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<td>Author(s)</td>
<td>Mochizuki, Takuro</td>
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<tr>
<td>Citation</td>
<td>Journal of Topology (2016), 9(4): 1021-1073</td>
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<td>Issue Date</td>
<td>2016-12</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/227916">http://hdl.handle.net/227916</a></td>
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<td>This is the peer reviewed version of the following article: [Mochizuki, T. (2016), Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces. Journal of Topology, 9: 1021–1073], which has been published in final form at <a href="https://doi.org/10.1112/jtopol/jtw018">https://doi.org/10.1112/jtopol/jtw018</a>. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Self-Archiving.; The full-text file will be made open to the public on 1 SEP 2017 in accordance with publisher's 'Terms and Conditions for Self-Archiving'.; This is not the published version. Please cite only the published version. この論文は出版社版ではありません。引用の際には出版社版をご確認ご利用ください。</td>
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<td>Kyoto University</td>
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Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces

Takuro Mochizuki

Abstract

Let \((E, \bar{\nabla}_E, \theta)\) be a stable Higgs bundle of degree 0 on a compact connected Riemann surface. Once we fix a flat metric \(h_{\text{det}(E)}\) on the determinant of \(E\), we have the harmonic metrics \(h_t (t > 0)\) for the stable Higgs bundles \((E, \bar{\nabla}_E, t\theta)\) such that \(\det(h_t) = h_{\text{det}(E)}\). We study the behaviour of \(h_t\) when \(t\) goes to \(\infty\).

First, we show that the Hitchin equation is asymptotically decoupled under the assumption that the Higgs field is generically regular semisimple. We apply it to the study of the so called Hitchin WKB-problem.

Second, we study the convergence of the sequence \((E, \bar{\nabla}_E, \theta, h_t)\) in the case \(\text{rank} E = 2\). We introduce a rule to determine the parabolic weights of a “limiting configuration”, and we show the convergence of the sequence to the limiting configuration in an appropriate sense. The results can be appropriately generalized in the context of Higgs bundles with a Hermitian-Einstein metric on curves.

Keywords: harmonic bundle, asymptotic behaviour, asymptotic decoupling, Hitchin WKB-problem, limiting configuration, Hermitian-Einstein metric

MSC: 14H60, 53C07

1 Introduction

Let \(X\) be a compact connected Riemann surface. Let \((E, \bar{\nabla}_E, \theta)\) be a Higgs bundle of rank \(r\) on \(X\) with \(\text{deg}(E) = 0\). Let \(h\) be a Hermitian metric of \(E\). We have the Chern connection \(\nabla_h\) associated to \((E, \bar{\nabla}_E, h)\). Let \(R(h)\) denote the curvature of \(\nabla_h\). Let \(\theta^h\) denote the adjoint of \(\theta\) with respect to \(h\). Recall the celebrated Hitchin equation [6]:

\[ R(h) + [\theta, \theta^h] = 0 \]

If the Hitchin equation is satisfied, \(h\) is called a harmonic metric of \((E, \bar{\nabla}_E, \theta)\), and \((E, \bar{\nabla}_E, \theta, h)\) is called a harmonic bundle.

Remark 1.1 The Hitchin equation makes sense for a Higgs bundle with a Hermitian metric on any complex curve. In this introduction, \(X\) is assumed to be compact to simplify the explanation.

If \(\text{deg}(E)\) is not 0, we take a Hermitian metric \(h_{\text{det}(E)}\) of the determinant line bundle \(\text{det}(E)\), and we consider the Hermitian-Einstein condition \(R(h)^+ + [\theta, \theta^h] = 0\), where \(R(h)^+\) is the trace-free part of \(R(h)\). The condition is also called the Hitchin equation. In this introduction, we assume \(\text{deg}(E) = 0\) for simplicity.

If \(\text{rank} E = 1\), we always have \([\theta, \theta^h] = 0\). Hence, the Hitchin equation is reduced to \(R(h) = 0\), i.e., the metric \(h\) is flat with respect to the Chern connection. By the classical harmonic theory, we can always find such a harmonic metric in the rank one case, which is unique up to the multiplication of positive constants.

In the higher rank case, we fix a harmonic metric \(h_{\text{det} E}\) of \((\text{det}(E), \bar{\nabla}_{\text{det} E}, \text{tr} \theta)\). According to Hitchin [6] and Simpson [19], if the Higgs bundle \((E, \bar{\nabla}_E, \theta)\) is stable, we have a unique harmonic metric \(h\) of \((E, \bar{\nabla}_E, \theta)\) such that \(\det(h) = h_{\text{det} E}\).

For any non-zero complex number \(t\), the Higgs bundle \((E, \bar{\nabla}_E, t\theta)\) is also stable. We obtain a family of harmonic metrics \(h_t (t \in \mathbb{C}^*)\) satisfying \(\det(h_t) = h_{\text{det} E}\). It is easy to observe that \(h_{t_1} = h_{t_2}\) if \(|t_1| = |t_2|\). So, it is enough to consider the case where \(t\) are positive numbers.

Simpson studied the behaviour of \((E, \bar{\nabla}_E, \theta, h_t)\) when \(t \to 0\). (See [19, 20, 21, 22], for example.) He discovered the convergence to a polarized variation of Hodge structure, and he gave various applications of this interesting phenomena.
To state the claim more precisely, we introduce a condition for Higgs bundles. Let $(E, \nabla, h)$ be a Higgs bundle.

1.1 Generically regular semisimple Higgs bundles

Let $(E, \nabla, h)$ be a Higgs bundle. The additional condition $[\theta, \theta^\dagger] = 0$ implies that, at least locally, we have a flat decomposition $(E, \nabla, h) = \bigoplus_{i=1}^r (E_i, \nabla_i, h_i)$ into flat line bundles such that $\theta = \bigoplus \phi_i \cdot \text{id}_{E_i}$, where $\phi_i$ are holomorphic one forms. In [11], this kind of simplification seems to be called “decoupling” of the Hitchin equation.

Our first purpose is to show that if $t$ is sufficiently large, the Hitchin equation for $(E, \nabla_E, t\theta)$ is almost decoupled in some sense.

1.1.1 Generically regular semisimple Higgs bundles

To state the claim more precisely, we introduce a condition for Higgs bundles. Let $(E, \nabla_E, \theta)$ be a Higgs bundle on $X$. We have the associated coherent sheaf $M_{E, \theta}$ on the cotangent bundle $T^*X$. The support $\Sigma(E, \theta)$ is called the spectral curve of the Higgs bundle. The number of the points of $\rho(P) = T^*_P X \cap \Sigma(E, \theta)$ are finite for any $P \in X$. We say that the Higgs bundle $(E, \nabla_E, \theta)$ is generically regular semisimple if the following holds:

- We have a discrete subset $D \subset X$ such that $\rho(P) = \text{rank } E$ for any $P \in X \setminus D$.

Let $D(E, \theta)$ denote the set of the points $P \in X$ such that $\rho(P) < \text{rank } E$, which we call the discriminant of the Higgs bundle.

Suppose that $(E, \nabla_E, \theta, h)$ is generically regular semisimple. Then, the following holds for any point $P \in X \setminus D(E, \theta)$ with a small neighbourhood $U_P$ of $P$:

- We have holomorphic 1-forms $\phi_{P,1}, \ldots, \phi_{P,r}$ on $U_P$ and a decomposition of the Higgs bundle

$$(E, \nabla_E, \theta)|_{U_P} = \bigoplus_{i=1}^r (E_{P,i}, \nabla_{E_{P,i}}, \phi_{P,i} \cdot \text{id}_{E_{P,i}}),$$

where we assume that rank $E_i = 1$ ($i = 1, \ldots, r$), and that $\phi_{P,i} - \phi_{P,j}$ ($i \neq j$) have no zero.

1.1.2 Asymptotic decoupling

Let $(E, \nabla_E, \theta)$ be a stable Higgs bundle of degree 0 on $X$. Suppose that it is generically regular semisimple. We take any Kähler metric $g_X$ of $X$. For any local section $s$ of $\text{End}(E) \otimes \Omega^{0,q}$, we have the function $|s|_{h_t, g_X} : X \rightarrow \mathbb{R}$, where $|s|_{h_t, g_X}(P)$ ($P \in X$) are the norm of $s|_P$ with respect to $h_t$ and $g_X$. We have the asymptotic decoupling in the following sense.
Theorem 1.2 (Theorem 2.9) We take any neighbourhood $N$ of the discriminant $D(E, \theta)$. Then, there exist positive constants $C_0$ and $\epsilon_0$ such that the following holds on $X \setminus N$:

$$|R(h_t)|_{h_t,g_X} = |t|^2 |[\theta, \theta_{h_t}]|_{h_t,g_X} \leq C_0 \exp(-\epsilon_0 t).$$

The constants $C_0$ and $\epsilon_0$ may depend only on $(X, g_X)$, $N$ and $\Sigma(E, \theta)$.

We also have the family of flat connections $\mathbb{D}_{h_t}^1 := \nabla_{h_t} + t\theta + (\theta h_t)^1_{h_t}$, which are correctly associated to the harmonic bundles $(E, \overline{\nabla}_E, t\theta, h_t)$. Let us describe that we have nice approximations of these connections.

Let $P$ be any point of $X \setminus D(E, \theta)$. We take a small neighbourhood $U_P$ of $P$ in $X \setminus D(E, \theta)$. We have a decomposition of the Higgs bundle $(E, \overline{\nabla}_E, \theta)_{U_P} = \bigoplus_{i=1}^r (E_{P,i}, \overline{\nabla}_{E_{P,i}}, \theta_{P,i})$, where rank $E_{P,i} = 1$. Let $h_{t,E_{P,i}}$ be the restriction of $h_t$ to $E_{P,i}$. By taking the direct sum, we obtain a Hermitian metric $h_{t,P,0} := \bigoplus_{i=1}^r h_{t,E_{P,i}}$ of $E_{U_P}$. Note that Theorem 1.2 implies the following.

Lemma 1.3 We have positive constants $C'_{P,0}$ and $\epsilon'_{P,0}$ such that the following estimate holds for any local sections $u_i$ and $u_j$ ($i \neq j$) of $E_{P,i}$ and $E_{P,j}$:

$$|h_t(u_i, u_j)| \leq C'_{P,0} \exp(-\epsilon'_{P,0} t)|u_i|_{h_t}|u_j|_{h_t}.$$

In particular, we have a constant $K_P > 1$ such that $K_P^{-1} h_{t,P,0} \leq h_{t|U_P} \leq K_P h_{t,P,0}$ for any $t > 1$.

By varying $P \in X \setminus D(E, \theta)$ and by gluing $h_{t,P,0}$, we obtain a family of Hermitian metrics $h_{t,0}$ ($t > 1$) on $E|_{X \setminus D(E, \theta)}$. We have the Chern connection $\nabla_{t,0}$ of $(E|_{X \setminus D(E, \theta)}, h_{t,0})$. Let $(\theta h_{t,0})_{t,0}$ denote the adjoint of $t\theta|_{X \setminus D(E, \theta)}$ with respect to $h_{t,0}$. We set $\mathbb{D}_{h_{t,0}}^1 := \nabla_{t,0} + t\theta + (\theta h_{t,0})_{t,0}$.

Theorem 1.4 (Proposition 2.12, Corollary 2.13) Take any neighbourhood $N$ of $D(E, \theta)$. Then, we have a constant $K > 1$ such that $K^{-1} h_{t,0}|_{X \setminus N} \leq h_{t|X \setminus N} \leq K h_{t,0}|_{X \setminus N}$ for any $t > 1$. We also have positive constants $C_1$ and $\epsilon_1$ such that the following holds on $X \setminus N$:

$$|R(h_{t,0})|_{h_{t,0},g_X} \leq C_1 \exp(-\epsilon_1 t), \quad \left|\mathbb{D}_{h_{t,0}}^1 - \mathbb{D}_{h_{t,0}}^1\right|_{h_{t,0},g_X} \leq C_1 \exp(-\epsilon_1 t).$$

The constants $C_1$ and $\epsilon_1$ may depend only on $(X, g_X)$, $N$ and $\Sigma(E, \theta)$.

We note that we can obtain these estimates in an elementary way which is standard in the study of the asymptotic behaviour of harmonic bundles around the singularity, pioneered by Simpson [20], and pursued further by the author [13, 16]. We also emphasize that we can obtain these kinds of estimates without the assumption that harmonic bundles are given on a compact Riemann surface. Indeed, we shall study harmonic bundles given on discs in §2, which is clearly enough for the above estimates on relatively compact regions.

Finally, we remark that the estimates can be generalized in the case $\text{deg}(E) \neq 0$, i.e., in the context of Higgs bundles with Hermitian-Einstein metrics on curves. (See §2.5.)

1.1.3 Hitchin WKB-problem

Together with a rather standard argument of singular perturbations, we can apply Theorem 1.4 to the Hitchin WKB-problem in [7].

We recall a notation in [7]. Let $V$ be an $r$-dimensional complex vector space. For Hermitian metrics $h_1, h_2$, we can take a base $e_1, \ldots, e_r$ of $V$ which is orthogonal with respect to both $h_i$ ($i = 1, 2$). We have the real numbers $\kappa_j$ $(j = 1, \ldots, r)$ determined by $\kappa_j := \log|e_j|_{h_1} - \log|e_j|_{h_2}$. We impose $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_r$. Then, we set

$$\bar{d}(h_1, h_2) := (\kappa_1, \ldots, \kappa_r) \in \mathbb{R}^r.$$

Let us return to the study on the family of harmonic bundles $(E, \overline{\nabla}_E, t\theta, h_t)$ ($t > 0$) for a stable Higgs bundle $(E, \overline{\nabla}_E, \theta)$ of rank $r$ with $\text{deg}(E) = 0$ on a compact Riemann surface $X$, which is generically regular semisimple. We take a universal covering $\pi : Y \rightarrow X \setminus D(E, \theta)$. Then, we have the decomposition of the Higgs bundle $\pi^*(E, \overline{\nabla}_E, \theta) = \bigoplus_{i=1}^r (E_{i}, \overline{\nabla}_{E_{i}}, \phi_i \text{id}_{E_{i}})$, where $\phi_i$ are holomorphic 1-forms. We have rank $E_{i} = 1$, and $\phi_i - \phi_j$ ($i \neq j$) have no zeroes.
Let $[0, 1]$ denote the closed interval $\{0 \leq s \leq 1\}$. Let $\gamma : [0, 1] \rightarrow Y$ be a $C^\infty$-path. We have the expressions $\gamma^*(\phi_i) = a_i \, ds$ where $a_i$ are $C^\infty$-functions on $[0, 1]$. The path $\gamma$ is called non-critical if $\text{Re} a_i(s) \neq \text{Re} a_j(s)$ $(i \neq j)$ for any $s$. In that case, we may assume $\text{Re} a_i(s) < \text{Re} a_j(s)$ $(i < j)$. We set

$$a_i := - \int_0^1 \text{Re}(a_i) \, ds.$$

We have the families of Hermitian metrics $h_{t, \gamma(k)}$ $(t > 0)$ on the fibers $E_{\gamma(k)}$ $(k = 0, 1)$, induced by the harmonic metrics $h_t$. Let $\Pi_{\gamma,t} : E_{\gamma(0)} \rightarrow E_{\gamma(1)}$ denote the parallel transport of the flat connection $\nabla_h^t$ along $\gamma$. Let $\Pi_{\gamma,t}^* h_{t, \gamma(1)}$ denote the family of Hermitian metrics on $E_{\gamma(0)}$ induced by $h_{t, \gamma(1)}$ and $\Pi_{\gamma,t}$.

**Theorem 1.5 (Theorem 2.17)** If $\gamma$ is non-critical, there exist positive constants $C_2$ and $\epsilon_2$ such that the following holds:

$$\left| \frac{1}{t} d(h_{t, \gamma(0)}, \Pi_{\gamma,t}^* h_{t, \gamma(1)}) - (2\alpha_1, \ldots, 2\alpha_r) \right| \leq C_2 \exp(-\epsilon_2 t)$$

The constants $C_2$ and $\epsilon_2$ may depend only on $X$, $\phi_1, \ldots, \phi_r$ and $\gamma$.

The theorem was conjectured in [7], and a different version of the problem called the Riemann-Hilbert WKB-problem was studied in detail. Some cases were verified in [3]. (See [3, 7] for the precise statements.) We emphasize that we can obtain this kind of estimate for more general families of harmonic bundles given on complex curves which are not necessarily compact.

### 1.2 Convergence to the limiting configuration

#### 1.2.1 Limiting configuration

Let $(E, \mathcal{D}_E, \theta)$ be a stable Higgs bundle of rank $r$ with $\text{deg}(E) = 0$ on a compact Riemann surface $X$, which is generically regular semisimple. We have the family of harmonic bundles $(E, \mathcal{D}_E, \theta, h_t)$ $(t > 0)$. Let $\nabla_t$ denote the Chern connection of $(E, \mathcal{D}_E, h_t)$. By Theorem 1.2, we can take a sub-sequence $t_i \rightarrow \infty$ such that the sequence of vector bundles with a Hermitian metric and a unitary connection $(E, \nabla_{t_i}, h_{t_i})|_{X \setminus D(E, \theta)}$ converges in some sense to a vector bundle with a Hermitian metric and a unitary flat connection $(E_\infty, \nabla_\infty, h_\infty)$ on $X \setminus D(E, \theta)$. It is interesting to determine the flat connection $\nabla_\infty$. Following [11, 12], such a limit (or the associated parabolic bundle) is called a limiting configuration of the Higgs bundle $(E, \mathcal{D}_E, \theta)$ in this paper. Note that it is not clear, in general, whether $\nabla_\infty$ is independent of the choice of a sub-sequence. In this paper, we shall study the case rank $E = 2$ by assuming that $\theta$ is generically regular semisimple, but without assuming that the zeroes of $\det(\theta) - (tr \theta)^2/4$ are simple. See §1.2.6 for a remark on the case where the Higgs bundle is not generically regular semisimple.

**Remark 1.6** The higher rank case has not yet been studied in general. Under some assumptions on the spectral curves, it looks possible to generalize the method in [11, 12] and our method in this paper. See also [3] for some Toda-like cases.

#### 1.2.2 The case where the spectral curve is smooth irreducible and simply ramified over $X$

In [11, 12], Mazzeo, Swoboda, Weiss, and Witt studied the case under the assumption that rank $E = 2$ and that the zeroes of $\det(\theta) - (tr \theta)^2/4$ are simple, i.e., the orders of the zeroes of the quadratic differential $\det(\theta) - (tr \theta)^2/4$ are at most 1, inspired by the work of Gaiotto, Moore and Neitzke [5]. (We remark that in [11, 12], the condition $\text{deg}(E) = 0$ is not imposed.) Let us describe the limiting configuration in this case. It is enough to consider the case $\text{tr}(\theta) = 0$. Under the assumption, the spectral curve $\tilde{X} := \Sigma(E, \theta)$ is smooth and connected. The natural projection $\pi : \tilde{X} \rightarrow X$ is the ramified covering of degree 2. The ramification index is at most 2. The discriminant $D(E, \theta)$ is exactly the set of the points on which $\pi$ is ramified. Let $\iota : \tilde{X} \rightarrow T^* \tilde{X}$ denote the inclusion. We have the line bundle $\tilde{L}$ on $\tilde{X}$ such that $M_{E, \theta} \simeq \iota_* \tilde{L}$. We have $E \simeq \pi_* \tilde{L}$.

We have the unique non-trivial involution $\rho : \tilde{X} \rightarrow \tilde{X}$ over $X$, i.e., $\pi \circ \rho = \pi$, $\rho \circ \rho = \text{id}_\tilde{X}$ and $\rho \neq \text{id}_\tilde{X}$. We have the line bundle $\rho^* \tilde{L}$ on $\tilde{X}$. We have a natural inclusion of $\mathcal{O}_{\tilde{X}}$-modules $\rho^* E \rightarrow \tilde{L} \oplus \rho^* \tilde{L}$, and the cokernel is isomorphic to the structure sheaf of $\tilde{D}(E, \theta) := \pi^{-1} D(E, \theta)$. We have $\tilde{L} \oplus \rho^* \tilde{L} \simeq \rho^* \det(E) \otimes \mathcal{O}_{\tilde{X}}(\tilde{D}(E, \theta))$.

Because $\text{deg}(\tilde{L}) = |\text{deg}(\tilde{D}(E, \theta))|/2 = 0$, we have a Hermitian metric $h_{\tilde{L}}^{\text{lim}}$ of $\tilde{L}|_{\tilde{X} \setminus \tilde{D}(E, \theta)}$ with the following property.
The Chern connection of $(\tilde{L}|_{X \setminus \mathcal{D}(E, \theta)}, h_{L}^{\text{lim}})$ is flat.

Let $\tilde{P} \in \tilde{D}(E, \theta)$. Let $e_{\tilde{P}}$ be a local frame of $\tilde{L}$ around $\tilde{P}$. Let $(\tilde{U}_{\tilde{P}}, w)$ be a holomorphic coordinate system around $\tilde{P}$ with $w(\tilde{P}) = 0$. Then, $\left| \log(|w|h_{L}^{\text{lim}}(e_{\tilde{P}}, e_{\tilde{P}})) \right|$ is bounded on $\tilde{U}_{\tilde{P}} \setminus \tilde{P}$.

We have $h_{L}^{\text{lim}} \otimes \rho^{*}h_{L}^{\text{lim}} = \pi^{*}h_{\det(E)}$ on $\tilde{X} \setminus \tilde{D}(E, \theta)$.

Such $h_{L}^{\text{lim}}$ is uniquely determined. We have the induced Hermitian metric $h_{E, \theta}^{\text{lim}}$ of $E|_{X \setminus \mathcal{D}(E, \theta)} = \pi_{*}(\tilde{L}|_{X \setminus \mathcal{D}(E, \theta)})$. We have the Chern connection $\nabla_{E, \theta}^{\text{lim}}$ of $(E|_{X \setminus \mathcal{D}(E, \theta)}, h_{E, \theta}^{\text{lim}})$. This is the limiting configuration of $(E, \mathcal{D}(E, \theta))$ in this case. Interestingly, in [11, 12], it is proved that the family $(E, \mathcal{D}(E, \theta))_{|X \setminus \mathcal{D}(E, \theta)}$ ($t > 0$) is convergent to $(E|_{X \setminus \mathcal{D}(E, \theta)}$, $\mathcal{D}_{E}, h_{E, \theta}^{\text{lim}}$) in an appropriate sense.

### 1.2.3 Limiting configuration in the general case

It is natural to study the case where $\theta$ is generically regular semisimple, but the zeroes of $\det(\theta) - (\text{tr} \theta)^{2}/4$ are not necessarily simple. We may assume $\text{tr} \theta = 0$. It is enough to consider the case where the spectral curve of the Higgs bundle $(E, \mathcal{D}(E, \theta))$ is reducible to the two components, i.e., we have holomorphic 1-forms $\omega \neq 0$ such that $\Sigma(E, \theta) = \text{Im}(\omega) \cup \text{Im}(-\omega)$. Indeed, if the spectral curve is irreducible, we have only to consider the pull back of the Higgs bundle by a ramified covering map of degree 2 given by the normalization of the spectral curve. (See §1.2.5. See §4.3.2 and §5.1.3 for more details.) Let $Z(\omega)$ denote the zero set of $\omega$, which is equal to the discriminant $D(E, \theta)$.

We have two line bundles $L_{i}$ ($i = 1, 2$) with an inclusion $\iota_{E}: E \to L_{1} \oplus L_{2}$ with the following property:

- Set $\theta_{L_{1} \oplus L_{2}} := \omega \text{id}_{L_{1}} \oplus (-\omega) \text{id}_{L_{2}}$. Then, we have $\theta_{L_{1} \oplus L_{2}} \circ \iota_{E} = \iota_{E} \circ \theta$.

- The restriction $\iota_{E}|_{X \setminus Z(\omega)}$ is an isomorphism.

- The induced morphisms $E \to L_{i}$ are surjective.

Set $d_{i} := \deg(L_{i})$. We may assume that $d_{1} \leq d_{2}$. For each $P \in Z(\omega)$, let $m_{P}$ denote the order of the zero of $\omega$ at $P$. Namely, for a holomorphic coordinate system $(U_{P}, z)$ around $P$ with $z(P) = 0$, we have $\omega|_{U_{P}} = g_{P} \cdot z^{m_{P}}dz$ with $g_{E}(P) \neq 0$. The support of the cokernel $(L_{1} \oplus L_{2})/E$ is contained in $Z(\omega)$. For each $P \in Z(\omega)$, let $\ell_{P}$ denote the length of the stalk of $(L_{1} \oplus L_{2})/E$ at $P$. Note that we have $L_{1} \otimes L_{2} = \det(E) \otimes \mathcal{O}(\sum_{P \in Z(\omega)} \ell_{P}P)$.

In particular, we have

$$d_{1} + d_{2} - \sum_{P \in Z(\omega)} \ell_{P} = \deg(E) = 0.$$ 

Let $L_{i}'$ denote the kernel of $E \to L_{2}$. Similarly, let $L_{2}'$ denote the kernel of $E \to L_{1}$. Proper subbundles of $E$ preserved by $\theta$ are only $L_{1}'$ and $L_{2}'$. Hence, the stability condition for $(E, \theta)$ is equivalent to $\deg(L_{i}') < 0$ ($i = 1, 2$). It is equivalent to $d_{i} = \deg(L_{i}) > 0$ ($i = 1, 2$).

To give the limiting configuration of $(E, \mathcal{D}(E, \theta))$, we would like to give “parabolic weights” of $L_{i}$ ($i = 1, 2$) at $Z(\omega)$. In §1.2.2, all the parabolic weights are 1/2. In the general case, it turns out that the parabolic weights are given by the following rule.

For each $P \in Z(\omega)$, we consider the piecewise linear function $\chi_{P}: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\leq 0}$ given by

$$\chi_{P}(a) := \begin{cases} (m_{P} + 1)a - \ell_{P}/2 & (a \leq \ell_{P}/2(m_{P} + 1)) \\ 0 & (a > \ell_{P}/2(m_{P} + 1)) \end{cases}$$

We set $\chi_{E, \theta}(a) := \sum_{P \in Z(\omega)} \chi_{P}(a)$. Then, we have a unique $0 \leq a_{E, \theta} < \max_{P \in Z(\omega)} \{\ell_{P}/2(m_{P} + 1)\}$ such that $d_{1} + \chi_{E, \theta}(a_{E, \theta}) = 0$.

We determine the parabolic weights of $L_{1}$ at $P \in Z(\omega)$ as $-\chi_{P}(a_{E, \theta})$, and the parabolic weights of $L_{2}$ at $P \in Z(\omega)$ as $\chi_{P}(a_{E, \theta}) + \ell_{P}$. We have associated singular Hermitian metrics on $L_{i}$, i.e., we consider Hermitian metrics $h_{L_{i}}^{\text{lim}}$ ($i = 1, 2$) on $L_{i}|_{X \setminus Z(\omega)}$ satisfying the following conditions.

- The Chern connections of $(L_{i}|_{X \setminus Z(\omega)}, h_{L_{i}}^{\text{lim}})$ are flat.
• Let $P \in Z(\omega)$. Let $v_{P,i}$ be local frames of $L_i$ around $P$. Let $(U_P, z)$ be a holomorphic coordinate neighbourhood around $P$ with $z(P) = 0$. Then, the functions

$$\log(|z|^{-2\rho_\alpha(E,h_i)h_i^\lim_L(v_{P,1},v_{P,1})}), \quad \log(|z|^{2\rho_\alpha(E,h_i)+2\rho_{P}h_i^\lim_L(v_{P,2},v_{P,2})})$$

are bounded on $U_P \setminus P$.

• We have $h_i^\lim_L \otimes h_i^\lim_L = h_{\det(E)}$.

We obtain the Hermitian metric $h_i^\lim_E = h_i^\lim_L \oplus h_i^\lim_L$ of $E|_{X\setminus Z(\omega)}$, and the Chern connection $\nabla_i^\lim_E$. 

Remark 1.7 We still have the ambiguity for $h_i^\lim_L$. Namely, if $h_i^\lim_L (i = 1, 2)$ satisfy the above conditions, $\alpha h_i^\lim_L$ and $\alpha^{-1} h_i^\lim_L$ also satisfy the above conditions for any $\alpha > 0$. In §1.2.2, we have an additional condition $\rho^* h_i^\lim_L = h_i^\lim_L$ under the natural isomorphism $\rho^* L_1 \simeq L_2$, with which the metrics are uniquely determined. But, in general, it seems that we do not have such an extra condition.

The metric $h_i^\lim_E$ is also characterized as a harmonic metric of the polystable Higgs bundle $(E, \tilde{\mathcal{J}}_E, \theta)|_{X\setminus Z(\omega)}$ adapted to the parabolic structure given above.

Although we have the ambiguity of the metrics, the connection $\nabla_i^\lim_E$ is uniquely determined. 

Remark 1.8 Let us consider the case where $m_P = \ell_P = 1$ for any $P \in Z(\omega)$, and $d_1 = d_2 = |Z(\omega)|/2$. Then, we have $a_{E,\theta} = 0$, and hence $-\chi_P(a_{E,\theta}) = \chi_P(a_{E,\theta}) + \ell_P = 1/2$. These are the parabolic weights appeared in §1.2.2.

Remark 1.9 It might be instructive to mention that if we have $\deg(L_1) = \deg(L_2)$ then we have $a_{E,\theta} = 0$, and $\chi_P(a_{E,\theta}) = \chi_P(a_{E,\theta}) + \ell_P = 1/2$. 

1.2.4 Convergence to the limiting configuration

Suppose that $(E, \tilde{\mathcal{J}}_E, \theta)$ is stable of degree 0 with rank $E = 2$ and that $\theta$ is generically regular semisimple. For simplicity, we assume $\tr \theta = 0$. Let $h_t (t > 0)$ denote the harmonic metric of the Higgs bundle $(E, \tilde{\mathcal{J}}_E, \theta_t)$ with $\det(h_t) = h_{\det(E)}$.

For any $\alpha > 0$, let $\Psi_{\alpha}$ be the automorphism of $L_1 \oplus L_2$ given by $\Psi_{\alpha} = \alpha \text{id}_{L_1} \oplus \alpha^{-1} \text{id}_{L_2}$. Let $\Psi_{\alpha}^* h_t$ be the metric of $E|_{X\setminus D(\theta)}$ given by $h_t(\Psi_{\alpha} s_1, \Psi_{\alpha} s_2)$ for local sections $s_j (j = 1, 2)$ of $E|_{X\setminus D(\theta)}$. We take any point $Q \in X \setminus D(E, \theta)$ and a frame $v_Q$ of $L_1|_Q$, and we put

$$\gamma(t, Q) := \left(\frac{h_t^\lim(v_Q, v_Q)}{h_t(v_Q)}\right)^{1/2}.$$ 

The following theorem is our second main result in this paper.

Theorem 1.10 (The degree 0 case of Theorem 5.1) When $t$ goes to $\infty$, the sequence $\Psi_{\alpha}^* h_t$ converges to $h_i^\lim$ in the $C^\infty$-sense on any compact subset in $X \setminus Z(\omega)$. In particular, the sequence of the connections $\nabla_i$ converges to $\nabla_i^\lim$. (See §1.2.3 for $h_i^\lim_E$ and $\nabla_i^\lim_E$.)

For the proof of Theorem 1.5, we need to study the global property of $(E, \tilde{\mathcal{J}}_E, \theta)$, in contrast to what we can obtain the estimates in §1.1.2 locally at any point of $X \setminus D(E, \theta)$. A key is the construction of a family of Hermitian metrics $h_t^0$ of $E$ such that the $L^p$-norms of $R(h_t^0) + [\theta_t, (\theta_t)]_{\theta_t}$ are bounded, for which we naturally encounter the above parabolic weights for the limiting configuration. We have the family of the self-adjoint endomorphisms $k_i$ of $(E, h_t^0)$ such that $h_t(u_1, u_2) = h_t^0(k_i u_1, u_2)$ for any local sections $u_i$ ($i = 1, 2$). By applying a variant of the arguments used in [15] with the tools given in [19], we shall observe that the sequence $\kappa_i k_i$, converges in some sense for an appropriate sequence $\kappa_i > 0$. Then, the claim of the theorem follows.

We note that for the construction of the family of metrics $h_t^0$, we use the general theory of wild harmonic bundles. We need a family of harmonic metrics given on a neighbourhood of the discriminant locus, for which we apply the Kobayashi-Hitchin correspondence for wild harmonic bundles on curves given in [2] (see [20] for
the tame case). On the basis of the general results on the asymptotic behaviour of wild harmonic bundles studied in [20] and [16], we can deduce rather detailed properties of the family of the harmonic metrics as in Proposition 3.17 and Proposition 3.19.

At this moment, it is not clear to the author whether we could directly use the argument in [11] to prove Theorem 1.10 or its variant in our setting. But, we should note that we do not study the order of the convergence in Theorem 1.10. In contrast, when all the zeroes of det \( \theta - (\text{tr} \theta)^2/4 \) are simple, the method in [11] is strong enough to give the order of the convergence.

**Remark 1.11** Let \( (E, \overline{\nabla} E, \theta) \) be a stable Higgs bundle of rank 2 on \( X \) such that \( \theta \) is generically regular semisimple, but that \( \text{deg}(E) \) is not necessarily 0. We fix a Hermitian metric \( h_{\det(E)} \) of \( \det(E) \). For each \( t > 0 \), we have the Hermitian-Einstein metric \( h_{HE}^t \) of \( (E, \overline{\nabla} E, \theta) \) such that \( \det(h_{HE}^t) = h_{\det E}^t \), according to [6] and [19]. Here, the Hermitian-Einstein condition means the trace-free part of \( R(h_{HE}^t) + [\theta, (\theta)_{h_{HE}^t}] \) is 0. We can study the behaviour of \( h_{HE}^t \) (\( t \to \infty \)) by using Theorem 1.10. (See Theorem 5.1.) Suppose that \( \text{deg}(E) = 2m \) for an integer \( m \). We can easily reduce the study on the behaviour of \( h_{HE}^t \) to the study on the behaviour of harmonic metrics for \( (E, \overline{\nabla} E, \theta) \otimes L \). Suppose that \( \text{deg}(E) \) is odd. We take any (unramified) covering map \( \varphi : X' \longrightarrow X \) of degree 2. Then, the degree of \( \varphi^*(E, \overline{\nabla} E, \theta) \) is even. Hence, the study can be reduced to the degree 0 case. (See also §5.2.)

### 1.2.5 Symmetric case and irreducible case

Let us explain that we can obtain a stronger result if the Higgs bundle is equipped with an extra symmetry. Suppose that \( X \) is equipped with a non-trivial involution \( \rho \), i.e., \( \rho : X \longrightarrow X \) is a holomorphic automorphism such that \( \rho \circ \rho = \text{id}_X \) and \( \rho \neq \text{id}_X \). Let \( (E, \overline{\nabla} E, \theta) \) be a stable Higgs bundle of degree 0 on \( X \) with \( \text{tr} \theta = 0 \) such that the spectral curve is reducible, i.e., \( \Sigma(E, \theta) = \text{Im}(\omega) \cup \text{Im}(-\omega) \) for a holomorphic one form \( \omega \neq 0 \) on \( X \). We impose the following conditions.

- We have \( \rho^* \omega = -\omega \).
- The Higgs bundle \( (E, \overline{\nabla} E, \theta) \) is equivariant with respect to the action of \( \{1, \rho\} \) on \( X \).

We have the induced isomorphism \( \rho^* \det(E) \cong \det(E) \). We naturally have \( \rho^* h_{\det(E)} = h_{\det(E)} \).

Let \( L_i \) (\( i = 1, 2 \)) be as in §1.2.3. We naturally have the isomorphisms \( \rho^* L_i \cong L_2 \) and \( \rho^* L_2 \cong L_1 \), which are compatible with the isomorphism \( \rho^* E \cong E \). Because \( \text{deg}(L_1) = \text{deg}(L_2) \), we have \( a_{E, \theta} = 0 \) and \( -\chi P(a_{E, \theta}) = \chi P(a_{E, \theta}) + \ell P = \ell P/2 \) for any \( P \in \mathbb{Z}(\omega) \). We can uniquely determine the metrics \( h_{L_i}^\text{lim} \) by imposing the extra condition \( \rho^* h_{L_i}^\text{lim} = h_{L_2}^\text{lim} \).

**Theorem 1.12** (The degree 0 case of Theorem 5.3) For \( t > 0 \), let \( h_t \) be the harmonic metrics of the Higgs bundles \( (E, \overline{\nabla} E, \theta) \) such that \( \det(h_t) = h_{\det(E)} \). Then, the sequence \( h_t \) is convergent to \( h_{E, \theta}^\text{lim} = h_{L_1}^\text{lim} \oplus h_{L_2}^\text{lim} \) on any compact subset in \( X \setminus \mathbb{Z}(\omega) \).

Let us remark that we can apply Theorem 1.12 to the case where the spectral curve is irreducible and the Higgs field is generically regular semisimple. Let \( X' \) be a compact connected Riemann surface. Let \( (E', \overline{\nabla} E', \theta') \) be a stable Higgs bundle of rank 2 with \( \text{deg}(E') = 0 \) on \( X' \) such that (i) the spectral curve \( \Sigma(E', \theta') \) is irreducible, (ii) \( \text{tr} \theta' = 0 \), (iii) \( \theta' \) is generically regular semisimple. We take the normalization \( \kappa : X \longrightarrow \Sigma(E', \theta') \). Let \( p : X \longrightarrow X' \) be the morphism obtained as the composite of \( \kappa \) and the projection \( \Sigma(E', \theta') \longrightarrow X' \). We set \( (E, \overline{\nabla} E, \theta) = p^*(E', \overline{\nabla} E', \theta') \). We have the involution on \( \Sigma(E', \theta') \) induced by the multiplication of \( -1 \) on the cotangent bundle \( T^* X' \). It induces an involution \( \rho \) on \( X \). We fix a Hermitian metric \( h_{\det(E')} \) of \( \det(E') \).

If \( (E, \overline{\nabla} E, \theta) \) is polystable, we can easily observe that the harmonic metrics \( h_t^\prime \) for the Higgs bundles \( (E', \overline{\nabla} E', \theta') \) are independent of \( t \). (See §4.3.2.) We formally set \( h_{E', \theta}^\text{lim} := h_t^\prime \).

If \( (E, \overline{\nabla} E, \theta) \) is stable, the above symmetry conditions for \( (E, \overline{\nabla} E, \theta) \) and \( \rho \) are satisfied. The pull back \( h_{\det(E')} = p^* h_{\det(E')} \) satisfies the condition \( \rho^* h_{\det(E')} = h_{\det(E')} \). We have the metrics \( h_{L_i}^\text{lim} \) and \( h_{E, \theta}^\text{lim} = h_{L_1}^\text{lim} \oplus h_{L_2}^\text{lim} \) as above. Because we have \( \rho^* h_{E, \theta}^\text{lim} = h_{E, \theta}^\text{lim} \), we have a unique Hermitian metric \( h_{E, \theta}^\text{lim} \) on \( X' \setminus D(E', \theta') \) such that \( \rho^* h_{E, \theta}^\text{lim} = h_{E, \theta}^\text{lim} \). We also have a characterization of \( h_{E, \theta}^\text{lim} \) as a harmonic metric for the stable filtered
Higgs bundles on \((X', D(E', \theta'))\) induced by the limiting configuration of \((E, \overline{\partial}_E, \theta)\). (See §4.3.2.) We have the following direct corollary.

**Corollary 1.13 (The degree 0 case of Corollary 5.2)** For \(t > 0\), let \(h_1^{t} \) be the harmonic metrics of the Higgs bundles \((E', \overline{\partial}_{E'}, t\theta')\) such that \(\det(h_1^{t}) = h_{\det(E')}.\) Then, the sequence \(h_1^{t}\) is convergent to \(h_{E',g'}^{\text{pol}}\) on any compact subset in \(X' \setminus D(E', \theta')\).

### 1.2.6 Complement

Let \((E, \overline{\partial}_E, \theta, h)\) be a Higgs bundle of rank \(2\). Suppose that \(\theta\) is nilpotent, i.e., \(\Sigma(E, \theta)\) is the 0-section in \(\mathcal{T}^* X\). Then, we can easily observe that \(\lim_{t \to \infty}(E, \overline{\partial}_E, t\theta, h_t)\) is convergent to a polarized complex variation of Hodge structure. We sketch it. We have a polystable Higgs bundle \((E_\infty, \overline{\partial}_{E_\infty}, \theta_\infty)\) obtained as the limit \(\lim_{t \to \infty}(E, \overline{\partial}_E, t\theta)\) in the coarse moduli space of semistable Higgs bundles of degree 0 on \(X\). Note that the norms \(\|R(\nabla_{h_t})\|_{h_t, g_X}\) (\(t > 0\)) are uniformly bounded, where \(g_X\) is a Kähler metric of \(X\). (See Proposition 2.1, for example.) Hence, we obtain a harmonic metric \(h_\infty\) of \((E_\infty, \overline{\partial}_{E_\infty}, \theta_\infty)\) as the limit of a convergent subsequence \(h_{t_i}\) \((t_i \to \infty)\). If \((E_\infty, \overline{\partial}_{E_\infty}, \theta_\infty)\) is not stable, \(\theta_\infty\) is trivial, \((E_\infty, \overline{\partial}_{E_\infty})\) is a direct sum of line bundles, and \(h_\infty\) is a flat metric. If \((E_\infty, \overline{\partial}_{E_\infty}, \theta_\infty)\) is stable, then as in the case of the limit for \(t \to 0\) studied by Simpson, we can observe that \((E_\infty, \overline{\partial}_{E_\infty}, \theta_\infty)\) is a Hodge bundle, and \(h_\infty\) is equivariant with respect to the natural grading \(S^1\)-action. Hence, \((E_\infty, \overline{\partial}_{E_\infty}, \theta_\infty, h_\infty)\) comes from a polarized complex variation of Hodge structure.

In view of the moduli theoretic picture [6, 22], we can see it as follows. The energies of the family of \((E, \overline{\partial}_E, \theta, h_{t_i})\) \((t_i > 0)\) are bounded. So, the family is relatively compact in the moduli space of harmonic bundles. Hence, when \(t\) goes to \(\infty\), \((E, \overline{\partial}_E, \theta, h_t)\) goes to a fixed point in the moduli space induced by the natural \(\mathbb{C}^*\)-action on the moduli space of Higgs bundles. It means that \((E, \overline{\partial}_E, \theta, h_t)\) is convergent to a polarized complex variation of Hodge structure.

If \((E, \overline{\partial}_E, \theta)\) is not generically regular semisimple but rank \(E = 2\), then the study of \(\lim_{t \to \infty}(E, \overline{\partial}_E, t\theta, h_t)\) is easily reduced to the above case.

### 1.3 Acknowledgement

The author thanks the referee for his thorough reading and valuable remarks. This study has grown out of my effort to understand (part of) the intriguing works [3], [5], [7] and [11, 12]. I thank Carlos Simpson for inspiring discussions on many occasions. Indeed, I am motivated to this study by his question several years ago. I owe a lot to his philosophy and his techniques. I also thank him for many valuable comments to improve this manuscript, and for his patience in reading the preliminary versions of this manuscript. I hope that this study would be useful for the project in [7]. I learned the attractive study [11, 12] from the stimulating talk given by Frederik Witt in the conference “The Geometry, Topology and Physics of Moduli Spaces of Higgs Bundles” held at the Institute for Mathematical Sciences, National University of Singapore in 2014, for which the visit was supported by the Institute. I thank Claude Sabbah for his kindness and for many discussions. I am grateful to Yoshihumi Tsuchimoto and Akira Ishii for their constant encouragement. I thank Szilard Szabo for his interest to this study and some discussions.

This work was partially supported by the Grant-in-Aid for Scientific Research (S) (No. 24224001) and the Grant-in-Aid for Scientific Research (C) (No. 15K04843), Japan Society for the Promotion of Science.

### 2 Asymptotic decoupling

#### 2.1 Simpson’s main estimate for harmonic bundles on discs

Let \((E, \overline{\partial}_E, \theta, h)\) be a harmonic bundle on a complex curve \(X\). We have the spectral curve \(\Sigma(E, \theta)\) in the cotangent bundle of \(X\). We have an estimate of the norm of \(E, \overline{\partial}_E, \theta, h)\) depending only on \(\Sigma(E, \theta)\) (Proposition 2.10). If \(\Sigma(E, \theta)\) is decomposed into a disjoint union \(\Sigma_1 \sqcup \Sigma_2\), then we have the corresponding decomposition of the Higgs bundle \((E, \overline{\partial}_E, \theta) = (E_1, \overline{\partial}_{E_1}, \theta_1) \sqcup (E_2, \overline{\partial}_{E_2}, \theta_2)\). The Hermitian product of sections of \(E_1\) and \(E_2\) given by \(h\) should be small. We have such estimates depending only on \(\Sigma(E, \theta)\) (Corollary 2.6). Because we use the
arguments essentially given in [20], such estimates are called Simpson’s main estimate. It is enough to consider the case where \( X \) is a disc.

### 2.1.1 Estimate of the sup norm in terms of the eigenvalues

For \( R > 0 \), we set \( \Delta(R) := \{ z \in \mathbb{C} \mid |z| < R \} \). We consider a harmonic bundle \((E, \overline{\partial}E, \theta, h)\) of rank \( r \) on \( \Delta(R) \).

We have the description \( \theta = f \, dz \), where \( f \) is a holomorphic endomorphism of \( E \). Fix \( M > 0 \), and suppose the following.

- For any \( P \in \Delta(R) \), the eigenvalues \( \gamma \) of \( f|_P \) satisfy \(|\gamma| < M\).

We recall the following proposition, for which a proof was given in [14], for example. We include an outline of the proof for the convenience of the readers.

**Proposition 2.1** Fix \( 0 < R_1 < R \). Then, we have \( C_1, C_2 > 0 \) depending only on \( r, R_1, R \) such that

\[
|f|_h \leq C_1 M + C_2
\]

holds on \( \Delta(R_1) \).

**Proof** Let \( f_h^\dagger \) denote the adjoint of \( f \) with respect to \( h \). As in [20], we have the following inequality on \( \Delta(R) \) (see also Lemma 2.4 below):

\[
-\partial_z \partial_{\overline{z}} \log |f|_h^2 \leq -\frac{\|f \cdot f_h^\dagger\|_h^2}{|f|_h^2}
\]

For any \( P \in \Delta(R) \), we set \( g(P) := \sum_{i=1}^r |\alpha_i|^2 \), where \( \alpha_1, \ldots, \alpha_r \) are the eigenvalues of \( f|_P \). There exists a constant \( C_3 > 0 \) depending only on \( r \) such that

\[
\|f \cdot f_h^\dagger|_P\|_h \geq C_3 (|f|_h^2 - g(P)).
\]

We obtain

\[
-\partial_z \partial_{\overline{z}} \log |f|_h^2(P) \leq -C_3 \frac{\|f|_h^2 - g(P)\|^2}{|f|_h^2}.
\]

Note that if \( |f|_h^2 \geq g(P) \) for some \( P \), then we have

\[
- \frac{\|f|_h^2 - g(P)\|^2}{|f|_h^2} \leq - \frac{1}{4} |f|_h^2.
\]

For any positive number \( B \), we have

\[
-\partial_z \partial_{\overline{z}} \log \frac{B}{(R^2 - |z|^2)^2} = -\frac{2R^2}{B} \frac{B}{(R^2 - |z|^2)^2}.
\]

Take \( B \) satisfying \( B \geq 8 R^2 / C_3^2 \) and \( B \geq 2 R^4 r M^2 \). Then, we use an idea in the proof of a lemma of Ahlfors [1].

Set \( \mathcal{Z} := \{ P \in \Delta(R) \mid |f|_h^2 > B(R^2 - |z(P)|^2)^{-2} \} \). Suppose that \( \mathcal{Z} \neq \emptyset \), and we shall derive a contradiction.

We have \( |f|_h^2 > BR^{-2} \geq 2 r M^2 > g(P) \) for any \( P \in \mathcal{Z} \). Hence, the following inequality holds on \( \mathcal{Z} \):

\[
-\partial_z \partial_{\overline{z}} \left( \log |f|_h^2 - \log (B(R^2 - |z|^2)^{-2}) \right) \leq - \frac{C_3^2}{4} \left( |f|_h^2 - B(R^2 - |z|^2)^{-2} \right) \leq 0
\]

Note that \( \mathcal{Z} \) is relatively compact in \( \Delta(R) \). So, we have \( |f|_h^2 = B(R^2 - |z|^2)^{-2} \) on the boundary of \( \mathcal{Z} \), and hence we obtain \( |f|_h^2 \leq B(R^2 - |z|^2)^{-2} \) on \( \mathcal{Z} \). But, it contradicts with the choice of \( \mathcal{Z} \). So, we obtain \( \mathcal{Z} = \emptyset \). Namely, we have \( |f|_h^2 \leq B(R^2 - |z|^2)^{-2} \) on \( \Delta(R) \). Then, we obtain the claim of the proposition. \( \blacksquare \)
2.1.2 Asymptotic orthogonality

We continue to use the notation in §2.1.1. Suppose that we have a finite subset $S \subset \mathbb{C}$ and a decomposition $(E, \theta) = \bigoplus_{\alpha \in S}(E_\alpha, \theta_\alpha)$. We have the expression $\theta_\alpha = f_\alpha dz$ for each $\alpha$, where $f_\alpha$ is a holomorphic endomorphism of $E_\alpha$.

Assumption 2.2 Fix $C_{10} > 1$, and we impose the following conditions.

- We set $d := \min\{ |\alpha - \beta| \mid \alpha, \beta \in S, \alpha \neq \beta \}$. Then, we have $d \geq 1$ and $M \leq C_{10}d$. Here, $M$ is the constant in the beginning of §2.1.1.

- For any $P \in \Delta(R)$, the eigenvalues $\gamma$ of $f_{\alpha|P}$ satisfy $|\gamma - \alpha| \leq d/100$.

Let $\pi_\alpha$ be the projection of $E$ onto $E_\alpha$ with respect to the above decomposition. Let $\pi'_\alpha$ denote the orthogonal projection of $E$ onto $E_\alpha$. We set $\rho_\alpha := \pi_\alpha - \pi'_\alpha$. The following proposition is a variant of the estimates given in [20] and [13, 16] for harmonic bundles given on punctured discs or the products of punctured discs, but it is more useful for the purpose in this paper.

Proposition 2.3 Let $R_1$ be as in Proposition 2.1. Fix $0 < R_2 < R_1$. There exist positive constants $\epsilon_0$ and $C_{11}$ depending only on $R, R_1, R_2, r$ and $C_{10}$, such that $|\rho_\alpha|_h \leq C_{11} \exp(-\epsilon_0d)$ on $\Delta(R_2)$.

Proof We recall a general inequality for holomorphic sections of $\text{End}(E)$ which commutes with $\theta$, for the convenience of the readers.

Lemma 2.4 Let $s$ be any holomorphic section of $\text{End}(E)$ such that $|\theta, s| = 0$. Then, we have the following inequality:

$$-\partial_z \partial_{\bar{z}} \log |s|_h^2 \leq \frac{|f_{1, s}|_h^2}{|s|_h^2}$$

Proof We have the equality $-\partial_z \partial_{\bar{z}} |s|_h^2 = -|\partial_z s, h(s, \partial_{\bar{z}} s)|_h^2 - h(s, \partial_{\bar{z}} \partial_{\bar{z}} s)$. Hence, we obtain the following:

$$-\partial_z \partial_{\bar{z}} \log |s|_h^2 = -\frac{\partial_z |s|_h^2}{|s|_h^2} + \frac{\partial_{\bar{z}} |s|_h^2}{|s|_h^2} + h(s, \partial_{\bar{z}} \partial_{\bar{z}} s) - \frac{|\partial_z s, h(s, \partial_{\bar{z}} s)|_h^2}{|s|_h^2}$$

$$\leq -\frac{h(s, \partial_{\bar{z}} \partial_{\bar{z}} s)}{|s|_h^2} - \frac{h(s, \partial_{\bar{z}} \partial_{\bar{z}} s)}{|s|_h^2} = -\frac{|f_{1, s}|_h^2}{|s|_h^2}.$$  (2)

By the Hitchin equation, we have $R(h) + |\theta, h| = R(h) + [f, f_{1}] dz d\bar{z} = 0$, i.e., $R(h) = [f, f_{1}] dz d\bar{z}$. We obtain the following from (2):

$$-\partial_z \partial_{\bar{z}} \log |s|_h^2 \leq \frac{-h(s, [f_{1}, s])}{|s|_h^2}$$

Because $[f, s] = 0$, we obtain

$$-\partial_z \partial_{\bar{z}} \log |s|_h^2 \leq \frac{-h(s, [f_{1}, s])}{|s|_h^2} = \frac{-h(f_{1}, s) |f_{1}|_h^2}{|s|_h^2} = \frac{|f_{1, s}|_h^2}{|s|_h^2}$$

Thus, we obtain Lemma 2.4.

Let $r_\alpha := \text{rank } E_\alpha$. Because $\partial_{\bar{z}} \partial_{\bar{z}} \log r_\alpha = 0$, we obtain the following inequality on $\Delta(R)$ from (1):

$$-\partial_z \partial_{\bar{z}} \log (|\pi_\alpha|_h^2 / r_\alpha) \leq \frac{|f_{1, \pi_\alpha}|_h^2}{|\pi_\alpha|_h^2}$$

According to Proposition 2.1 and Lemma 2.7 below, there exists a positive constant $C_{12}$ depending only on $R, R_1, r$ and $C_{10}$ such that we have $|\pi_\alpha|_h \leq C_{12}$ and $|\rho_\alpha|_h \leq C_{12}$ on $\Delta(R_1)$. We set $k_\alpha := \log(|\pi_\alpha|_h^2 / r_\alpha)$. Note
that $|\pi_\alpha|^2 = |\pi_\alpha'|^2 + |\rho_\alpha|^2$ and $|\pi_\alpha'|^2 = r_\alpha$. Hence, we have $k_\alpha = \log(1 + |\rho_\alpha|^2/r_\alpha)$. There exists a positive constant $C_{13}$ depending only on $R, R_1, r$ and $C_{10}$ such that $C_{13} |\rho_\alpha|^2 \leq k_\alpha \leq C_{13} |\rho_\alpha|^2$ on $\Delta(R_1)$.

According to Lemma 2.8 below, there exists a constant $\epsilon_1 > 0$ depending only on $R, R_1, r, C_{10}$ such that we have $\|[f_h^\dagger, \pi_\alpha]\|_h \geq \epsilon_1 d(\rho_\alpha)|_{h}$ on $\Delta(R_1)$. Hence, we have a constant $\epsilon_2 > 0$ depending only on $R, R_1, r, C_{10}$ such that the following holds on $\Delta(R_1)$:

$$-\partial_2 \partial_3 k_\alpha \leq -\epsilon_2 d^2 k_\alpha$$

For any positive number $\epsilon_3 > 0$, we have the following on $\Delta(R_1)$:

$$-\partial_2 \partial_3 \exp(\epsilon_3 d|z|^2) = -(d \epsilon_3 + d^2|z|^2 \epsilon_3^2) \exp(\epsilon_3 d|z|^2) \geq -(d \epsilon_3 + d^2 \epsilon_3^2 R_1^2) \exp(\epsilon_3 d|z|^2)$$

Take $\epsilon_3 > 0$ such that $\epsilon_3 \leq (\epsilon_2 R_1^{-1})/2$ and $\epsilon_3 \leq d \epsilon_2/2$. Then, we have

$$-\partial_2 \partial_3 \exp(\epsilon_3 d|z|^2) \geq -\epsilon_2 d^2 \exp(\epsilon_3 d|z|^2).$$

We take $C_{14} > 0$ depending only on $R, R_1, r, C_{10}$ such that $k_\alpha(P) < C_{14}$ for $|z(P)| = R_1$. We set

$$Z := \left\{ P \in \Delta(R_1) \big| k_\alpha(P) > C_{14} \exp(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2)(P) \right\}.$$ 

Suppose that $Z$ is non-empty. By the choice of $C_{14}$, $Z$ is relatively compact in $\Delta(R_1)$. So, we have $k_\alpha = C_{14} \exp(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2)$ on the boundary of $Z$. We also have the following inequality on $Z$:

$$-\partial_2 \partial_3 \left( k_\alpha - C_{14} \exp(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2) \right) \leq -\epsilon_2 d^2 \left( k_\alpha - C_{14} \exp(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2) \right) \leq 0$$

We obtain $k_\alpha \leq C_{14} \exp(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2)$ on $Z$, which contradicts with the construction of $Z$. Hence, we obtain that $Z = \emptyset$, i.e., $k_\alpha \leq C_{14} \exp(\epsilon_3 d|z|^2 - \epsilon_3 d R_1^2) = C_{14} \exp(-\epsilon_3(R_1^2 - |z|^2) d)$ holds on $\Delta(R_1)$. We obtain the following on $\Delta(R_2)$:

$$k_\alpha \leq C_{14} \exp(-\epsilon_3(R_1^2 - R_2^2) d)$$

Then, by setting $C_{11} := C_{14} > 0$ and $\epsilon_0 := \epsilon_3(R_1^2 - R_2^2) > 0$, we obtain the claim of Proposition 2.3.

For any endomorphism $g$ of $E$, let $g_h^\dagger$ denote the adjoint of $g$ with respect to $h$. We also denote it by $g^\dagger$ if there is no risk of confusion.

**Corollary 2.5** There exist positive constants $C_{20}$ and $\epsilon_{20}$, depending only on $R, R_1, R_2, r$ and $C_{10}$ such that $\|[f_h, (\pi_\alpha)]\|_h = \|[f_h^\dagger, \pi_\alpha]\|_h \leq C_{20} \exp(-\epsilon_{20} d)$ on $\Delta(R_2)$.

**Proof** We have $|\rho_\alpha|_h = |\pi_\alpha - \pi_\alpha'| \leq C_{11} \exp(-\epsilon_0 d)$. We also have $|(\rho_\alpha)|_h = |(\pi_\alpha)^\dagger - \pi_\alpha'|_h \leq C_{11} \exp(-\epsilon_0 d)$. We obtain

$$\|[f_h, \pi_\alpha]\|_h = \|[f_h, \pi_\alpha^\dagger - \pi_\alpha]\|_h \leq \|[f_h, \rho_\alpha]\|_h + \|[f_h, \rho_\alpha]\|_h \leq 2(C_1 M + C_2) C_{11} \exp(-\epsilon_0 d).$$

Then, we obtain the desired estimate.

**Corollary 2.6** There exist positive constants $C_{21}$ and $\epsilon_{21}$ depending only on $R, R_1, R_2, r$ and $C_{10}$, such that the following holds:

- Take any $\alpha, \beta \in S$ with $\alpha \neq \beta$. Let $s_\alpha$ and $s_\beta$ be local sections of $E_\alpha$ and $E_\beta$. Then, we have

$$|h(s_\alpha, s_\beta)| \leq C_{21} \exp(-\epsilon_{21} d) \cdot |s_\alpha|_h \cdot |s_\beta|_h$$

**Proof** We have $h(s_\alpha, s_\beta) = h(\pi_\alpha s_\alpha, s_\beta) = h(s_\alpha, \pi_\alpha^\dagger s_\beta) = h(s_\alpha, (\pi_\alpha^\dagger - \pi_\alpha) s_\beta)$. We have $\pi_\alpha^\dagger - \pi_\alpha = \rho_\alpha^\dagger - \rho_\alpha$, where $\rho_\alpha^\dagger$ denote the adjoint of $\rho_\alpha$ with respect to $h$. Then, we obtain (3) from Proposition 2.3.
2.1.3 Appendix: Preliminary from linear algebra

Fix positive constants $G_i \ (i = 1, 2)$. Fix a positive integer $r > 0$. Let $U$ be any $r$-dimensional $\mathbb{C}$-vector space with a Hermitian metric $h$. Let $f$ be an endomorphism on $U$. Suppose that we have a finite subset $S \subset \mathbb{C}$ and a decomposition $(U, f) = \bigoplus_{\alpha \in S} (U_\alpha, f_\alpha)$, which is not necessarily orthogonal. We impose the following conditions.

- We set $d := \min \{ |\alpha - \beta| \mid \alpha, \beta \in S, \alpha \neq \beta \}$. Then, we have $d \geq 1$ and $|f|_h \leq G_1 d + G_2$.
- The eigenvalues $\gamma$ of $f_\alpha$ satisfy $|\gamma - \alpha| \leq d/100$.

Let $\pi_\alpha$ be the projection with respect to the decomposition $U = \bigoplus U_\alpha$. Let $\pi'_\alpha$ denote the orthogonal projection of $U$ onto $U_\alpha$. We set $\rho_\alpha := \pi_\alpha - \pi'_\alpha$. To clarify the argument, we recall the following lemma from [20] which was used in [13, 16].

**Lemma 2.7** There exists a positive constant $B_1$ depending only on $r$ and $G_i \ (i = 1, 2)$ such that $$|\pi_\alpha|_h \leq B_1, \quad |\rho_\alpha|_h \leq B_1.$$  

**Proof** Because $\rho_\alpha$ and $\pi'_\alpha$ are orthogonal, we have $|\rho_\alpha|_h \leq |\pi_\alpha|_h$. Hence, it is enough to obtain the estimate for $\pi_\alpha$. Let $\text{id}_U$ denote the identity on $U$. We have $$\pi_\alpha = \frac{1}{2\pi \sqrt{-1}} \int_{\gamma_\alpha} (\zeta \text{id}_U - f)^{-1} d\zeta.$$  

Here, $\gamma_\alpha$ denotes the loop $\gamma_\alpha(\theta) = \alpha + d \sqrt{-1} \theta / 10 (0 \leq \theta \leq 2\pi)$. There exist positive constants $B_1 (i = 2, 3)$ depending only on $r$ and $G_i \ (i = 1, 2)$ such that $$|\gamma_\alpha(\theta) \text{id}_U - f|^{-1} \leq d^{-1} B_2 \left( (G_1 + d^{-1} G_2)^\epsilon + 1 \right) \leq d^{-1} B_3.$$  

Thus, we obtain the claim of the lemma.

Let $f_h^1$ denote the adjoint of $f$ with respect to $h$. To clarify our argument, we recall the following lemma from [20] which was used in [13, 16].

**Lemma 2.8** We have $\delta > 0$ depending only on $r$ and $G_i \ (i = 1, 2)$ such that $|\{f_h^1, \pi_\alpha\}|_h \geq \delta \cdot d \cdot |\rho_\alpha|_h$.

**Proof** We take a numbering of the elements of $S$, i.e., $S = \{ \alpha_1, \ldots, \alpha_m \}$. We impose $\alpha_1 = \alpha$. We set $F_j(U) := \bigoplus_{i \leq j} U_{\alpha_i}$ and $F_{\leq j}(U) := \bigoplus_{i \leq j} U_{\alpha_i}$. Let $U_j^h$ be the orthogonal complement of $F_{<j}(U)$ in $F_j(U)$. We have the orthogonal decomposition $F_j(U) = \bigoplus_{i \leq j} U_i^h$. Because $f(F_j) \subset F_j$, we have the decomposition $f = \sum_{i \leq j} f_{ij}$, where $f_{ij} : U_i^h \rightarrow U_i^h$. As the adjoint, we have $f_h^1 = \sum_{i \leq j} (f_{ij})_h^* h : U_i^h \rightarrow U_i^h$. Then, we set $f_h^1 = (f_{ij})_h^* h$. Then, we have $f_h^1 = \sum_{i \leq j} f_{ij}^* h$, where $f_{ij}^* h : U_j^h \rightarrow U_i^h$.

For each $i$, we take an orthonormal basis $e_1^{(i)}, \ldots, e_{r_i}^{(i)}$ of $U_i^h$ for which $f_{ij}^* h$ is represented by a lower triangular matrix $A_i$. Let $\Gamma_i \in \text{End}(U_i^h)$ be determined by $\Gamma_i(e_{k}^{(i)}) = \alpha_k^{(i)} e_{k}^{(i)}$, where $\alpha_k^{(i)}$ denote the $(k, k)$-entry of $A_i$. Then, $f_h^1 - \Gamma_i$ is nilpotent. We put $\Gamma = \sum \Gamma_i \in \text{End}(U)$. Then, $f_h^1 - \Gamma$ is nilpotent, and we have $|f_h^1 - \Gamma|_h \leq G_1 d + G_2$.

Let $\Pi$ denote the orthogonal projection of $\text{End}(E) = \bigoplus \text{Hom}(U_i^h, U_j^h)$ onto $\bigoplus_{j > i} \text{Hom}(U_j^h, U_i^h)$. Let $F_1$ be the endomorphism of $\bigoplus_{j > i} \text{Hom}(U_i^h, U_j^h)$ given by $F_1(B) := \Pi(\{f_h^1(B), B\})$. Let $F_2$ be the endomorphism of $\bigoplus_{j > i} \text{Hom}(U_i^h, U_j^h)$ given by $F_2(B) := [\Gamma, B] = \Pi([\Gamma, B])$.

Let us observe that $F_1 - F_2$ is nilpotent. We have the nilpotent endomorphism $\tilde{F}$ on $\text{End}(E)$ given by $\tilde{F}(B) = [f_h^1 - \Gamma, B]$, which preserves $\bigoplus_{i < j} \text{Hom}(U_j^h, U_i^h)$. Then, $F_1 - F_2$ is equal to the endomorphism on $\bigoplus_{i < j} \text{Hom}(U_j^h, U_i^h) \simeq \text{End}(E) / \bigoplus_{i < j} \text{Hom}(U_j^h, U_i^h)$ induced by $\tilde{F}$, and hence $F_1 - F_2$ is nilpotent.

There exists a constant $B_{11} > 0$ depending only on $r$ such that $|F_1 - F_2|_h \leq B_{11}(G_1 d + G_2)$. Moreover, $F_2$ is invertible, and there exists a constant $B_{12} > 0$ depending only on $r$ such that $|F_2^{-1}|_h \leq B_{12} d^{-1}$. Hence, we obtain that $|F_2^{-1}|_h \leq B_{13} d^{-1}(1 + (G_1 + G_2/d)^\epsilon)$ for a constant $B_{13} > 0$ depending only on $r$. 

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Note that we have \( \Pi([f^\dagger_h, \pi'_\alpha]) = 0 \). Hence, we have the following:

\[
||f^\dagger_h, \pi'_\alpha||_h \geq ||\Pi([f^\dagger_h, \pi'_\alpha])||_h = ||\Pi(f^\dagger_h, \rho_\alpha)||_h = ||F_1(\rho_\alpha)||_h \geq B_{13}^{-1}d(1 + (G_1 + G_2d^{-1})^r)^{-1}|\rho_\alpha|h
\]

Thus, we obtain the claim of the lemma. \qed

2.2 Asymptotic decoupling of harmonic bundles on discs

We continue to use the setting in §2.1. We further impose the following condition:

- rank \( E_\alpha = 1 \) for each \( \alpha \in \mathcal{S} \).

In other words, we assume to have holomorphic functions \( g_\alpha (\alpha \in \mathcal{S}) \) on \( \Delta(R) \) such that \( \theta = \bigoplus g_\alpha \operatorname{id}_{E_\alpha} \, dz \).

(Recall the generically regular semisimple condition in §1.1.1.) By the condition, we have \( |g_\alpha(P) - \alpha| \leq d/100 \).

In this setting, we explain that Simpson’s main estimate implies the asymptotic decoupling of the Hitchin equation.

Let \( g_C \) denote the Euclidean metric \( dz \, d\bar{z} \) of \( \mathbb{C} \). For any section \( s \) of \( \operatorname{End}(E) \otimes \Omega^{p,q} \) on \( \Delta(R') \) \( (R' > 0) \), let \( |s|_{h,g_C} \) denote the function on \( \Delta(R') \) by taking the norm of \( s \) at each \( P \in \Delta(R') \) with respect to \( h \) and \( g_C \).

2.2.1 Decay of the curvatures

Let \( R(h) \) denote the curvature of the Chern connection of \( (E, \mathcal{I}_E, h) \). We obtain the following “asymptotic decoupling” of the Hitchin equation \( R(h) + [\theta, \theta^\dagger_h] = 0 \).

**Theorem 2.9** There exist positive constants \( C_{30} \) and \( \epsilon_{30} \), depending only on \( R, R_1, R_2, r \) and \( C_{10} \) such that

\[
|R(h)|_{h,g_C} = ||[\theta, \theta^\dagger_h]|_{h,g_C} \leq C_{30} \exp(-\epsilon_{30}d) \text{ on } \Delta(R_2).
\]

**Proof** It is enough to obtain the estimate for \( [\theta, \theta^\dagger_h] \). We have the decomposition \( f^\dagger_h = \sum_{\alpha, \beta} \pi_\alpha \circ f^\dagger_h \circ \pi_\beta \). By Corollary 2.5, if \( \alpha \neq \beta \), we have \( C_{31} > 0 \), depending only on \( R, R_1, R_2, r \) and \( C_{10} \), such that

\[
||\pi_\alpha \circ f^\dagger_h \circ \pi_\beta||_h = ||[\pi_\alpha, f^\dagger_h] \circ \pi_\beta||_h \leq C_{31} \exp(-\epsilon_{30}d).
\]

For \( \alpha \neq \beta \), we have \( [f_\alpha, \pi_\beta \circ f^\dagger_h \circ \pi_\beta] = 0 \). Because \( \text{rank } E_\alpha = 1 \), we also have \( [f_\alpha, \pi_\alpha \circ f^\dagger_h \circ \pi_\alpha] = 0 \). Then, we obtain the estimate for \( [\theta, \theta^\dagger_h] \). \qed

2.2.2 The connections and the projections

Let \( \mathcal{I}_E \) denote the \((1,0)\)-part of the Chern connection associated to \( h \) and \( \mathcal{I}_E \). According to Proposition 2.3, the decomposition \( E = \bigoplus E_\alpha \) is almost orthogonal. Let us see that such an almost orthogonality holds at the level of the first derivative in the sense that \( \mathcal{I}_E \pi_\alpha \) is very small.

**Proposition 2.10** Take \( 0 < R_3 < R_2 \). There exist positive constants \( \epsilon_{40} \) and \( C_{40} \) depending only on \( R, R_1, R_2, R_3, r \) and \( C_{10} \) such that

\[
||\mathcal{I}_E \pi_\alpha||_{h,g_C} = ||\mathcal{I}_E \pi^\dagger_\alpha||_{h,g_C} \leq C_{40} \exp(-\epsilon_{40}d) \text{ on } \Delta(R_3).
\]

**Proof** It is enough to obtain the estimate for \( \mathcal{I}_E \pi_\alpha \). In the following, the constants may depend only on \( R, R_1, R_2, r \) and \( C_{10} \). We have \( C_{41} > 0 \) such that the following holds on \( \Delta(R_2) \):

\[
||\mathcal{I}_E \partial_E \pi_\alpha||_{h,g_C} = ||[R(h), \pi_\alpha]|_{h,g_C} \leq C_{41} \exp(-\epsilon_{30}d).
\]

Because \( \partial_E \pi^\dagger_\alpha = 0 \), we have the following on \( \Delta(R_2) \):

\[
||\mathcal{I}_E \partial_E (\pi_\alpha - \pi^\dagger_\alpha)||_{h,g_C} \leq C_{41} \exp(-\epsilon_{30}d)
\]

We already have \( ||\pi_\alpha - \pi^\dagger_\alpha||_h \leq C_{42} \exp(-\epsilon_0d) \) for a constant \( C_{42} > 0 \). We may assume \( \epsilon_{30} < \epsilon_0 \). Hence, we have a constant \( C_{43} > 0 \) such that \( ||\pi_\alpha - \pi^\dagger_\alpha||_{h,g_C, L_p^2} \leq C_{43} \exp(-\epsilon_{30}d) \) for a large \( p > 1 \). Here, \( ||\cdot||_{h,g_C, L_p^2} \) denote the \( L^2 \)-norm with respect to \( h \) and \( g_C \) on \( \Delta(R_2) \). Then, we obtain the claim of the lemma. \qed

Recall that \( \pi'_\alpha \) denote the orthogonal projection of \( E \) onto \( E_\alpha \). Let us see that it is almost holomorphic.
Proposition 2.11 We have a positive constant $C_{50}$, depending only on $R$, $R_1$, $R_2$, $R_3$, $r$ and $C_{10}$ such that the following holds on $\Delta(R_3)$:

$$|\mathcal{J}_E \pi'_\alpha|_{h,gc} = |\partial E \pi'_\alpha|_{h,gc} \leq C_{50} \exp(-\epsilon_{40} d).$$

**Proof** It is enough to prove the estimate for $\mathcal{J}_E \pi'_\alpha$. Let $E = E_\alpha \oplus E_\alpha^\perp$ denote the orthogonal decomposition. We may naturally regard $\rho_\alpha := \pi_\alpha - \pi'_\alpha$ as a morphism $E_\alpha^\perp \to E_\alpha$. We may also regard $\rho_\alpha^\dagger := \pi_\alpha^\dagger - \pi'_\alpha$ as a morphism $E_\alpha \to E_\alpha^\perp$.

We have the induced holomorphic structure on $E_\alpha^\perp \simeq E/E_\alpha$. Let $\overline{\mathcal{J}}_E^{(0)}$ be the holomorphic structure on $E_\alpha \oplus E_\alpha^\perp$ obtained as the direct sum. We have $\overline{\mathcal{J}}_E = \overline{\mathcal{J}}_E^{(0)} + \kappa$, where $\kappa$ is a section of $\text{Hom}(E_\alpha^\perp, E_\alpha) \otimes \Omega^{0,1}$. We have

$$\overline{\mathcal{J}}_E(\pi_\alpha - \pi'_\alpha) = \overline{\mathcal{J}}_E \rho_\alpha - \overline{\mathcal{J}}_E \rho_\alpha^\dagger = \overline{\mathcal{J}}_E^{(0)} \rho_\alpha - \overline{\mathcal{J}}_E^{(0)} \rho_\alpha^\dagger - \kappa \circ \rho_\alpha^\dagger + \rho_\alpha \circ \kappa$$

(4)

Note that $\overline{\mathcal{J}}_E^{(0)}(\rho_\alpha)$, $\overline{\mathcal{J}}_E^{(0)}(\rho_\alpha^\dagger)$, $\kappa \circ \rho_\alpha^\dagger$ and $\rho_\alpha \circ \kappa$ are sections of $\text{Hom}(E_\alpha^\perp, E_\alpha) \otimes \Omega^{0,1}$, $\text{Hom}(E_\alpha, E_\alpha^\perp) \otimes \Omega^{0,1}$, $\text{Hom}(E_\alpha, E_\alpha^\perp) \otimes \Omega^{0,1}$ and $\text{Hom}(E_\alpha^\perp, E_\alpha^\perp) \otimes \Omega^{0,1}$, respectively. Note that the bundles are orthogonal with respect to $h$ and $g_c$. In general, for any orthogonal decomposition of bundles $\mathcal{V} = \bigoplus \mathcal{V}_i$ and for any section $s = \sum s_i$ of $\mathcal{V}$, the norms of $s_i$ are smaller than the norm of $s$. Hence, we obtain the following from (4):

$$|\overline{\mathcal{J}}_E \rho_\alpha|_{h,gc} \leq |\overline{\mathcal{J}}_E (\pi_\alpha - \pi'_\alpha)|_{h,gc} = |\overline{\mathcal{J}}_E (\pi'_\alpha)|_{h,gc}$$

(5)

We also have $\overline{\mathcal{J}}_E \pi'_\alpha = -\overline{\mathcal{J}}_E \rho_\alpha = -\overline{\mathcal{J}}_E^{(0)} \rho_\alpha$. Hence, we obtain $|\overline{\mathcal{J}}_E \pi'_\alpha|_{h,gc} \leq |\overline{\mathcal{J}}_E (\pi'_\alpha)|_{h,gc}$ from (5). Then, the claim of Proposition 2.11 follows from the estimate in Proposition 2.10.

2.2.3 The decay of the curvatures on the line bundles

Let $\overline{\mathcal{J}}_\alpha$ denote the holomorphic structure of $E_\alpha$. Let $h_\alpha$ be the restriction of $h$ to $E_\alpha$. Let $\partial_\alpha$ denote the (1,0)-part of the Chern connection of $(E_\alpha, \overline{\mathcal{J}}_\alpha, h_\alpha)$. Let $R(h_\alpha)$ denote the curvature of the connection $\overline{\mathcal{J}}_\alpha + \partial_\alpha$. We have $\partial_\alpha s = \pi'_\alpha \circ \partial E s$ and $\overline{\mathcal{J}}_\alpha s = \overline{\mathcal{J}}_E s$ for any section $s$ of $E_\alpha$.

**Proposition 2.12** We have a positive constant $C_{60}$ depending only on $R$, $R_1$, $R_2$, $R_3$, $r$ and $C_{10}$ such that $|R(h_\alpha)|_{h_\alpha,gc} \leq C_{60} \exp(-\epsilon_{40} d)$ on $\Delta(R_3)$.

**Proof** In the following, the constants may depend only on $R$, $R_1$, $R_2$, $R_3$, $r$ and $C_{10}$. Let $s$ be any section of $E_\alpha$. We have

$$\overline{\mathcal{J}}_E \circ \pi'_\alpha(\partial E s) = \overline{\mathcal{J}}_E \circ \pi'_\alpha \circ \partial E(\pi_\alpha s) = \overline{\mathcal{J}}_E \circ \pi'_\alpha(\partial E s) + \overline{\mathcal{J}}_E \circ \pi'_\alpha \circ (\partial E \pi_\alpha)(s) = \pi_\alpha(\overline{\mathcal{J}}_E \partial E s) + \overline{\mathcal{J}}_E (\pi'_\alpha(\partial E \pi_\alpha) s).$$

We also have

$$\pi'_\alpha \circ \partial_\alpha \circ \overline{\mathcal{J}}_E s = \pi'_\alpha \circ \partial_\alpha(\overline{\mathcal{J}}_E s) = \pi'_\alpha \partial_\alpha \overline{\mathcal{J}}_E s + \pi'_\alpha \partial_\alpha(\overline{\mathcal{J}}_E s).$$

Hence, it is enough to obtain an estimate of $\overline{\mathcal{J}}_E(\pi'_\alpha \circ \partial E \pi_\alpha) = \overline{\mathcal{J}}_E(\pi'_\alpha) \circ \partial E(\pi_\alpha) + \pi'_\alpha \circ \partial E \pi_\alpha$. By Proposition 2.10 and Proposition 2.11, we have $C_{61} > 0$ such that $|\overline{\mathcal{J}}_E(\pi'_\alpha) \circ \partial E(\pi_\alpha)|_{h,gc} \leq C_{61} \exp(-\epsilon_{40} d)$. We also have $|\pi'_\alpha \circ \partial_\alpha \circ \overline{\mathcal{J}}_E s|_{h,gc} = |\pi'_\alpha \circ [R(h), \pi_\alpha]|_{h,gc} \leq C_{62} \exp(-\epsilon_{40} d)$ for a constant $C_{62} > 0$. Then, the claim of the lemma follows.

2.2.4 Approximation of the flat connections

We consider the flat connection $\mathbb{D}^\dagger := \mathcal{J}_E + \partial E + \theta + \theta^\dagger$ on $E$. We also have the connection $\mathbb{D}^\dagger_0$ on $E = \bigoplus_{\alpha \in S} E_\alpha$ given by

$$\mathbb{D}^\dagger_0 = \bigoplus_{\alpha \in S} \left( (\partial_\alpha + \partial_\alpha) + (g_\alpha dz + \overline{\alpha}_\alpha d\overline{z}) \text{id}_{E_\alpha} \right).$$

Here, $g_\alpha$ are holomorphic functions such that $\theta = \bigoplus g_\alpha \text{id}_{E_\alpha} dz$, as in the beginning of §2.2.
Proposition 2.10 and our choice of $\alpha E$. We have a constant $C_{10}$, depending only on $R, R_1, R_2, R_3, r$ and $C_{10}$ such that $|D^4 - D_0^4|_{h,gc} \leq C_{10} \exp(-4d_0) \exp(\Delta(R_3))$.

Proof It is enough to obtain an estimate of $\partial E - \Theta_{\alpha E}$ $\partial_{\alpha E}$. We have a constant $C_{10} > 0$, depending only on $R, R_1, R_2, R_3, r$ and $C_{10}$ such that $|\pi_{\alpha} \circ (\partial E \pi_{\alpha})|_{h,gc} \leq C_{10} \exp(-4d_0)$. Hence, for any section $s$ of $E_{\alpha}$, we have

$$|\pi_{\alpha} \circ \partial_E s - \partial_{\alpha} s|_{h,gc} = |\pi'_{\alpha} \circ \pi_{\alpha} \circ \partial_E s - \pi'_{\alpha} \partial_E (\pi_{\alpha} s)|_{h,gc} = |\pi'_{\alpha} \circ \partial_E \pi_{\alpha} (s)|_{h,gc} \leq C_{10} \exp(-4d_0)|s|_{h}.$$

Then, the claim of the lemma follows.

2.2.5 Higher derivatives

We take a number $\{\alpha_1, \ldots, \alpha_r\}$ on $S$. For each $i$, we can take a holomorphic frame $u_i$ of $E_{\alpha_i}$ such that $|u_i|_h = 1$ and $\partial_{\alpha_i} u_i|_{h,gc} \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0)$ on $\Delta(R_3)$ for some positive constants $C_{10}$ and $\epsilon_{\alpha_i}$, depending only on $R, R_1, R_2, R_3, r$ and $C_{10}$. Let us sketch how to obtain such sections in an elementary way. For a real coordinate $z = x + \sqrt{-1}y$, we can take a section $s_i$ of $E_{\alpha_i}$ on $\Delta(R_2)$ such that $(\overline{\partial}_{\alpha_i} + \partial_{\alpha_i})s_i = s_i \cdot \nu_i dx = s_i \cdot \nu_i (dz + dz)/2$ such that $\nu_i(x, 0) = 0$. The curvature form is given by $\partial_\nu \nu_i dy dx$. By Proposition 2.12, we have $|\nu_i| \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0)$. Take $R_2'$ such that $R_2 < R_2' < R_3$. We can take a function $\rho_i$ on $\Delta(R_2')$ satisfying $d\rho_i = \nu_i/2$. We have $|\rho_i| \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0)$ and $|d\rho_i|_{L^p(\Delta(R_2'))} \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0)$ for any $p \geq 1$. We set $\nu_i := s_i e^{-\rho_i}$ on $\Delta(R_2')$. Then, we have $\overline{\partial}_{\alpha_i} \nu_i = 0$ and $\partial_{\alpha_i} \nu_i = 1$. As in the Hitchin equation can be large, we give a rather detailed argument.

Because $H_{ij} = (h_{ij})$ is the curvature of $\Theta_{ij}$ and $\Theta_{ij} = 0$, we have $\partial H_{ij} = 0$ for $i \neq j$. We have $\partial u = (u \Theta_i dz, \ldots, u \Theta_r dz)$. Let $\Theta^\dagger$ be the $r$-square matrix valued function given by $\Theta^\dagger = \overline{\Theta}^{-1} \Theta$. We have $\partial^\dagger u = u \partial^\dagger \Theta^\dagger d\overline{z}$.

Because $H_{ij} = |u_i|^2_h$, we have $|\log H_{ij}| \leq 2C_{10} \exp(-\epsilon_{\alpha_i}d_0)$, as remarked above. We also have positive constants $C_{10}$ and $\epsilon_{\alpha_i}$ depending only on $R, R_1, R_2, R_3, r$ and $C_{10}$ such that the following holds:

$$|H_{ij}| \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0), \quad (i \neq j)$$

$$|\partial_{\overline{z}} H_{ij}| = |\partial_{\overline{z}} H_{ij}| \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0)$$

$$|\partial_{\overline{z}} H_{ij}| \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0)$$

Indeed, (6) follows from Corollary 2.6. We have $\partial h_{ij} = h ((\partial E \pi_{\alpha_i}) u_i, u_j) + h (\pi_{\alpha_i} \partial_E u_i, u_j)$. As in the proof of Corollary 2.13, we have $|\pi_{\alpha_i} \partial_E u_i - \partial_{\alpha_i} u_i|_{h,gc} \leq C_{10} \exp(-\epsilon_{\alpha_i}d_0)|u_i|_{h}$. Then, we obtain (7) from Proposition 2.10 and our choice of $u_i$. We obtain (8) from the estimate for the curvature $\overline{\partial}(\Theta^\dagger \partial H)$ and (7).

Lemma 2.14 Take any $\ell = (\ell_1, \ell_2) \in (Z_{\geq 0} \times Z_{\geq 0}) \setminus \{(0, 0)\}$ and $R_4 < R_3$. We also take any $p > 1$. Then, we have positive constants $C_{85,\ell,p}$ and $\epsilon_{83,\ell,p}$ depending only on $R, R_1, R_2, R_3, R_4$ and $C_{10}$. Let $\ell$ and $p$ be such that the following holds:

$$|\partial_{\overline{z}} H_{\Delta(R_4)}|_{L^p_{\ell,p}} \leq C_{85,\ell,p} \exp(-\epsilon_{83,\ell,p}d_0).$$

Proof The proof is given in a standard inductive argument using the Hitchin equation and the elliptic regularity. Because $\Theta_{ii}$ in the Hitchin equation can be large, we give a rather detailed argument.

Let us consider the case $\ell = (2, 0)$. By (7) and (8), we have positive constants $C_{84}$ and $\epsilon_{85}$ such that $|\partial_{\overline{z}} H_{\Delta(R_4)}|_{L^p_{\ell,p}} \leq C_{84} \exp(-\epsilon_{85}d_0)$. So, we obtain $|\partial_{\overline{z}} \partial_{\overline{z}} H_{\Delta(R_4)}|_{L^p} \leq C_{85} \exp(-\epsilon_{85}d_0)$ for some $C_{85} > 0$ and $\epsilon_{85} > 0$. Similarly, we can obtain the estimate for $|\partial_{\overline{z}} \partial_{\overline{z}} H_{\Delta(R_4)}|_{L^{p'}}$. 

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Suppose the claim has already been proved for any $R_4$ and $p$ if $\ell_1 + \ell_2 < k$. We consider $\ell = (\ell_1, \ell_2)$ satisfying $\ell_1 + \ell_2 = k$ and $\ell_1 > 0$. The Hitchin equation is described as follows:

$$H^{-1}\partial_x \partial_z H - H^{-1}\partial_x H \cdot \bar{H}^{-1} \partial_z H - [\Theta, \bar{H}^{-1}(\bar{\Theta})\bar{H}] = 0$$ (9)

For each $\ell$, we have $C_{s0,\ell}$ depending on $C_{10}$ and $\ell$, such that the following holds:

$$|\partial_x^s(\Theta_{\alpha})| + |\partial_x^s(\bar{\bar{\Theta}}_{\alpha})| \leq C_{s0,\ell}d$$

Applying $\partial_x^{s-1} \partial_z^{s-1}$ to (9), we obtain the following:

$$H^{-1}\partial_x^s \partial_z^s H - \Delta H - [\Theta, \bar{H}^{-1}(\bar{\Theta})\bar{H}] = 0$$

Here, $G$ is expressed as a linear combination of terms which contains derivatives of $H$. Hence, we have $\|G_{\Delta(R_4)}\|_{L^p} \leq C_{s7} \exp(-\epsilon_{s7}d)$. We also have $\|\partial_x^{s-1} \Theta, \bar{H}^{-1}(\partial_x^{s-1} \Theta)\|_{\Delta(R_4)} \leq C_{s8} \exp(-\epsilon_{s8}d)$, by (6) and (9). Hence, we obtain the desired estimate if $\ell_1 + \ell_2 = k$ and $\ell_1 \neq 0$. We can deal with the cases $\ell = (k, 0), (0, k)$ by using the elliptic regularity. Then, by an inductive argument, we can obtain the claim of Lemma 2.14.

**Corollary 2.15** Take any $\ell = (\ell_1, \ell_2) \in (\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0}) \setminus \{(0, 0)\}$ and $R_4 < R_3$. Then, we have positive constants $C_{s0,\ell}$ and $\epsilon_{s0,\ell}$ depending only on $R, R_1, R_2, R_3, R_4, r, C_{10}$ and $\ell$ such that the following holds:

$$\sup_{\Delta(R_4)} |\partial_x^s \partial_z^s H| \leq C_{s0,\ell} \exp(-\epsilon_{s0,\ell}d).$$

### 2.3 Hitchin WKB-problem

#### 2.3.1 Preliminary

Let $V$ be an $r$-dimensional complex vector space. For Hermitian metrics $h_j$ ($j = 1, 2$), we can take a base $e_1, \ldots, e_r$ of $V$ which is orthogonal with respect to both $h_1$ and $h_2$. We have the real numbers $\alpha_i$ determined by $\kappa_i := \log |e_i|_{h_2} - \log |e_i|_{h_1}$. We impose $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_r$. Then, we set

$$d(h_1, h_2) := (\kappa_1, \ldots, \kappa_r) \in \mathbb{R}^r.$$

Let $V_j$ ($j = 1, 2$) be $r$-dimensional complex vector spaces with Hermitian metrics $h_j$. Let $f : V_1 \rightarrow V_2$ be a linear isomorphism. We define the Hermitian metric $f^* h_2$ on $V_1$ by $f^* h_2(u, v) = h_2(f(u), f(v))$. We recall the following lemma from [7], which can be easily proved.

**Lemma 2.16** We have orthonormal frames $q(s) = (q_1(s), \ldots, q_r(s))$ ($s = 0, 1$) on $V_s$ such that $f(q_j(0)) = e^{\beta_j} q_j(1)$ ($j = 1, \ldots, r$), where $\beta_j$ are real numbers satisfying $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_r$. In this case, we have

$$d(h_1, f^* h_2) = (\beta_1, \ldots, \beta_r).$$

We set $\|f\|_{\text{op}} := \sup \{ |f(u)|_{h_2} | u \in V_1, |u|_{h_1} = 1 \}$. By Lemma 2.16, we have $\beta_1 = \log \|f\|_{\text{op}}$. We also have

$$\sum_{j=1}^k \beta_j = \log \|\bigwedge^k f\|_{\text{op}},$$

where $\bigwedge^k f : \bigwedge^k V_1 \rightarrow \bigwedge^k V_2$ are the induced morphisms. Hence, we have the following formula, as noted in [7]:

$$\beta_k = \log \|\bigwedge^k f\|_{\text{op}} - \log \|\bigwedge^{k-1} f\|_{\text{op}}.$$
2.3.2 Hitchin WKB-problem

Let $X$ be any complex curve. Let $\phi_i (i=1, \ldots, r)$ be holomorphic 1-forms on $X$. Let $[0, 1] := \{ s \in \mathbb{R} | 0 \leq s \leq 1 \}$ be the closed interval. Let $\gamma : [0, 1] \rightarrow X$ be a $C^1$-map. Suppose that it is a non-critical path in the sense of [7], i.e., the following holds:

- At any point $s \in [0, 1]$, we have $\gamma^* \text{Re}(\phi_i)_s \neq \gamma^* \text{Re}(\phi_j)_s$ ($i \neq j$).

We have the expression $\gamma^*(\phi_i)_x = a_i(s) ds$ for some $C^\infty$-functions $a_i : [0, 1] \rightarrow \mathbb{C}$. We may assume $\text{Re} a_i(s) < \text{Re} a_j(s)$ ($i < j$) for any $s$. We set

$$\alpha_i := -\int_0^1 \text{Re}(a_i) ds.$$ 

We have $\alpha_1 > \alpha_2 > \cdots > \alpha_r$.

Let $(E, \bar{\nabla}_E, h)$ be a harmonic bundle of rank $r$ on $X$. We suppose the following.

- We have the decomposition $(E, \bar{\nabla}_E, \theta) = \bigoplus_{i=1}^r (E_i, \bar{\nabla}_{E_i}, t_i \phi_i id_{E_i})$ for some $t > 0$, where rank $E_i = 1$.

We have the associated flat connection $\mathbb{D}^1 = \bar{\nabla}_E + \partial_E + \theta + \theta^\dagger$ on $E$. Let $\Pi_{\gamma} : E_{\gamma(0)} \rightarrow E_{\gamma(1)}$ be the parallel transport of $\mathbb{D}^1$ along $\gamma$. We have the metrics $h_{\gamma(0)}$ on the fibers $E_{\gamma(k)}$ ($k = 0, 1$) induced by $h$. We obtain the metric $\Pi_{\gamma}^* h_{\gamma(1)}$ on $E_{\gamma(0)}$ induced by $h_{\gamma(1)}$ and $\Pi_{\gamma}$.

The following theorem was conjectured in [7], and some cases were verified in [3].

**Theorem 2.17** There exist positive constants $t_0$, $\epsilon_0$ and $C_0$, which may depend only on $X$, $\phi_1, \ldots, \phi_r$ and $\gamma$, such that the following holds if $t \geq t_0$:

$$\left| \frac{1}{t} \tilde{d}(h_{\gamma(0)}, \Pi_{\gamma}^* h_{\gamma(1)}) - (2\alpha_1, \ldots, 2\alpha_r) \right| \leq C_0 \exp(-\epsilon_0 t)$$

**Proof** In the following, the constants $C_i$ and $\epsilon_i$ may depend only on $X$, $\phi_1, \ldots, \phi_r$ and $\gamma$, unless otherwise specified.

We can take finite points $s_0 = 0 < s_1 < \cdots < s_{N-1} < s_N = 1$ and coordinate neighbourhoods $(U_k, z_k)$ around $\gamma(s_k)$ such that the following holds:

- We can take $R_k > 0$ ($k = 0, \ldots, N$) such that $U_k$ contains the disc $\Delta_k(R_k) := \{|z_k| < R_k\}$ and that $\Delta_k(R_k/2) := \{|z_k| < R_k/2\}$ ($k = 0, \ldots, N$) give a covering of $\gamma([0, 1])$.

- We have the expressions $\phi_i|_{U_k} = f_{k,i} dz_k$. Set $d_k := \min \{|f_{k,i}(0) - f_{k,i}(0)|, |i \neq j| \}$. Then, we have $|f_{k,i}(z_k) - f_{k,i}(0)| \leq d_k/100$ on $\Delta_k(R_k)$.

If $t_0$ is sufficiently large, we have $t_0 d_k \geq 1$ for any $k$. Then, Assumption 2.2 is satisfied for the harmonic bundles $(E, \bar{\nabla}_E, \theta, h)|_{\Delta_k(R_k)}$ if $t \geq t_0$. So, we can apply the results in §2.1–2.2 to each $(E, \bar{\nabla}_E, \theta, h)|_{\Delta_k(R_k)}$.

Let $h_{E_i}$ be the restriction of $h$ to $E_i$. Let $\nabla_i$ be the Chern connection of $(E_i, \bar{\nabla}_{E_i}, h_{E_i})$, and we have the following connection on $E$:

$$\mathbb{D}^1_{\gamma} = \bigoplus_{i=1}^r (\nabla_i + t(\phi_i + \overline{\phi_i}) id_{E_i})$$

We fix a Kähler metric $g_X$ of $X$. By Corollary 2.13, we have the following estimate on the union of $\Delta_k(R_k/2)$ with respect to $h$ and $g_X$ for some positive constants $C_1$ and $\epsilon_1$:

$$|\mathbb{D}^1 - \mathbb{D}^1_{\gamma}|_{h, g_X} \leq C_1 \exp(-\epsilon_1 t)$$

We have the vector bundle $\gamma^* E = \bigoplus_{i=1}^r \gamma^* E_i$ with the metric $\gamma^* h$ and the connections $\gamma^* \mathbb{D}^1$ and $\gamma^* \mathbb{D}^1_i$. We take any orthonormal frames $u_i$ of $\gamma^* E_i$ such that $\gamma^* \nabla_i u_i = 0$. They give a frame $u = (u_1, \ldots, u_r)$ of $\gamma^* E$. We have the following estimates for some positive constants $C_2$ and $\epsilon_2$:

$$|\gamma^* h(u_i, u_j)| \leq C_2 \exp(-\epsilon_2 t), \quad (i \neq j).$$
The connection \( \gamma^* \mathbb{D}_I^1 \) is represented by the diagonal matrix \( A_{ds} \) with respect to the frame \( u \), where the \((i, i)\)-entry of \( A \) is \( 2t \Re a_i(s) \). Hence, we have

\[
\gamma^* \mathbb{D}_I^1 u = u(A(s) + B_0(s) + B_1(s)) \, ds
\]

where \( B_m(s) \) (\( m = 0, 1 \)) are \( r \)-square matrix valued \( C^\infty \)-functions such that (i) \( B_0(s)_{ij} = 0 \) (\( i \neq j \)) and \( B_1(s)_{ij} = 0 \) (\( i = j \)), (ii) there exist \( c_2 > 0 \) and \( C_3 > 0 \) such that

\[
|B_m(s)| \leq C_3 \exp(-c_3 t) \quad (m = 0, 1).
\]

We may assume that \( C_3 \exp(-c_3 t) \) is sufficiently small for any \( t \geq t_0 \) if \( t_0 \) is sufficiently large. Then, applying Corollary 2.19 in \$2.4\) below, we have a \( C^1 \)-function \( G : [0, 1] \rightarrow M_r(\mathbb{C}) \) and a \( C^0 \)-function \( H : [0, 1] \rightarrow M_r(\mathbb{C}) \) such that the following holds for some positive constants \( C_4 \) and \( c_4 \):

- The \( C^1 \)-norm of \( G \) is dominated by \( C_4 \exp(-c_4 t) \).
- \( H(s) \) are diagonal, and the \( C^0 \)-norm of \( H \) is dominated by \( C_4 \exp(-c_4 t) \).
- The connection \( \gamma^* \mathbb{D}_I^1 \) is represented by \( (A + B_0 + H) \, ds \) with respect to the frame \( v := u(I + G) \), where \( I \) denotes the \( r \)-square identity matrix.

The frame \( v \exp\left(-\int_{s}^{0} (A + B_0 + H) \, d\tau \right) \) of \( \gamma^* E \) is flat with respect to \( \gamma^* \mathbb{D}_I^1 \). The parallel transport \( \Pi_{\gamma} \) of \( \gamma^* \mathbb{D}_I^1 \) from \( E_{\gamma(0)} \) to \( E_{\gamma(1)} \) is represented by the following matrix with respect to \( u(0) \) and \( u(1) \):

\[
(I + G(1)) \exp\left(-\int_{0}^{1} (A + B_0 + H) \, d\tau \right)(I + G(0))^{-1}
\]

Here, \( I \) denotes the \( r \)-square identity matrix.

Let \( p(s) = (p_1(s), \ldots, p_r(s)) \) \((s = 0, 1)\) be the orthonormal frames of \( E_{\gamma(s)} \), induced by the frames \( u(s) \) and the Gram-Schmidt process. We have \( p(s) = u(s) \cdot (I + K(s)) \), where we have positive constants \( C_5 \) and \( c_5 \) such that \( |K(s)| \leq C_5 \exp(-c_5 t) \). Let \( L(s) \) \((s = 0, 1)\) be determined by \( L(s) = (I + K(s))(I + G(s)) - I \). Then, \( \Pi_{\gamma} \) is represented by the following matrix with respect to the orthonormal bases \( p(0) \) and \( p(1) \):

\[
Z_{\gamma} := (I + L(1)) \exp\left(-\int_{0}^{1} (A + B_0 + H) \, d\tau \right)(I + L(0))^{-1}
\]

For any \( r \)-square matrix \( Y \), we set

\[
\|Y\|_{op} := \sup\left\{ |Yv| \mid v \in \mathbb{C}^r, |v| = 1 \right\}.
\]

We clearly have \( \|Y_1 Y_2\|_{op} \leq \|Y_1\|_{op}\|Y_2\|_{op} \). So, we have positive constants \( C_6 \) and \( c_6 \), such that the following holds:

\[
\log \|Z_{\gamma}\|_{op} \leq \log \|\exp\left(-\int_{0}^{1} (A + B_0 + H) \, d\tau \right)\|_{op} + \rho_1, \quad |\rho_1| \leq C_6 \exp(-c_6 t)
\]

We also have positive constants \( C_7 \) and \( c_7 \), such that the following holds:

\[
\log \|\exp\left(-\int_{0}^{1} (A + B_0 + H) \, d\tau \right)\|_{op} \leq 2\alpha_1 + \rho_2, \quad |\rho_2| \leq C_7 \exp(-c_7 t)
\]

We also have positive constants \( C_8 \) and \( c_8 \), such that the following holds:

\[
|\Pi_{\gamma}p_1(0)|_{h_{\gamma(1)}} = \exp(2t\alpha_1) \cdot (1 + \rho_3), \quad |\rho_3| \leq C_8 \exp(-c_8 t)
\]

Hence, we have positive constants \( C_9 \) and \( c_9 \), such that the following holds:

\[
\log \|Z_{\gamma}\|_{op} \geq 2\alpha_1 + \rho_4, \quad |\rho_4| \leq C_9 \exp(-c_9 t)
\]
Therefore, we have positive constants $C_{10}$ and $\epsilon_{10}$, such that the following holds:

$$\left| \log \|Z_\gamma\|_{op} - 2t\alpha_1 \right| \leq C_{10} \exp(-\epsilon_{10}t)$$

By applying the argument to $\bigwedge^k Z_\gamma$, we obtain positive constants $C_{11}$ and $\epsilon_{11}$, such that the following holds for any $k$:

$$\left| \log \|\bigwedge^k Z_\gamma\|_{op} - 2t \sum_{j=1}^k \alpha_j \right| \leq C_{11} \exp(-\epsilon_{11}t) \quad (12)$$

By using (11), we can deduce the claim of the theorem from (12).

\[\square\]

### 2.4 Appendix: A singular perturbation theory

We explain a singular perturbation theory which is available in our situation, and which seems slightly different from those in [3] and [7]. Note that we applied Corollary 2.19 in the proof of Theorem 2.17 to find a family of small gauge transforms with which the family of connections $\gamma^*D$ on $[0,1]$ are transformed to the connections whose connection matrices are diagonal.

#### 2.4.1 Preliminary

We set $[0,1] := \{ s \in \mathbb{R} \mid 0 \leq s \leq 1 \}$. For any non-negative integer $\ell$, let $C^\ell([0,1])$ denote the space of $C^\ell$-functions on $[0,1]$. We set $\|f\|_0 := \sup_{s \in [0,1]} |f(s)|$ for any $f \in C^0([0,1])$.

Let $M_r(\mathbb{C})$ denote the space of $r$-square matrices. Let $M_r(\mathbb{C})_0$ denote the space of the $r$-square diagonal matrices, i.e., $M_r(\mathbb{C})_0 = \{ (a_{ij}) \in M_r(\mathbb{C}) \mid a_{ij} = 0 \ (i \neq j) \}$. Let $M_r(\mathbb{C})_1$ denote the set of the off-diagonal matrices, i.e., $M_r(\mathbb{C})_1 := \{ (a_{ij}) \in M_r(\mathbb{C}) \mid a_{ij} = 0 \ (i = j) \}$. We have $M_r(\mathbb{C}) = M_r(\mathbb{C})_0 \oplus M_r(\mathbb{C})_1$.

For any non-negative integer $\ell$, let $C^\ell([0,1], M_r(\mathbb{C}))$ denote the space of $C^\ell$-maps $X : [0,1] \rightarrow M_r(\mathbb{C})$. Similarly, let $C^\ell([0,1], M_r(\mathbb{C})_\kappa)$ $(\kappa = 0,1)$ be the space of $C^\ell$-maps $X : [0,1] \rightarrow M_r(\mathbb{C})_\kappa$. We set $\|X\|_0 := \sup_{s \in [0,1]} \|X(s)\|$ for $X \in C^0([0,1], M_r(\mathbb{C}))$.

#### 2.4.2 Statement

Fix $C_0 > 0$. We consider $a_j, b_j \in C^0([0,1])$ $(j = 1, \ldots, r)$ satisfying the following conditions:

- $\text{Re} a_1(s) < \text{Re} a_2(s) < \cdots < \text{Re} a_r(s)$ for any $s$.
- $|b_j(s)| \leq C_0$ for any $s$.

For any $t \geq 0$, we put $a_j^t(s) := ta_j(s) + b_j(s)$. Let $A^t$ denote the $r$-square matrix whose $(j,j)$-entries are $a_j^t$.

**Proposition 2.18** There exist constants $C_1 > 0$ and $\epsilon_1 > 0$, depending only on $C_0$, such that the following holds:

- For any $t \geq 0$ and any $B \in C^0([0,1], M_r(\mathbb{C})_1)$ satisfying $\|B\|_0 \leq \epsilon_1$, we can take $G^t \in C^1([0,1], M_r(\mathbb{C})_1)$ and $H^t \in C^0([0,1], M_r(\mathbb{C})_0)$ such that (i) $\|G^t\|_0 + \|\partial_s G^t + [A^t, G^t]\|_0 + \|H^t\|_0 \leq C_1\|B\|_0$, (ii) we have

$$A^t + B = (I + G^t)^{-1}(A^t + H^t)(I + G^t) + (I + G^t)^{-1}\partial_s G^t.$$  \quad (13)

Here, $I$ denotes the $r$-square identity matrix.

We shall prove the proposition in §2.4.3–2.4.4. Indeed, we shall give a more refined claim (see Corollary 2.23).

We give a reformulation of Proposition 2.18. Recall that when we have a vector bundle $V$ with a connection $\nabla$ and a frame $w = (w_1, \ldots, w_m)$, we have the matrix-valued 1-form $A = (A_{ij})$ determined by $\nabla w_j = \sum A_{ij} w_i$, and we describe the relation by $\nabla w = w A$. 

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Corollary 2.19 Let $E$ be a $C^1$-vector bundle on $[0,1]$ with a frame $v = (v_1, \ldots, v_r)$. Let $B \in C^0([0,1], M_r(C_1))$ satisfying \(\|B\|_0 \leq \epsilon_1\). Take $t \geq 0$. Let $\nabla^t$ be the connection on $E$ given as follows:
\[
\nabla^t v = v(A^t(s) + B(s))ds.
\]
Then, we can take $G^t \in C^1([0,1], M_r(C_1))$ and $H^t \in C^0([0,1], M_r(C_1))$ such that (i) \(\|G^t\|_0 + \|\partial_s G^t + [A^t, G^t]\|_0 + \|H^t\|_0 \leq C_1 \|B\|_0\), (ii) for the frame $u^t = v \cdot (1 + G^t)^{-1}$, we have $\nabla^t u^t = u^t \cdot (A^t + H^t)ds$. Here, $\epsilon_1$ and $C_1$ are constants as in Proposition 2.18.

Remark 2.20 In Proposition 2.18, the off-diagonal part $B$ is required to be sufficiently small. It is satisfied in the Hitchin WKB-problem if $t$ is sufficiently large, as observed in Corollary 2.13. However, it is not satisfied in the Riemann-Hilbert WKB-problem [7] even if $t$ is large, in general. But, we may still apply Proposition 2.18 after dividing the path to shorter paths $\dag$ as did in [7]. It also seems possible to apply Proposition 2.18 to the WKB-problem for family of $\lambda$-connections without going to small paths.

2.4.3 Some linear maps

Let $C^1([0,1], M_r(C_1))_0$ denote the subspace of $C^1([0,1], M_r(C_1))$ which consists of the functions $X = (X_{i,j}) : [0,1] \to M_r(C_1)$ such that $X_{i,j}(1) = 0$ ($i < j$) and $X_{i,j}(0) = 0$ ($i > j$). We have the linear map $D_0^t : C^1([0,1], M_r(C_1))_0 \to C^0([0,1], M_r(C_1))_0$ given by
\[
D_0^t(X) := \partial_s X + [A^t, X], \quad \text{i.e., } D_0^t(X)_{i,j} = \partial_s X_{i,j} + (\alpha_i - \alpha_j) X_{i,j}.
\]

For $i \neq j$, we put
\[
F_{i,j}^t(s) := \int_s^1 (\alpha_i(\tau) - \alpha_j(\tau)) d\tau.
\]

We have the map $I_0^t : C^0([0,1], M_r(C_1)) \to C^1([0,1], M_r(C_1))_0$ given as follows:
\[
I_0^t(X)(i,j) := \int_0^1 \exp(-F_{i,j}^t(s) + F_{i,j}^t(\tau)) X_{i,j}(\tau) d\tau \quad (i < j)
\]
\[
\int_0^1 \exp(-F_{i,j}^t(s) + F_{i,j}^t(\tau)) X_{i,j}(\tau) d\tau \quad (i > j)
\]

Together with the identity map on $C^0([0,1], M_r(C_1))_0$, we also obtain the following maps:
\[
I_0^t : C^0([0,1], M_r(C_1))_0 \oplus C^0([0,1], M_r(C_1)) \to C^0([0,1], M_r(C_1))_0 \oplus C^1([0,1], M_r(C_1))_0
\]
\[
D_0^t : C^0([0,1], M_r(C_1))_0 \oplus C^1([0,1], M_r(C_1))_0 \to C^0([0,1], M_r(C_1))_0 \oplus C^0([0,1], M_r(C_1))_0
\]

Then, $I_0^t$ and $D_0^t$ are mutually inverse.

Lemma 2.21 There exists a constant $K_1 > 0$, depending only on $C_0$, such that the following holds for any $t \geq 0$ and for any $(Z, W) \in C^0([0,1], M_r(C_1))_0 \oplus C^0([0,1], M_r(C_1))_0$:
\[
\|I_0^t(Z, W)\|_0 \leq K_1 \|Z, W\|_0.
\]

Proof The estimate for $(i,i)$-entries are obvious by the construction. Let us consider the estimate for $(i,j)$-entries $(i \neq j)$. We set $Q_{i,j}^t(s) := t \int_0^1 (a_i(\tau) - a_j(\tau)) d\tau$ and $R_{i,j}(s) := \int_0^1 (b_i(\tau) - b_j(\tau)) d\tau$. We have $F_{i,j}^t = Q_{i,j}^t + R_{i,j}$. Because $\text{Re}(a_k) < \text{Re}(a_\ell)$ ($k < \ell$), we have the following for $s_1 \leq s_2$:
\[
\text{Re} Q_{i,j}^t(s_1) - \text{Re} Q_{i,j}^t(s_2) \geq 0 \quad (i < j), \quad \text{Re} Q_{i,j}^t(s_1) - \text{Re} Q_{i,j}^t(s_2) \leq 0 \quad (i > j).
\]

Hence, we have a constant $K_1'$, depending only on $C_0$, such that $\left|\exp(-F_{i,j}^t(s) + F_{i,j}^t(\tau))\right| \leq K_1'$ holds in the cases (i) $s \leq \tau$ and $i < j$, (ii) $s \geq \tau$ and $i > j$. Then, we obtain the following in the case $i < j$:
\[
\left|I_0^t(X)_{i,j}\right| \leq \int_s^1 \left|\exp(-F_{i,j}^t(s) + F_{i,j}^t(\tau)) X_{i,j}(\tau)\right| d\tau \leq K_1' \int_s^1 \left|X_{i,j}(\tau)\right| d\tau \leq K_1' \|X_{i,j}\|_0
\]

Hence, we obtain $\|I_0^t(X)_{i,j}\|_0 \leq K_1' \|X_{i,j}\|_0$ in the case $i < j$. Similarly, we obtain $\|I_0^t(X)_{i,j}\|_0 \leq K_1' \|X_{i,j}\|_0$ in the case $i > j$.

This was remarked by C. Simpson.
2.4.4 Proof of Proposition 2.18

We take a small number $\epsilon > 0$. We set

$$\mathcal{H}_\epsilon := \{ H \in C^0([0,1], M_r(C)_0) \mid \|H\|_0 \leq \epsilon \}, \quad \mathcal{G}_\epsilon^r := \{ G \in C^1([0,1], M_r(C)_1)_0 \mid \|D_0^t G\|_0 \leq \epsilon \}.$$

Note that $\|G\|_0 \leq K_1 \|D_0^t G\|_0$ for any $G \in \mathcal{G}_\epsilon^r$. Let $I \in M_r(C)$ denote the identity matrix. If $\epsilon K_1 < 1/2$, then $(I + G)(s)$ is invertible for any $G \in \mathcal{G}_\epsilon^r$. So, we have the maps $J^t : \mathcal{H}_\epsilon \times \mathcal{G}_\epsilon^r \rightarrow C^0([0,1], M_r(C)_0) \oplus C^0([0,1], M_r(C)_1)$ given by

$$J^t(H,G) := (I + G)^{-1}(A^t + H)(I + G) + (I + G)^{-1} \partial_s G - A^t.$$

Let $T_{(H,G)} J^t$ denote the derivative at $(H,G)$. We have $T_{(0,0)} J^t(X,Y) = X + D_0^t Y$. More generally, we have

$$T_{(H,G)} J^t(X,Y) = (I + G)^{-1}X(I + G) + (I + G)^{-1}(A^t + H)Y - (I + G)^{-1}Y(I + G)^{-1}(A^t + H)(I + G) - (I + G)^{-1}Y(I + G)^{-1} \partial_s G + (I + G)^{-1} \partial_s Y$$

(14)

We regard $T_{(H,G)} J^t$ as maps $C^0([0,1], M_r(C)_0) \oplus C^1([0,1], M_r(C)_1)$ to $C^0([0,1], M_r(C)_0) \oplus C^0([0,1], M_r(C)_1)$. We obtain the following family of endomorphisms on $C^0([0,1], M_r(C)_0) \oplus C^0([0,1], M_r(C)_1)$:

$$T_{(H,G)} J^t \circ (T_{(0,0)} J^t)^{-1}, \quad (H,G) \in \mathcal{H}_\epsilon \times \mathcal{G}_\epsilon^r.$$

Lemma 2.22 There exists a constant $C_2 > 0$, depending only on $C_0$, such that the following holds for any $t \geq 0$ and any $(Z,W) \in C^0([0,1], M_r(C)_0) \oplus C^0([0,1], M_r(C)_1)$:

$$\|T_{(H,G)} J^t \circ (T_{(0,0)} J^t)^{-1}(Z,W) - (Z,W)\|_0 \leq C_2(\|H\|_0 + \|D_0^t G\|_0)(\|Z\|_0 + \|W\|_0)$$

Proof We consider

$$(I + G)^{-1}(A^t)Y - (I + G)^{-1}Y(I + G)^{-1}(A^t)(I + G) - (I + G)^{-1}Y(I + G)^{-1} \partial_s G + (I + G)^{-1} \partial_s Y - D_0^t Y.$$ \hspace{0.5cm} (15)

We have $(I + G)^{-1} \partial_s Y = (I + G)^{-1}D_0^t Y - (I + G)^{-1}[A^t, Y]$. Hence, (15) is rewritten as

$$(I + G)^{-1}(A^t)Y - (I + G)^{-1}Y(I + G)^{-1}(A^t)(I + G) - (I + G)^{-1}Y(I + G)^{-1} \partial_s G - (I + G)^{-1}[A^t, Y] + ((I + G)^{-1} - I)D_0^t Y.$$ \hspace{0.5cm} (16)

It is equal to the following:

$$(I + G)^{-1}Y(A^t) - (I + G)^{-1}Y(I + G)^{-1}(A^t)(I + G) - (I + G)^{-1}Y(I + G)^{-1} \partial_s G + ((I + G)^{-1} - I)D_0^t Y.$$ \hspace{0.5cm} (17)

We have the following:

$$- (I + G)^{-1}Y(I + G)^{-1}(A^t)(I + G) - (I + G)^{-1}Y(I + G)^{-1} \partial_s G$$

$$= - (I + G)^{-1}Y(I + G)^{-1}(A^t) - (I + G)^{-1}Y(I + G)^{-1}(\partial_s G + (A^t)G)$$

$$= -(I + G)^{-1}Y(I + G)^{-1}(A^t) - (I + G)^{-1}Y(I + G)^{-1}G(A^t) - (I + G)^{-1}Y(I + G)^{-1}D_0^t G.$$ \hspace{0.5cm} (18)

Hence, (17) is equal to the following:

$$- (I + G)^{-1}Y(I + G)^{-1}(A^t) - (I + G)^{-1}Y(I + G)^{-1}G(A^t)$$

$$- (I + G)^{-1}Y(I + G)^{-1}D_0^t G + ((I + G)^{-1} - I)D_0^t Y$$

$$= -(I + G)^{-1}Y(I + G)^{-1}D_0^t G + ((I + G)^{-1} - I)D_0^t Y.$$ \hspace{0.5cm} (19)
In all, we obtain the following:

\[
T_{(H^i,G)}^{-1}(Z,W) - (Z,W) = -(I + G)^{-1} T_0(W)(I + G)^{-1} D_0 G - ((I + G)^{-1} - I) W \\
+ (I + G)^{-1} Z(I + G) - Z + [(I + G)^{-1} H(I + G), (I + G)^{-1} T_0(W)]
\]

(20)

Because \(\|G\|_0 \leq K_1 \|D_0^2 G\|_0\), we have \(\|(I + G)^{-1} - I\|_0 \leq C_3 \|D_0^2 G\|_0\) and \(\|(I + G)^{-1} X(I + G) - X\|_0 \leq C_4 \|D_0^2 G\|_0 \|X\|_0\) for positive constants \(C_i\) (\(i = 3, 4\)) depending only on \(C_0\). We also have the following for a positive constant \(C_5\) depending only on \(C_0\):

\[
\left\|\left[ (I + G)^{-1} H(I + G), (I + G)^{-1} Y \right]\right\|_0 \leq C_5 \left( \|D_0^2 G\|_0 + \|H\|_0 \right) \|D_0^2 Y\|_0
\]

Hence, we obtain the claim of the lemma.

Corollary 2.23 There exist positive constants \(\epsilon_{10} > 0\) and \(C_{10} > 0\), depending only on \(C_0\), with the following property:

- For any \(t \geq 0\) and any \(B \in C^0([0,1], M_r(C_1))\) such that \(\|B\|_0 \leq \epsilon_{10}\), we have a unique \((H^t, G^t) \in H_c \times G_c^t\) such that \(J^t(H^t, G^t) = B\) and that \(\|H^t\|_0 + \|D_0^2 G^t\|_0 \leq C_{10} \|B\|_0\).

Proof We set \(\Xi_c := \{G \in C^0([0,1], M_r(C_1)) \mid \|G\|_0 \leq \epsilon\} \subset C^0([0,1], M_r(C_1))\). We have the bijections \(I_0^t : \Xi_c \sim G_c^t\) for any \(t \geq 0\). We consider the maps

\[
\mathcal{F}^t := J^t \circ (T_{(0,0)} J^t)^{-1} : H_c \times \Xi_c \rightarrow C^0([0,1], M_r(C_0)) \subset C^0([0,1], M_r(C_1)).
\]

Let \(T \mathcal{F} : C^0([0,1], M_r(C_1)) \rightarrow C^0([0,1], M_r(C))\) denote the derivative of \(\mathcal{F}^t\). According to Lemma 2.22, we have a positive constant \(C_3\) such that the operator norms of \(T_{(H_c, G_c)} \mathcal{F} - \text{id}\) are dominated by \(C_3 \left( \|H\|_0 + \|G\|_0\right)\) for any \((H, G) \in H_c \times \Xi_c\). The constant \(C_3\) may depend only on \(C_0\), and the estimate is uniform for \(t\). By the inverse function theorem (see [8], for instance), there exist positive constants \(\epsilon_{10}\) and \(C_{10}\), depending only on \(C_0\), with the following property:

- For any \(t \geq 0\) and any \(B \in C^0([0,1], M_r(C_1))\) such that \(\|B\|_0 \leq \epsilon_{10}\), we have a unique \((H^t, G^t) \in H_c \times \Xi_c\) such that \(\mathcal{F}^t(H^t, G^t) = B\) and that \(\|H^t\|_0 + \|G^t\|_0 \leq C_{10} \|B\|_0\).

By setting \(G^t := (T_{(0,0)} J^t)^{-1} G\), we obtain the claim of Corollary 2.23. We also finish the proof of Proposition 2.18.

2.5 Appendix: The case of Hermitian-Einstein metrics

For \(R > 0\), we set \(\Delta(R) := \{z \in \mathbb{C} \mid |z| < R\}\). Let \((E, \overline{\mathcal{E}}_E, \theta)\) be a Higgs bundle of rank \(r\) on \(\Delta(R)\). We fix a Hermitian metric \(h_{\text{det}(E)}\) on \(\text{det}(E)\). Let \(h\) be a Hermitian-Einstein metric of \((E, \overline{\mathcal{E}}_E, \theta)\) such that \(\det(h) = h_{\text{det}(E)}\), i.e., \(R(h)^{\perp} + [\theta, \theta]^{\perp} = 0\), where \(R(h)^{\perp}\) denotes the trace-free part of the curvature \(R(h)\). We have an obvious generalization of the results in §2.1 and §2.2, which we state explicitly in this subsection.

We have a \(C^\infty\)-function \(\nu\) such that \(\overline{\partial} \nu = R(h_{\text{det}(E)})/r\) such that the \(L^p\)-norm of \(\nu\) is dominated by \(C_{0,p} \|R(h_{\text{det}(E)})\|_\infty (p > 1)\), where \(\|R(h_{\text{det}(E)})\|_\infty\) denotes the sup norm of \(R(h_{\text{det}(E)})\) with respect to the Euclidean metric, and \(C_{0,p}\) denotes the constant depending on \(p\) and the radius \(R\). Then, the metric \(\tilde{h} = he^{-\nu}\) is a harmonic metric for the Higgs bundle \((E, \overline{\mathcal{E}}_E, \theta)\). Note that \(h\) and \(\tilde{h}\) induce the same metrics on the vector bundles \(\text{End}(E) \otimes \Omega^{p,q}\).

We have the description \(\theta = f dz\). Let \(M\) be the constant as in §2.1.1. We obtain the following from Proposition 2.1.

Corollary 2.24 Fix \(0 < R_1 < R\). Then, we have \(C_1, C_2 > 0\) depending only on \(r, R_1, R\) such that \(|f|_h \leq C_1 M + C_2\) holds on \(\Delta(R_1)\).

We impose the conditions as in Assumption 2.2. We obtain the following from Proposition 2.3.
Corollary 2.25 Let $R_1$ be as in Proposition 2.1. Fix $0 < R_2 < R_1$. There exist positive constants $\epsilon_0$ and $C_{11}$ depending only on $R$, $R_1$, $R_2$, $r$ and $C_{10}$, such that $|\rho_\alpha| \leq C_{11} \exp(-\epsilon_0d)$ on $\Delta(R_2)$.

We impose the condition that rank $E_\alpha = 1$ as in \S 2.2. We have $R(\tilde{h}) = R(h)^{\perp}$ and $\theta_h^{|} = \theta_{\tilde{h}}^{|}$. Hence, we obtain the following from Theorem 2.9.

Corollary 2.26 There exist positive constants $C_{30}$ and $\epsilon_{30}$, depending only on $R$, $R_1$, $R_2$, $r$ and $C_{10}$ such that $|R(h)^{\perp}|_{\rho_\alpha} \leq \epsilon_{30} \exp(-\epsilon_0d)$ on $\Delta(R_2)$.

Because $h$ and $\tilde{h}$ give the same connection on $\text{End}(E)$ as the Chern connections, we obtain the following from Proposition 2.10 and Proposition 2.11.

Corollary 2.27 Take $0 < R_3 < R_2$. There exist positive constants $\epsilon_{40}$ and $C_{10}$ depending only on $R$, $R_1$, $R_2$, $R_3$, $r$ and $C_{10}$ such that $|\partial_{E,h} \pi_\alpha|_{\rho_\alpha} = |\tilde{\partial}_{E,h} \pi_\alpha|_{\rho_\alpha} \leq C_{40} \exp(-\epsilon_{40d})$ on $\Delta(R_3)$.

Let $h_\alpha$ (resp. $h_{0,\alpha}$) denote the restriction of $h$ (resp. $\tilde{h}$) to $L_\alpha$. Because $h_\alpha = \tilde{h}_\alpha e^{\nu/r}$, we obtain the following from Proposition 2.12.

Corollary 2.28 We have a positive constant $C_{60}$ depending only on $R$, $R_1$, $R_2$, $R_3$, $r$ and $C_{10}$ such that $|R(h_\alpha) - R(h_{0,\alpha})|_{\rho_\alpha} \leq C_{60} \exp(-\epsilon_{40d})$ on $\Delta(R_3)$.

We have the projectively flat connection $\tilde{D}^{1} := \tilde{\partial}_{E} + \partial_{E,h} + \theta + \theta_{\tilde{h}}$. They are related as $D^{1} = \tilde{D}^{1} + \partial \nu \cdot \text{id}_{E}$. We have the Chern connections $\tilde{\partial}_{\alpha} + \partial_{\alpha,h}$ on $E_\alpha$ induced by $h_\alpha$. Similarly, we have the Chern connections $\tilde{\partial}_{\alpha} + \partial_{\alpha,h}$ on $E_\alpha$ induced by $\tilde{h}_\alpha$. They are related as $\partial_{\alpha,h} = \partial_{\alpha,h} + \partial \nu \cdot \text{id}_{E_\alpha}$. Hence, we obtain the following from Corollary 2.13.

Corollary 2.29 We set $D^{1} := \bigoplus (\tilde{\partial}_{\alpha} + \partial_{\alpha,h} + (g_\alpha dz + \bar{g}_\alpha d\bar{z}) \cdot \text{id}_{E_\alpha})$ as in \S 2.2.4. Then, there exists a positive constant $C_{70}$, depending only on $R$, $R_1$, $R_2$, $R_3$, $r$ and $C_{10}$ such that $|\tilde{D}^{1} - D^{1}|_{\rho_\alpha} \leq C_{70} \exp(-\epsilon_{40d})$ holds on $\Delta(R_3)$.

We also have an obvious generalization of the estimates of the higher derivatives in \S 2.2.5, which we omit to describe.

3 Local models

3.1 Review on unramifiedly good filtered Higgs bundles

3.1.1 Filtered bundles on curves

Let $X$ be a complex curve with a discrete subset $D$. We recall the concept of filtered bundles on $(X, D)$ [20]. Let $\mathbb{R}^{D}$ denote the set of maps $D \rightarrow \mathbb{R}$. Elements of $\mathbb{R}^{D}$ are denoted by $a = (a_P | P \in D)$. Let $O_X(*)$ be the sheaf of meromorphic functions on $X$ whose poles are contained in $D$. Let $E$ be a locally free $O_X(*)$-module of rank $r$. A filtered bundle over $E$ is a family of coherent $O_X$-submodules $\mathcal{P}_aE = (\mathcal{P}_aE \subset E | a \in \mathbb{R}^{D})$ with the following property.

- $\mathcal{P}_aE$ are lattices of $E$, i.e., $\mathcal{P}_aE \otimes_{O_X} O_X(*) = E$.

- The stalk of $\mathcal{P}_aE$ at $P \in D$ depends only on $a_{(P)} \in \mathbb{R}$. We denote it by $\mathcal{P}^{P}_{a_{{(P)}}}(E_P)$.

- We have $\mathcal{P}^{P}_{a_{{(P)}}}(E_P) \subset \mathcal{P}^{P}_{b_{{(P)}}}(E_P)$ for $a \leq b$. For any $a \in \mathbb{R}$, there exists $\epsilon > 0$ such that $\mathcal{P}^{P}_{a}(E_P) = \mathcal{P}^{P}_{a + \epsilon}(E_P)$.

- For $n \in \mathbb{Z}$, we have $\mathcal{P}^{P}_{a}(E_P) \otimes_{O_X} O_X(nP) = \mathcal{P}^{P}_{a + n}(E_P)$.

Such $\mathcal{P}_aE$ is called a filtered bundle over $E$ on $(X, D)$. For any $O_X(*)$-submodule $\mathcal{G} \subset E$, we have the induced filtered bundle $\mathcal{P}_a\mathcal{G}$ over $\mathcal{G}$ given by $\mathcal{P}_a\mathcal{G} = \mathcal{P}_aE \cap \mathcal{G}$ in $E$. 23
Parabolic structure Let \( \mathcal{P} \mathcal{E} \) be a filtered bundle on \((X, D)\). Let \( 0 \in \mathbb{R}^D \) denote the element such that the \( P \)-th component of \( 0 \) are 0 for any \( P \in D \). We have the locally free \( O_X \)-module \( \mathcal{P} \mathcal{E} \) on \( X \). For each \( Q \in X \), let \( \mathcal{P} \mathcal{E}_Q \) denote the fiber of the vector bundle \( \mathcal{P} \mathcal{E} \) over \( Q \). For \( P \in D \) and for \(-1 < a \leq 0\), we set
\[
F_a^P(\mathcal{P} \mathcal{E}_P) := \text{Im} \left( (P \mathcal{E}_P) \longrightarrow \mathcal{P} \mathcal{E}_P \right).
\]
In this way, we obtain a filtration \( F^P \) of \( \mathcal{P} \mathcal{E}_P \) indexed by \( \{-1 < a \leq 0\} \). We have \( F_0^P(\mathcal{P} \mathcal{E}_P) = \mathcal{P} \mathcal{E}_P \). Note that we have the natural identification \( \mathcal{P} \mathcal{E}_P \cong F_a^P(\mathcal{P} \mathcal{E}_P)/F_{a+1}^P(\mathcal{P} \mathcal{E}_P) \) because \( \bar{\mathcal{P}}_{a+1}^P(\mathcal{E}) = \bar{\mathcal{P}}_a^P(\mathcal{E}) \otimes \bar{O}_{X,P} \otimes \mathcal{O}_{X,P}(nP) \), and hence we have \( F_{-1}^P(\mathcal{P} \mathcal{E}_P) = 0 \). For any \(-1 < a \leq 0\), there exists \( \epsilon > 0 \) such that \( F_a^P(\mathcal{P} \mathcal{E}_P) = F_{a+\epsilon}^P(\mathcal{P} \mathcal{E}_P) \).

\subsection{3.1.2 Unramifiedly good filtered Higgs bundles}

Let \( X \) and \( D \) be as in §3.1.1. Let \( \mathcal{P} \mathcal{E} \) be a filtered bundle on \((X, D)\). Let \( \theta \) be a Higgs field of \( \mathcal{E} \), i.e., \( \theta \) is a holomorphic section of \( \text{End}(\mathcal{E}) \otimes \Omega_X \). The filtered bundle with a Higgs field \((\mathcal{P} \mathcal{E}, \theta)\) is called unramifiedly good at \( P \), if the following holds.

- We have a finite subset \( \mathcal{I}(P) \subset \mathcal{O}_X(*D)_P \) and a decomposition of the stalk \( \mathcal{P}_a^P \mathcal{E}_P = \bigoplus_{a \in \mathcal{I}(P)} \mathcal{P}_a^P \mathcal{E}_{P,a} \) such that
\[
(\theta - da d\bar{a})\mathcal{P}_a^P \mathcal{E}_{P,a} \subset \mathcal{P}_a^P \mathcal{E}_{P,a} \otimes \Omega_X^1(\log D)_P.
\]

Here, \( da \) denotes the exterior derivative of \( a \in \mathcal{I}(P) \subset \mathcal{O}_X(*D)_P \). We have \( \mathcal{P}_a^P \mathcal{E}_{P,a} \subset \mathcal{P}_b^P \mathcal{E}_{P,a} \) for \( a \leq b \).

The induced map \( \mathcal{I}(P) \rightarrow \mathcal{O}_X(*D)_P/\mathcal{O}_{X,P} \) is assumed to be injective.

If \((\mathcal{P} \mathcal{E}, \theta)\) is unramifiedly good at any \( P \in D \), it is called an unramifiedly good filtered Higgs bundle.

\begin{remark}
More generally, a filtered bundle with a Higgs field \((\mathcal{P} \mathcal{E}, \theta)\) is called good at \( P \) if we have a neighbourhood \( X_P \) of \( P \), a Galois covering map \( X_P \rightarrow X_P \), and such a family of filtrations \( \{F^P\} \) is called a parabolic structure on the vector bundle \( \mathcal{P} \mathcal{E} \) along \( D \). We can easily observe that filtered bundles on \((X, D)\) are equivalent to vector bundles on \( X \) with a parabolic structure along \( D \).
\end{remark}

\subsection{3.1.3 Unramified wild harmonic bundles and the associated unramifiedly good filtered Higgs bundles}

Let \( X \) and \( D \) be as in §3.1.1. Let \((E, \bar{\mathcal{O}}_E, \theta, h)\) be a harmonic bundle on \( X \setminus D \). It is called wild and unramified over \((X, D)\) if the following holds for any \( P \in D \).

- Let \((U_P, z)\) be a holomorphic coordinate neighbourhood of \( P \) in \( X \) with \( z(P) = 0 \). Then, we have a finite subset \( \mathcal{I}(P) \subset z^{-1} \mathbb{C}[z^{-1}] \) and a decomposition
\[
(E, \bar{\mathcal{O}}_E, \theta)|_{U_P \setminus \{P\}} = \bigoplus_{a \in \mathcal{I}(P)} (E_a, \bar{\mathcal{O}}_{E_a}, \theta_a),
\]
and the coefficients \( a_{a,j}(z) \) of the characteristic polynomials \( \det(t \text{id} - g_a) = \sum a_{a,j}(z)t^j \) are holomorphic at \( z = 0 \), where \( g_a \in \text{End}(E_a) \) are determined by \( \theta_a - da d\bar{a} = g_a dz/z \).

A unramified wild harmonic bundle is called tame if \( \mathcal{I}(P) = \{0\} \) in \( \mathcal{O}_X(*D)_P/\mathcal{O}_{X,P} \) at each \( P \in D \).

The same case of the following proposition was established in [20], and generalized to the wild case in [16].

\begin{proposition}
Let \((E, \bar{\mathcal{O}}_E, \theta, h)\) be an unramified wild harmonic bundle on \((X, D)\). Then, we have an unramifiedly good filtered Higgs bundle \((\mathcal{P}^b \mathcal{E}, \theta)\) on \((X, D)\) with the following property.

- We have \((\bar{\mathcal{P}}^a_\mathcal{E}, \theta)|_{X \setminus D} = (E, \theta)\) for any \( a \in \mathbb{R}^D \).

- Let \( U \) be any open subset of \( X \). Let \( f \) be a holomorphic section of \( E \) on \( U \setminus D \). Then, \( f \) is a section of \( \bar{\mathcal{P}}^b_\mathcal{E} \) on \( U \) if and only if \( |f|_h = O(|z_P|^{-a}) \) for any \( a > 0 \) around each \( P \in U \cap D \), where \( z_P \) denotes a holomorphic coordinate around \( P \) with \( z(P) = 0 \).
\end{proposition}

In this situation, \( h \) is called adapted to the filtered bundle \( \bar{\mathcal{P}}^b_\mathcal{E} \).
3.1.4 Filtered Higgs bundles on compact Riemann surfaces

Suppose that $X$ is a compact connected Riemann surface with a finite subset $D$. Let $(\mathcal{P}_*, \mathcal{E}, \theta)$ be an unramifiedly good filtered Higgs bundle on $(X, D)$. Let $\mathcal{G} \subset \mathcal{E}$ be any locally free $\mathcal{O}_X(\ast D)$-submodule. We set $\mathcal{P}_a \mathcal{G} := \mathcal{P}_a \mathcal{E} \cap \mathcal{G}$ in $\mathcal{E}$ for any $a \in \mathbb{R}^D$, and then we obtain a filtered bundle $\mathcal{P}_a \mathcal{G}$ over $\mathcal{G}$. The degree of $\mathcal{P}_a \mathcal{G}$ is defined as follows [9, 10] for any $b \in \mathbb{R}^D$:

$$\deg(\mathcal{P}_a \mathcal{G}) = \deg(\mathcal{P}_b \mathcal{G}) - \sum_{P \in D} \sum_{b_P - 1 < a_P \leq b_P} a \cdot \dim_{\mathbb{C}}(\mathcal{P}_a^P(\mathcal{G}_P)/\mathcal{P}^{P}_{<a} \mathcal{G}_P))$$  \hspace{1cm} (21)

Here, we set $\mathcal{P}_c^P \mathcal{G}_P = \bigcup_{c<a} \mathcal{P}_c^P \mathcal{G}_P$, and we regard $\mathcal{P}_a^P(\mathcal{G}_P)/\mathcal{P}^{P}_{<a} \mathcal{G}_P)$ as finite dimensional $\mathbb{C}$-vector spaces in a natural way. It is easy to see that the right hand side of (21) is independent of the choice of $b$. The unramifiedly good filtered Higgs bundle $(\mathcal{P}_*, \mathcal{E}, \theta)$ is called stable (resp. semistable) if the following inequality holds for any locally free $\mathcal{O}_X(\ast D)$-submodule $\mathcal{G} \subset \mathcal{E}$ with (i) $\theta \mathcal{G} \subset \mathcal{G} \otimes \Omega^1_X$, (ii) $0 < \text{rank} \mathcal{G} < \text{rank} \mathcal{E}$:

$$\frac{\deg(\mathcal{P}_a \mathcal{G})}{\text{rank} \mathcal{G}} < \frac{\deg(\mathcal{P}_b \mathcal{E})}{\text{rank} \mathcal{E}} \quad \text{(resp.} \quad \frac{\deg(\mathcal{P}_a \mathcal{G})}{\text{rank} \mathcal{G}} \leq \frac{\deg(\mathcal{P}_b \mathcal{E})}{\text{rank} \mathcal{E}})$$

The unramifiedly good filtered Higgs bundle $(\mathcal{P}_*, \mathcal{E}, \theta)$ on $(X, D)$ is called polystable if it is the direct sum of stable ones $(\mathcal{P}_*, \mathcal{E}_i, \theta_i)$ ($i = 1, \ldots, m$) with $\deg(\mathcal{P}_* \mathcal{E}_i) = \deg(\mathcal{P}_* \mathcal{E})$. In the following proposition, the tame case was proved by Simpson [20], and see [16] for the wild case, for example.

**Proposition 3.3** Let $(E, \mathcal{G}_P, \theta, h)$ be an unramifiedly good wild harmonic bundle on $(X, D)$. Then, the associated filtered bundle $(\mathcal{P}_h^b E, \theta)$ on $(X, D)$ is poly-stable with $\deg(\mathcal{P}_h^b E) = 0$.

In the following proposition, the tame case was proved by Simpson [20], and the wild case was proved by Biquard and Boalch [2].

**Proposition 3.4** Let $(\mathcal{P}_*, \mathcal{E}, \theta)$ be an unramifiedly good filtered Higgs bundle on $(X, D)$ with $\deg(\mathcal{P}_* \mathcal{E}) = 0$. If $(\mathcal{P}_*, \mathcal{E}, \theta)$ is stable, we have a harmonic metric $h$ of $(E, \theta) = (E, \theta)|_{X \setminus D}$ with an isomorphism $(\mathcal{P}_h^b E, \theta) \simeq (\mathcal{P}_* \mathcal{E}, \theta)$.

Such a metric $h$ is unique up to the multiplication of positive constants.

3.1.5 Filtered bundles of rank one on curves

Let $X$ be any complex curve with a discrete subset $D$. Let $L$ be a line bundle on $X$. Suppose that a tuple of real numbers $b = (b_P | P \in D)$ is attached. Then, we have the filtered bundle $\mathcal{P}_b L$ over the meromorphic bundle $L(\ast D)$ given as follows. For $a = (a_P | P \in D) \in \mathbb{R}^D$, we have the integers $n(a_P) (P \in D)$ such that $a_P - 1 < n(a_P) + b_P \leq a_P$, and we set

$$\mathcal{P}_a^b L := L \left( \sum_{P \in D} n(a_P) P \right).$$

If $X$ is compact, we have $\deg(\mathcal{P}_a^b L) = \deg(L) - \sum_{P \in D} b_P$.

3.1.6 Comparison of filtered bundles

Let $X$ be a compact connected Riemann surface with a finite subset $D$. Let $\mathcal{P}, \mathcal{E}$ be a filtered bundle over a locally free $\mathcal{O}_X(\ast D)$-module $\mathcal{E}$ on $(X, D)$. Fix a point $P \in D$. Set $D_1 := D \setminus P$. For any $b \in \mathbb{R}$, let $\mathcal{P}_b^P \mathcal{E}$ denote the $\mathcal{O}_X(\ast D_1)$-module such that $\mathcal{P}_b^P \mathcal{E}(\ast P) = \mathcal{E}$ and the stalk of $\mathcal{P}_b^P \mathcal{E}$ at $P$ is $\mathcal{P}_b^P \mathcal{E}_P$. For any $b \in \mathbb{R}$ and $c \in \mathbb{R}^{D_1}$, we set $\mathcal{P}_c(\mathcal{P}_b^P \mathcal{E}) := \mathcal{P}_{(b,c)}^P \mathcal{E}$. For any $b \in \mathbb{R}$, we have the filtered bundle $\mathcal{P}_c(\mathcal{P}_b^P \mathcal{E}) = (\mathcal{P}_c \mathcal{P}_b^P \mathcal{E} | c \in \mathbb{R}^{D_1})$ over $\mathcal{P}_b^P \mathcal{E}$ on $(X, D_1)$.

We have the set $\{a | 0 < a \leq 1, \mathcal{P}_a^P \mathcal{E}/\mathcal{P}_{<a}^P \mathcal{E} \neq 0\} = \{a_1, \ldots, a_t\}$. We assume $a_i < a_{i+1}$. The following observation is given in [9].
Lemma 3.5 We have the following equality:

$$\int_0^1 \deg(P_*(P^0_\ell E)) \, db = \deg(P_*(P^0_\ell E)) - \sum_{i=1}^\ell (a_i - 1) \dim(P^0_{a_i}(E_P)/P^0_{\leq a_i}(E_P))$$

(22)

Here, we regard \( \deg(P_*(P^0_\ell E)) \) as a measurable function in variable \( b \). In particular, we have \( \deg(P_*E) = \int_0^1 \deg(P_*(P^0_\ell E)) \, db \).

**Proof** We obtain (22) by a direct computation. Because \( P^0_{a_i}(E_P)/P^0_{\leq a_i}(E_P) \simeq P^0_{a_i-1}(E_P)/P^0_{\leq a_i-1}(E_P) \), we can observe the right hand side of (22) is equal to \( \deg(P_*E) \).

We recall a lemma for the comparison of filtered bundles for the convenience of the readers.

**Lemma 3.6** Let \( \mathcal{E} \) be a locally free \( \mathcal{O}_X(*D) \)-module. Let \( P_i\mathcal{E} \) \((i = 1, 2)\) be filtered bundles over \( \mathcal{E} \) such that \( P^0_a\mathcal{E} \subset P^0_a\mathcal{E} \) for any \( a \in \mathbb{R}^D \). If \( \deg(P^0_1\mathcal{E}) = \deg(P^0_2\mathcal{E}) \), then we have \( P^0_a\mathcal{E} = P^0_a\mathcal{E} \) for any \( a \in \mathbb{R}^D \).

**Proof** We use an induction on \( |D| \). If \( D = \emptyset \), the claim is clear. Take \( P \in D \), and set \( D_1 := D \setminus \{P\} \). We obtain the filtered bundles \( P^0_i\mathcal{E} \) \((i = 1, 2)\) for any \( b \in \mathbb{R} \) as above. By using Lemma 3.5, we can easily deduce \( \deg(P^0_1\mathcal{E}) = \deg(P^0_2\mathcal{E}) \) for any \( b \in \mathbb{R} \) from the equality \( \deg(P^0_1\mathcal{E}) = \deg(P^0_2\mathcal{E}) \). By the hypothesis of the induction, we obtain \( P^0_c(P^0_1\mathcal{E}) = P^0_c(P^0_2\mathcal{E}) \) for any \( c \). Then, the claim of the lemma follows.

### 3.2 Filtered Higgs bundles of rank 2 on \((\mathbb{P}^1, \infty)\)

Let \( \zeta \) be the standard coordinate on \( \mathbb{C} \subset \mathbb{P}^1 \). We set \( \tilde{V} = \mathcal{O}_{\mathbb{P}^1}(\infty)\tilde{v}_1 \oplus \mathcal{O}_{\mathbb{P}^1}(\infty)\tilde{v}_2 \). Take a non-zero complex number \( \alpha \) and a positive integer \( m \). We have the Higgs field \( \tilde{\theta} \) on \( \tilde{V} \) given by

$$\tilde{\theta}(\tilde{v}_1, \tilde{v}_2) = (\tilde{v}_1, \tilde{v}_2) \left( \begin{array}{cc} \alpha \zeta^m d\zeta & 0 \\ 0 & -\alpha \zeta^m d\zeta \end{array} \right) .$$

Fix \( 0 \leq \ell \leq m \). Let \( \tilde{E}_\ell \subset \tilde{V} \) be the \( \mathcal{O}_{\mathbb{P}^1}(\infty) \)-submodule generated by \( \tilde{e}_1 = \tilde{v}_1 + \tilde{v}_2 \) and \( \tilde{e}_2 = \zeta^\ell \tilde{v}_2 \). Because \( \tilde{\theta}(\tilde{E}_\ell) \subset \tilde{E}_\ell \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1} \), we obtain the meromorphic Higgs bundle \((\tilde{E}_\ell, \tilde{\theta})\). We have

$$\tilde{\theta}(\tilde{e}_1, \tilde{e}_2) = (\tilde{e}_1, \tilde{e}_2) \left( \begin{array}{cc} \alpha \zeta^m d\zeta & 0 \\ -2\alpha \zeta^m - \ell d\zeta & -\alpha \zeta^m d\zeta \end{array} \right) .$$

Take \( c \in \mathbb{R} \), and set \( c_1 := c \) and \( c_2 := -c - \ell \). We have the parabolic Higgs bundle \((P^\ell_\mathcal{E}(\tilde{E}_\ell), \tilde{\theta})\) on \((\mathbb{P}^1, \infty)\) with \( \deg(P^\ell_\mathcal{E}(\tilde{E}_\ell)) = 0 \) given as follows

$$P^\ell_\mathcal{E}(\tilde{E}_\ell)_{|[b \setminus \infty]} = \left( \mathcal{O}_{\mathbb{P}^1}([b - c_1] \infty)\tilde{v}_1 \oplus \mathcal{O}_{\mathbb{P}^1}([b - c_2] \infty)\tilde{v}_2 \right)_{|[b \setminus \infty]} .$$

Here, \([x] := \max\{n \in \mathbb{Z} \mid n \leq x\} \). We shall impose that \( c_1 \geq c_2 \), i.e., \( c \geq -\ell/2 \).

#### 3.2.1 Stability condition

We set \( L_i := \mathcal{O}_{\mathbb{P}^1}(\infty)\tilde{v}_i \subset \tilde{V} \) \((i = 1, 2)\). We have \( \tilde{E}_\ell \cap L_i = \mathcal{O}_{\mathbb{P}^1}(\infty) \cdot \zeta^\ell \tilde{v}_i \). We have the induced parabolic bundle \( P^\ell_*(\tilde{E}_\ell \cap L_i) \). We have \( \deg(P^\ell_*(\tilde{E}_\ell \cap L_i)) = -\ell - c_i \). Hence, \((P^\ell_*(\tilde{E}_\ell), \tilde{\theta})\) is stable if and only if \(-\ell - c_i < 0 \) \((i = 1, 2)\), i.e., \( c_i < 0 \) \((i = 1, 2)\). We also have that \((P^\ell_*(\tilde{E}_\ell), \tilde{\theta})\) is semistable if and only if \(-\ell - c_1 = 0 \) or \(-\ell - c_2 = 0 \), i.e., \( c_1 = 0, -\ell \). Under the assumption \( c_i \geq c_2 \), the semistability is equivalent to \( c_1 = 0 \).

#### 3.2.2 The determinant bundles

We have \( \det(\tilde{E}_\ell) = \mathcal{O}_{\mathbb{P}^1}(\infty) \cdot \tilde{e}_1 \Lambda \tilde{e}_2 \) \(= \mathcal{O}_{\mathbb{P}^1}(\infty) \cdot \zeta^\ell \tilde{v}_1 \Lambda \tilde{v}_2 \). The induced filtered bundle \( \det(P^\ell_*(\tilde{E}_\ell)) \) over \( \det(\tilde{E}_\ell) \) is equal to \( P^\ell_*(\mathcal{O}_{\mathbb{P}^1} \cdot \tilde{e}_1 \Lambda \tilde{e}_2) \), where \( P^\ell_0(\mathcal{O}_{\mathbb{P}^1} \cdot \tilde{e}_1 \Lambda \tilde{e}_2) = \mathcal{O}_{\mathbb{P}^1}([b] \cdot \tilde{e}_1 \Lambda \tilde{e}_2) \) for \([b] := \max\{n \in \mathbb{Z} \mid n \leq b\} \). In particular, they are independent of \( c \). We fix the Hermitian metric \( h_{\text{det}(\tilde{E}_\ell)} \) of \( \det(\tilde{E}_\ell) \) given by \( h_{\text{det}(\tilde{E}_\ell)}(\tilde{e}_1 \Lambda \tilde{e}_2, \tilde{e}_1 \Lambda \tilde{e}_2) = 1 \).
3.2.3 Harmonic metrics in the case $\ell = 0$

If $\ell = 0$, $(P^*_c \tilde{E}_{0}, \tilde{\theta})$ cannot be stable. And, $(P^*_c \tilde{E}_{0}, \tilde{\theta})$ is semistable if and only if $c = 0$. Indeed, it is polystable in that case, i.e., we have the decomposition $(P^*_c \tilde{E}_{0}, \tilde{\theta}) = (P^*_c L_{1}, \tilde{\theta}_1) \oplus (P^*_c L_{2}, \tilde{\theta}_2)$, where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are the multiplications of $\alpha\zeta d\zeta$ and $-\alpha\zeta d\zeta$, respectively. We have the harmonic metrics $h_{L_i}$ for $(P^*_c L_{i}, \tilde{\theta}_i)$. We impose $h_{L_1} \otimes h_{L_2} = h_{\det(E_0)}$. Then, we have a harmonic metric $h_{E_0} = h_{L_1} \oplus h_{L_2}$ for $(P^*_c \tilde{E}_{0}, \tilde{\theta})$. Note that we have an ambiguity given by automorphisms of $(P^*_c \tilde{E}_{0}, \tilde{\theta})$, i.e., for any $\alpha > 0$, $\alpha h_{L_1} \oplus \alpha^{-1} h_{L_2}$ is also a harmonic metric for $(P^*_c \tilde{E}_{0}, \tilde{\theta})$. We also note that $h_{L_i}(\tilde{v}_i, \tilde{v}_i)$ are constants.

3.2.4 Harmonic metrics in the case $\ell > 0$ and their homogeneous property

Suppose that $\ell > 0$. According to Proposition 3.4, if the unramifiedly good filtered Higgs bundle $(P^*_c \tilde{E}_{\ell}, \tilde{\theta})$ is stable, i.e., $-\ell < c < 0$, we have the harmonic metric $h_{c, \ell}$ of $(\tilde{E}_{\ell}, \tilde{\theta})|_{c}$ adapted to the filtered Higgs bundle $(P^*_c \tilde{E}_{\ell}, \tilde{\theta})$ such that $\det(h_{c, \ell}) = h_{\det(E_{\ell})}$.

For any $\tau \in \mathbb{C}^*$, let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be given by $\varphi(\zeta) = \tau^2 \zeta$. We have the isomorphism $\Phi_\tau : \varphi_* \tilde{E}_{\ell} \simeq \tilde{E}_{\ell}$ given by $\tau^{\ell} \varphi_* \tilde{v}_1 \leftrightarrow \tilde{v}_1$ and $\tau^{-\ell} \varphi_* \tilde{v}_2 \leftrightarrow \tilde{v}_2$. Under the isomorphism, we have $\tau^{\ell} \varphi_* \tilde{v}_i \leftrightarrow \tilde{v}_i$ ($i = 1, 2$). Hence, $\Phi_\tau$ induces an isomorphism of the filtered bundles $\varphi_* \mathcal{P}_c \tilde{E}_{\ell} \simeq \mathcal{P}_c \tilde{E}_{\ell}$. Under the isomorphism, we have $\varphi_* \tilde{\theta} = \tau^{2(m+1)} \tilde{\theta}$. Hence, the Hermitian metric $\varphi_* h_{c, \ell}$ gives a harmonic metric of $(P^*_c \tilde{E}_{\ell}, \tau^{2(m+1)} \tilde{\theta})$. Note that $\det(\varphi_* h_{c, \ell}) = h_{\det \tilde{E}_{\ell}}$.

**Proposition 3.7** If $|\tau| = 1$, we have $\varphi_* h_{c, \ell} = h_{c, \ell}$.

**Proof** If $|\tau| = 1$, both the Hermitian metrics $\varphi_* h_{c, \ell}$ and $h_{c, \ell}$ are harmonic metrics of $(\tilde{E}_{\ell}, \tilde{\theta})$. They are adapted to the filtered bundle $P^*_c \tilde{E}_{\ell}$, and they satisfy $\det(h_{c, \ell}) = \det(\varphi_* h_{c, \ell}) = h_{\det \tilde{E}_{\ell}}$. Hence, we have $h_{c, \ell} = \varphi_* h_{c, \ell}$ by the uniqueness.

**Corollary 3.8** The functions $h_{c, \ell}(\tilde{v}_i, \tilde{v}_i)$ ($i = 1, 2$) depend only on $|\zeta|$. The function $\zeta^{-\ell} h_{c, \ell}(\tilde{v}_1, \tilde{v}_2)$ depends only on $|\zeta|$. The functions $h_{c, \ell}(\tilde{v}_i, \tilde{v}_j)$ ($i, j \in \{1, 2\}$) depend only on $|\zeta|$.

**Remark 3.9** We clearly have the homogeneous property of the harmonic metrics such as Corollary 3.8 even in the case $\ell = 0$.

3.2.5 The norms of $\tilde{v}_j$

By the norm estimate of wild harmonic bundles [16], we have constants $C_{c, \ell} > 0$ ($i = 1, 2$) depending on $c$ such that $C_{1, c} |\zeta|^c \leq |\tilde{v}_i|_{h_{c, \ell}} \leq C_{2, c} |\zeta|^c$, where $c_1 = c$ and $c_2 = -c - \ell$. Let us refine it in our situation.

**Proposition 3.10** There exist positive constants $b_c$, $C_3$ and $\epsilon_3$ such that the following holds on $\{ |\zeta| > 1 \}$:

\begin{align}
|\log |\tilde{v}_1|_{h_{c, \ell}} - \log(b_c |\zeta|^c)| & \leq C_3 \exp(-\epsilon_3 |\zeta|^{m+1}) \\
|\log |\tilde{v}_2|_{h_{c, \ell}} - \log(b_c^{-1} |\zeta|^c)| & \leq C_3 \exp(-\epsilon_3 |\zeta|^{m+1}) \\
|\zeta \partial_{\zeta} \log |\tilde{v}_1|_{h_{c, \ell}} - c_i/2| & \leq C_3 \exp(-\epsilon_3 |\zeta|^{m+1}) \quad (i = 1, 2)
\end{align}

Here, $b_c$ may depend on $c$, but $C_3$ and $\epsilon_3$ are independent of $c$.

**Proof** Let $\eta := \zeta^{-1}$ be the coordinate around $\infty$. Set $U_{\infty} = \{ |\eta| \leq 1 \} \subset \mathbb{P}^1$.

**Lemma 3.11** There exist $C_4 > 0$ and $\epsilon_4 > 0$ which are independent of $c$, such that the following holds on $U_{\infty} \setminus \{ \infty \}$:

\[ |\partial_{\eta} \partial_{\eta} \log |\tilde{v}_j|_{h_{c, \ell}}| \leq C_4 \exp(-\epsilon_4 |\eta|^{-m-1}) \]

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Proof Let $h_{L_j,e,f}$ denote the restriction of $h_{e,f}$ to $L_j$. Let $R(h_{L_j,e,f})$ denote the curvature of the Chern connection of $(L_j, h_{L_j,e,f})$. Let $g_C$ be the Euclidean metric $d\zeta d\zeta^*$. By Proposition 2.12, we have positive constants $C_3$ and $\epsilon_3$ which are independent of $c$, such that the following holds:

$$|R(h_{L_j,e,f})|_{h_{L_j,e,f},g_C} \leq C_3 \exp(-\epsilon_3|\zeta|^{m+1})$$

Because $\overline{\partial}\log|\tilde{v}_1|_{h_{e,f}} = R(h_{L_j,e,f})$, we obtain the claim of the lemma.

We set $Y_\eta := \{\tilde{\eta} \in C \mid |\tilde{\eta}| < 1\}$. For any $\kappa < 1$, we have the isomorphism $\Psi_\kappa : Y_\eta \rightarrow \{\eta| < \kappa\}$ given by $\Psi_\kappa(\tilde{\eta}) = \kappa \tilde{\eta}$. We have the following on $Y_\eta \setminus \{0\}$:

$$|\partial_\eta \partial_{\bar{\eta}} \Psi_\kappa^{*}(\log|\tilde{v}_1|_{h_{e,f}})| \leq C_4 \kappa^2 \exp(-\epsilon_4 \kappa^{-m-1}|\tilde{\eta}|^{-m-1})$$

Take a large $p > 1$. For each $c$ and $\kappa$, we can take an $\mathbb{R}$-valued $L^p$-function $G_{\kappa, c}$ on a neighbourhood of the closure of $Y_\eta$ such that (i) $G_{\kappa, c}$ is a function of $|\eta|$, (ii) there exist positive constants $C_5$ and $\epsilon_5$ such that the $L^p_C$-norm of $G_{\kappa, c}$ on $Y_\eta$ is dominated by $C_5 \exp(-\epsilon_5 \kappa^{-m-1})$, (iii) $G_{\kappa, c}(0) = 0$, (iv) the following holds on $Y_\eta$:

$$\partial_\eta \partial_{\bar{\eta}} \Psi_\kappa^{*}(\log|\tilde{v}_1|_{h_{e,f}}) = G_{\kappa, c}$$

Because $\Psi_\kappa^{*}(\log|\tilde{v}_1|_{h_{e,f}}) - G_{\kappa, c} - \log|\tilde{\eta}|^{-c_1}$ is bounded, we have a holomorphic function $g_{\kappa, c}$ on $Y_\eta$ such that $\Psi_\kappa^{*}(\log|\tilde{v}_1|_{h_{e,f}}) - G_{\kappa, c} - \log|\tilde{\eta}|^{-c_1} = \text{Re}(g_{\kappa, c})$. Because $\Psi_\kappa^{*}(\log|\tilde{v}_1|_{h_{e,f}}) - G_{\kappa, c} - \log|\tilde{\eta}|^{-c_1}$ depends only on $|\eta|$, we obtain that $g_{\kappa, c}$ is constant. We also obtain

$$\tilde{\eta} \partial_{\bar{\eta}} \Psi_\kappa^{*}(\log|\tilde{v}_1|_{h_{e,f}}) + c_1/2 - \tilde{\eta} \partial_{\bar{\eta}} G_{\kappa, c} = 0.$$ 

Hence, on $\{|\eta| < \kappa\}$, the function $\log|\tilde{v}_1|_{h_{e,f}} - (\Psi_\kappa^{-1})^{*} G_{\kappa, c} - \log|\eta|^{-c_1}$ is a constant. We have positive constants $C_6$ and $\epsilon_6$ such that the following holds on $\{\kappa/2 < |\eta| < \kappa\}$:

$$|G_{\kappa, c} - (\Psi_\kappa^{-1})^{*} G_{\kappa, c}| \leq C_6 \exp(-\epsilon_6 |\eta|^{-m-1})$$

$$|\partial_{\bar{\eta}} G_{\kappa, c}| \leq C_6 \exp(-\epsilon_6 |\eta|^{-m-1})$$

We also obtain that the function $F := \log|\tilde{v}_1|_{h_{e,f}} - \log|\eta|^{-c_1}$ gives a continuous function on $\{|\eta| < 1\}$. We set $b_c := \exp(F(0))$. Then, we obtain (23), and (25) for $i = 1$. By using $|\tilde{v}_1 \wedge \tilde{v}_2|_{h_{e,f}} = |e|^{-\ell}$ and the estimate (26) below, we obtain the estimate (24). We also obtain (25) for $i = 2$ with a similar argument.

3.2.6 Asymptotic orthogonality and some complements

As studied in [16], we have the following estimate on $\{|\zeta| \geq 1\}$, which also follows from Corollary 2.6:

$$|h_{e,f}(\tilde{v}_1, \tilde{v}_2)| \leq K \exp(-\delta |\zeta|^{m+1}) \cdot |\tilde{v}_1|_{h_{e,f}} \cdot |\tilde{v}_2|_{h_{e,f}} \leq K' \exp(-\delta |\zeta|^{m+1})$$

(26)

Here, $K, K'$ and $\delta$ are positive constants which are independent of $c$.

Let $\partial_{e,f}$ denote the holomorphic structure of $E_{e,f}$. Let $\partial_{e,f}$ denote the $(1,0)$-part of the Chern connection of $(E_{e,f}, h_{e,f})$. Let $g_C$ denote the Euclidean metric of $C_\zeta$.

Lemma 3.12 We have positive constants $K_1$ which are independent of $c$, such that $|\zeta \partial_{e,f} \tilde{v}_1|_{h_{e,f},g_C} \leq K_1 \cdot |	ilde{v}_1|_{h_{e,f}}$ on $\{|\zeta| \geq 1\}$.

Proof In this proof, the constants $K_1$ and $\delta_i$ are independent of $c$. To simplify the description, we denote $h_{e,f}$ by $h$. By the asymptotic orthogonality (26), we have a positive constant $K_2$ such that the following holds on $\{|\zeta| \geq 1\}$:

$$|\partial_{e,f} \tilde{v}_1|_{h,g_C} \leq K_2 \left( |\tilde{v}_1|^{-1}_h \cdot |h(\partial_{e,f} \tilde{v}_1, \tilde{v}_1)|_{g_C} + |\tilde{v}_2|^{-1}_h \cdot |h(\partial_{e,f} \tilde{v}_1, \tilde{v}_2)|_{g_C} \right)$$

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Lemma 3.13 We have positive constants $K_4$ and $\delta_4$ such that
$$|\partial_{c,\ell}\pi_1|_{h,gc} \leq K_4\exp(-\delta_4|\zeta|^{m+1}).$$  
(28)
By (26) and Proposition 2.3, we have positive constants $K_5$ and $\delta_5$ such that
$$|h(\pi_1(\partial_{c,\ell}\tilde{v}_1),\tilde{v}_2)| \leq K_5\exp(-\delta_5|\zeta|^{m+1})|\partial_{c,\ell}\tilde{v}_1|_{h,gc}.  
(29)$$
Hence, we obtain the following on $\{|\zeta| \geq 1\}$:
$$|\partial_{c,\ell}\tilde{v}_1|_{h,gc} \leq K_2(K_4|\zeta|^{-1} + K_4\exp(-\delta_4|\zeta|^{m+1}))|\tilde{v}_1|_h + K_2K_5\exp(-\delta_5|\zeta|^{m+1})|\partial_{c,\ell}\tilde{v}_1|_{h,gc}.  
(30)$$
Hence, we obtain the desired estimate for $\partial_{c,\ell}\tilde{v}_1$. Similarly, we obtain the estimate for $\partial_{c,\ell}\tilde{v}_2$.

3.2.7 Convergence of some sequences

Set $t := \tau^{2(m+1)}$ for $\tau \in \mathbb{C}^*$. We use the notation in §3.2.4. We have the family of the harmonic metrics $h_{t,c,\ell} := \varphi_{t,\ell}^* h_{c,\ell}$ for $(P_{c,\ell}^* E_{\ell}, t\theta)$ satisfying $\det(h_{t,c,\ell}) = h_{4\text{det}(E)}$.

Under the isomorphism $\Phi_\tau : \varphi_{t,\ell}^* E_{\ell} \simeq E_{\ell}$, we have $\tau^i \varphi_{t,\ell}^* \tilde{v}_i \leftrightarrow \tilde{v}_i$ $(i = 1, 2)$. Take any $T > 0$. We have positive constants $C$ and $\epsilon$, which are independent of $c$ and $T$, such that the following holds on $\{|\zeta| \geq T\}$ for any $t$ satisfying $|t| > T^{-1}$:
$$|\log|\tilde{v}_1|_{h_{t,c,\ell}} - \log(b_c|t|^{(t+2c)/2(m+1)}|\zeta|^{c})| \leq C\exp(-\epsilon|\zeta|^{m+1}|t|)  
(31)$$

The desired estimate holds for $h_{t,c,\ell}$. Hence, we obtain the estimate for $h_{t,c,\ell}$ for $h_{t,c,\ell}$. Hence, we obtain the desired estimate for $h_{t,c,\ell}$. Hence, we obtain the desired estimate for $h_{t,c,\ell}$.
For any $\gamma > 0$, $\exp(h(\Psi u))$.

We continue to use the notation in §3.3.1 Continuity of $b_{c,\ell}$ with respect to the parabolic weights

**Proposition 3.15** $b_c$ is continuous with respect to $c$.

**Proof** Fix $-\ell < c^0 < 0$. It is enough to study the continuity at $c^0$. We give a preliminary. Take a neighbourhood $U$ of $c^0$. We have a family of Hermitian metrics $h^0_{c,\ell}$ ($c \in U$) of $E\varepsilon^*$ satisfying the following conditions:

1. We have $h^0_{c,0} = h_{c,0}$ on $\mathbb{C}$.

2. We have $h^0_{c,\ell}(\bar{v}_1, \bar{v}_2) = h^0_{c,\ell}(\bar{v}_1, \bar{v}_2)$.

3. We have $h^0_{c,\ell}(\bar{v}_1, \bar{v}_1) = h^0_{c,\ell}(\bar{v}_1, \bar{v}_1)|\xi|^{c-\ell}$ and $h^0_{c,\ell}(\bar{v}_2, \bar{v}_2) = h^0_{c,\ell}(\bar{v}_2, \bar{v}_2)|\xi|^{c+\ell}$ on $\{ |\xi| \geq 1 \}$.

4. We have $\|\partial h^0_{c,\ell}\|_{h^0_{c,\ell}} = |\xi|^{\ell}$.

5. We have $\lim_{c \to c^0} h^0_{c,\ell} = h^0_{c^0,\ell}$ in the $C^\infty$-sense on any compact subset in $\mathbb{C}$.

Note that the conditions are compatible on $\{ |\xi| \geq 1 \}$. We fix a $C^\infty$-Kähler metric $g_{\mathbb{P}^1}$ of $\mathbb{P}^1$.

**Lemma 3.16** We have a constant $C > 0$ such that the following holds on $\mathbb{P}^1 \setminus \{ \infty \}$ for any $c \in U$:

$$|R(h^0_{c,\ell})|_{h^0_{c,\ell}, g_{\mathbb{P}^1}} \leq C, \quad |[\partial_h, \partial_{h^0_{c,\ell}}]|_{h^0_{c,\ell}, g_{\mathbb{P}^1}} \leq C$$

**Proof** In the proof, $C_i$ and $\epsilon_i$ are positive constants, which are independent of $c$. It is enough to consider the issue on $\{ |\xi| \geq 1 \}$. The estimate for $[\partial_h, \partial_{h^0_{c,\ell}}]$ follows from (26) and the construction of $h^0_{c,\ell}$. Let us study the estimate for $R(h^0_{c,\ell})$.

Let $H$ be the $M_2(\mathbb{C})$-valued function on $\{ |\xi| \geq 1 \}$ given by $H_{ij} = h_{c,\ell}(\bar{v}_i, \bar{v}_j)$. By Lemma 3.13, we have positive constants $C_1$ and $\epsilon_1$ such that the following holds:

$$|\partial H_{12}|_{g_{\mathbb{P}^1}} \leq C_1 \exp(-\epsilon_1 |\xi|^{-m-1})$$

Set $c_1 = c^0$ and $c_2 = -c^0 - \ell$. By Lemma 3.12, we have a positive constant $C_2$ such that the following holds:

$$|\partial H_{1i}|_{g_{\mathbb{P}^1}} \leq C_2 |\xi|^{-2c_1^{0}-1} (i = 1, 2)$$
Let $\Gamma_c$ be the $M_2(\mathbb{C})$-valued function given as $\Gamma_{c,11} = |\xi|^{-c_{10}}$, $\Gamma_{c,22} = |\xi|^{-c_{11}}$, and $\Gamma_{c,ij} = 0 (i \neq j)$. Then, $R(h_{0,c})$ on $\{|\xi| \geq 1\}$ is represented by the following matrix with respect to the frame $(\bar{v}_1, \bar{v}_2)$:

$$
\bar{\Theta}((\bar{\Gamma}_c,H)_{c,0})^{-1}\partial(\bar{\Gamma}_c,H) = \bar{\Theta}(\Gamma_c^{-1}H^{-1}) \cdot \Gamma_c^{-1} \partial \Gamma \cdot \bar{\Theta}(\Gamma_c^{-1}H^{-1})^{-1} (\Gamma_c^{-1} \partial \Gamma_c) \partial \bar{\Theta}(\Gamma_c)
+ \bar{\Theta}_c^{-1}(\Gamma_c^{-1}H^{-1}) \partial \Gamma_c \cdot \bar{\Theta}(\Gamma_c^{-1}H^{-1}) \partial \bar{\Theta}(\Gamma_c) + \Gamma_c^{-1}(\partial \bar{\Theta}(\Gamma_c) \cdot \Gamma_c)
$$

Because $|R(h_{0,c})|_{h_{0,c},0} \leq C_3 \exp(-\epsilon_3|\eta|^{-m-1})$, we have

$$
|\bar{\Theta}(\Gamma_c^{-1}H^{-1})|_{\eta} \leq C_4 \exp(-\epsilon_4|\eta|^{-m-1}),
$$

and hence $|\Gamma_c^{-1} \bar{\Theta}(\Gamma_c^{-1}H^{-1}) \cdot \Gamma_c_{1,1}| \leq C_5 \exp(-\epsilon_5|\eta|^{-m-1})$. We also have

$$
\bar{\Theta}(\Gamma_c^{-1}H^{-1}) \cdot \Gamma_c^{-1} \partial \Gamma_c \cdot \bar{\Theta}(\Gamma_c) = -\left[(\bar{\Theta}(\Gamma_c^{-1}H^{-1}) \cdot \Gamma_c^{-1} \partial \Gamma_c) \cdot \bar{\Theta}(\Gamma_c)ight]
$$

Because the off-diagonal part of $\bar{\Theta}(\Gamma_c)$ and $\bar{\Theta}(\Gamma_c^{-1}H^{-1})$ are dominated by $C_6 \exp(-\epsilon_6|\eta|^{-m-1})$, the term (32) is dominated by $C_7 \exp(-\epsilon_7|\eta|^{-m-1}) d\zeta d\bar{\zeta}$. Similarly,

$$
\bar{\Theta}_c^{-1}(\partial \Gamma_c \cdot \Gamma_c^{-1}H^{-1}) \partial \bar{\Theta}(\Gamma_c) = -\left[(\partial \Gamma_c \cdot \Gamma_c^{-1}H^{-1}) \partial \bar{\Theta}(\Gamma_c)ight]
$$

is dominated by $C_8 \exp(-\epsilon_8|\eta|^{-m-1}) d\zeta d\bar{\zeta}$. Then, the claim of the lemma follows.

We have the self-adjoint endomorphisms $k_c$ of $(\hat{E}_1, b_{0,c})$ determined by $h_{0,c}(u_1, u_2) = h_{0,c}^0(k_c u_1, u_2)$. Note that $k_c$ are bounded with respect to $h_{0,c}^0$, although the estimate might depend on $c$ at this stage. We also remark that $\text{Tr} k_c(P) \geq 1$ at any $P \in \mathbb{C}$. The claim of Proposition 3.15 is deduced from the following proposition.

**Proposition 3.17** We have the convergence $k_c \rightarrow \text{id} (c \rightarrow 0)$ uniformly on $\mathbb{C}$.

**Proof** Take a large $p > 1$. Let $||k_c||_{h_{0,c}^0,0}^{0,p} L_p$ be the $L^p$-norm of $k_c$ with respect to $h_{0,c}^0$ and $g_{p,1}$. We set $s_c := k_c / ||k_c||_{h_{0,c}^0,0}^{0,p} L_p$. According to [19, Lemma 3.1], we have the following inequality on $\mathbb{C}$:

$$
\sqrt{-1} \Lambda_{g_{p,1}} \bar{\partial} \partial \text{Tr}(s_c) \leq \Lambda_{g_{p,1}} \text{Tr}\left((R(h_{0,c}^0) + [\bar{\Theta}(\Gamma_c^{-1}H^{-1}) \cdot \Gamma_c]) s_c\right)
$$

We recall the following general lemma, which is a variant of [20, Lemma 2.2].

**Lemma 3.18** Let $\varphi$ and $g$ be bounded $\mathbb{R}$-valued $C^\infty$-functions on a punctured disc $\{x \in \mathbb{C} | 0 < |x| < 1\}$. Suppose that $-\partial_x \varphi \leq g$ holds on $\{x \in \mathbb{C} | 0 < |x| < 1\}$. Then, the inequality holds on $\{x \in \mathbb{C} | |x| < 1\}$ in the sense of distributions.

**Proof** We give only an outline of the proof. We take a $C^\infty$-function $\rho : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\rho(t) = 1 (t \leq 1)$ and $\rho(t) = 0 (t \geq 2)$. For any $N > 0$, we set $\chi_N(x) = \rho(-N^{-1} \log |x|)$. Note that $\partial \chi_N$, $\bar{\partial} \chi_N$, and $\partial \bar{\partial} \chi_N$ are bounded with respect to the Poincaré metric $(\log |x|^2)^{-1}|x|^{-2} dx d\bar{x}$, which are uniformly for $N$.

Let $f$ be any $\mathbb{R}$-valued test function on $\{|x| < 1\}$. The claim of the lemma is the following inequality:

$$
\int_{|x| < 1} -\varphi \partial_x \partial \chi_N f |dx d\bar{x}| \leq \int_{|x| < 1} g \cdot f |dx d\bar{x}|
$$

By the assumption, we have $\int_{|x| < 1} -\varphi \partial_x \partial \chi_N f |dx d\bar{x}| \leq \int_{|x| < 1} g \cdot \chi_N f |dx d\bar{x}|$. It is enough to prove that $\lim_{N \rightarrow \infty} \int_{|x| < 1} \varphi \partial \bar{\partial} (\chi_N f) - \chi_N \partial \bar{\partial} (f) = 0$. It follows from the uniform boundedness of $\partial \chi_N$, $\bar{\partial} \chi_N$, and $\partial \bar{\partial} \chi_N$ with respect to the Poincaré metric.

In particular, the inequality (33) holds on $P^1$. The right hand side of (33) is uniformly bounded in $L^p$. We can find $L^2_\mu$-functions $M_c$ on $P^1$ and constants $C_i > 0$ ($i = 10, 11$) such that the following holds for any $c \in U$:

$$
\Lambda_{g_{p,1}} \bar{\partial} \text{Tr}(s_c) - M_c \leq C_{10}, \sup |M_c| \leq C_{11}
$$
Let $s_c$ be the pull back of $s$, which is weakly convergent in $L^2_0$ locally on $\mathbb{P}^1 \setminus \{\infty\}$. Let $s_\infty$ denote the limit. The sequence $s_c$ converges to $s_\infty$ almost everywhere. By the uniform boundedness (34), we have the boundedness of $s_\infty$. It also implies $\|s_\infty\|_{L^p} = \lim |s_c|_{L^p} = 1$. In particular, $s_\infty \neq 0$. By the construction, we have $(\bar{\partial} + \bar{\theta})s_\infty = 0$. Hence, $s_\infty$ gives a non-zero endomorphism of the stable filtered Higgs bundle $(P_\ell^0 E_\ell, \bar{\theta})$. It implies that $s_\infty$ is the multiplication of a non-zero complex number. In particular, $\det(s_\infty) \neq 0$. It implies that $\|k_c\|_{L^p,k^0_\ell}\ell\{\infty\}$ are bounded. We obtain that $\|k_c\|_{L^p,k^0_\ell}\ell\{\infty\}$ is bounded.

It implies that the sequence $k_c$ is weakly convergent in $L^2_0$ locally on $\mathbb{P}^1 \setminus \{\infty\}$. Then, the limit $k_\infty$ is the multiplication of a non-zero positive number. Because $\det(k_c) = 0$ for any $c$, we obtain that $k_\infty = \operatorname{id}$. It means that $k_c$ is weakly convergent to $\operatorname{id}$ in $L^2_0$ locally on $\mathbb{P}^1 \setminus \{\infty\}$.

By using [19, Proposition 3.1], we obtain $\sqrt{-1} \Lambda_{g_1} \bar{\partial}\bar{\partial}(\operatorname{Tr}(k_c) - 2) \leq C_{16}$ (c ∈ U) for a constant $C_{16} > 0$. By [19, Proposition 2.1], we obtain $\sup_{\mathbb{P}^1} |\operatorname{Tr}(k_c) - 2| \leq C_{17} \int_{\mathbb{P}^1} |\operatorname{Tr}(k_c) - 2| d\nu$ (c ∈ U) for a constant $C_{17} > 0$. Note that we always have $\operatorname{Tr}(k_c) - 2 \geq 0$, and $\operatorname{Tr}(k_c) - 2 = 0$ implies that $k_c = \operatorname{id}$. Then, we obtain the uniform convergence $\operatorname{Tr}(k_c) - 2 \rightarrow 0$ (c → $c_0$). It implies the uniform convergence $k_c \rightarrow \operatorname{id}$ (c → $c_0$). Thus, the claims of Proposition 3.17 and Proposition 3.15 are proved.

**3.3.2 Behaviour of $b_c$ when $c \rightarrow 0$**

Recall that we have the following description on $|\zeta| > 1$:

$$\log|\tilde{v}_1|_{h_{c,e}} = c \log|\zeta| + \log b_c + \rho_{1,c}, \quad \log|\tilde{v}_2|_{h_{c,e}} = -(\ell + c) \log|\zeta| - \log b_c + \rho_{2,c}.$$  

According to Proposition 3.10, we have $|\rho_{1,c}| \leq C_{30} \exp(-c_{30}|\zeta|^{m+1})$ (i = 1, 2) for positive constants $C_{30}$ and $c_{30}$ which are independent of $c$.

**Proposition 3.19** When $c \rightarrow 0$, we have $b_c \rightarrow \infty$. We also have the uniform convergences $\rho_{i,c} \rightarrow 0$ (i = 1, 2).

**Proof** First, let us study the convergence of $\rho_{i,c}$. We begin with a preliminary. We set $Y_0 := C_w \times \mathbb{P}^1$ and $D_0 := C_w \times \{\infty\}$. Let $p : Y_0 \rightarrow \mathbb{P}^1$ be the projection. We have the locally free $\mathcal{O}_{Y_0}(\ast D_0)$-module $p^*(L_1 \oplus L_2)$. Let $\pi_1, \pi_2$ be the pull back of $\zeta$. We have the $\mathcal{O}_Y(\ast D_Y)$-submodule $\mathcal{E}_\ell \subset p^*(L_1 \oplus L_2)$ generated by $\pi_1 = \pi_1 + w \cdot \pi_2$ and $\pi_2 = \zeta^\ell \pi_2$.  

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The restriction of $E_\ell$ to $\{w\} \times \mathbb{P}^1$ is denoted by $E_{\ell,w}$. For each $w$, we may naturally regard $E_{\ell,w}$ as a subsheaf of $\tilde{V} = L_1 \oplus L_2$. We have $\tilde{\theta}(E_{\ell,w}) \subset E_{\ell,w} \otimes \Omega^1_{\mathbb{P}^1}$. So, we have the family of Higgs bundles $(E_{\ell,w}, \tilde{\theta})$. The restrictions of $\tau_i$ and $\tau_j$ to $\{w\} \times \mathbb{P}^1$ are denoted by $\tau_{i,w}$ and $\tau_{j,w}$, respectively. We have

$$\tilde{\theta}(\tau_{1,w}, \tau_{2,w}) = (\alpha \zeta^m d\zeta, 0), \quad \tilde{\theta}(\tau_{1,w}, \tau_{2,w}) = (\alpha \zeta^m d\zeta, 0).$$

We have a natural isomorphism $E_{\ell,0} \simeq L_1 \oplus \xi^* L_2$. For $w \neq 0$, we pick $w^{1/2}$, then we have the isomorphism $\Phi_w : (E_{\ell,w}, \tilde{\theta}) \simeq (E_{\ell}, \tilde{\theta})$ given by $w^{-1/2} \tau_{1,w} \leftrightarrow e_1$ and $w^{1/2} \tau_{2,w} \leftrightarrow e_2$, i.e., $w^{-1/2} \tau_{1,w} \leftrightarrow v_1$ and $w^{1/2} \tau_{2,w} \leftrightarrow v_2$.

We take a $C^\infty$-metric $\tilde{h}$ of $E_{\ell,0} \times C_\xi$, satisfying the following conditions:

- On $C_w \times \{s \geq 1\}$, we have $\tilde{h}(\tau_1, \tau_1) = |s|^{-2|w|} \tilde{h}(\tau_2, \tau_2) = |s|^{-2|e|}$, and $\tilde{h}(\tau_1, \tau_2) = 0$.
- On $\{w = 0\} \times C_\xi$, we have $\tilde{h}(\tau_{1,0}, \tau_{1,0}) = 1, \tilde{h}(\tau_{2,0}, \tau_{2,0}) = 1$, and $\tilde{h}(\tau_{1,0}, \tau_{2,0}) = 0$.

The restriction of $\tilde{h}$ to $\{w\} \times \mathbb{P}^1$ is denoted by $\tilde{h}_w$. We take a small $\delta > 0$, and consider $U_w := \{|w| \leq \delta\}$. We have the following uniform boundedness on $C_\xi$:

$$|R(\tilde{h}_w) + \tilde{\theta}, \tilde{\theta}|_{\tilde{h}_w, g^p} \leq C_{31} \quad (w \in U_w).$$

Moreover, we have the uniform convergence $\lim_{|w| \to 0} \left(R(\tilde{h}_w) + [\tilde{\theta}, \tilde{\theta}]_{\tilde{h}_w} \right) = 0$.

For any $c$ satisfying $-\delta < c < 0$, we have the self-adjoint endomorphism $k_c$ of $(E_{\ell,0}, \tilde{\theta}_e)$ determined by $\Phi_c, \Phi_c(1, u_2) = \tilde{h}_e(k_c u_1, u_2)$. Take a large $p > 1$. Let $\|k_c\|_{\tilde{h}_e, g^p, L^p}$ be the $L^p$-norm of $k_c$ with respect to $\tilde{h}_e$ and $g^p$. We set $s_c := k_c/\|k_c\|_{\tilde{h}_e, g^p, L^p}$.

Suppose that $|\rho_{1,c}| + |\rho_{2,c}|$ is not uniformly convergent to 0 when $c \to 0$, and we shall deduce a contradiction. Under the assumption, we have a positive number $\delta > 0$ and a subsequence $c^j \to 0$ such that

$$\sup_{|s| > 1} (|\rho_{1,c^j}| + |\rho_{2,c^j}|) \geq \delta. \quad (35)$$

By the argument in the proof of Proposition 3.17, we can assume that $s_{c^j}$ weakly converges to a non-zero endomorphism $s_{\infty}$ of $E_{\ell,0}$ in $L^2_{\tilde{h}}$ locally on $\mathbb{P}^1 \setminus \{\infty\}$, such that (i) $(\tilde{\theta} + \tilde{\theta}_\infty) s_{\infty} = 0$, (ii) $s_{\infty}$ is bounded with respect to $\tilde{h}_0$. So, we have an endomorphism of the poly-stable parabolic Higgs bundle $(\tilde{\theta}_{\infty}, \tilde{h}_0)$. We obtain that $s_{\infty} = \alpha_1 \text{id}_{e_1} \oplus \alpha_2 \text{id}_{e_2}$. Here, $\alpha_i$ are non-negative real numbers, and $(\alpha_1, \alpha_2) \neq (0, 0)$. Suppose that $\alpha_1 \neq 0$. We have the following uniform convergence on any compact subset in $C$.

$$\lim_{|\xi| \to \infty} (\tilde{h}_{\xi, e_{c^j}}(\tilde{v}_1, \tilde{v}_2)) |\xi|^{-2c^j} f \|k_{c^j}\|_{\tilde{h}_{\xi, g^p}, L^p}^{-1} = 1.$$

It implies that $\rho_{1,c^j}$ is convergent to 0 on any compact subset in $\{|\xi| \geq 1\}$. Together with the uniform estimate $|\rho_{1,c^j}| \leq C_{30} \exp(-c_{30} |\xi|^{m+1})$, we obtain the uniform convergence $\lim_{|\xi| \to \infty} \sup_{|\xi| > 1} |\rho_{1,c^j}| = 0$. We have $|\tilde{v}_1 \wedge \tilde{v}_2|_{h_{c^j}} = |\xi|^{-c^j}$. By (26), we have the following estimate:

$$\left| \tilde{v}_1_{h_{c^j}} \cdot \tilde{v}_2_{h_{c^j}} - |\xi|^{-c^j} \right| \leq C_{32} \exp(-c_{32} |\xi|^{m+1})$$

Here, the constants $C_{32}$ and $c_{32}$ are independent of $c$. Then, we can deduce the uniform convergence $\rho_{2,c^j} \to 0$. Hence, we obtain $\lim_{|\xi| \to \infty} \sup_{|\xi| > 1} |\rho_{i,c^j}| = 0 (i = 1, 2)$ in the case $\alpha_1 \neq 0$. Similarly, we can deduce the uniform convergences $\lim_{|\xi| \to \infty} \sup_{|\xi| > 1} |\rho_{i,c^j}| = 0 (i = 1, 2)$ in the case $\alpha_2 \neq 0$. But, it contradicts with (35). Thus, we can conclude the uniform convergences $\lim_{c \to 0} \rho_{i,c} = 0 (i = 1, 2)$. 33
Let us study the divergence of $b_c$. By the convergence of $\rho_{n,c}$, we have the following on $\{||\zeta|| \geq 1\}$:

$$\lim_{c \to 0} \left( b_{c}^{-1} |\vec{v}_1|_{h_{c,t}} \right) = 1, \quad \lim_{c \to 0} \left( b_{c} |\vec{v}_2|_{h_{c,t}} \right) = ||\zeta||^{-t}. \quad (36)$$

Suppose that there exists a subsequence $c^i \to 0$ such that $b_{c^i}$ are bounded, and we shall deduce a contradiction. We may assume the convergence $b_{c^i} \to \tilde{b}$. We shall give a detailed argument in the case $\tilde{b} = 0$. Later, we shall sketch the argument for the simpler case $\tilde{b} \neq 0$.

We set $Y_1 := C_x \times \mathbb{P}^1$ and $D_1 := C_x \times \{\infty\}$. Let $p_1 : Y_1 \to \mathbb{P}^1$ be the projection. We have the locally free $\mathcal{O}_{Y_1}(\ast D_1)$-module $p_!^1(L_1 \oplus L_2)$. The pull back of $v_i$ are denoted by $\tilde{v}_i$. Let $\hat{E}_{c^i} \subset p_!^1(L_1 \oplus L_2)$ generated by $\hat{f}_1 = x \hat{v}_1 + \hat{v}_2$ and $\hat{f}_2 = \zeta^t \hat{v}_1$. The restriction of $\tilde{E}_{c^i}$ to $\{x\} \times \mathbb{P}^1$ is denoted by $\tilde{E}_{c^i,t}$. The restriction of $\tilde{v}_i$ and $\tilde{f}_i$ to $\{x\} \times \mathbb{P}^1$ are denoted by $\tilde{v}_{i,x}$ and $\tilde{f}_{i,x}$. We have

$$\tilde{\theta}(\tilde{v}_{1,x}, \tilde{v}_{2,x}) = \left( \begin{array}{cc} \alpha \zeta^m d\zeta & 0 \\ 0 & -\alpha \zeta^m d\zeta \end{array} \right)$$

$$\tilde{\theta}(\tilde{f}_{1,x}, \tilde{f}_{2,x}) = \left( \begin{array}{cc} \alpha \zeta^m d\zeta & 0 \\ 2\alpha \zeta^{m-t} d\zeta & -\alpha \zeta^m d\zeta \end{array} \right)$$

We have $\tilde{E}_{c^i,0} \simeq \zeta^t L_1 \oplus L_2$. For $x \neq 0$, taking $x^{1/2}$, we have the isomorphism $\Psi_x : (\tilde{E}_{c^i,x}, \tilde{\theta}) \simeq (\tilde{E}_{c^i}, \tilde{\theta})$ given by the correspondences $x^{-1/2} \tilde{f}_{1,x} \leftrightarrow v_1 + v_2$ and $x^{1/2} \tilde{f}_{2,x} \leftrightarrow \zeta^t v_1$, i.e., $x^{1/2} \tilde{v}_{1} \leftrightarrow v_1$ and $x^{-1/2} \tilde{v}_2 \leftrightarrow v_2$.

We have the isomorphisms of holomorphic vector bundles $\Upsilon_x : \tilde{E}_{c^i,0} \simeq \tilde{E}_{c^i,x}$ given by $\Upsilon_x(\tilde{f}_{i,0}) = \tilde{f}_{i,x}$. We shall implicitly identify the vector bundles $\tilde{E}_{c^i,0}$ and $\tilde{E}_{c^i,x}$ by $\Upsilon_x$ in the following argument.

Let $x_i := \tilde{h}_{c^i}$. By the assumption, we have $\lim_{i \to \infty} x_i = 0$. We obtain the harmonic metrics $\Psi^*_x h_{c^i,t}$ on $(\tilde{E}_{c^i,x}, \theta)$. We take a family of $C^\infty$-Hermitian metrics $h^0_{x_i}$ of $\tilde{E}_{c^i,x}|_{C_\zeta}$ satisfying the following conditions:

- $h^0_0(\tilde{v}_{1,0}, \tilde{v}_{1,0}) = 1$, $h^0_0(\tilde{v}_{2,0}, \tilde{v}_{2,0}) = ||\zeta||^{-2t}$, and $h^0_0(\tilde{v}_{1,0}, \tilde{v}_{2,0}) = 0$.
- $\Upsilon_x^* h^0_{x_i} \to h^0_0$ in the $C^\infty$-sense on any compact subset in $C_\zeta$.
- We have $h^0_{x_i}(\tilde{v}_{1,x}, \tilde{v}_{1,x}) = ||\zeta||^{2t}$, $h^0_{x_i}(\tilde{v}_{2,x}, \tilde{v}_{2,x}) = ||\zeta||^{-2t-2t}$ and $h^0_{x_i}(\tilde{v}_{1,x}, \tilde{v}_{2,x}) = 0$ on $\{||\zeta|| > 1\}$.

Let $k_i$ be the self-adjoint endomorphism of $(\tilde{E}_{c^i,x}, h^0_{x_i})$ determined by $\Psi^*_x h_{c^i,t} = h^0_{x_i} \cdot k_i$. By (36), we have the convergence of $\Upsilon^*_x k_i$ to the identity on any compact subset in $\{||\zeta|| > 1\}$. We have the convergence of $\Upsilon^*_x k_i^{-1}$ to the identity on any compact subset in $\{||\zeta|| > 1\}$.

Let us study the convergence of $\Upsilon^*_x k_i$ and $\Upsilon^*_x k_i^{-1}$ on $\{||\zeta|| < 2\}$. We have the following uniform estimate:

$$|\text{R}(h^0_{x_i,t})|_{h^0_{x_i,t,9\zeta}} \leq C_{42}.$$ 

Hence, we have a constant $C_{43} > 0$ which is independent of $c^i$ such that $-\partial_{\zeta} \partial_{\overline{\zeta}} \log |\tilde{f}_{1}|_{h^0_{x_i,t}} \leq C_{43}$ on $\{||\zeta|| < 2\}$. Namely, we have

$$-\partial_{\zeta} \partial_{\overline{\zeta}} \log (|\tilde{f}_{1}|_{h^0_{x_i,t}} - C_{43})^2 \leq 0$$

We have already known the uniform boundedness of $|\tilde{f}_{1}|_{h^0_{x_i,t}}$ on $\{1 < ||\zeta|| < 2\}$. Then, we obtain the uniform boundedness of $|\tilde{f}_{2}|_{h^0_{x_i,t}}$ on $\{||\zeta|| < 2\}$. Similarly, we obtain the uniform boundedness of $|\tilde{f}_{2}|_{h^0_{x_i,t}}$ on $\{||\zeta|| < 2\}$. Hence, $k_i$ and $k_i^{-1}$ are uniformly bounded on $\{||\zeta|| < 2\}$.

Then, we may assume that the sequence $\Upsilon^*_x k_i$ is weakly convergent to $k_\infty$ in $L^2_{c^i}$ on any compact subset in $C_\zeta$. We have the convergence of $\Upsilon^*_x k_i^{-1}$ to $k^{-1}_\infty$. We obtain a harmonic metric $h_{c^i} = h^0_{c^i} \cdot k_i$. By the construction, we have $P^0_{c^i} \tilde{E}_{c^i,0} \subset P^\infty_{c^i} \tilde{E}_{c^i,0}$. Because $\text{deg}(P^0_{c^i} \tilde{E}_{c^i,0}) = \text{deg}(P^\infty_{c^i} \tilde{E}_{c^i,0}) = 0$, we obtain $P^0_{c^i} \tilde{E}_{c^i,0} = P^\infty_{c^i} \tilde{E}_{c^i,0}$ by Lemma 3.6. But, $(P^0_{c^i} \tilde{E}_{c^i,0}, \tilde{\theta})$ is not poly-stable because $P^0_{c^i} L_2 \simeq \mathcal{O}_{2}(\ell)$ and $\text{deg} P^0_{c^i} L_2 = \ell > 0$. Hence, we have deduced a contradiction from the assumption $b_c \to 0$.  

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Let us give a sketch of the argument in the case \( \lim b_c = b \neq 0 \). Under the assumption, the sequence \( h_{c,T} \) is convergent on \( \{ |ζ| \geq 1 \} \). As in the previous case, by taking a subsequence, we may assume that the sequence \( h_{c,T} \) is weakly convergent in \( L^p_{\text{loc}} \) on any compact subset in \( C_0 \), and the limit \( h_\infty \) is a harmonic metric of \((\tilde{E}_t, \theta)\) which is adapted to the filtered bundle \( P^*_{\text{loc}} \tilde{E}_t \). But, \((P^*_{\text{loc}} \tilde{E}_t, \theta)\) is not poly-stable. Hence, we obtain a contradiction even in the case \( b \neq 0 \).

### 3.3.3 Convergence of some sequences

Let \( t_i \) be a sequence of positive numbers such that \( t_i \to \infty \). According to Proposition 3.15 and Proposition 3.19, we can take a sequence of negative numbers \( c_i \) such that \( c_i \to 0 \) and \( b_c t_i^{1/(m+1)} = 1 \). We set \( t_i := t_i^{1/(m+1)} \).

We use the notation in §3.2.4. We have the isomorphisms \( \Phi_{\tau_i} : \varphi_{\tau_i} \tilde{E}_i \simeq \tilde{E}_i \). Let \( h_i \) denote the Hermitian metric of \( \tilde{E}_i \) induced by \( \Phi_{\tau_i} \) and \( \varphi_{\tau_i} h_{c_i,T} \). Take any \( T > 0 \). We have the following estimates on \( \{ |ζ| \geq T \} \) for any \( i \) such that \( t_i^{1/(m+1)} T > 1 \), where \( C \) and \( \epsilon \) are positive constants independent of \( i \) and \( T \):

\[
\left| \log|\tilde{v}_1|_{h_i} - \log(|ζ|^\epsilon t_i^{\epsilon/(m+1)} t_i) \right| \leq C \exp(-\epsilon |ζ|^{m+1} t_i) \\
\left| \log|\tilde{v}_2|_{h_i} - \log(|ζ|^{-\epsilon} t_i^{-\epsilon/(m+1)} t_i) \right| \leq C \exp(-\epsilon |ζ|^{m+1} t_i) \\
|h_i(\tilde{v}_1, \tilde{v}_2)| \leq C \exp(-\epsilon |ζ|^{m+1} t_i) \cdot |\tilde{v}_1|_{h_i} \cdot |\tilde{v}_2|_{h_i} \\

\]

Let \( h_{\text{lim}} \) be the Hermitian metric of \( \tilde{V}_{|ζ|} \) given by

\[
h_{\text{lim}}(\tilde{v}_1, \tilde{v}_1) = 1, \quad h_{\text{lim}}(\tilde{v}_1, \tilde{v}_2) = |ζ|^{-2\ell}, \quad h_{\text{lim}}(\tilde{v}_1, \tilde{v}_2) = 0.

For any \( \gamma > 0 \), the automorphism \( \Psi_\gamma \) on \( \tilde{V}_{|ζ|} \) is given by \( \Psi_\gamma = \gamma \text{id}_{L_i} \circ \gamma^{-1} \text{id}_{L_2} \). We define \( \Psi_\gamma h_i(u_1, u_2) := h_i(\Psi_\gamma u_1, \Psi_\gamma u_2) \). The following is clear.

**Proposition 3.20** Set \( \gamma_i := t_i^{-\epsilon/(m+1)} \). Then, we have the convergence \( \lim_{i \to \infty} \Psi_\gamma h_i = h_{\text{lim}} \) on \( C_\ast \).

### 3.4 Complement

We use the notation in §3.2. Take \( 0 < \kappa < 1 \). We take a \( C^\infty \)-function \( \rho : \mathbb{C} \to \mathbb{R}_{\geq 0} \) satisfying \( \rho(0) = 1 \) (\( |ζ| \leq 1/2 \)) and \( \rho(0) = 0 \) (\( |ζ| \geq 1 \)). We set

\[
u_1 := -(\kappa \rho(ζ) + |ζ|^{2\ell})^{-1} e_2 + e_1, \quad \nu_2 := e_2.
\]

They give a \( C^\infty \)-frame of \( \tilde{E}_t \). Note that we have \( \nu_1 = v_1 \) and \( \nu_2 = \zeta^\ell v_2 \) on \( \{ |ζ| \geq 1 \} \).

We take a large integer \( L \). Let \( h_{\kappa} \) be the Hermitian metric of \( \tilde{E}_t \) determined by the following conditions:

\[
u_2|_{h_{\kappa}} = \kappa^L, \quad \nu_1|_{h_{\kappa}} = \kappa^{-L}, \quad h_{\kappa}(u_1, u_2) = 0
\]

Let \( \nabla_\kappa \) be the Chern connection of \((\tilde{E}_t, h_{\kappa})\), and let \( R(h_{\kappa}) \) denote the curvature of \( \nabla_\kappa \). Let \( \tilde{\nabla}_\kappa \) denote the adjoint of \( \tilde{\theta} \) with respect to \( h_{\kappa} \). Let \( \tilde{\nabla}_\kappa \) denote the adjoint of \( \tilde{\nabla}_\kappa \) with respect to \( h_{\kappa} \).

**Lemma 3.21** On \( \{ |ζ| \leq 1 \} \), we have \( |R(h_{\kappa})|_{h_{\kappa}, g_{\kappa}} \leq C_{50} \kappa^L \) and \( |[\tilde{\nabla}_\kappa, \tilde{\nabla}_\kappa]|_{h_{\kappa}, g_{\kappa}} \leq C_{50} \kappa^L \) for a constant \( C_{50} > 0 \). On \( \{ |ζ| \geq 1 \} \), we have \( R(h_{\kappa}) = [\tilde{\nabla}_\kappa, \tilde{\nabla}_\kappa] = 0 \).

**Proof** In the proof, \( O(\kappa^L) \) denotes functions which are dominated by \( C_{50} \kappa^L \) for a constant \( C_{51} > 0 \). The equalities on \( \{ |ζ| \geq 1 \} \) are clear. Let us argue the estimates on \( \{ |ζ| \leq 1 \} \). We have \( \tilde{\nabla}_\kappa(u_2) = 0 \), and

\[
\tilde{\nabla}_\kappa(u_1) = -\frac{(k \rho(ζ) + |ζ|^{2\ell})^2 (\kappa^{-L} u_2)}{(k \rho(ζ) + |ζ|^{2\ell})^2 (\kappa^{-L} u_2)}.
\]

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We have
\[ \nabla_\kappa (\kappa^L u_1, \kappa^{-L} u_2) = (\kappa^L u_1, \kappa^{-L} u_2) A, \quad A = \begin{pmatrix} 0 & \partial(\Omega^L \rho)_{\kappa^{2L+1}}^2 \\ \frac{-\partial(\Omega^L \rho)_{\kappa^{2L+1}}^2} {\kappa^{2L+1}} & 0 \end{pmatrix} \]

It is easy to obtain the following estimate on \(|\zeta| \leq 1|
\[ \partial(\zeta^L \rho)_{\kappa^{2L+1}}^2 = O(\kappa^L) d\zeta, \quad \partial(\zeta^L \rho)_{\kappa^{2L+1}}^2 = O(\kappa^L) d\zeta \]
We also have the following on \(|\zeta| \leq 1|
\[ d\left( \frac{\partial(\zeta^L \rho)_{\kappa^{2L+1}}^2}{\kappa^{2L+1}} \right) = O(\kappa^L) d\zeta d\tilde{\zeta} \]
Hence, we have the following on \(|\zeta| \leq 1|
\[ dA + A \wedge A = O(\kappa^L) d\zeta d\tilde{\zeta} \]
It implies the estimate for \(R(h_\kappa)\) on \(|\zeta| \leq 1|

The Higgs field \( \tilde{\theta} \) is represented as follows:
\[ \tilde{\theta}(\kappa^L u_1, \kappa^{-L} u_2) = (\kappa^L u_1, \kappa^{-L} u_2) \begin{pmatrix} \alpha \zeta^m d\zeta \\ \frac{-2\kappa^{2L+1} \alpha \zeta^m}{\kappa^{2L+1}} d\zeta \end{pmatrix} \]

The adjoint \( \tilde{\theta}_u^L \) is represented as follows:
\[ \tilde{\theta}_u^L(\kappa^L u_1, \kappa^{-L} u_2) = (\kappa^L u_1, \kappa^{-L} u_2) \begin{pmatrix} \frac{-2\kappa^{2L+1} \alpha \zeta^m}{\kappa^{2L+1}} d\zeta \\ \alpha \zeta^m d\zeta \end{pmatrix} \]

Hence, \([\tilde{\theta}, \tilde{\theta}_u^L] \) is represented as follows:
\[ [\tilde{\theta}, \tilde{\theta}_u^L](\kappa^L u_1, \kappa^{-L} u_2) = (\kappa^L u_1, \kappa^{-L} u_2) A_2 \]
\[ A_2 := \begin{pmatrix} 4\kappa^{4L+2} |\alpha|^2 \zeta \rho(\zeta)^2 \frac{|\zeta|^{2(m-\ell)}} {\kappa \rho(\zeta) + |\zeta|^{2L}} d\zeta d\tilde{\zeta} \\ 4\kappa^{4L+2} |\alpha|^2 \zeta \rho(\zeta) \frac{|\zeta|^{2m-\ell}} {\kappa \rho(\zeta) + |\zeta|^{2L}} d\zeta d\tilde{\zeta} \\ -4\kappa^{2L+1} |\alpha|^2 \zeta \rho(\zeta) \frac{|\zeta|^{2m-\ell}} {\kappa \rho(\zeta) + |\zeta|^{2L}} d\zeta d\tilde{\zeta} \\ -4\kappa^{4L+2} |\alpha|^2 \zeta \rho(\zeta)^2 \frac{|\zeta|^{2(m-\ell)}} {\kappa \rho(\zeta) + |\zeta|^{2L}} d\zeta d\tilde{\zeta} \end{pmatrix} \]
In particular, we have the following on \(|\zeta| \leq 1|
\[ A_2 = O\left( \frac{\kappa^{4L+2} \rho(\zeta)|\zeta|^{2(m-\ell)}} {\kappa \rho(\zeta) + |\zeta|^{2L}} + \frac{\kappa^{2L+1} \rho(\zeta)|\zeta|^{2m-\ell}} {\kappa \rho(\zeta) + |\zeta|^{2L}} \right) d\zeta d\tilde{\zeta} = O(\kappa^L) d\zeta d\tilde{\zeta} \]
Thus, we are done.

### 3.4.1 Convergence of some sequences

Suppose that we are given a number \(a > \ell/2(m+1). \) Set \( \nu := a - \ell/2(m+1) > 0. \) Let \( t_i \) be a sequence of positive numbers such that \( t_i \to \infty. \) We set \( \kappa_i := t_i^{-\nu/L}, \) for which we have \( \kappa_i \to 0. \) We have the Hermitian metric \( h_{\kappa_i} \) of \( \tilde{E}_\ell \) as above.

We use the notation in §3.2.4. We set \( \tau_i := t_i^{1/2(m+1)}. \) We have the isomorphisms \( \Phi_{\tau_i} : \varphi_{\tau_i}^* \tilde{E}_\ell \simeq \tilde{E}_\ell. \) Let \( h_i \) be the Hermitian metric of \( \tilde{E}_\ell \) induced by \( \varphi_{\tau_i}^* h_{\kappa_i} \) and \( \Phi_{\tau_i}. \)
Lemma 4.2. We have local frames $v_i$ of $L_{\pm}$ around $\Delta$. Let $h_{\text{lim}}$ be the Hermitian metric of $V_{|\mathbb{C}^*}$ given by $h_{\text{lim}}(\tilde{v}_1, \tilde{v}_2) = 1$, $h_{\text{lim}}(\tilde{v}_1, \tilde{v}_1) = |\zeta|^{-2\ell}$, $h_{\text{lim}}(\tilde{v}_2, \tilde{v}_2) = 0$.

For any $\gamma > 0$, the automorphism $\Psi_\gamma$ on $V_{|\mathbb{C}^*}$ is given by $\Psi_\gamma = i \cdot \gamma \cdot (\Psi \circ \gamma^{-1}) \cdot (\Psi \circ \gamma^{-1})$. We define $\Psi_\gamma^* h_i(u_1, u_2) := h_i(\Psi_\gamma u_1, \Psi_\gamma u_2)$. The following is clear.

Proposition 3.22 Set $\gamma := t_i^{-\alpha}$. Then, we have the convergence $\lim_{t_i \to \infty} \Psi_\gamma^* h_i = h_{\text{lim}}$ on $\mathbb{C}^*$.

4 Limiting configurations

4.1 A description of generically regular semisimple Higgs bundles of rank 2

Let $X$ be a connected complex curve. Let $(E, \bar{\Omega}_E, \bar{\theta})$ be a Higgs bundle on $X$ of rank 2 which is generically regular semisimple. For simplicity, we assume that $\text{tr}(\bar{\theta}) = 0$. Let us consider the case that the spectral curve $\Sigma(E, \bar{\theta})$ is reducible, i.e., we have a holomorphic 1-form $\omega \neq 0$ on $X$ such that $\Sigma(E, \bar{\theta})$ is the union of the images of $\omega$ and $-\omega$. Such $\omega$ is determined up to the multiplication of $\pm 1$.

Remark 4.1 Suppose that $\text{tr}(\bar{\theta}) \neq 0$. Then, we set $\theta' := \theta - (\text{tr}(\bar{\theta})/2) \cdot \mathbb{id}_E$. Then, the Higgs bundle $(E, \bar{\Omega}_E, \theta')$ satisfies $\text{tr}(\bar{\theta}') = 0$. Hence, it is enough to study Higgs bundles such that the trace of the Higgs field is 0.

Suppose that $\Sigma(E, \bar{\theta})$ is irreducible. We take a normalization $\varphi : X \to E \to \Sigma(E, \bar{\theta})$. We have the induced morphism $\varphi_1 : X \to E$. The Higgs bundle $\varphi_1^*(E, \bar{\Omega}_E, \theta)$ satisfies the above condition. If $X$ is compact and $(E, \bar{\Omega}_E, \theta)$ is stable, then $\varphi_1^*(E, \bar{\Omega}_E, \theta)$ is stable or poly-stable. So, the study on the irreducible case can be reduced to the reducible case. (See also §4.3.2.)

Let $Z(\omega)$ denote the zero set of $\omega$. We have the decomposition of the $\mathcal{O}_X(\ast Z(\omega))$-module $E \otimes \mathcal{O}_X(\ast Z(\omega)) = L_{\omega} \oplus L_{-\omega}$ corresponding to the decomposition of the spectral curve, i.e., $\omega = -\omega$ · $\mathbb{id}_L \oplus (\omega)$ · $\mathbb{id}_L$. We can regard $E$ as an $\mathcal{O}_X$-submodule of $L_{\omega} \oplus L_{-\omega}$.

We obtain the $\mathcal{O}_X$-module $\det(L_{\omega} \oplus L_{-\omega})/\det(E)$ whose supports are contained in $Z(\omega)$. For each $P \in Z(\omega)$, let $\ell_P$ denote the length of the stalk of $\det(L_{\omega} \oplus L_{-\omega})/\det(E)$ at $P$. Because we have the exact sequence $0 \to L_{\omega} \cap E \to E \to L_{-\omega} \to 0$,

we have $L_{\omega} \cap E = L_{\omega}(-\sum_{P \in Z(\omega)} \ell_P \cdot P)$ in $L_{\omega} \oplus L_{-\omega}$. Similarly, we have $L_{-\omega} \cap E = L_{-\omega}(-\sum_{P \in Z(\omega)} \ell_P \cdot P)$ in $L_{\omega} \oplus L_{-\omega}$.

Local description Let $(U, z)$ be a small holomorphic coordinate neighbourhood around $P$ satisfying $z(P) = 0$. Let $sp$ be any frame of $\det(E)|_{U_P}$.

Lemma 4.2 We have local frames $v_{+\pm}$ of $L_{+\pm}$ around $P$ such that (i) $e_1 = v_+ + v_-$ and $e_2 = z^{\ell_P} v_-$ is a frame of $E|_{U_P}$, (ii) $e_1 \cap e_2 = sp$.

Proof If $\ell_P > 0$, we have $E = L_{\omega} \oplus L_{-\omega}$ on $U_P$, and hence the claim is clear. We shall consider the case $\ell_P > 0$. We omit to distinguish the restriction to $U_P$. We take a frame $e'_2$ of $z^{\ell_P} L_{-\omega}$. We take a section $e_1'$ of $E$ such that $e_1'$ and $e_2'$ give a frame of $E$. We have the unique decomposition $e_1' = e_1'' + e_2''$, where $e_1''$ are sections of $L_{+\omega}$. By the construction, $L_{+\omega}$ is generated by the images of $e_1''$ and $e_2''$. Because $\ell_P > 0$, we can observe that the image of $v_{+\pm}^i$ generates $L_{+\omega}$, i.e., $v_{+\pm}^i$ are frames of $L_{+\omega}$. We may assume that $e_2' = z^{\ell_P} v_{-\pm}^i$. 37
Let $g$ be the holomorphic function determined by $e_1 \wedge e_2 = g \cdot s$. Because $g(P) \neq 0$, we can take a holomorphic function $g_1$ such that $g_1^2 = g$. We set $e_i := g_1^{-1} \cdot e_i$ and $v_\pm := g_1^{-1} v_\pm$. Then, they satisfy the desired condition.

Let $m_P$ denote the order of zero of $\omega$ at $P$, i.e., we have $\omega = z^{m_P} g_P(z) \, dz$ on $U_P$ for a holomorphic function $g_P(z)$ with $g_P(0) \neq 0$. For the frame $(e_1, e_2)$, the Higgs field $\theta_{|U_P}$ is described as

$$\theta(e_1, e_2) = (e_1, e_2) \left( \begin{array}{cc} \omega & 0 \\ -2z^{-\ell_P} \omega & -\omega \end{array} \right).$$

In particular, we have $\ell_P \leq m_P$.

### 4.2 The limiting configurations of stable Higgs bundles

Let $X$ be a compact connected Riemann surface. Let $(E, \overline{\partial} E, \theta)$ be a Higgs bundle of rank 2 on $X$ such that (i) $(E, \overline{\partial} E, \theta)$ is stable, (ii) $(E, \overline{\partial} E, \theta)$ is generically regular semisimple, (iii) $\text{tr}(\theta) = 0$. We assume that the spectral curve of $\theta$ is reducible. We obtain a holomorphic 1-form $\omega \neq 0$ and the line bundles $L_{\pm, \omega}$ as in §4.1. We impose that $\deg(L_{\omega}) \leq \deg(L_{-\omega})$. We set $L_1 := L_{\omega}$ and $L_2 := L_{-\omega}$.

We set $d_i := \deg(L_i)$ $(i = 1, 2)$. We have

$$d_1 + d_2 - \sum_{P \in \Omega(\omega)} \ell_P = \deg(E).$$

The stability condition for $(E, \theta)$ is equivalent to the inequalities

$$\deg(L_j \cap E) = d_j - \sum_{P \in \Omega(\omega)} \ell_P < \deg(E)/2 \quad (j = 1, 2),$$

i.e.,

$$d_j - \deg(E)/2 > 0 \quad (j = 1, 2).$$

The local scaling factor $a_P$ of $(E, \partial, \theta)$ at $P \in \Omega(\omega)$ is defined as follows:

$$a_P := \frac{\ell_P}{2(m_P + 1)}.$$

We have the functions $\chi_P : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\leq 0}$ $(P \in \Omega(\omega))$ given as follows:

$$\chi_P(a) := \begin{cases} (m_P + 1)(a - a_P) & (0 \leq a \leq a_P) \\ 0 & (a \geq a_P) \end{cases}$$

We set $\chi_{E, \theta}(a) := \sum_{P \in \Omega(\omega)} \chi_P(a)$.

**Lemma 4.3** We have the unique number $a_{E, \theta}$ satisfying the conditions

$$d_1 - \frac{\deg(E)}{2} + \chi_{E, \theta}(a_{E, \theta}) = 0, \quad 0 \leq a_{E, \theta} < \max \{ a_P \mid P \in \Omega(\omega) \}.$$

**Proof** The function $\chi_{E, \theta}$ is strictly increasing for $0 \leq a \leq a_1 := \max \{ a_P \mid P \in \Omega(\omega) \}$, and we have $\chi_{E, \theta}(a) = 0$ for $a \geq a_1$. Hence, we have $d_1 - \frac{\deg(E)}{2} + \chi_{E, \theta}(a_1) = d_1 - \frac{\deg(E)}{2} > 0$. We have $\chi_{E, \theta}(0) = -\sum_{P \in \Omega(\omega)} \ell_P/2$. Because of $d_1 + d_2 - \deg(E) - \sum_{P \in \Omega(\omega)} \ell_P = 0$ and $d_1 \leq d_2$, we have $d_1 - \frac{\deg(E)}{2} + \chi_{E, \theta}(0) \leq 0$. Hence, the claim of the lemma follows.

As in §3.1.5, the numbers $-\chi_P(a_{E, \theta})$ $(P \in \Omega(\omega))$ give a parabolic structure on the line bundle $L_1$. For any $c = (c_P \mid P \in \Omega(\omega)) \in \mathbb{R}^{\Omega(\omega)}$, let $n(c) = (n_P(c) \mid P \in \Omega(\omega)) \in \mathbb{Z}^{\Omega(\omega)}$ be determined by the condition

$$c_P - 1 < n_P(c) - \chi_P(a_{E, \theta}) \leq c_P.$$ 

Then, we set $P_{\lim}^c(L_1) = L_1 \left( \sum_{P \in \Omega(\omega)} n_P(c) P \right)$. We obtain a filtered bundle $P_{\lim}^c(L_1) = (P_{\lim}^c(L_1) \mid c \in \mathbb{R}^{\Omega(\omega)})$ over the meromorphic line bundle $L_1 \ast \mathbb{Z}(\omega)$. The parabolic degree of $P_{\lim}^c(L_1)$ is

$$d_1 - \sum_{P \in \Omega(\omega)} (-\chi_P(a_{E, \theta})) = d_1 + \chi_{E, \theta}(a_{E, \theta}) = \frac{\deg(E)}{2}.$$
Similarly, the numbers \( \chi_P(a, E_\theta) + \ell_P \) \( (P \in Z(\omega)) \) determine a filtered bundle over \( L_2(\ast Z(\omega)) \). The parabolic degree of \( \mathcal{P}_s^\lim L_2 \) is

\[
d_2 = \sum_{P \in Z(\omega)} (\chi_P(a, E_\theta) + \ell_P) = d_1 - \frac{\deg(E)}{2} + d_2 - \sum_{P \in Z(\omega)} \ell_P = \frac{\deg(E)}{2}.
\]

The direct sum \( \mathcal{P}_s^\lim L_1 \oplus \mathcal{P}_s^\lim L_2 \) is called the limiting configuration of \( (E, \Theta_E, \theta) \).

### 4.2.1 Hermitian metrics of the limiting configuration

We fix a Hermitian metric \( h_{\det(E)} \) on the determinant line bundle \( \det(E) \). We have the 2-form \( R(h_{\det(E)}) \) obtained as the curvature of the Chern connection of \( (\det(E), h_{\det(E)}) \).

**Lemma 4.4** We have Hermitian metrics \( h_{L_j}^\lim \) of the holomorphic line bundles \( L_j|_{X \setminus Z(\omega)} \) satisfying the following conditions:

- The curvature of the Chern connection \( \nabla_j^\lim \) of \( (L_j|_{X \setminus Z(\omega)}, h_{L_j}^\lim) \) is equal to \( R(h_{\det(E)})/2 \).
- For each \( P \in Z(\omega) \), let \((U_P, z_P)\) be any holomorphic coordinate neighbourhood of \( P \) with \( z_P(P) = 0 \). Then, \( |z_P|^{-2\nu_0(\omega, s)} h_{L_1,j}^\lim |_{L_1,j}\{U_P\} \) and \( |z_P|^{2\nu_0(\omega, s) + 2\nu_j} h_{L_2,j}^\lim |_{L_2,j}\{U_P\} \) induce Hermitian metrics of \( C^\infty \)-class on \( L_1|_{U_P} \) and \( L_2|_{U_P} \), respectively.
- Under the isomorphism \( \det(E)|_{X \setminus Z(\omega)} \simeq (L_1 \otimes L_2)|_{X \setminus Z(\omega)} \), we have \( h_{L_1}^\lim \otimes h_{L_2}^\lim = h_{\det(E)} \) on \( X \setminus Z(\omega) \).

**Proof** Because this is standard, we give only a sketch of the proof. We can take Hermitian metrics \( h_{L_j} \) \( (j = 1, 2) \) of \( L_j|_{X \setminus Z(\omega)} \) such that \( |z_P|^{-2\nu_0(\omega, s)} h_{L_1,j} |_{L_1,j}\{U_P\} \) and \( |z_P|^{2\nu_0(\omega, s) + 2\nu_j} h_{L_2,j}^\lim |_{L_2,j}\{U_P\} \) induce Hermitian metrics of \( C^\infty \)-class on \( L_1|_{U_P} \) and \( L_2|_{U_P} \), respectively. Let \( R(h_{L_j}) \) denote the curvature form of \( (L_j|_{X \setminus Z(\omega)}, h_{L_j}) \). They naturally induce 2-forms of \( C^\infty \)-class on \( X \), which are also denoted by \( R(h_{L_j}) \). Because \( \frac{\sqrt{\pi}}{2} \int R(h_{L_j}) \) is equal to the parabolic degree of \( \mathcal{P}_s^\lim L_j \), we have \( \int \left( R(h_{L_j}) - R(h_{\det(E)})/2 \right) = 0 \). We have \( C^\infty \)-functions \( \rho_j \) \( (j = 1, 2) \) on \( X \) such that \( \overline{\partial} \rho_j = R(h_{L_j}) - R(h_{\det(E)})/2 \). We set \( h_{L_j}^{(1)} := e^{-\rho_j} h_{L_j} \). Then, we have \( R(h_{L_j}^{(1)}) = R(h_{\det(E)})/2 \). Because \( \det(E) = L_1 \otimes L_2 \otimes \mathcal{O}_X(-\sum \ell_P P) \), the tensor product \( h_{L_1}^{(1)} \otimes h_{L_2}^{(2)} \) induces a \( C^\infty \)-Hermitian metric of \( \det(E) \). By comparison of the curvature, we have \( h_{L_1}^{(1)} \otimes h_{L_2}^{(2)} = \alpha \cdot h_{\det(E)} \) for a positive constant \( \alpha \). Hence, we obtain Hermitian metrics \( h_{L_j}^\lim \) with the desired property by adjusting \( h_{L_j}^{(1)} \) for example by setting \( h_{L_1}^\lim = \alpha^{-1} h_{L_1}^{(1)} \) and \( h_{L_2}^\lim = h_{L_2}^{(2)} \).

Such a metric \( h_{L_1}^\lim \otimes h_{L_2}^\lim \) is also called the limiting configuration of \( (E, \Theta_E, \theta) \). Note that we have the ambiguity of the actions of automorphisms \( \alpha \cdot \id_{L_1} \oplus \alpha^{-1} \id_{L_2} \) \( (\alpha > 0) \), i.e., the pair of \( \alpha \cdot h_{L_1}^\lim \) and \( \alpha^{-1} \cdot h_{L_2}^\lim \) satisfies the conditions in Lemma 4.4. But, the induced Chern connection \( \nabla_{E,\theta}^\lim \) := \( \nabla_1^\lim \oplus \nabla_2^\lim \) on \( E|_{X \setminus Z(\omega)} \) is well defined. Note that \( \nabla_{E,\theta}^\lim \) is projectively flat, whose curvature is given by the multiplication of \( R(h_{\det(E)})/2 \).

**Remark 4.5** The metric \( h_{E,\theta}^\lim \) is also characterized as a Hermitian-Einstein metric for the Higgs bundle \( (L_1, t_\omega) \oplus (L_2, -t_\omega) \) adapted to the filtered bundle \( \mathcal{P}_s^\lim L_1 \oplus \mathcal{P}_s^\lim L_2 \) for any \( t \neq 0 \) such that \( \det(h_{E,\theta}^\lim) = h_{\det(E)} \). Because \( (\mathcal{P}_s^\lim L_1, t_\omega) \oplus (\mathcal{P}_s^\lim L_2, -t_\omega) \) is polystable, we have the ambiguity of the metric \( h_{E,\theta}^\lim \) by the automorphisms \( \alpha \cdot \id_{L_1} \oplus \alpha^{-1} \id_{L_2} \), as usual.

### 4.3 The limiting configuration in complementary cases

Let \( X \) be a compact connected Riemann surface. Let \( (E, \Theta_E, \theta) \) be any Higgs bundle of rank 2 on \( X \) such that (i) \( (E, \Theta_E, \theta) \) is generically regular semisimple, (ii) \( \tr(\theta) = 0 \). We give limiting configurations in some complementary cases.

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4.3.1 Polystable Higgs bundles

Suppose that \((E, \mathcal{J}_E, \theta)\) is polystable. Then, we have the decomposition \((E, \theta) = (L_\omega, \omega) \oplus (L_{-\omega}, -\omega)\). In particular, the spectral curve is reducible, and we have \(\ell_P = 0\) in the description in \(\S4.1\). In this case, we set \((E, \theta)\) as the limiting configuration.

We take a Hermitian metric \(h_{\det(E)}\) on the line bundle \(\det(E)\). We have Hermitian metrics \(h^\lim_{L_{\pm \omega}}\) on \(L_{\pm \omega}\) such that \(R(h^\lim_{L_{\pm \omega}}) = R(h_{\det(E)})/2\). We set \(h^\lim_{E} := h^\lim_{L_{\omega}} \oplus h^\lim_{L_{-\omega}}\) such that \(h^\lim_{E \ominus} = h_{\det(E)}\). The metric \(h^\lim\) is also called limiting configuration. It is characterized as a Hermitian-Einstein metric on the Higgs bundle \((E, \mathcal{J}_E, \theta)\) such that \(\det(h^\lim) = h_{\det(E)}\). We have the ambiguity of the metric \(h^\lim\) caused by automorphisms of \((E, \theta)\).

The limiting configuration can also be given as a filtered bundle on \((X, Z(\omega))\) as in the case of stable Higgs bundles \(\S4.2\). We consider the trivial parabolic structures for \(L_{\pm \omega}\) at \(P \in Z(\omega)\). Namely, for any \(c = (c_P \mid P \in Z(\omega)) \in \mathbb{R}^{Z(\omega)}\), let \(n(c) = (n_P(c)) \in \mathbb{Z}^{Z(\omega)}\) be determined by the condition \(c_P - 1 < n_P(c) \leq c_P\). Then, we set \(P^\lim_c(L_{\pm \omega}) = L_{\pm \omega}(\sum_{P \in Z(\omega)} n_P(c)P)\). The parabolic degrees of \(P^\lim_c(L_{\pm \omega})\) is \(\deg(E)/2\). The filtered bundle \(P^\lim_c L_\omega \oplus P^\lim_c L_{-\omega}\) is called the limiting configuration of \((E, \mathcal{J}_E, \theta)\).

**Remark 4.6** The metric \(h^\lim_{E, \theta}\) can be characterized as a Hermitian-Einstein metric for \((E, \theta)|_{X \setminus Z(\omega)}\) adapted to the filtered bundle such that \(\det(h^\lim_{E, \theta}) = h_{\det(E)}\).

4.3.2 The case where the spectral curve is irreducible

Suppose that the spectral curve \(\Sigma(E, \theta)\) is irreducible. It implies that the Higgs bundle \((E, \mathcal{J}_E, \theta)\) is stable. We take a normalization \(\tilde{X} \longrightarrow \Sigma(E, \theta)\). We have the induced morphism \(p : \tilde{X} \longrightarrow X\), which is a ramified covering of degree 2. We have the involution of \(\Sigma(E, \theta)\) induced by the multiplication of \(-1\) on the cotangent bundle \(T^*X\). It induces an involution \(\rho\) of \(\tilde{X}\) over \(X\). We can regard \(X\) as the quotient space of \(\tilde{X}\) by the action of the group \(\{1, \rho\}\).

We set \((\tilde{E}, \tilde{\mathcal{J}}_{\tilde{E}}, \tilde{\theta}) := p^*(E, \mathcal{J}_E, \theta)\). The spectral curve of \((\tilde{E}, \tilde{\theta})\) is reducible, i.e., it is the union \(\operatorname{Im}(\tilde{\omega}) \cup \operatorname{Im}(\tilde{\omega}^-)\) for a holomorphic one form \(\tilde{\omega}\) on \(\tilde{X}\). We have a natural isomorphism \(\rho^*(\tilde{E}, \tilde{\mathcal{J}}_{\tilde{E}}, \tilde{\theta}) \simeq (\tilde{E}, \tilde{\mathcal{J}}_{\tilde{E}}, \tilde{\theta})\). By the construction, we have \(\rho^*\tilde{\omega} = -\tilde{\omega}\). We have \(Z(\tilde{\omega}) = p^{-1}D(E, \theta)\). We have the line bundles \(\tilde{L}_\omega\) and \(\tilde{L}_{-\omega}\) on \(\tilde{X}\) with an inclusion \(\tilde{E} \longrightarrow \tilde{L}_\omega \oplus \tilde{L}_{-\omega}\) as in \(\S4.1\). Because \(\rho^*\tilde{\omega} = -\tilde{\omega}\), we have natural isomorphisms \(\rho^*\tilde{L}_{\pm \omega} \simeq \tilde{L}_{\mp \omega}\) such that the following is commutative:

\[
\begin{align*}
\rho^*\tilde{E} & \longrightarrow \rho^*\tilde{L}_\omega \oplus \rho^*\tilde{L}_{-\omega} \\
\downarrow & \downarrow \\
\tilde{E} & \longrightarrow \tilde{L}_\omega \oplus \tilde{L}_{-\omega}
\end{align*}
\]

Note that we have already known that \((\tilde{E}, \tilde{\mathcal{J}}_{\tilde{E}}, \tilde{\theta})\) is poly-stable. Indeed, because \((E, \mathcal{J}_E, \theta)\) is stable, we have a Hermitian-Einstein metric \(h^{HE}\) for \((E, \mathcal{J}_E, \theta)\). The pull back \(p^*h^{HE}\) is a Hermitian-Einstein metric for \((\tilde{E}, \tilde{\mathcal{J}}_{\tilde{E}}, \tilde{\theta})\), which implies the poly-stability of the Higgs bundle.

**Stable case** When \((\tilde{E}, \tilde{\mathcal{J}}_{\tilde{E}}, \tilde{\theta})\) is stable, we obtain the limiting configuration for \((\tilde{E}, \tilde{\mathcal{J}}_{\tilde{E}}, \tilde{\theta})\) as a filtered bundle on \((X, Z(\tilde{\omega}))\) by the procedure in \(\S4.2\). Namely, we obtain the filtered line bundles \(P^\lim_c L_{\pm \omega}\) as in \(\S4.2\). In this case, we have \(\deg(\tilde{L}_\omega) = \deg(\tilde{L}_{-\omega})\) which implies that \(a_{\tilde{E}, \tilde{\theta}} = 0\) and that the parabolic weights are \(-\chi_P(a_{\tilde{E}, \tilde{\theta}}) = \chi_P(a_{E, \theta}) + \ell_P = \ell_P/2\). Because \(\ell_P = \ell_P(p)\), the isomorphisms \(\rho^*\tilde{L}_{\pm \omega} \simeq \tilde{L}_{\mp \omega}\) induce the isomorphisms of filtered line bundles \(\rho^* P^\lim_c L_{\pm \omega} \simeq P^\lim_c L_{\pm \omega}\). We have the Hermitian metrics \(h^\lim_{L_{\pm \omega}}\) of \(\tilde{L}_{\pm \omega}|_{\tilde{X} \setminus Z(\tilde{\omega})}\) as in Lemma 4.4. We may also impose the condition \(\rho^* h^\lim_{L_{\pm \omega}} = h^\lim_{L_{\pm \omega}}\) with which the metrics \(h^\lim_{L_{\pm \omega}}\) are uniquely determined.

We set \(h^\lim_{E, \theta} := h^\lim_{L_{\omega}} \oplus h^\lim_{L_{-\omega}}\) on \(E|_{X \setminus D(E, \theta)}\). Because \(\rho^* h^\lim_{E, \theta} = h^\lim_{E, \theta}\), we have the Hermitian metric \(h^\lim_{E, \theta}\) of \(E|_{X \setminus D(E, \theta)}\) such that \(p^* h^\lim_{E, \theta} = h^\lim_{E, \theta}\). The metric \(h^\lim_{E, \theta}\) and the associated Chern connection \(\nabla^\lim_{E, \theta}\) of \(E|_{X \setminus D(E, \theta)}\) are uniquely determined.
Remark 4.7 The metric $h_{\mathrm{lim}}^E,\theta$ is also characterized as follows. Because the filtered bundle $\mathcal{P}^\lim_*\bar{L}_\omega \oplus \mathcal{P}^\lim_*\bar{L}_{-\omega}$ is equivariant with respect to the action of $\{1, \rho\}$, we have the filtered bundle $\mathcal{P}^\lim_*E$ on $(X, D(E, \theta))$ obtained as the descent of $\mathcal{P}^\lim_*L_\omega \oplus \mathcal{P}^\lim_*L_{-\omega}$. (See §4.3.3 below for the descent of filtered bundles in this situation.) We can easily observe that the filtered Higgs bundles $(\mathcal{P}^\lim_*E, \theta, \rho)$ $(t \neq 0)$ are stable, and that the parabolic degree is $\deg(E)$. The metric $h_{\mathrm{lim}}^E,\theta$ is a unique Hermitian-Einstein metric for the Higgs bundle $(E, \mathcal{J}_E, \theta)|_{X \setminus D(E, \theta)}$ for any $t$, such that $\det(h_{\mathrm{lim}}^E,\theta) = h_{\det(E)}$ adapted to the filtered bundle $\mathcal{P}^\lim_*E$.

Polystable case We have $\bar{E} = \bar{L}_\omega \oplus \bar{L}_{-\omega}$. As in §4.3.1, we take Hermitian metrics $h_{\mathrm{lim}}^E,\theta$ of $\bar{L}_\omega$ satisfying $R(h_{\mathrm{lim}}^E,\theta) = p^*R(h_{\det(E)})/2$ and $h_{\mathrm{lim}}^E,\theta \otimes h_{\mathrm{lim}}^E,\theta = p^*h_{\det(E)}$, and we set $h_{\mathrm{lim}}^{E,\theta} := h_{\mathrm{lim}}^E \oplus h_{\mathrm{lim}}^{E,\theta}$. We also impose the condition $\rho^*h_{\mathrm{lim}}^E,\theta = h_{\mathrm{lim}}^E,\theta$ with which the metrics $h_{\mathrm{lim}}^E,\theta$ and $h_{\mathrm{lim}}^{E,\theta}$ are uniquely determined. Because $\rho^*h_{\mathrm{lim}}^{E,\theta}$, we have a unique Hermitian metric $h_{\mathrm{lim}}^{E,\theta}$ of $E$ such that $p^*h_{\mathrm{lim}}^{E,\theta} = h_{\mathrm{lim}}^E$.

The metric $h_{\mathrm{lim}}^{E,\theta}$ is also characterized as follows.

Lemma 4.8 For any $t \neq 0$, $h_{\mathrm{lim}}^{E,\theta}$ is a unique Hermitian-Einstein metric for the stable Higgs bundle $(E, \mathcal{J}_E, t\theta)$ such that $\det(h_{\mathrm{lim}}^{E,\theta}) = h_{\det(E)}$. In particular, the Hermitian-Einstein metrics for the Higgs bundles $(E, \mathcal{J}_E, t\theta)$ are independent of $t$.

Proof The claim is clear by the construction of $h_{\mathrm{lim}}^{E,\theta}$.

Remark 4.9 We have the filtered bundle $\mathcal{P}^\lim_*\bar{E}$ as in §4.3.1. Because it is equivariant with respect to $\rho$, we obtain a filtered bundle $\mathcal{P}^\lim_*E$ obtained as the descent of $\mathcal{P}^\lim_*\bar{E}$. The metric $h_{\mathrm{lim}}^{E,\theta}$ is also characterized as a unique Hermitian metric for the stable Higgs bundle $(\mathcal{P}^\lim_*E, t\theta)$ $(t \neq 0)$.

4.3.3 Descent (Appendix)

Let $\bar{X}, X, p, \rho, D(E, \theta)$ and $Z(\omega)$ be as in §4.3.2. Let $\bar{V}$ be a locally free $O_{\bar{X}}(\ast Z(\omega))$-module of finite rank which is equivariant with respect to $\{1, \rho\}$, i.e., we are given an isomorphism $\Phi : \rho^*\bar{V} \simeq V$ such that $\Phi \circ \rho^*\Phi = \text{id}$. Let $\mathcal{P}_*\bar{V}$ be a filtered bundle over $\bar{V}$, which is equivariant with respect to $\{1, \rho\}$, i.e., $\rho^*\mathcal{P}_*\bar{V} = \mathcal{P}_*\bar{V}$ for any $a \in \mathbb{R}Z(\omega)$ under the above isomorphism. In this case, the descent of $\mathcal{P}_*\bar{V}$ is given as follows. By the equivariance of $\bar{V}$, we have a locally free $O_X(\ast D(E, \theta))$-module $V$ with an isomorphism $\rho^*V \simeq \bar{V}$. It is also described as follows. We have the locally free $O_X(\ast D(E, \theta))$-module $p_!\bar{V}$. It is equivariant with respect to $\{1, \rho\}$, where the action of $\{1, \rho\}$ on $X$ is trivial. Then, $V$ is the invariant part of $p_!\bar{V}$ with respect to the action. For any $P \in D(E, \theta)$, let $q(P)$ denote the number of the set $\rho^{-1}(P)$, which are 1 or 2. Let $a = (a_P | P \in D(E, \theta)) \in \mathbb{R}D(E, \theta)$. For $Q \in \rho^{-1}(P)$, we set $\tilde{a}_Q := q(P) \cdot a_P$. We obtain the locally free $O_X$-module $\mathcal{P}_a(\bar{V})$. It is equivariant with respect to $\{1, \rho\}$. We obtain the locally free $O_X$-module $p_!\mathcal{P}_a\bar{V}$. It is equivariant with respect to $\{1, \rho\}$. The invariant part is denoted by $\mathcal{P}_aV$. Thus, we obtain a filtered bundle $\mathcal{P}_*V$ over $V$, which is the descent of $\mathcal{P}_*\bar{V}$.

5 Convergence to the limiting configurations

5.1 Statements

5.1.1 General case

Let $(E, \mathcal{J}_E, \theta)$ be a stable Higgs bundle of rank 2 on a compact connected Riemann surface $X$, such that (i) $(E, \mathcal{J}_E, \theta)$ is generically regular semisimple, (ii) the spectral curve is reducible, (iii) $\text{tr}(\theta) = 0$. We fix a Hermitian metric $h_{\det(E)}$ of $E$. We use the notation in §4.2. We have the limiting configuration $\mathcal{P}^\lim_*L_1 \oplus \mathcal{P}^\lim_*L_2$. We take Hermitian metrics $h_{\mathrm{lim}}^{L_j}$ for the parabolic line bundle $\mathcal{P}^\lim_*L_j$ satisfying the condition in Lemma 4.4. We set $h_{\mathrm{lim}}^{E,\theta} := h_{\mathrm{lim}}^{L_1} \otimes h_{\mathrm{lim}}^{L_2}$. We have the associated Chern connection $\nabla_{\mathrm{E,lim}}^{E,\theta}$ of $E_{|X \setminus Z(\omega)}$, which is projectively flat.

For any $t > 0$, the Higgs bundle $(E, \mathcal{J}_E, t\theta)$ is also stable. We have the Hermitian-Einstein metrics $h_t$ of the Higgs bundles $(E, \mathcal{J}_E, t\theta)$, i.e., $R(h_t) + [t\theta, t\theta]_{h_t}$ is equal to the multiplication of $R(h_{\det(E)})/2$ according
to Hitchin [6] and Simpson [19]. We impose that \( \det(h_t) = h_{\det(E)} \). We have the Chern connection \( \nabla_{h_t} \) of \((E, \mathcal{J}_E, h_t)\).

For any \( \gamma > 0 \), let \( \Psi_\gamma \) denote the automorphism of \( L_1 \oplus L_2 \) given by \( \Psi_\gamma = \gamma \id_{L_1} \oplus \gamma^{-1} \id_{L_2} \). We define the metric \( \Psi_\gamma^* h_t \) of \( E|_{X\setminus Z(\omega)} \) by \( \Psi_\gamma^* h_t(u_1, u_2) = h_t(\Psi_\gamma u_1, \Psi_\gamma u_2) \) for local sections \( u_i \) of \( E|_{X\setminus Z(\omega)} \).

Take any \( Q \in X \setminus Z(\omega) \). Let \( v_Q \) be any frame of \( L_1|_Q \). We set

\[
\gamma(t, Q) := \left( \frac{h_{\text{lim}}(v_Q, v_Q)}{h_t(v_Q, v_Q)} \right)^{1/2}
\]

We shall prove the following theorem in \( \S 5.2 - \S 5.4 \).

**Theorem 5.1** When \( t \) goes to \( \infty \), the sequence \( \Psi_\gamma^* h_t \) converges to \( h_{\text{lim}, \theta} \) in the \( C^\infty \)-sense on any compact subset in \( X \setminus Z(\omega) \).

In particular, we obtain the following convergence of unitary connections \( \nabla_{h_t} \).

**Corollary 5.2** The sequence \( \nabla_{h_t} \) \( (t \to \infty) \) converges to \( \nabla_{\text{lim}, \theta} \) on any compact subset in \( X \setminus Z(\omega) \).

**Proof** If \( \deg(E) = 0 \) and \( \text{R}(h_{\det(E)}) = 0 \), according to the estimates in \( \S 2.2 \), it is enough to prove that the sequence of the Chern connections of \( (L_j|_{X\setminus Z(\omega)}, h_{L_j}) \) converges to the Chern connection of \( (L_j|_{X\setminus Z(\omega)}, h_{\text{lim}}^{L_j}) \).

It follows from Theorem 5.1.

Let us consider the case where \( \deg(E) \) is even. We have a holomorphic line bundle \( L_0 \) on \( X \) with an isomorphism \( \det(E) \simeq L_0 \otimes L_0 \). We have the Hermitian metric \( h_{L_0} \) of \( L_0 \) such that \( h_{\det(E)} = h_{L_0} \otimes h_{L_0} \) under the isomorphism. Then, \( h_{L_0} \otimes h_{E^{-1}} \) on \( E \otimes L_0^{-1} \) is a harmonic metric of \( (E \otimes L_0^{-1}, \mathcal{J}_E \otimes L_0^{-1}, \theta, h_{L_0} \otimes h^{-1}_{E^{-1}}) \). We have the convergence of \( \nabla_{h_{L_0}} \otimes h^{-1}_{E^{-1}} \) to the Chern connection of \( (L_1 \otimes L_0^{-1})|_{X\setminus Z(\omega)}, h_{\text{lim}}^{L_1} \otimes h^{-1}_{E^{-1}} \) by the consideration in the case \( \deg(E) = 0 \) and \( h_{\det(E)} = 0 \). Hence, we obtain the convergence of \( \nabla_{h_t} \) to \( \nabla_{\text{lim}, \theta} \) in the case where \( \deg(E) \) is even.

Let us consider the case where \( \deg(E) \) is odd. We take a covering map \( p : \tilde{X} \to X \) of degree 2 such that \( \tilde{X} \) is connected. Note that \( p^*(E, \mathcal{J}_E, \theta) \) is stable because \( \deg(p^*(L_1)) = \deg(p^*(E))/2 > 0 \). Because \( \deg(p^*E) \) is even, we have the convergence of \( p^*\nabla_{h_t} \) to \( p^*\nabla_{\text{lim}, \theta} \). Hence, we obtain the convergence of \( \nabla_{h_t} \) to \( \nabla_{\text{lim}, \theta} \).

### 5.1.2 Symmetric case

We can deduce a stronger result if \( X \) and \( (E, \mathcal{J}_E, \theta) \) are equipped with an extra symmetry. Suppose that \( X \) is equipped with a holomorphic non-trivial involution \( \rho \), i.e., \( \rho \) is an automorphism of \( X \) such that \( \rho \circ \rho = \id_X \) and \( \rho \neq \id_X \). Let \((E, \mathcal{J}_E, \theta)\) be as in \( \S 5.1.1 \). We impose the following additional conditions.

- \((E, \mathcal{J}_E, \theta)\) is equivariant with respect to the action of \( \{ \id_X, \rho \} \). Namely, we have an isomorphism \( v_\rho : \rho^*(E, \mathcal{J}_E, \theta) \simeq (E, \mathcal{J}_E, \theta) \) such that \( \rho^* v_\rho \circ v_\rho = \id_X \).
- We have \( \rho^* \omega = -\omega \).

We impose the condition \( \rho^* h_{\det(E)} = h_{\det(E)} \) to the metric \( h_{\det(E)} \) under the induced isomorphism \( \rho^* \det(E) \simeq \det(E) \).

The conditions imply that we have natural isomorphisms \( \rho^* L_1 \simeq L_2 \) and \( \rho^* L_2 \simeq L_1 \) which are compatible with \( \rho^* E \simeq E \). Because \( \deg(L_1) = \deg(L_2) \), we have \( a_{E, \theta} = 0 \) and \( -\chi_P(a_{E, \theta}) = \chi_P(a_{E, \theta}) + \ell_P = \ell_P/2 \) for any \( P \in Z(\omega) \). We also have \( \ell_P = \ell_{\rho(P)} \). So, we have natural isomorphisms \( \rho^* P_{\text{lim}}^{E, \theta} L_1 \simeq P_{\text{lim}}^{E, \theta} L_2 \) and \( \rho^* P_{\text{lim}}^{E, \theta} L_2 \simeq P_{\text{lim}}^{E, \theta} L_1 \) compatible with the isomorphism \( \rho^* E \simeq E \). We can impose the additional condition \( \rho^* h_{\text{lim}}^{E, \theta} = h_{\text{lim}}^{E, \theta} \) to the conditions in Lemma 4.4, with which the metrics \( h_{\text{lim}}^{E, \theta} \) are uniquely determined. We shall prove the following theorem in \( \S 5.5 \).

**Theorem 5.3** Suppose the symmetric property of \( (E, \mathcal{J}_E, \theta) \) as above. For any \( t > 0 \), let \( h_t \) be the Hermitian-Einstein metric for \( (E, \mathcal{J}_E, \theta) \) satisfying \( \det(h_t) = h_{\det(E)} \). Then, when \( t \) goes to \( \infty \), the sequence \( h_t \) is convergent to \( h_{\text{lim}, \theta} \) in the \( C^\infty \)-sense on any compact subset in \( X \setminus Z(\omega) \).
5.1.3 The case where the spectral curve is irreducible

As a complement, we explain how to deduce the results in the case where the spectral curve is irreducible, from Theorem 5.3.

Let \((E, \bar{\mathcal{E}}_E, \theta)\) be a Higgs bundle of rank 2 such that (i) \((E, \bar{\mathcal{E}}_E, \theta)\) is generically regular semisimple, (ii) the spectral curve \(\Sigma(E, \theta)\) is irreducible, (iii) \(\text{tr}(\theta) = 0\). Note that the Higgs bundle is stable. We fix a Hermitian metric \(h_{\text{det}(E)}\) on \(\det(E)\). We have the Hermitian-Einstein metrics \(h_{t}\) of \((E, \bar{\mathcal{E}}_E, \theta)\) such that \(\det(h_{t}) = h_{\text{det}(E)}\). Recall that we have constructed a Hermitian metric \(h_{E, \theta}^{\lim}\) of \(E_{|X \setminus D(E, \theta)}\) in §4.3.2.

**Corollary 5.4** When \(t \to \infty\), the sequence \(h_{t}\) converges to \(h_{E, \theta}^{\lim}\) in the \(C^\infty\)-sense on any compact subset in \(X \setminus D(E, \theta)\).

**Proof** We use the notation in §4.3.2. If the Higgs bundle \((\bar{E}, \bar{\mathcal{E}}_{\bar{E}}, \bar{\theta})\) is polystable, then the metric \(h_{t}\) are independent of \(t\) and equal to \(h_{E, \theta}^{\lim}\) as remarked in Lemma 4.8. Let us consider the case where \((\bar{E}, \bar{\mathcal{E}}_{\bar{E}}, \bar{\theta})\) is stable. Then, by Theorem 5.3, \(p^*h_{t}\) is convergent to \(h_{E, \theta}^{\lim}\) in the \(C^\infty\)-sense on any compact subset in \(\bar{X} \setminus Z(\bar{\varnothing})\). Hence, we obtain the convergence of \(h_{t}\) to \(h_{E, \theta}^{\lim}\). 

### 5.2 A reduction for the proof of Theorem 5.1

Let us observe that for the proof of Theorem 5.1 it is enough to consider the case where \(\deg(E) = 0\) and \(R(h_{\text{det}(E)}) = 0\). The argument already appeared in the proof of Corollary 5.2.

**Lemma 5.5** Suppose that we have already proved the claim of Theorem 5.1 in the case where \(\deg(E) = 0\) and \(R(h_{\text{det}(E)}) = 0\). Then, we obtain the claim of Theorem 5.1 in the general case.

**Proof** Let us consider the case where \(\deg(E)\) is even. We have a holomorphic line bundle \(L_0\) with an isomorphism \(L_0^{\otimes 2} \simeq \det(E)\). We have a Hermitian metric \(h_{L_0}\) on \(L_0\) such that \(h_{L_0} \otimes h_{L_0} = h_{\text{det}(E)}\). By the assumption and the construction of the limiting configuration, we obtain that the sequence \(\Psi^{\ast}_{\gamma(t,Q)}(h_t)\) converges to \(h_{E, \theta}^{\lim} \otimes h_{L_0}^{-1}\) in the \(C^\infty\)-sense on any compact subset in \(X \setminus Z(\varnothing)\). Hence, we obtain the convergence of \(\Psi^{\ast}_{\gamma(t,Q)}(h_t)\) to \(h_{E, \theta}^{\lim}\). Let us consider the case where \(\deg(E)\) is odd. We take a covering \(p : \tilde{X} \to X\) of degree 2 such that \(\tilde{X}\) is connected. Note that \(p^*(E, \bar{\mathcal{E}}_E, \theta)\) is stable because \(\deg(p^*L_i) = \deg(p^*E)/2 > 0\). We take \(\tilde{Q}\) such that \(p(\tilde{Q}) = Q\). Because \(\deg(p^*E)\) is even, we obtain that the sequence \(\Psi^{\ast}_{\gamma(t,Q)}(p^*h_t)\) is convergent to \(p^*h_{E, \theta}^{\lim}\) in the \(C^\infty\)-sense on any compact subset in \(\tilde{X} \setminus p^{-1}D(E, \theta)\). Hence, we obtain the convergence of \(\Psi^{\ast}_{\gamma(t,Q)}h_t\) to \(h_{E, \theta}^{\lim}\). Thus, the proof of Lemma 5.5 is finished.

It remains to prove Theorem 5.1 in the case where \(\deg(E) = 0\) and \(R(h_{\text{det}(E)}) = 0\), which will be established in §5.3–5.4.

### 5.3 Construction of approximate solutions

Let \(X\) and \((E, \bar{\mathcal{E}}_E, \theta)\) be the Higgs bundle as in §5.1.1. We impose \(\deg(E) = 0\). We fix a flat metric \(h_{\text{det}(E)}\) on \(\det(E)\).

#### 5.3.1 Rescaling around the zeroes

Let \(P \in Z(\varnothing)\). We take a holomorphic coordinate system \((U_P, z)\) such that the eigenvalues of \(\theta\) are \(\pm d(z^{mp+1}) = \pm (mp + 1)z^{mp}dz\). Such \(z\) is determined up to the multiplication of a \((mp + 1)\)-th square root of 1. We take frames \(e_i\) of \(L_i|U_P\) (\(i = 1, 2\)) such that (i) \(e_1 = e_1 + v_2\) and \(e_2 = z^{mp}v_2\) give a frame of \(E|U_P\), (ii) \(|e_1 \wedge e_2|_{h_{\text{det}(E)}} = 1\).

We use the Higgs bundle \((\bar{E}_\ell, \bar{\theta})\) in §3 by setting \(\alpha = mp + 1\). Let \(\varphi_{\ell} : U_P \to \mathbb{C}\) be given by \(\varphi_{\ell}(z) = \zeta^{1/(mp+1)}z = \zeta\). We have

\[
\varphi_{\ell}^*\bar{\theta}(\varphi_{\ell}^*\bar{e}_1, \varphi_{\ell}^*\bar{e}_2) = (\varphi_{\ell}^*\bar{e}_1, \varphi_{\ell}^*\bar{e}_2) \begin{pmatrix}
1 & t^{\alpha^z}d
-2\alpha t \cdot t^{-\ell_P/(mp+1)}z^{mp-\ell_P}dz & 0
\end{pmatrix}.
\]
Hence, we have the isomorphism \( \varphi^*_t (\tilde{E}_t, \tilde{\theta}) \simeq (E, t\theta)|_{U_p} \) given by the following correspondence:

\[
\begin{align*}
    t^{r/2(mP+1)}\varphi^*_t e_1 & \leftrightarrow e_1, \\
    t^{-r/2(mP+1)}\varphi^*_t e_2 & \leftrightarrow e_2
\end{align*}
\]

Moreover, we have \( t^{r/2(mP+1)}\varphi^*_t v_j \leftrightarrow v_j \).

### 5.3.2 Local constructions around the zeroes

In the following, for a given positive function \( \nu \), let \( O(\nu) \) denote a function \( f \) such that \( |f| \leq C\nu \), where \( C \) is a positive constant independent of \( t \). Let \( \epsilon \) denote small positive numbers which are independent of \( t \).

Let \( P \in \mathbb{Z}(\omega) \). Suppose that \( a_{E,\theta} - \ell_P/2(m_P + 1) < 0 \). We have the harmonic metric \( h_{x_P(a_{E,\theta}), \ell_P} \) of \((\tilde{E}_t, \tilde{\theta})\) as in §3.3. We obtain harmonic metrics \( h^0_{\ell_P} := \varphi^*_t h_{x_P(a_{E,\theta}), \ell_P} \) of \((E, \overline{\partial}E, t\theta)|_{U_p} \). By construction, we obtain the following from Proposition 3.10:

**Lemma 5.6** Take \( R_{1, P} > 0 \) such that \( \{ |z| \leq R_{1, P} \} \subset U_p \). Take \( 0 < R_{2, P} < R_{1, P} \). On \( \{ R_{2, P} \leq |z| < R_{1, P} \} \subset U_p \), we have

\[
\begin{align*}
|v_1|_{h^0_{\ell_P}} &= t^{a_{E,\theta}} |\chi_P(a_{E,\theta})| b_{\chi_P(a_{E,\theta})} \left( 1 + O\left( \exp(-\epsilon|z|^{m_P+1}t) \right) \right)
|v_2|_{h^0_{\ell_P}} &= t^{-a_{E,\theta}} |\chi_P(a_{E,\theta})|^{-1} b_{\chi_P(a_{E,\theta})}^{-1} \left( 1 + O\left( \exp(-\epsilon|z|^{m_P+1}t) \right) \right)
\end{align*}
\]

\[
h^0_{\ell_P}(v_1, v_2) = O\left( \exp(-\epsilon|z|^{m_P+1}t) \right)
\]

We also have the following lemma.

**Lemma 5.7** On \( \{ R_{2, P} \leq |z| \leq R_{1, P} \} \), we have the following:

\[
\begin{align*}
\partial \log |v_1|_{h^0_{\ell_P}}^2 &= \chi_P(a_{E,\theta}) dz/z + O(\exp(-\epsilon t)) \quad dz = O(1) \quad dz \\
\partial \log |v_2|_{h^0_{\ell_P}}^2 &= -(\ell_P + \chi_P(a_{E,\theta})) dz/z + O(\exp(-\epsilon t)) \quad dz = O(1) \quad dz \\
\overline{\partial} \partial \log |v_j|_{h^0_{\ell_P}}^2 &= O(\exp(-\epsilon t)) \quad dz \quad (j = 1, 2) \\
\partial h^0_{\ell_P}(v_1, v_2) &= O(\exp(-\epsilon t)) \quad dz, \quad \overline{\partial} h^0_{\ell_P}(v_1, v_2) = O(\exp(-\epsilon t)) \quad dz \\
\end{align*}
\]

**Proof** According to Proposition 3.10, we have \( \partial \partial \log |v_1|_{h^0_{\ell_P}}^2 - \chi_P(a_{E,\theta}) = O(\exp(-\epsilon|z|^{m_P+1}t)) \). Hence, we obtain the estimate for \( \partial \log |v_1|_{h^0_{\ell_P}}^2 \) on the domain. We obtain the estimate for \( \partial \log |v_2|_{h^0_{\ell_P}}^2 \) in a similar way. We obtain the estimate for \( \overline{\partial} \partial \log |v_j|_{h^0_{\ell_P}}^2 \) from Lemma 3.11. We obtain the estimate for \( \partial h^0_{\ell_P}(v_1, v_2) \) and \( \overline{\partial} h^0_{\ell_P}(v_1, v_2) \) from Lemma 3.13.

Suppose that \( a_{E,\theta} - \ell_P/2(m_P + 1) = 0 \). If \( \ell_P = a_{E,\theta} = 0 \), we set \( h_{t_1, P} := \varphi^*_t h_{E_0} \), where \( h_{E_0} \) is the harmonic metric given in §3.2.3. Suppose \( \ell_P > 0 \). According to Proposition 3.15 and Proposition 3.19, for a given sequence \( t_i \to \infty \), we can take a sequence of negative numbers \( c_i \to 0 \) such that

\[
\frac{\log b_{c_i}}{-c_i} = \frac{\log t_i}{m_P + 1}, \quad \text{i.e.,} \quad b_{c_i} t_i^{c_i/(m_P+1)} = 1
\]

We obtain the sequence of harmonic metrics \( h_{t_i, P} := \varphi^*_t h_{c_i, \ell_P} \) of \((E, \overline{\partial}E, t_i\theta)|_{U_p} \), where \( h_{c_i, \ell_P} \) are given as in §3.3. By Proposition 3.10, we have the following.
Lemma 5.8 Take $0 < R_{2,P} < R_{1,P}$ as in Lemma 5.6. On $\{R_{2,P} \leq |z| \leq R_{1,P}\}$, we have the following estimates:

\[
\begin{align*}
|v_1|_{h_{t_i}^0}^2 &= t_i^{α_{E,θ}}|z|^2t_i\left(1 + O(\exp(-ε|z|^{m_P+1}t_i))\right) \\
|v_2|_{h_{t_i}^0}^2 &= t_i^{α_{E,θ}}|z|^{-εt_P}\left(1 + O(\exp(-ε|z|^{m_P+1}t_i))\right) \\
h_{t_i}^0(v_1, v_2) &= O(\exp(-ε|z|^{m_P+1}t_i))
\end{align*}
\]

As in the case of Lemma 5.7, we have the following.

Lemma 5.9 Take $0 < R_{2,P} < R_{1,P}$. On $\{R_{2,P} \leq |z| \leq R_{1,P}\}$, we have the following:

\[
\begin{align*}
\partial \log |v_1|_{h_{t_i}^0}^2 &= c_i dz/z + O(\exp(-εt_i)) \, dz = O(1) \, dz \\
\partial \log |v_2|_{h_{t_i}^0}^2 &= -(c_i + ℓ_P) dz/z + O(\exp(-εt_i)) \, dz = O(1) \, dz \\
\overline{∂} \log |v_1|_{h_{t_i}^0}^2 &= O(\exp(-εt_i)) \, dz dz \\
\overline{∂}h_{t_i}^0(v_1, v_2) &= O(\exp(-εt_i)) \, dz dz \\
\overline{∂}h_{t_i}^0(v_1, v_2) &= O(\exp(-εt_i)) \, dz dz
\end{align*}
\]

Suppose that $j_P := a_{E,θ} - ℓ_P/2(1 + m_P) > 0$. We use the notation in §3.4. For a given sequence $t_i \to \infty$, we set $κ_i := t_i^{-1/(m_P+1)}$. We have the Hermitian metrics $h_{κ_i}$ of $E^\ell_i$ as in §3.4. We obtain the Hermitian metrics $h_{t_i, P}^0 := φ_{t_i}^* h_{κ_i}$ of $E_{t_i}$. By construction, we have the following on $\{t_i^{-1/(m_P+1)} \leq |z| < 1\}$:

\[
|v_1|_{h_{t_i}^0} = t_i^{α_{E,θ}}, \quad |v_2|_{h_{t_i}^0} = t_i^{-α_{E,θ}}|z|^{-εt_P}, \quad h_{t_i, P}^0(v_1, v_2) = 0
\]

Lemma 5.10 Take $0 < ε < (m_P + 1)j_P$. We have the following estimate:

\[
R(h_{t_i}^0) = O\left(t_i^{-j_P + εP/(m_P+1)}\right)|z|^{|P|^2} \, dz dz
\]

\[
[t_i θ, (t_i θ)^\dagger]_{h_{t_i}^0} = O\left(t_i^{-j_P + εP/(m_P+1)}\right)|z|^{|P|^2} \, dz dz
\]

**Proof** Because we have $R(h_{t_i}^0) = [θ_i, (t_i θ_i)^\dagger]_{h_{t_i}^0} = 0$ on $\{|z| \geq t_i^{-1/(m_P+1)}\}$, it is enough to argue the estimates on $|z| < t_i^{-1/(m_P+1)}$. We have the following:

\[
R(h_{t_i}^0) = O\left(t_i^{-j_P + 2/(m_P+1)}\right)dz dz, \quad [t_i θ_i, (t_i θ_i)^\dagger]_{h_{t_i}^0} = O\left(t_i^{-j_P + 2/(m_P+1)}\right)dz dz
\]

Both of them are dominated by

\[
O\left(t_i^{-j_P + 2/(m_P+1)}\right)|z|^{2εP/(m_P+1)} \, dz dz = O\left(t_i^{-j_P + εP/(m_P+1)}\right)|z|^{|P|^2} \, dz dz
\]

Thus, we obtain the claim of the lemma.
5.3.3 Global construction

We take a Kähler metric $g_X$ of $X$. Let $t_i$ be any sequence of positive numbers going to $\infty$. We set $\beta_i := \ell_i^{a_{E,\omega}}$. We shall construct a family of Hermitian metrics $h^0_{t_i}$ of $(E, \overline{\nabla}_E, \theta)$ with the following property:

- There exists $p > 1$ such that the $L^p$-norms of $R(h^0_{t_i}) + [t_i \theta, (t_i \theta)^\dagger_{h^0_{t_i}}]$ with respect to $g_X$ and $h^0_{t_i}$ are bounded.
- There exists $C > 0$ such that $C^{-1} h^0_{t_i} \leq h^0_{t_i, P} \leq C h^0_{t_i}$ on the neighbourhood $U_P$ for each $P \in Z(\omega)$, and that $C^{-1} h^0_{t_i} \leq \Psi^{t_i, lim}_{\beta_i} h^0_{E, \theta} \leq C h^0_{t_i}$ on $X \setminus \bigcup_{P \in Z(\omega)} U_P$.

Let $P \in Z(\omega)$. We take $0 < R_2, R_1 > 0$ such that $\{|s| \leq R_1, R_2 \} \subset U_P$. We take a function $\rho_P : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\rho_P(s) = 1$ $(s \leq R_2)$ and $\rho_P(s) = 0$ $(s \geq R_1)$. On $\{R_2 \leq |s| \leq R_1 \} \subset U_P$, we define $h^0_{t_i}$ by the following conditions:

$$
\log h^0_{t_i}(v_j, v_j) = \rho_P(|s|) \log h^0_{t_i, P}(v_j, v_j) + (1 - \rho_P(|s|)) \log h^{lim}_{E, \theta}(\Psi_{\beta_i, t_i}(v_j, v_j) \quad (j = 1, 2)
$$

$$
h^0_{t_i}(v_1, v_2) = \rho_P(|s|) h^0_{t_i, P}(v_1, v_2)
$$

Note that $\log h^0_{t_i, P}(v_j, v_j) - \log h^{lim}_{E, \theta}(\Psi_{\beta_i, t_i}(v_j, v_j)$ are uniformly bounded on $\{|s| \leq R_1, R_2 \}$. On $\{|s| \leq R_2, R_1 \} \setminus \{R_2 \leq |s| \leq R_1 \}$, we set $h^0_{t_i} := h^0_{t_i, P}$. On $X \setminus \bigcup_{P \in Z(\omega)} \{|s| \leq R_1, R_2 \}$, we set $h^0_{t_i} := \Psi^{t_i, lim}_{\beta_i} h^0_{E, \theta}$. Then, we can check that the family of the Hermitian metrics $h^0_{t_i}$ has the desired property by using the estimates in §5.3.2.

The following lemma is clear by the construction and Proposition 3.14, Proposition 3.20 and Proposition 3.22.

Lemma 5.11 The sequence of Hermitian metrics $\Psi^{t_i, lim}_{\beta_i, t_i, P}(\omega) \setminus Z(\omega)$ is convergent in the $C^\infty$-sense on any compact subset in $X \setminus Z(\omega)$. For the limit $\lim_{i \to \infty}$, the decomposition $L_1 \oplus L_2$ is orthogonal. There exists $M_1 > 0$ such that $M_1^{-1} h^{lim}_{E, \theta} \leq h^0_{t_i} \leq M_1 h^{lim}_{E, \theta}$. In particular, there exists $M_2 > 0$ with the following property.

- For any neighbourhood $N$ of $Z(\omega)$, there exists $i_0(N)$ such that $M_2^{-1} h^{lim}_{E, \theta} \leq \Psi^{lim}_{t_i, \beta_i} h^0_{t_i} \leq M_2 h^{lim}_{E, \theta}$ on $X \setminus N$ for any $i \geq i_0(N)$.

Let $\rho_i$ be the self-adjoint endomorphisms of $(E|_{X \setminus Z(\omega)}, h^{lim}_{E, \theta})$ determined by $\Psi^{lim}_{t_i, \beta_i} h^0_{t_i}(u, v) = h^{lim}_{E, \theta}(\rho_i u, v)$ for any local sections $u$ and $v$. We also have the following.

Lemma 5.12 The sequence $\rho_i$ are convergent in the $C^\infty$-sense with respect to $h^{lim}_{E, \theta}$ on any compact subset in $X \setminus Z(\omega)$. The limit $\rho_\infty$ preserves the decomposition $L_1 \oplus L_2$. We have the boundedness of $\rho_\infty$ and $\rho_\infty^{-1}$ with respect to $h^{lim}_{E, \theta}$.

5.4 Proof of Theorem 5.1

We continue to use the notation in §5.3.

5.4.1 Boundedness of a modified sequence

Let $t_i \to \infty$ be any sequence. It is enough to prove that we can take a subsequence $t'_i$ such that the sequence $\Psi^{t'_i, lim}_{E, \theta}$ converges to $h^{lim}_{E, \theta}$.

Let $\Delta_X$ be the Laplacian with respect to the Kähler metric $g_X$ of $X$. We construct the family of Hermitian metrics $h^0_{t_i}$ on $E$ as in §5.3. Let $k_i$ be the self-adjoint endomorphism of $(E, h^0_{t_i})$ determined by $h^0_{t_i} = h^0_{t_i} + k_i$, i.e., $h^0_{t_i}(u, v) = h^0_{t_i}(k_i u, v)$ for local sections $u$ and $v$. According to [19, Proposition 3.1], we have the following on $X$:  

$$
\Delta_X \text{Tr}(k_i) \leq \left| \text{A}_{g_X} \text{Tr} \left( k_i \cdot (R(h^0_{t_i}) + [t_i \theta, (t_i \theta)^\dagger_{h^0_{t_i}}] \right) \right|
$$


Let $p > 1$ be as in §5.3.3, i.e., the $L^p$-norms of $R(h_{i_t}^0) + [t, \theta, (t, \theta)^\dagger h_{i_t}^0]$ with respect to $g_X$ and $h_{i_t}^0$ are bounded. We take $q > 1$ such that $p^{-1} + q^{-1} < 1$. Set $r := (p^{-1} + q^{-1})^{-1}$. Let $\nu_i := \|k_i\|_{L^r, h_{i_t}^0, g_X}$ be the $L^r$-norm of $k_i$ with respect to $h_{i_t}^0$ and $g_X$. Set $s_i := \nu_i^{-1} k_i$. We have

$$\Delta_X \text{Tr}(s_i) \leq \left| A_{g_X} \text{Tr}\left(s_i \cdot \left(R(h_{i_t}^0) + [t, \theta, (t, \theta)^\dagger h_{i_t}^0]\right)\right) \right|$$

(37)

The $L^r$-norms of the right hand side in (37) are bounded for $i$. So we have a constant $C_1 > 0$ and $L^2$-functions $G_i$ such that

$$\Delta_X \left(\text{Tr}(s_i) - G_i\right) \leq C_1, \quad \|G_i\|_{L^2} \leq C_1.$$

Hence, we have $C_2 > 0$ such that $\sup_X |s_i|_{h_{i_t}^0} \leq C_2$ holds for any $i$. Again, according to [19, Proposition 3.1], we have the following:

$$\Delta_X \text{Tr}(s_i) = \sqrt{-1} \Delta_X \text{Tr}\left(s_i \cdot \left(R(h_{i_t}^0) + [t, \theta, (t, \theta)^\dagger h_{i_t}^0]\right)\right) - |s_i|^{-1/2} (\bar{\partial} + t, \theta) s_i |h_{i_t}^0, g_X|^2$$

We have a constant $C_3 > 0$ such that $|\bar{\partial} E, t, \theta) s_i |^2 |h_{i_t}^0, g_X| \leq C_3 |s_i|^{-1/2} (\bar{\partial} E, t, \theta) s_i |^2 |h_{i_t}^0, g_X|$. Hence, we obtain the following for a constant $C_4 > 0$:

$$\int_X |\bar{\partial} E, t, \theta) s_i |^2 |h_{i_t}^0, g_X| + \int_X |t, \theta, s_i |^2 |h_{i_t}^0, g_X| \leq C_4$$

5.4.2 Weak convergence of a subsequence

We consider the sequences of metrics $\bar{h}_{i_t} := \Psi_{\rho_i^{-1}} h_{i_t}$ and $\bar{h}_{i_t}^0 := \Psi_{\rho_i^{-1}} h_{i_t}^0$ on $E_{1, X} \times Z(\omega)$. Let $\bar{k}_i$ be the self-adjoint endomorphism of $(E_{1, X} \times Z(\omega), \bar{h}_{i_t}^0)$ determined by $\bar{k}_i := \bar{h}_{i_t}^0 \bar{k}_i$. We have $\|\bar{k}_i\|_{L^q, \bar{h}_{i_t}^0, g_X} = \nu_i$. Set $\pi_i := \nu_i^{-1} \bar{k}_i$. We have

$$\sup_X |\pi_i|_{\bar{h}_{i_t}^0} \leq C_2.$$  

(38)

We also have

$$\int_X |\bar{\partial} E, t, \theta) \pi_i |^2 |h_{i_t}^0, g_X| + \int_X |t, \theta, \pi_i |^2 |h_{i_t}^0, g_X| \leq C_4$$  

(39)

By Lemma 5.11 and (39), we may assume that the sequence $\pi_i$ is weakly convergent in $L^2$ on any compact subset in $X \setminus Z(\omega)$ with respect to $g_X$ and $h_{i_t}^0$. Let $\pi_\infty$ denote the weak limit.

Lemma 5.13 $\pi_\infty$ is bounded with respect to $h_{i_t}^0$. We have $\pi_\infty \neq 0$.

Proof By Lemma 5.12 and (38), there exists $M_3 > 0$ with the following property.

• For any neighbourhood $N$ of $Z(\omega)$, there exists $i_3(N)$ such that $|\pi_i|_{h_{i_t}^0} \leq M_3$ on $X \setminus N$ for any $i \geq i_3(N)$.

Hence, we have the boundedness $|\pi_\infty|_{h_{i_t}^0} \leq M_3$.

Take a small $\delta > 0$. By (38), we have a small neighbourhood $N_1$ of $Z(\omega)$ such that

$$\int_{X \setminus N_1} |\pi_i|_{\bar{h}_{i_t}^0}^2 d\text{vol}_g \geq 1 - \delta > 0$$

Hence, we obtain $\int_{X \setminus N_1} |\pi_\infty|_{h_{i_t}^0}^q d\text{vol}_g > 0$. In particular, we have $\pi_\infty \neq 0$.

Lemma 5.14 We have $[\pi_\infty, \theta] = 0$. In particular, $\pi_\infty$ preserves the decomposition $L_1 \oplus L_2$.

Proof We have $\lim_{i \to \infty} \int_X |\pi_i, \theta |^2 |h_{i_t}^0, g_X| = 0$ from (38), which implies the claim of the lemma.
5.4.3 Modification

Let $\rho_1$ be as in Lemma 5.12. We set $\bar{s}_1 := \rho_1 \circ s_i$. It is self-adjoint with respect to $h^0_{E, \theta}$, and we have $\nu_1^{-1} \Psi^*_{\beta_i^0}, h_{t_i} = h^0_{E, \theta} \bar{s}_1$. The sequence $\bar{s}_1$ is weakly convergent in $L^2_E$ on any compact subset in $X \setminus Z(\omega)$. Let $\bar{s}_1^\infty$ denote the weak limit. We have $\bar{s}_1^\infty = \rho_\infty \circ \bar{s}_1$. We obtain the following from Lemma 5.12, Lemma 5.13 and Lemma 5.14.

**Lemma 5.15** $\bar{s}_1^\infty$ is bounded with respect to $h^0_{E, \theta}$. We have $\bar{s}_1^\infty \neq 0$ and $[\bar{s}_1^\infty, \theta] = 0$.

**Lemma 5.16** We have $\bar{D}_E \overline{s}_1^\infty = 0$.

**Proof** Applying [19, Proposition 3.1] to $h^0_{E, \theta}$ and $\nu_1^{-1} \Psi^*_{\beta_i^0}, h_{t_i}$, we obtain the following on $X \setminus Z(\omega)$:

$$
|\bar{s}_1^\infty|^{1/2} \bar{D}_E \overline{s}_1^\infty \leq |\bar{s}_1^\infty|^{1/2} (\bar{D} + t_i \theta) \bar{s}_1^\infty \leq \sqrt{-1} \Lambda_{g_{E, \bar{s}_1}} \bar{D} \bar{s}_1^\infty
$$

We take a $C^\infty$-function $\mu : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ such that $\mu(s) = 1$ ($s \leq 1$) and $\mu(s) = 0$ ($s \geq 2$). For any sufficiently large real number $T$, let $\chi_T : X \rightarrow \mathbb{R}_{\geq 0}$ be the $C^\infty$-function such that (i) $\chi_T \equiv 1$ on $X \setminus \bigcup_{P \in Z(\omega)} U_P$, (ii) $\chi_T(z) = \mu(-T^{-1} \log|z|)$ on the coordinate neighbourhoods $(U_P, z)$ for $P \in Z(\omega)$.

We have a constant $M_4 > 0$ with the following property.

- For any neighbourhood $N$ of $Z(\omega)$, there exists $i_4(N)$ such that $|\bar{s}_1^\infty|_{h^0_{E, \theta}} \leq M_4$ on $X \setminus N$ for any $i \geq i_4(N)$.

Then, we have a constant $C_{10}$ such that for any fixed $T > 0$ the following holds for a large $i$:

$$
\int \chi_T |\bar{D}_E \overline{s}_1^\infty|_{h^0_{E, \theta}}^2 \, \text{dvol}_g \leq C_{10} \int \chi_T |\bar{s}_1^\infty|^{1/2} \bar{D}_E \overline{s}_1^\infty |_{h^0_{E, \theta}}^2 \, \text{dvol}_g \leq C_{10} \left| \int \chi_T \bar{D} \bar{s}_1^\infty \right| = C_{10} \left| \int (\bar{D} \chi_T) \cdot \bar{s}_1^\infty \right| (40)
$$

By taking the limit for $i \rightarrow \infty$, we obtain

$$
\int \chi_T |\bar{D}_E \overline{s}_1^\infty|_{h^0_{E, \theta}}^2 \, \text{dvol}_g \leq C_{10} \left| \int (\bar{D} \chi_T) \cdot \bar{s}_1^\infty \right|
$$

Note that we have already known the boundedness of $\text{Tr}(\bar{s}_1^\infty)$. We also have the uniform boundedness of $\bar{D} \chi_T$ with respect to the Poincaré like metric on $X \setminus Z(\omega)$. Hence, by taking the limit for $T \rightarrow \infty$, we obtain

$$
\int |\bar{D}_E \overline{s}_1^\infty|_{h^0_{E, \theta}}^2 \, \text{dvol}_g \leq 0
$$

Hence, we obtain $\bar{D}_E \overline{s}_1^\infty = 0$.

By Lemma 5.15 and Lemma 5.16, we have $\overline{s}_1^\infty = \alpha_1 \text{id}_{L_1} \oplus \alpha_2 \text{id}_{L_2}$ for non-negative real numbers $\alpha_i$ ($i = 1, 2$). We have $(\alpha_1, \alpha_2) \neq (0, 0)$.

5.4.4 End of the proof of Theorem 5.1

Suppose that $\alpha_1 \neq 0$. Let $u_{j}$ be local frames of $L_j$ ($j = 1, 2$) on a relatively compact open subset in $X \setminus Z(\omega)$. We set $\gamma_1 := \beta_i^0 \nu_i^{-1/2}, \alpha_i^{-1/2}$. Then, we have the following:

$$
\lim_{i \rightarrow \infty} |u_1|_{h_{t_i}^1} \gamma_i = |u_1|_{h_{t_i}^0}
$$

By the asymptotic orthogonality in §2, we have the following on $X \setminus N$, where $N$ is any neighbourhood of $Z(\omega)$:

$$
|u_1 \wedge u_2|^2_{\det(E)} = |u_1|_{h_{t_i}^1}^2 |u_2|_{h_{t_i}^1}^2 - |h_{t_i}(u_1, u_2)|^2 = |u_1|_{h_{t_i}^1}^2 |u_2|_{h_{t_i}^1}^2 \cdot \left(1 + O(\exp(-\epsilon_i))\right)
$$
We also have $|u_1 \wedge u_2|_{\det(E)} = |u_1|_{h_{E,0}^{\lim}} \cdot |u_2|_{h_{E,0}^{\lim}}$. Hence, we obtain the following from (41):

$$\lim_{i \to \infty} |u_2|_{h_i}^\gamma \gamma^{-1} = |u_2|_{h_E^{\lim}}$$

(42)

We obtain the following for $j = 1, 2$ from (41) and (42):

$$\lim_{i \to \infty} \Psi^* \gamma_i h_{t_i}(u_j, u_j) = h_{E,\theta}^{\lim}(u_j, u_j).$$

We also have the following on $X \setminus N$, where $N$ is any neighbourhood of $Z(\omega)$:

$$\Psi^* \gamma_i h_{t_i}(u_1, u_2) = h_{t_i}(u_1, u_2) = O(\exp(-\epsilon t_i)) \cdot |u_1|_{h_{t_i}} \cdot |u_2|_{h_{t_i}} = O(\exp(-\epsilon t_i)) \cdot |u_1|_{\Psi_i^* h_{t_i}} \cdot |u_2|_{\Psi_i^* h_{t_i}}.$$

So, we obtain $\lim_{i \to \infty} \Psi^* \gamma_i h_{t_i}(u_1, u_2) = 0 = h_{E,\theta}^{\lim}(u_1, u_2)$. Hence, we have the convergence of $\Psi^* \gamma_i h_{t_i}$ to $h_{E,\theta}^{\lim}$ in $C_0$ on any compact subset in $X \setminus Z(\omega)$.

By the construction of the sequence $\gamma(t_i, Q)$, we have $\lim_{i \to \infty} \gamma(t_i, Q) \cdot \gamma_i^{-1} = 1$. Hence, we have the convergence of $\Psi^* \gamma(t_i, Q)_{h_{t_i}}$ to $h_{E,\theta}^{\lim}$ in $C_0$ on any compact subset in $X \setminus Z(\omega)$. We can obtain the convergence of the higher derivative from Corollary 2.15. Thus, we are done in the case $\alpha_1 \neq 0$.

We can argue the case $\alpha_2 \neq 0$ in a similar way. Thus, the proof of Theorem 5.1 is finished.

5.5 Proof of Theorem 5.3

5.5.1 Preliminary

Let $(E, \mathcal{D}_E, \theta)$ be any stable Higgs bundle of rank 2 on $X$ such that (i) $(E, \mathcal{D}_E, \theta)$ is generically regular semi-simple, (ii) the spectral curve $\Sigma(E, \theta)$ is reducible, (iii) $\text{tr}(\theta) = 0$. We have holomorphic line bundles $L_i$ ($i = 1, 2$) with an inclusion $E \rightarrow L_1 \oplus L_2$ as in §4.2. We assume that $\text{deg}(L_1) = \text{deg}(L_2)$. Then, we have $a_{E,\theta} = 0$ and $-\chi_P(a_{E,\theta}) = \chi_P(a_{E,\theta}) + \ell_P = \ell_P/2$.

We fix a Hermitian metric $h_{E,\theta}$ of $\det(E)$. For any $t > 0$, we have Hermitian-Einstein metrics $h_t$ for $(E, \mathcal{D}_E, \theta)$ such that $\text{det}(h_t) = h_{E,\theta}$. We have the metric $h_{E,\theta}^{\lim}$ as in §5.1.1.

We take any sequence $t_i \rightarrow \infty$. Let $k_i^{(2)}$ be the self-adjoint endomorphisms of $(E|X \setminus Z(\omega), h_{E,\theta}^{\lim})$ determined by $h_{t_i} = h_{E,\theta}^{\lim} \cdot k_i^{(2)}$.

Lemma 5.17 After going to a subsequence $\{i(p)\} \subset \{i\}$ there exists a sequence of positive numbers $\nu_i(p)$ such that the sequence $\nu_i^{-1} h_{E,\theta}^{\lim} k_i^{(2)}$ weakly converges to a morphism $\alpha_1 \cdot \text{id}_{L_1} \oplus \alpha_2 \cdot \text{id}_{L_2}$ in $L_1^2$ locally on $X \setminus Z(\omega)$ for non-negative real numbers $\alpha_1$ ($j = 1, 2$) with $(\alpha_1, \alpha_2) \neq (0, 0)$.

We take any sequence $t_i \rightarrow \infty$. It is enough to prove that we can take a subsequence $t_i'$ such that $h_{t_i'}$ converges to $h_{E,\theta}^{\lim}$ on any compact subsets in $X \setminus Z(\omega)$.

Let $k_i^{(2)}$ be the self-adjoint endomorphisms of $(E|X \setminus Z(\omega), h_{E,\theta}^{\lim})$ determined by $h_{t_i} = h_{E,\theta}^{\lim} \cdot k_i^{(2)}$, as in §5.5.1. Because $\rho^* h_{E,\theta}^{\lim} = h_{E,\theta}^{\lim}$ and $\rho^* h_{t_i} = h_{t_i}$, we have $\rho^* k_i^{(2)} = k_i^{(2)}$. As remarked in Lemma 5.17, by going to a subsequence, we may assume to have a sequence of positive numbers $\nu_i$ such that the sequence $\nu_i^{-1} k_i^{(2)}$ is
weakly convergent to $\alpha_1 \text{id}_{L_1} \oplus \alpha_2 \text{id}_{L_2}$ in $L^2_1$ locally on $X \setminus Z(\omega)$, where $\alpha_i$ are non-negative numbers such that $(\alpha_1, \alpha_2) \neq (0, 0)$. Because $\rho^*(\nu_i^{-1}k_i^{(2)}) = \nu_i^{-1}k_i^{(2)}$, $\rho^*(L_1) = L_2$ and $\rho^*(L_2) = L_1$, we have $\alpha_1 = \alpha_2$. In particular, $\alpha_1 \cdot \alpha_2 \neq 0$. Because $\det(k_i^{(2)}) = 1$, the sequence $\nu_i^{-1}$ converges to $\alpha_1 \cdot \alpha_2$. In particular, the sequences $\nu_i$ and $\nu_i^{-1}$ are bounded.

We take a subsequence $h_{i(p)}$ for which the sequence $k_i^{(2)}$ is convergent to $\beta \text{id}_{L_1} \oplus \beta \text{id}_{L_2}$ for a positive number $\beta$. But, we have $\det(k_{i(p)}^{(2)}) = 1$ and hence $\beta = 1$, i.e., $k_i^{(2)}$ is convergent to the identity $\text{id}$, indeed. Hence, we can conclude that the sequence $k_i^{(2)}$ is convergent to id, and the proof of Theorem 5.3 is finished.

References

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