ON THE NON-VANISHING CONJECTURE AND EXISTENCE OF LOG MINIMAL MODELS

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1. Introduction

Throughout this paper we will work over the complex number field.

When a variety is given, the minimal model theory indicates the existence of a "good" variety which is birational to the given variety. The minimal model theory consists of two conjectures: the minimal model conjecture and the abundance conjecture. These conjectures are known when the dimension of the given variety is not greater than 3. But these are not proved in full generality. On the other hand, the non-vanishing conjecture is also a very important conjecture and it is closely related to the minimal model theory.

In this article we focus on the minimal model conjecture and the non-vanishing conjecture. We explain the relation between these two conjectures and that these two conjectures can be reduced the simplest case of the non-vanishing conjecture, that is, the non-vanishing conjecture for smooth varieties. We also introduce my recent result on the non-vanishing conjecture.

2. The minimal model conjecture

In this section we define some notions and explain the minimal model conjecture.

We start with the simplest example of the minimal model theory.

Example 2.1. *Let X be a smooth projective surface. Then there is a sequence of birational morphisms*

$$X = X_0 \to X_1 \to \cdots \to X_l = X',$$

such that X' satisfies one of the following properties:

- *$K_{X'}$ is nef,*
- *X' is a minimal ruled surface, or*
- *$X' \simeq \mathbb{P}^2$.*

When $X' \simeq \mathbb{P}^2$, we regard X' as a Fano fibration $X' \to \operatorname{Spec}\mathbb{C}$. Thus we see that there are two possibilities: $K_{X'}$ is nef or X' has a Fano fibration.

Next we introduce threefold case.

Example 2.2. *Let X be a smooth projective threefold. Then there is a sequence of birational contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_l = X',$$

where X' may not be smooth, such that X' satisfies one of the following properties:

- *$K_{X'}$ is nef, or*
- *there is a contraction $X' \to Z$ with $\dim Z < \dim X'$ such that $-K_{X'}$ is ample over Z and the relative Picard number is one.*

Remark 2.3. When $K_{X'}$ is nef in Example 2.1 or Example 2.2, $K_{X'}$ is semi-ample. In particular X' has a Calabi–Yau fibration. This property is surface or threefold case of the abundance conjecture.

To find such X', we run the *Minimal model program* (MMP, for short). Today we can run the MMP for smooth projective varieties in any dimension, and moreover, the MMP can be considered in more general setting. So we can consider the minimal model theory in more general setting.

We define *log canonical pair*. It is the largest class on which we can consider the minimal model theory.

Definition 2.4. Let X be a normal projective variety and $\Delta = \sum d_i D_i$ be an $\mathbb{R}$-divisor on X such that $d_i \in [0, 1]$ for any i and $K_X + \Delta$ is $\mathbb{R}$-Cartier.

For any projective birational morphism $f : Y \to X$, we can write

$$K_Y = f^*(K_X + \Delta) + \sum_j a_j E_j,$$

where a_j are real numbers and E_j are distinct prime divisors. Then the pair (X, Δ) is called a *log canonical pair* if $a_j \geq -1$ for any f and any E_j.

Remark 2.5. We can define log canonical pairs for normal varieties that are not necessarily projective. But in this article we only deal with normal projective varieties and so all log canonical pairs in this article are assumed to be projective.

We briefly explain why we work in the framework of log canonical pairs. To show some results of the minimal model theory, including the main result of this article, inductive arguments on the dimension of the variety are very powerful tools. In particular the adjunction formula and the canonical bundle formula are frequently used. To work induction with these formulas, we need to consider pairs of varieties and divisors as well as varieties themselves. Log canonical pair is a suitable class to work induction on the dimension.

Next we define *log minimal models* and *Mori fiber spaces*.

Definition 2.6. Let (X, Δ) be a log canonical pair and $\phi : X \dashrightarrow X'$ be a birational map to a normal projective variety. We set $\Delta' = \phi_*\Delta + E$, where E is the reduced ϕ^{-1}-exceptional divisor on X'. Assume that

- X' is $\mathbb{Q}$-factorial, and
- for birational morphisms $p : W \to X$ and $q : W \to X'$ such that $\phi \circ p = q$, the divisor $F := p^*(K_X + \Delta) - q^*(K_{X'} + \Delta')$ is effective and it contains the birational transform of all ϕ-exceptional prime divisors in its support.

Then the pair (X', Δ') is called a *log minimal model* of (X, Δ) if $K_{X'}+\Delta'$ is nef. On the other hand, the pair (X', Δ') is called a *Mori fiber space* of (X, Δ) if there is a contraction $X' \to Z$ with $\dim Z < \dim X'$ such that $-(K_{X'} + \Delta')$ is ample over Z and the relative Picard number is one.

By the definition we can check that any log minimal model or Mori fiber space (X', Δ') is also a log canonical pair.

Now we can state the conjecture on existence of models.

Conjecture 2.7. *Let (X, Δ) be a log canonical pair. If $K_X + \Delta$ is pseudo-effective, (X, Δ) has a log minimal model. If $K_X + \Delta$ is not pseudo-effective, then (X, Δ) has a Mori fiber space.*

Conjecture 2.7 is known when

(i) $\dim X \leq 3$ (by Kawamata, Kollár, Matsuki, Mori, Shokurov and others).
(ii) $\dim X = 4$ (cf. [S], [B2]).
(iii) X is smooth and K_X is big (cf. [BCHM]), or
(iv) $K_X + \Delta$ is not pseudo-effective (cf. [BCHM]).

In [BCHM], they argue in the framework of Kawamata log terminal pairs, that are special case of log canonical pairs and include smooth varieties. So they proved in [BCHM] a slightly stronger result than (iii).

Thanks to (iv), we can rephrase Conjecture 2.7.

Conjecture 2.8 (Minimal model conjecture)**.** *Let (X, Δ) be a log canonical pair. If $K_X + \Delta$ is pseudo-effective, (X, Δ) has a log minimal model.*

3. The non-vanishing conjecture and the main result

In this section we explain the non-vanishing conjecture and the main result.

First we introduce the non-vanishing conjecture.

Conjecture 3.1 (Non-vanishing for log canonical pairs)**.** *Let (X, Δ) be a log canonical pair. If $K_X + \Delta$ is pseudo-effective, there is an effective $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R}} D$.*

Conjecture 3.1 is deeply linked to the minimal model theory. Indeed, fix a log canonical pair (X, Δ) such that $K_X + \Delta$ is pseudo-effective. Then Conjecture 2.8 for (X, Δ) and the abundance conjecture for a log minimal model of (X, Δ) implies Conjecture 3.1 for (X, Δ). In addition to this fact, it is known by Birkar [B1] that the following theorem holds.

Theorem 3.2 (cf. [B1, Theorem 1.4])**.** *Assume Conjecture 3.1 for all log canonical pairs of dimension $\leq n$. Then Conjecture 2.8 holds for all log canonical pairs of dimension $\leq n$.*

In other words, Conjecture 3.1 is stronger than Conjecture 2.8.

The following conjecture is the non-vanishing conjecture for smooth varieties, that is the simplest case of Conjecture 3.1.

Conjecture 3.3 (Non-vanishing for smooth varieties)**.** *Let X be a smooth projective variety. If K_X is pseudo-effective, then there is an effective $\mathbb{Q}$-divisor D such that $K_X \sim_{\mathbb{Q}} D$.*

With the main result of [BDPP], we can reword Conjecture 3.3 to the following:

Conjecture 3.4. *Let X be a smooth projective non-uniruled variety. Then there is $m > 0$ such that $\dim H^0(X, \mathcal{O}_X(mK_X)) > 0$.*

Now we state the main result of this article.

Theorem 3.5 (cf. [H1, Theorem 1.4])**.** *Assume Conjecture 3.3 for all smooth projective varieties of dimension n. Then Conjecture 2.8 and Conjecture 3.1 holds for all log canonical pairs of dimension $\leq n$.*

This theorem is known under the assumption of the abundance conjecutre for log canonical pairs of dimension $\leq n - 1$ (see [DHP] and [G2]). This result is a generalization of Theorem 3.2 and results of [DHP] and [G2].

The next two lemmas are key results for the proof of Theorem 3.5.

Lemma 3.6 (cf. [H1, Lemma 3.2]). *Assume Conjecture 3.3 for all smooth projective varieties of dimension n. Then Conjecture 3.3 holds for all smooth projective varieties of dimension $\leq n$.*

Lemma 3.7 (cf. [B1, Corollary 1.7]). *Fix an integer $d > 0$, and assume Conjecture 2.8 for all log canonical pairs of dimension $\leq d-1$. Let (X, Δ) be a d-dimensional log canonical pair such that $K_X + \Delta \sim_{\mathbb{R}} D$ for an effective $\mathbb{R}$-divisor D. Then Conjecture 2.8 holds for (X, Δ).*

With these lemmas we prove Theorem 3.5 by induction on n. In this article we write only the proof of Lemma 3.6.

Proof of Lemma 3.6. Assume Conjecture 3.3 for all smooth projective varieties of dimension n. Pick any $d \leq n$ and a smooth projective variety X of dimension d such that K_X is pseudo-effective. Let W be the product of X and an $(n-d)$-dimensional abelian variety, and let $f : W \to X$ be the projection. Then $K_W = f^*K_X$ and K_W is pseudo-effective. From our assumption Conjecture 3.3 holds for W, and therefore Conjecture 3.3 holds for X. So we are done. $\square$

4. On the non-vanishing conjecture

In this section we introduce a recent result on the non-vanishing conjecture.

Theorem 4.1 (cf. [H2]). *Let (X, Δ) be a log canonical pair. Assume that there is an $\mathbb{R}$-divisor $C \geq 0$ such that (X, C) is a log canonical pair and $K_X + C \equiv 0$. Then Conjecture 3.1 holds for (X, Δ).*

The theorem is known when $\Delta = C$ (cf. [G1], [CKP], [K]). By using the proof of Theorem 4.1, we can also prove the following:

Theorem 4.2. *Let (X, Δ) be a log canonical pair. Assume that X is rationally connected. Then Conjecture 3.1 holds for (X, Δ).*

Finally we give a comment on the non-vanishing conjecture. Almost all of known results on the non-vanishing conjecture are the case when the underlying variety is uniruled, in other words, when the canonical divisor is not pseudo-effective. Little is known about Conjecture 3.3 or the non-vanishing conjecture for non-uniruled varieties (cf. [LP]). One of difficulties is that when X is not uniruled it is hard to construct good fibrations like Mori fiber spaces. Because of this difficulty we can not directly apply induction on the dimension of varieties. Analytic techniques may be useful to tackle Conjecture 3.3.

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