NUMBER OF ISOLATED PERIODIC POINTS OF RATIONAL MAPS

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ABSTRACT. Let f be a rational self-map on a projective manifold. In general, the set of periodic points of f may have a positive dimension. We will discuss some estimates for the number of isolated periodic points. Although the question is of algebraic nature, the techniques we used are based on some recent progress in complex analysis of several variables. In this report, we will present the main results and approaches in this direction and try to give a guide for non-experts in the reading of the original articles.

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1. INTRODUCTION

Let X be a compact Kähler manifold of dimension k. Let $f : X \to X$ be a dominant holomorphic/meromorphic map or correspondence. By correspondences, we mean multi-valued maps. Define $f^n := f \circ \cdots \circ f$ (n times) *the iterate of order* n of f. Let Q_n be the set of *isolated* periodic points of period n of f, i.e. isolated fixed points of f^n, counting multiplicity. Here are the main questions we will discuss in this report. They are fundamental problems in complex dynamics in higher dimension.

Problem 1.1 (sub-problem of Problem 1.3). *Compute or estimate the cardinality of* Q_n.

Problem 1.2 (sub-problem of Problem 1.1). *Find a good upper bound for the cardinality of* Q_n.

Problem 1.3. *Study the distribution of* Q_n *when* n *goes to infinity.*

The current strategy used in complex dynamics is the following one.

Strategy for Problem 1.3 :

(1) Get a good upper bound for the cardinality Q_n (Problem 1.2).
(2) Construct a good family of periodic points using tools from dynamics or complex analysis.

These two steps together will give us a good lower bound for the cardinality of Q_n and then an equidistribution property for Q_n as n goes to infinity. We see that by using this strategy, a direct study for the lower bound of the cardinality of Q_n is not a priority.

Remark 1.4. Problem 1.2 is not completely solved even in dimension 2. Problem 1.3 is even open for holomorphic automorphisms on non-projective surfaces.

Our main contribution in this research direction can be summarized in the following statement that will be presented later in more details.

Theorem 1.5 (Dinh-Sibony, Dinh-Nguyen-Truong/-Vu). *Let f be a dominant meromorphic correspondence on a compact Kähler manifold X of dimension k. Let Q_n be the set of isolated periodic points of period n of f, counting multiplicity. Then*

- *$\#Q_n$ grows at most exponentially fast;*
- *The exponential growth of $\#Q_n$ is bounded by the algebraic entropy of f which is a finite non-negative number.*
- *In many cases, Q_n is asymptotically equidistributed with respect to a canonical invariant probability measure, as n goes to infinity.*

Note that in the setting of real smooth dynamical systems, $\#Q_n$ can grow arbitrarily fast, see e.g. the work by Kaloshin [19]. Meromorphic maps are more rigid but may have singularities and these maps are not continuous in general.

For the reader's convenience, we recall now briefly some notions related to meromorphic maps and correspondences that we use. Let $\pi_1, \pi_2 : X \times X \to X$ be the canonical projections. Let $\Gamma \subset X \times X$ be an analytic subset of dimension $k = \dim X$. If $\pi_i : \Gamma \to X$ are surjective (hence generically finite) for both $i = 1$ and 2, then Γ defines a *dominant meromorphic correspondence* f on X with graph Γ. More precisely, we have for $x \in X$

$$f(x) := \pi_2\big(\pi_1^{-1}(x) \cap \Gamma\big) \quad \text{and} \quad f^{-1}(x) := \pi_1\big(\pi_2^{-1}(x) \cap \Gamma\big).$$

This can be seen as a "multivalued meromorphic map". If moreover, $\pi_1 : \Gamma \to X$ is generically 1:1, then f is a *dominant meromorphic map*; and if $\pi_1 : \Gamma \to X$ is 1:1, then f is a *dominant holomorphic map*.

Recall that the iterate of order n of f is defined by $f^n := f \circ \cdots \circ f$ (n times) in a suitable Zariski open set where the map/correspondence is finite, and then we compactify the graph of the obtained map/correspondence in order to get a meromorphic map/correspondence on X. Periodic points of period n correspond to the intersection of the graph Γ_n of f^n with the diagonal Δ of $X \times X$. It may have a positive dimension. Let Q_n denote the set of isolated periodic points of period n, counting multiplicity. Recall also that when $\dim \Gamma_n \cap \Delta = 0$ one can compute $\#Q_n$ using the cohomology classes of Γ_n and Δ by the formula

$$\#Q_n = \{\Gamma_n\} \smile \{\Delta\}.$$

According to the classical Lefschetz fixed point theorem, the last cup-intersection can be computed using the action of the map/correspondence on the cohomology of X. Therefore, we can get an estimate on $\#Q_n$. The main difficulty for counting the points in Q_n in the general setting is the contribution of the positive dimension components of the set of periodic points $\Gamma_n \cap \Delta$. It is not difficult to construct correspondences such that $\Gamma_n \cap \Delta$ has a positive dimension. It is more delicate to get such examples for maps, see e.g. the recent work by Oguiso on Dolgachev's problem [20] and also [21].

2. Upper bound for the number of isolated periodic points

In this section, we will discuss Probblem 1.2. We need to introduce the fundamental notion of algebraic stability/unstability due to Fornæss-Sibony and the basic dynamical invariants such as the dynamical degrees and algebraic entropy.

Let $f : X \to X$ be a correspondence of graph $\Gamma \subset X \times X$ as above. If ϕ is a smooth (p, q)-form on X, we can define $f^*(\phi)$ by

$$f^*(\phi) := (\pi_1)_*\big(\pi_2^*(\phi) \wedge [\Gamma]\big).$$

This is an L^1 (p,q)-form on X, not continuous in general. Therefore, we cannot iterate this operation. However, since the Hodge cohomology groups can be defined using smooth or singular differential forms, the operator f^* induces a linear map on Hodge cohomology $f^* : H^{p,q}(X) \to H^{p,q}(X)$; we can iterate this one. Fornaess-Sibony observed that we don't have the *algebraic stability* : $(f^n)^* = (f^*)^n$ on $H^{p,q}(X)$ in general and such a property may be checked using some geometric criterium, see [16]. Nevertheless, we have the following general result.

Theorem 2.1 (Dinh-Sibony [8]). *For $0 \leq p \leq k$, the limit*

$$d_p := \lim_{n\to\infty} \left\|(f^n)^* : H^{p,p}(X) \to H^{p,p}(X)\right\|^{1/n}$$

exists, is finite and is a bi-meromorphic invariant, independent of the choice of $\| \ \|$. The topological entropy of f satisfies $h_t(f) \leq \max \log d_p < \infty$.

Remarks 2.2.
- If f is algebraically stable, then $(f^n)^* = (f^*)^n$ and therefore d_p is the spectral radius of $f^* : H^{p,p}(X) \to H^{p,p}(X)$.
- The proof uses tool from complex analysis : the regularization of positive closed (p,p)-currents (Demailly $p = 1$, Dinh-Sibony $p \geq 1$).
- More general version with invariant fibrations are obtained in [3, 4].

Definition 2.3. The limit d_p is the *dynamical degree* of order p of f and $h_a := \max \log d_p$ is the *algebraic entropy* of f. They are bi-meromorphic invariants.

The following general statement shows that the number of isolated periodic points grows at most exponentially fast.

Theorem 2.4 (Dinh-Nguyen-Truong [6]). *Let $f : X \to X$ be a dominant meromorphic correspondence/map as above and Q_n the set of isolated periodic points of period n, counting multiplicity. Then we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \#Q_n \leq h_a := \max \log d_p < \infty.$$

Remarks 2.5.
- The map f is an *Artin-Mazur map,* that is, the number of isolated periodic points grows at most exponentially fast.
- The *dynamical ζ-function* ζ_f associated with f is always analytic near 0, where

$$\zeta_f(z) := \sum_{n\geq 1} \frac{1}{n} (\#Q_n) z^n.$$

The following questions are open in general.

Problem 2.6. *Is the function ζ_f always rational ?*

Problem 2.7. *Define $L(f) := \{\Gamma\} \smile \{\Delta\}$ the Lefschetz number of f. Do we always have*

$$\#Q_n \leq L(f^n) + o(L(f^n)) \quad \textit{as} \quad n \to \infty\ ?$$

In the following result, the second assertion is related to the works by Favre, Iwasaki-Uehara, Jonsson-Reschke, Saito and Xie [15, 17, 18, 22, 24].

Theorem 2.8 (Dinh-Nguyen-Truong/-Vu [5, 7]). *We use the notations introduced above.*

(1) *Assume that the last dynamical degree d_k is strictly larger than the other dynamical degrees. Then*

$$L(f^n) = d_k^n + o(d_k^n) \quad \textit{and} \quad \#Q_n = d_k^n + o(d_k^n).$$

(2) *Assume that $\dim X = 2$ and that f is algebraically stable in the sense of Fornaess-Sibony. Assume also that $d_1 > d_2$. Then we have*

$$L(f^n) = d_1^n + o(d_1^n) \quad \textit{and} \quad \#Q_n \leq d_1^n + o(d_1^n).$$

(3) *Let f be a holomorphic map or finite correspondence whose action on cohomology is simple. Let d be the maximal dynamical degree. Then*

$$L(f^n) = d^n + o(d^n) \quad \textit{and} \quad \#Q_n \leq d^n + o(d^n).$$

Our strategy of the proof is as follows (the notation is different from the references).

- Consider the graph Γ_n of f^n as a positive closed current in $X \times X$.
- Consider the limits Γ_∞ in the sense of currents of Γ_n, properly normalized. So Γ_∞ is a positive closed current.
- Study the intersection of Γ_∞ with the diagonal Δ.
- Deduce properties of $Q_n \subset \Gamma_n \cap \Delta$.

In the general setting, the intersection of two varieties of dimension k in $X \times X$ is expected to be a finite set. However, it may have a positive dimension. The *dimension excess* of the intersection $\Gamma_n \cap \Delta$ is exactly the main difficulty in our study. The *ideal situation* is that the "dimension" of $\Gamma_\infty \cap \Delta$ is 0. Then using a *upper semi-continuity* for the intersection of currents, due to Siu and Dinh-Sibony, we get a control of the dimension excess for $\Gamma_n \cap \Delta$ when n goes to infinity.

3. Equidistribution of periodic points

In this section, we will present some results related to Problem 1.3. We have the complete answer to this question for large classes of maps/correspondences. The following results are valid for maps/correspondences with dominant topological degree.

Theorem 3.1 (Dinh-Sibony, Dinh-Nguyen-Truong [5, 10]). *Let $f : X \to X$ be a meromorphic map/correspondence such that the last topological degree d_k is strictly larger than the other ones. There is an invariant probability measure μ such that for x outside an explicit countable union $\mathscr{E}$ of analytic sets the fiber $f^{-n}(x)$ is equidistributed with respect to μ. More precisely, for $x \notin \mathscr{E}$*

$$\lim_{n\to\infty} \frac{1}{d_k^n} \sum_{a \in f^{-n}(x)} \delta_a = \mu,$$

where δ_a is the Dirac mass at a.

Theorem 3.2 (Lyubich, Briend-Duval for holomorphic maps on $\mathbb{P}^k$ [2], Dinh-Nguyen-Truong [5]). *Under the hypothesis of the last theorem, the isolated periodic points of period n are equidistributed with respect to μ. More precisely, we have*

$$\lim_{n\to\infty} \frac{1}{d_k^n} \sum_{a \in Q_n} \delta_a = \mu.$$

Furthermore, most of these points are repelling.

Consider now another family of maps. Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial automorphism. We extend it to a birational map $f : \mathbb{P}^k \to \mathbb{P}^k$. Let I_+ and I_- be the indeterminacy sets of f and f^{-1} at infinity.

Definition 3.3 (Sibony [23]). We say that f is of *Hénon-type* if

$$I_+ \cap I_- = \varnothing.$$

Note that a result by Friedland-Milnor says that in dimension $k = 2$ every automorphism is conjugated to a Hénon-type map or an elementary one.

Theorem 3.4 (Bedford-Lyubich-Smillie for $k = 2$ [1], Dinh-Sibony for $k \geq 2$ [13]). *There is a canonical invariant probability measure μ such that periodic points are equidistributed with respect to μ. More precisely,*

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{a\in Q_n} \delta_a = \mu,$$

where d is the largest dynamical degree. Moreover, most of these points are saddles.

In the proof of this theorem, we introduced a new method to get the equidistribution of periodic points. Let Γ_n be the graph of f^n in $X \times X$. Roughly, we need to show that

$$\lim_{n\to\infty} (\Gamma_n \cap \Delta) = \big(\lim_{n\to\infty} \Gamma_n\big) \cap \Delta.$$

Here, a normalization in the sense of currents is needed before taking the limits.

The main difficulty to get the last identity is that Γ_n does not intersect Δ (uniformly) transversally. In our approach, we prove and use an asymptotic transversality for the intersection between Γ_n and Δ. The idea is as follows. If $F : X \times X \to X \times X$ is given by

$$F\big(x_1, x_2\big) := \big(f(x_1), f^{-1}(x_2)\big)$$

then

$$\Gamma_n = F^{-n/2}(\Delta).$$

We lift F, Γ_n, Δ to suitable jet bundles over $X \times X$ and obtain $\widehat{F}, \widehat{\Gamma}_n, \widetilde{\Delta}$. Using dynamics of $\widehat{F}$, we prove the convergence of $\widehat{\Gamma}_n$ to some $\widehat{\Gamma}_\infty$ and

$$\widehat{\Gamma}_\infty \cap \widetilde{\Delta} = \varnothing.$$

The last property is exactly the asymptotic transversality for $\Gamma_n \cap \Delta$. Indeed, the above construction is so that Γ_n intersects Δ transversally if and only if $\widehat{\Gamma}_n \cap \widetilde{\Delta} = \varnothing$. We don't have this property in general but a similar property for Γ_∞ shows that the non-transversality of $\Gamma_n \cap \Delta$ is, in some sense, negligible when n tends to infinity.

4. Positive closed currents and intersection theory

In this last section, we will briefly present the notion of positive closed currents to non-experts. The theory was introduced by Oka and Lelong. We will consider here some geometric point of view rather than formal definitions in complex analysis that can be found in several references. We only consider currents on a compact Kähler manifold X of dimension k. Positive closed currents can be seen as natural objects to compactify the space of effective analytic cycles (effective algebraic cycles when X is algebraic).

Consider first the case of maximal bi-degree : positive closed (p,p)-currents with $p = k$. In this case, a positive closed (k,k)-current of mass 1 is just a probability measure. The following diagram shows that we can embed the space of effective 0-cycles of degree 1 into the space of probability measures. The last one is a compact subset of the space of distributions which is, by definition, the dual space of the space of smooth functions.

$$
\begin{aligned}
\big\{\text{effective 0-cycles of degree 1}\big\} \quad &\hookrightarrow \quad \big\{\text{probability measures on } X\big\} \\
&\qquad\ \text{which is a } \boxed{compact} \text{ space} \\
&\subset \quad \big\{\text{continuous functions on } X\big\}^\star \\
&\subset \quad \big\{\text{smooth functions on } X\big\}^\star \\
&=: \quad \big\{\text{distributions on } X\big\}.
\end{aligned}
$$

Concretely, if $\sum \lambda_i a_i$ is an effective 0-cycle of degree 1, i.e. the a_i's are points in X and the λ_i's are positive numbers of total 1, then it can be associated with a probability measure by

$$\boxed{\sum \lambda_i a_i \quad \longleftrightarrow \quad \sum \lambda_i \delta_{a_i}}.$$

Here, δ_{a_i} denotes the Dirac mass at a_i.

For the general case of bi-degree (p,p), we can embed the space of effective cycles of degree 1 and codimension p into the compact space of positive closed (p,p)-currents of mass 1. We will not give here the formal definition of positive closed current. A general (p,p)-current, not necessarily positive and closed, is a continuous linear form on the space of smooth differential $(k-p,k-p)$-forms. We have the following diagram :

$$
\begin{aligned}
&\big\{\text{effective cycles of degree 1 and codimension } p \text{ of } X\big\} \\
&\qquad \hookrightarrow \quad \big\{\text{positive closed } (p,p)\text{-currents of mass 1 on } X\big\} \\
&\qquad\qquad\quad \text{which is a } \boxed{compact} \text{ space} \\
&\qquad \subset \quad \big\{\text{continuous differential } (k-p,k-p)\text{-forms } \varphi \text{ on } X\big\}^\star \\
&\qquad \subset \quad \big\{\text{smooth differential } (k-p,k-p)\text{-forms } \varphi \text{ on } X\big\}^\star \\
&\qquad =: \quad \big\{(p,p)\text{-currents on } X\big\}.
\end{aligned}
$$

$$\boxed{\sum \lambda_i V_i \quad \longleftrightarrow \quad \sum \lambda_i \int_{z\in V_i}} \qquad \varphi \mapsto \sum \lambda_i \int_{z\in V_i} \varphi(z).$$

So for currents, we need to test differential forms instead of functions for measures. Positive closed currents are global objects and can be seen as generalized submanifolds. In order to use these currents, we need to develop a calculus, in particular, the theory of intersections which is an analytic counter-part of the intersection of cycles in algebraic geometry.

The theory of intersection of currents is well developed in the case of co-dimension 1, i.e. $p = 1$, which corresponds to hypersurfaces in the algebraic setting, thanks to the works by Chern-Levine-Nirenberg, Bedford-Taylor, Demailly, Fornaess-Sibony among others. For applications in dynamics, we need currents of any dimension. For this purpose, we developed a theory of intersection without dimension excess using super-potentials and also a theory of intersection with dimension excess using densities and tangent currents. Both of them are quite technical and give several applications in complex analysis, dynamics and foliations. We refer the reader to [9, 11, 12, 14] for more details.

Finally, we give below some selected references. The reader will find there a more exhaustive list of references for both pluripotential theory and complex dynamical systems.

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