# FROBENIUS MAPS AND ALGEBRAIC VARIETIES

AKIYOSHI SANNAI

ABSTRACT. Many characterization of abelian varieties in positive characteristic is given by several authors. In this article, we give a characterization of abelian varieties by the language of $F_*\mathcal{O}_X$ and its Poincare bundles.

Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $X$ be a smooth proper variety over $k$. When does $X$ satisfy the following property $(*)$?

$(*)$ $F_*\mathcal{O}_X \simeq \bigoplus_j M_j$ where $F : X \to X$ is the absolute Frobenius morphism and each $M_j$ is a line bundle.

For example, an arbitrary smooth proper toric variety satisfies this property $(*)$ (cf. [Achinger][Thomsen]). Thus there are many varieties which satisfy $(*)$. But every toric variety has negative Kodaira dimension. On the other hand,In [ST] we show that ordinary abelian varieties satisfy $(*)$ and this property gives the characterization of the ordinary abelian varieties.

**Theorem 0.1.** *Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $X$ be a smooth projective variety over $k$. Assume the following conditions.*

- *For infinitely many $e \in \mathbb{Z}_{>0}$, $F^e_*\mathcal{O}_X \simeq \bigoplus_j M_j$ where each $M_j$ is an invertible sheaf.*
- *$\kappa(X) \geq 0$ where $\kappa(X)$ is the Kodaira dimension of $X$.*

*Then $X$ is an ordinary abelian variety.*

By the Theorem above, it is natural to ask the same criterion holds only for $F_*O_X$. But in [ES], we gave the counter example for that problem.

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**Theorem 0.2.** *Let $X$ be a non-abelian smooth projective surface. Then $X$ satisfies the following conditions if and only if $X$ is an $F$-split Igusa surface:*

- *$F_*\mathcal{O}_X$ is isomorphic to direct sum of line bundles.*
- *$K_X$ is pseudo-effective.*

We also showed that if we consider $e > 1$, then the same characterization holds.

**Theorem 0.3.** *Let $X$ be a smooth projective variety over $k$ of characteristic $p > 2$ (resp. $p > 0$). $X$ is an ordinary abelian variety if and only if the following conditions hold:*

- *$F_*\mathcal{O}_X$ (resp. $F_*^2\mathcal{O}_X$) is isomorphic to a direct sum of line bundles.*
- *$K_X$ is pseudo-effective.*

Another natural question is how can we characteraze non ordinary abelian varieties by the language of $F_*\mathcal{O}_X$. In this article we give the characterization by using Poincare bundle on albanese varieties. For details of Poincare bundles and albanese maps, see [FGAex, Section 9].

**Theorem 0.4.** *Let $X$ be a smooth projective variety over $k$ of characteristic $p$. There is a canonical map $\Phi : \tilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$ with the following properties:*

- *$\widetilde{\mathcal{P}}_X \simeq p_{1*}\mathcal{P}_{X\times kerV}$ where $\mathcal{P}$ is the Poincare bundle on $X \times Pic^0 X$ and $V$ is the Verschiebung map on $Pic^0 X$.*
- *$\Phi$ is isomorphic if and only if $X$ is an abelian variety.*
- *$\Phi$ is generically isomorphic if and only if $X$ has maximal albanese dimension.*

We overview the proof of Theorem 0.4. At first, we construct the map $\Phi : \tilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$.

**Lemma 0.5.** *Let $X$ be a smooth projective variety over $k$ of characteristic $p$ and $\alpha_X : X \to \mathrm{Alb}(X)$ be the albanese map of $X$. Then we have*

$$\alpha^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \widetilde{\mathcal{P}}_X$$

*Proof.* $\alpha_X : X \to \mathrm{Alb}(X)$ induces the following map;

$$\alpha_X^* : \mathrm{Pic}^0\mathrm{Pic}^0\mathrm{Pic}^0_{red}(X) \simeq \mathrm{Pic}^0(X)$$

Here we can see that the kernel of the Verschiebung maps corresponds each other. Hence, we have

$$\alpha_X^*\mathrm{KerV}_{\mathrm{Pic}^0(X)} \simeq \mathrm{KerV}_{\mathrm{Pic}^0\mathrm{Pic}^0\mathrm{Pic}^0_{red}(X)}$$

. By considering the universal bundles on them, we get the assersion.

□

**Lemma 0.6.** *Let $X$ be a smooth projective variety over $k$ of characteristic $p$. There is a canonical map $\Phi : \widetilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$.*

*Proof.* We use the following theorem.

**Theorem 0.7** (Oda). *Let $f : X \to Y$ be an isogeny of abelian varieties over $k$. Set $\hat{f} : \hat{Y} \to \hat{X}$ to be the dual of $f$. Let $L \in \mathrm{Pic}^0(X)$. Then, $f$*

$$f_*L \simeq \mathrm{pr}_{1*}(\mathcal{P}_Y|_{Y\times \hat{f}^{-1}([L])})$$

*where $\mathcal{P}_Y$ is the normalized Poincare line bundle of $(Y, 0)$.*

By applying this theorem to $f = F$ and $L = \mathcal{O}_X$, we have $\Phi : \widetilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$ for any abelian varieties $X$. (Note that Verschiebung map is defined by the dual of relative Frobenius map.)
In general, we have the base change map $\alpha^*F_*\mathcal{O}_{\mathrm{Alb}(X)} \to F_*\mathcal{O}_X$. Then by the discussion above and lem 0.6,

$$\alpha^*F_*\mathcal{O}_{\mathrm{Alb}(X)} \simeq \alpha^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \widetilde{\mathcal{P}}_X$$

Hence, by composing them, we get the assersion.

□

**Claim 0.8.** *If $\Phi_X$ is isomorphism, then $X$ has maximal albanese dimension, namely* $\dim X = \dim \alpha(X)$

*Proof.* By Stein factrizaton theorem, we have the following decomposition;

$$\alpha : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} A = \mathrm{Alb}(X)$$

where $g, h$ is finite, and $g$ is purely inseparable, $h$ is separable, $Y, Z$ is normal.

We first see the map

$$\Phi_Y : \widetilde{\mathcal{P}_Y} \to F_*\mathcal{O}_Y$$

. Pulling back by $f$, we have

$$\widetilde{\Phi}_Y : f^*\widetilde{\mathcal{P}}_Y \xrightarrow{f^*\Phi_Y} f^*F_*\mathcal{O}_Y \to F_*\mathcal{O}_X$$

where the second map is base change map.
We also have

$$f^*\widetilde{\mathcal{P}}_Y \simeq f^*g^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \alpha^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \widetilde{\mathcal{P}}_X.$$

Hence, we see that $\widetilde{\Phi}_Y = \Phi_X$. Hence, by calculating the rank of $F_*\mathcal{O}$, we have $\dim X \le \dim Y$. This implies $\dim X = \dim Y = \dim \alpha(X)$.

□

**Claim 0.9.** *If $\Phi_X$ is isomorphism, then $\Phi_Y$ is also isomorphism.*

*Proof.* Since $X$ has maximal albanese dimension, $f$ is a birational contraction. Let $U, V$ be the isomophic locus of $f$ in $X, Y$ respectively. Then $U, V$ have codimension two in their ambient space. Here we have

$$F_*\mathcal{O}_Y|U \simeq F_*\mathcal{O}_X|V \simeq \mathcal{P}_X|V \simeq \mathcal{P}_Y|U$$

Since all the sheaf above are reflexive and all the varieties above are normal, and $U, V$ are codimension two subset, we get $\Phi_Y : \widetilde{\mathcal{P}_Y} \simeq F_*\mathcal{O}_Y$. □

**Claim 0.10.** *There is a decomposition*

$$Y \to Z_1 \to ...Z_i \to Z_{i+1}... \to Z$$

*such that $h_i : Z_i \to Z_{i+1}$ is a purely inseparable extension of height 1 of normal varieties. Futhermore, if $\Phi_Y$ is an isomorphism, then $\Phi_{Z_i}$ is also an isomorphism.*

*Proof.* The first assersion follows by taking field extensions of height 1 and normalizations in the fields. We see the second assersion. Since $Z_i$ is normal, $F_*\mathcal{O}_{Z_i}$ is reflexive. Hence we may assume $Z_{i+1}$ is a spectrum of a DVR. Since torsion free module over DVR is faithfully flat, $h_i^*$ is a faithfully flat functor. Hence by the induction starting from $Y$, we have the second assersion. □

By standard calculation of Grothendiek duality and Kunz's theorem, we have the following proposition.

**Proposition 0.11.** *Let $X$ be normal projective variety over $k$. Assume that $\Phi_X$ is an isomorphism. Then $X$ is smooth and $-(p-1)K_X$ is an effective divisor.*

We prove the main theorem for purelu inseparable extension of height 1.

**Proposition 0.12.** *Let $\alpha : Z \to A$ be the albanese map of normal variety $Z$. Assume that $\alpha$ is purely inseparable of height 1 and $\Phi_Z$ is an isomorphism. Then $\alpha$ is an isomorphism.*

*Proof.* By [Eke], any purely inseparable of height 1 morphism can be described by a quotient of 1-foliation. Namely, there is a 1-foliation $\mathcal{F} \subset \mathcal{T}_A$ such that $Z \simeq A/\mathcal{F}$ and $\omega_Z^p \simeq \alpha^*\omega_A \otimes (det\mathcal{F})^{(1-p)}$ Since $A$ is an abelian variety, $\mathcal{T}_A$ is trivial. This implies $det\mathcal{F} \subset \oplus\mathcal{O}_A$. Hence $det\mathcal{F})$ is negative line bundle, put $det\mathcal{F} \simeq \mathcal{O}(-D)$ for an effective divisor $D$. Then we have

$$\omega_Z^p \simeq \mathcal{O}_Z((p-1)D).$$

Prop 0.11 and this implies $\kappa(Z) = 0$. Hence by [HP], we get the assersion.

□

By claim 0.10 and Prop 0.12, $h$ is an isomorphism. Next, we show $g$ is an isomorphism.

**Claim 0.13.** *$g$ is an isomorphism.*

*Proof.* Put

$$K_Y = g^* K_A + R$$

where $R$ is the ramification divisor of separable morphism $g$. Since $K_A = 0$ and by Prop 0.11, we have $R = 0$.

Hence $\alpha : Y \to A$ is etale in codimension one. Then, by the Zariski–Nagata purity, $\alpha$ is etale. By [Mumford, Section 18, Theorem], $Y$ is also an abelian variety and we are done.

□

**Claim 0.14.** *$f$ is an isomorphism*

*Proof.* We can write

$$K_X = f^* K_A + E$$

where $E$ is an $f$-exceptional divisor. Since $A$ is terminal (cf. [KM, Section 2.3]), $E$ is effective. By Prop 0.12, we have $K_X \equiv 0$. Since $K_X \equiv 0$, we see that $E$ is $f$-nef. By the negativity lemma (cf. [KM, Lemma 3.39]), we see $E = 0$. Thus, the codimension of $\mathrm{Ex}(f)$ in $X$ is at least two. Since $A$ is smooth, $f$ is an isomorphism. □

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

*E-mail address*: `sannai@kurims.kyoto-u.ac.jp`