# ARITHMETIC DEGREES AND DYNAMICAL DEGREES OF SELF-MAPS OF ALGEBRAIC VARIETIES

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## 1. Introduction

I would like to thank the organizers of Kinosaki algebraic geometry symposium 2017. This is the report of my talk on the symposium. In this short note, I will summarize our recent results related to Silverman's conjecture on height growth along the orbits of self-maps. Let $X$ be a smooth projective variety and $f\colon X \dashrightarrow X$ a rational self-map, both defined over $\overline{\mathbb{Q}}$. Silverman introduced the notion of arithmetic degree in [12], which measures the arithmetic complexity of $f$-orbits.

**Definition 1.1.**

(1) We write $X_f(\overline{\mathbb{Q}})$ the set of $\overline{\mathbb{Q}}$-valued points of $X$ whose forward $f$-orbits are well-defined:

$$X_f(\overline{\mathbb{Q}}) = \{x \in X(\overline{\mathbb{Q}}) \mid f^n(x) \notin I_f \text{ for all } n \geq 0\}.$$

(2) Fix a Weil height function $h_X$ associated with an ample divisor on $X$. (A good references for height functions are [1, 6].) Let $x \in X_f(\overline{\mathbb{Q}})$. The *arithmetic degree* of $f$ at $x$ is

$$\alpha_f(x) = \lim_{n\to\infty} \max\{h_X(f^n(x)), 1\}^{1/n}$$

provided that the limit exists. Let $\overline{\alpha}_f(x)$ and $\underline{\alpha}_f(x)$ denote the limit sup and limit inf respectively. It is easy to prove that $\alpha_f(x), \overline{\alpha}_f(x), \underline{\alpha}_f(x)$ do not depend on the choice of ample height function $h_X$.

In [7], Kawaguchi and Silverman proved that when $f$ is a morphism, the arithmetic degree $\alpha_f(x)$ always exists and is equal to either 1 or the absolute value of one of the eigenvalues of $f^*\colon N^1(X)\otimes_{\mathbb{Z}}\mathbb{R} \longrightarrow N^1(X)\otimes_{\mathbb{Z}}\mathbb{R}$, where $N^1(X)$ is the group of divisors modulo numerical equivalence.

When $f$ is a polarized endomorphism, i.e. there exists an ample divisor $H$ such that $f^*H$ is linearly equivalent to $dH$ for some $d > 1$, $\alpha_f(x) = 1$ or $d$ and $\alpha_f(x) = 1$ if and only if the $f$-orbit of $x$ is finite.

To formulate this type statement for general $f$, we need to introduce the first dynamical degree of $f$. We define the pull-back homomorphism $f^*\colon N^1(X) \longrightarrow N^1(X)$ as follows. Take a resolution of indeterminacy

$g\colon Y \longrightarrow X$ of $f$ with $Y$ smooth. Then $f^*D = g_*((f \circ g)^*D)$ for every $D \in N^1(X)$. This is independent of the choice of the resolution.

**Definition 1.2.** The first dynamical degree $\delta_f$ of $f$ is:

$$\delta_f = \lim_{n\to\infty} \|(f^n)^*\colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}\|^{1/n}$$

where $\|\ \|$ is any norm on $\mathrm{End}_{\mathbb{R}}(N^1(X)_{\mathbb{R}})$. Note that $\delta_f \geq 1$ since $f$ is dominant and $(f^n)^*$ is a homomorphism of the $\mathbb{Z}$-module $N^1(X)$. We refer, e.g., to [2, 3, 5, 14] for basic properties of dynamical degrees.

In the above polarized situation, the first dynamical degree $\delta_f$ of $f$ is $d$. For a dominant rational map $f$, Silverman and Kawaguchi conjectured the following in [12], [8, Conjecture 6]:

**Conjecture 1.3** (KSC)**.** *Let $x \in X_f(\overline{\mathbb{Q}})$.*

(1) *The limit defining $\alpha_f(x)$ exists.*
(2) *The arithmetic degree $\alpha_f(x)$ is an algebraic integer.*
(3) *The collections of arithmetic degrees $\{\alpha_f(x) \mid x \in X_f(\overline{\mathbb{Q}})\}$ is a finite set.*
(4) *If the forward orbit $\mathcal{O}_f(x) = \{f^n(x) \mid n = 0, 1, 2, \ldots\}$ is Zariski dense in $X$, then $\alpha_f(x) = \delta_f$.*

This is the Kawaguchi-Silverman conjecture, and we abbreviate it as KSC. When $X$ is quasi-projective, arithmetic degrees and dynamical degrees can be defined by taking a smooth compactification of $X$ and we can consider Conjecture 1.3 for quasi-projective $X$.

## 2. MAIN THEOREMS

2.1. **Fundamental theorems on height growth along orbits.** In this subsection, $X$ is a smooth projective variety and $f\colon X \dashrightarrow X$ a dominant rational map, both defined over $\overline{\mathbb{Q}}$. Fix a Weil height function $h_X$ associated with an ample divisor on $X$. Let $h_X^+$ denote $\max\{1, h_X\}$. First theorem says that the arithmetic degrees are bounded above by the first dynamical degrees.

**Theorem 2.1** ([10, Theorem 1.4])**.** *For any $\epsilon > 0$, there exists $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C(\delta_f + \epsilon)^n h_X^+(P)$$

*for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$. In particular, for any $P \in X_f(\overline{\mathbb{Q}})$, we have*

$$\overline{\alpha}_f(P) \leq \delta_f.$$

If $f$ is a morphism, we have the following slightly stronger inequalities.

**Theorem 2.2** (c.f. [10, Theorem 1.6])**.** *Assume $f\colon X \longrightarrow X$ is a surjective morphism. Let $r = \dim N^1(X)_{\mathbb{R}}$ be the Picard number of $X$.*

(1) *When* $\delta_f = 1$, *there exists a constant* $C > 0$ *such that*

$$h_X^+(f^n(P)) \leq Cn^{2r} h_X^+(P)$$

*for all* $n \geq 1$ *and* $P \in X(\overline{\mathbb{Q}})$.

(2) *Assume that* $\delta_f > 1$. *Then there exists a constant* $C > 0$ *such that*

$$h_X^+(f^n(P)) \leq Cn^{r-1}\delta_f^n h_X^+(P)$$

*for all* $n \geq 1$ *and* $P \in X(\overline{\mathbb{Q}})$.

As a corollary of Theorem 2.1, we can define so called dynamical canonical height functions for self-rational maps having some nice properties.

**Corollary 2.3** ([10, Proposition 1.10])**.** *Assume* $\delta_f > 1$ *and there exists a nef* $\mathbb{R}$*-divisor* $H$ *on* $X$ *such that* $f^*H \equiv \delta_f H$. *Fix a height function* $h_H$ *associated with* $H$. *Then for any* $P \in X_f(\overline{\mathbb{Q}})$, *the limit*

$$\hat{h}_{X,f}(P) = \lim_{n\to\infty} \frac{h_H(f^n(P))}{\delta_f^n}$$

*converges or diverges to* $-\infty$.

*Proof.* We take a resolution of indeterminacy $p\colon Y \longrightarrow X$ of $f$ so that $p$ is an isomorphism outside the indeterminacy locus $I_f$ of $f$:

$$\begin{array}{ccccc} & & Y & & \\ & \overset{p}{\swarrow} & & \overset{g}{\searrow} & \\ X & & \overset{f}{\dashrightarrow} & & X. \end{array}$$

Write $g = f \circ p$. By negativity lemma, $p^*p_*g^*H - g^*H$ is a $p$-exceptional effective divisor on $Y$. Then as in the proof of [8, Proposition 21], we have $h_H \circ f \leq h_{f^*H} + O(1)$ on $X \setminus I_f$ where $h_H$ and $h_{f^*H}$ are height functions associated with $H$ and $f^*H$. Fix an ample height $h_X$ on $X$. Since $f^*H \equiv \delta_f H$, we have $h_{f^*H} - \delta_f h_H = O\left(\sqrt{h_X^+}\right)$. Thus, we have

$$h_H \circ f \leq \delta_f h_H + O\left(\sqrt{h_X^+}\right) \quad \text{on } X \setminus I_f.$$

Write $B = h_H \circ f - \delta_f h_H$. Then, for any $P \in X_f$,

$$\begin{aligned} h_H(f^n(x)) &= \sum_{k=0}^{n-1} \delta_f^{n-1-k}\big(h_H(f^{k+1}(P)) - \delta_f h_H(f^k(P))\big) + \delta_f^n h_H(P) \\ &= \sum_{k=0}^{n-1} \delta_f^{n-1-k} B(f^k(P)) + \delta_f^n h_H(P). \end{aligned}$$

Take $\epsilon > 0$ so that $\sqrt{\delta_f + \epsilon} < \delta_f$. By Theorem 2.1, there exists $C > 0$ such that $B(f^k(P)) \leq C\sqrt{\delta_f + \epsilon}^k$ for all $k \geq 0$. Set

$$a_k = \frac{B(f^k(P))}{\sqrt{\delta_f + \epsilon}^k}.$$

Note that $a_k$ is bounded above. Then

$$\begin{aligned}
&\frac{h_H(f^n(P))}{\delta_f^n} \\
&= h_H(P) + \sum_{k=0}^{n-1} \frac{B(f^k(P))}{\delta_f^{k+1}} \\
&= h_H(P) + \frac{1}{\delta_f} \sum_{k=0}^{n-1} a_k \left( \frac{\sqrt{\delta_f + \epsilon}}{\delta_f} \right)^k \\
&= h_H(P) + \frac{1}{\delta_f} \left\{ \sum_{\substack{0 \leq k \leq n-1 \\ a_k \geq 0}} a_k \left( \frac{\sqrt{\delta_f + \epsilon}}{\delta_f} \right)^k - \sum_{\substack{0 \leq k \leq n-1 \\ a_k < 0}} (-a_k) \left( \frac{\sqrt{\delta_f + \epsilon}}{\delta_f} \right)^k \right\}.
\end{aligned}$$

The first summation in the bracket is convergent since $a_k$ is bounded above and the second summation is monotonically increasing. Hence, the claim follows.

□

I do not know whether the diverging to $-\infty$ case actually happens or not.

2.2. **Some cases KSC holds.** All materials in this subsection are joint work with Kaoru Sano and Takahiro Shibata.

**Theorem 2.4** ([11, Theorem 1.3, 1.4])**.**

(1) *Let $X$ a smooth projective surface over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ a surjective endomorphism on $X$. Then Conjecture 1.3 holds for $f$.*
(2) *Let $X$ be a smooth projective irrational surface over $\overline{\mathbb{Q}}$. Let $f\colon X \dashrightarrow X$ be a birational automorphism on $X$. Then Conjecture 1.3 holds for $f$.*

**Remark 2.5.** KSC for automorphisms of surfaces had already been proven by Kawaguchi.

A dominant rational self-map might have no Zariski dense orbits. For instance, all self-maps of varieties with positive Kodaira dimension cannot have a Zariski dense orbit because they preserve the Iitaka fibration. So asking whether a self-map has a point whose arithmetic degree is equal to the dynamical degree is another question. We prove that such a point always exists for any surjective endomorphism on any smooth projective variety.

**Theorem 2.6** ([11, Theorem 1.6])**.** *Let $X$ be a smooth projective variety over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ a surjective endomorphism on $X$. Then there exists a point $x \in X(\overline{\mathbb{Q}})$ such that $\alpha_f(x) = \delta_f$.*

*Proof.* Since $\overline{\alpha}_f(x) \le \delta_f$, it is enough to prove $\delta_f \le \underline{\alpha}_f(x)$. We may assume $\delta_f > 1$ since when $\delta_f = 1$ there is nothing to prove. Since $f^*\colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$ preserves the nef cone, by the theorem of Perron-Frobenius-Birkhoff, there is a nef $\mathbb{R}$-divisor $D \not\equiv 0$ such that $f^*D \equiv \delta_f D$. We fix a height function $h_D$ associated with $D$. Fix an ample divisor $H$ on $V$ and take a height function $h_H$ associated with $H$ with $h_H \ge 1$. Define

$$\hat{h}_D(Q) = \lim_{n\to\infty} \frac{h_D(f^n(Q))}{\delta_f^n}$$

for any $Q \in V(K)$. This limit exists and we have

$$\hat{h}_D = h_D + O(\sqrt{h_H}).$$

Since $D \not\equiv 0$, there exists a curve $C \subset X$ such that $(D \cdot C) > 0$. Since

$$\hat{h}_D|_{C(K)} = h_{D|_C} + O\left(\sqrt{h_{H|_C}}\right)$$

and $D|_C$ is an ample $\mathbb{R}$-divisor class, we have

$$\hat{h}_D|_{C(\overline{\mathbb{Q}})} \not\le 0.$$

Here we use the fact that an ample height is not bounded. Therefore, there exists a point $x \in C(\overline{\mathbb{Q}}) \subset V(\overline{\mathbb{Q}})$ such that $\hat{h}_D(x) > 0$. By the definition of $\hat{h}_D$, we have $\underline{\alpha}_f(x) \ge \delta_f$. □

If $f$ is an automorphism, we can construct a "large" collection of points whose orbits have full arithmetic complexity.

**Theorem 2.7** ([11, Theorem 1.7])**.** *Let $X$ be a smooth projective variety over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ an automorphism. Then there exists a subset $S \subset X(\overline{\mathbb{Q}})$ that satisfies the following conditions.*

(1) *For every $x \in S$, $\alpha_f(x) = \delta_f$.*
(2) *For $x, y \in S$ with $x \ne y$, $\mathcal{O}_f(x) \cap \mathcal{O}_f(y) = \emptyset$.*
(3) *$S$ is Zariski dense in $X$.*

## 3. Do the arithmetic degrees are controled by geometry?

Let us consider the following question:

What is the set $A(f) = \{\alpha_f(x) \mid x \in X_f(\overline{\mathbb{Q}})\}$ ?

When $X$ is smooth projective and $f$ is a morphism, by a theorem due to Kawaguchi-Silverman, we know that

$$A(f) \subset \{1\} \cup \{|\alpha| \mid \alpha \text{ is an eigenvalue of } f^*|_{N^1(X)}\}.$$

By Theorem 2.6, we know $\delta_f \in A(f)$. To my knowledge, these are the only things that we can say about $A(f)$ so far. It is very interesting to describe the set $A(f)$ in terms of the geometry of $f$.

If we assume KSC for possibly singular varieties, the geometry of $f$ determines the set $A(f)$. For a point $x \in X$, consider the closure of the orbit $\overline{O_f(x)}$. Take an irreducible component $Z$ of $\overline{O_f(x)}$ that is $f$-periodic with period $n$. Then $\alpha_f(x) = \alpha_{f^n|_Z}(f^l(x))^{1/n}$ where $l$ is a positive integer such that $f^l(x) \in Z$. If KSC holds for $Z$ and $f^n|_Z$, we have $\alpha_f(x) = \delta_{f^n|_Z}^{1/n}$.

The next question is:

Which absolute values of eigenvalues of $f^*|_{N^1(X)}$ are realized as the arithmetic degrees of a point?

We now gather examples to get insight on this question. Here is a list of our achievement in this direction up to the present time.

(1) When $X$ is a smooth projective surface and $f$ is a surjective morphism, I have calculated the set $A(f)$ completely.
(2) When $X$ is an algebraic torus (so possibly non-projective) and $f$ is an isogeny, Silverman determined the set $A(f)$ in terms of the minimal polynomial of $f$ as an element of $\mathrm{End}(X)$ [12]. Lin generalized Silverman's result to toric varieties and monomial maps case [9].
(3) More generally, when $X$ is a semi-abelian variety and $f$ is a self-morphism, Kaoru Sano and I have determined the set $A(f)$ recently.

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