{L_2}$ for a positive number β. But, we have $\det(k_{i(p)}^{(2)}) = 1$ and hence $\beta = 1$, i.e., $k_{i(p)}^{(2)}$ is convergent to the identity id, indeed. Hence, we can conclude that the sequence $k_i^{(2)}$ is convergent to id, and the proof of Theorem 5.3 is finished. ∎

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