ON THE STRUCTURE OF LOCAL AND GLOBAL SINGULARITIES: SHOKUROV'S CONJECTURE

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Abstract. We survey some recent results on the structure of log canonical pairs (X, Δ) with $-(K_X + \Delta)$ nef. In particular, we motivate and explain a conjecture due to Shokurov concerning upper bounds for the number of components of the divisor Δ and we illustrate the proof of this conjecture given by Brown, M^c^Kernan, Zong, and the author.

Contents

In order to classify algebraic varieties from a birational viewpoint, it is unavoidable to deal with singularities. For example, this phenomenon can already be observed in dimension two when constructing the canonical model of a surface of general type: then ADE singularities make their appearance. In dimension three and higher, this is an even more common phenomenon, as singularities immediately appear when running the Minimall Model Program, via divisorial contractions and flips.

When studying the structure of algebraic varieties, it is desirable to find bounds for those numerical quantities that are naturally associated to the geometric structures under scrutiny. These bounds can then be used to show that certain structures are not attainable, or characterize those varieties that attain certain special values of such quantities – usually those values closest to the bounds.

In the context of the classification, actually, it is often more convenient to work with a slightly more general type of objects, namely, pairs (X, Δ), where X is an algebraic variety and Δ is a sum of prime divisors with

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coefficients in $(0, 1]$. Such pairs appear quite naturally: for example, given U a quasi-projective variety, taking X to be a smooth compactification with $\Delta = X \setminus U$ a simple normal crossing boundary at infinity – which exists, at least in characteristic 0, by Hironaka's resolution of singularities. Another motivation for working with pairs comes form adjunction theory: when X is a mildly singular hypersurface in a mildly singular variety Y, then often the classical adjunction formula $(K_Y + X)|_X = K_X$ fails to hold. One then needs a correction term in the form of an effective divisor, that is, the Adjunction formula looks like $(K_Y + X)|_X = K_X + \Delta$, for some $\Delta \geq 0$ on X.

When working with pairs (X, Δ), we are usually interested in understanding their singularities, that is, the singularities of the underlying variety X, those of the components of Δ and the interaction of these two. It is then natural to wonder what kind of restrictions different types of singularities impose on the structure of (X, Δ). For example, one may ask the following very basic set of questions.

Question 0.1. *Is it possible to bound the number of components of Δ in terms of the type of singularities that we impose?*
Is it possible to understand anything about the way components of Δ interact among themselves and with X just in terms of the type of singularities of the pair? For example, is it possible to describe the combinatorial structure of the components of Δ and their intersections (that is, its dual complex)?

These are fundamental questions in the study of birational geometry, since in order to answer those we need to understand both the local and the global structure of the singularities of a given pair.

It is not too hard to see that if we allow only mild singularities – of kawamata log terminal or log canonical type (klt and lc, in short) – then around a given closed point on X the sum of the coefficients of Δ cannot be greater than the dimension of X and when the bound is attained then Δ is a sum of smooth components intersecting as transversely as possible – up to passing to a quasi-étale cover, see Section 2.

In more generality, when we consider a projective variety X at large and not just a local situation setting it is not possible to find analogous bounds. This is already clear in dimension 1, as explained in Section 2. Nonetheless, if we are willing to bound the positivity of the divisor $K_X + \Delta$ then it is actually possible to find a positive answer to the first part of Question 0.1 and also to partially describe the structure of those components of Δ having coefficient one, in any dimension. For a more general study of the combinatorial structure of $\lfloor \Delta \rfloor$ in a large class of cases, the reader can consult [11].

The aim of this note is to explain what this positive answer actually is and what are the reasons behind it. The material explained here originates from the work in [3].

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Conventions. We work over a field of characteristic zero, which is algebraically closed, unless otherwise stated.

Let X be a proper normal variety. $\rho(X)$ is the rank of the Picard group of X. We denote the class group, the group of Weil divisors modulo linear equivalence, by $A_{n-1}(X)$.

We will follow the terminology from [13].

1. Local and global singularities

As we explained above, we want to describe some features of the structure of certain types of singularities both in a local and global framework. What we are actually going to do is using local information about the singularities of pairs to draw conclusions about the global structure of pairs.

As a first task, we will define these two competing frameworks.

Local Setting: $(x \in X, \Delta)$, where $x \in X$ is a (pointed) germ of a normal variety and $\Delta = \sum_i a_i D_i$ is a Weil divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier at x and (X, Δ) is log canonical at x.

Global Setting: (X, Δ) where X is a normal proper variety and $\Delta = \sum_i a_i D_i$ is a Weil divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and (X, Δ) is log canonical.

For the definition of log canonical (klt, dlt) singularities of pairs, the reader can consult [13, §2.3].

2. Classical bounds

We have seen in the introduction that one question property that we may want to answer is whether or not we can bound the number of components of log canonical divisors on a given variety X. An even simpler question is whether we can do that locally around a point $x \in X$. Given that when we work with log canonical pairs (X, Δ) the coefficients of Δ may vary in $(0, 1]$, then rather than counting the components of Δ it is more convenient to weight each component by its coefficient in Δ.

Definition 2.1. *Let $(x \in X, \Delta)$ be the germ of a log pair. A local decomposition of Δ is an expression of the form*

$$\sum a_i D_i \leq \Delta,$$

where $D_i \geq 0$ are $\mathbb{Z}$-Weil, $\mathbb{Q}$-Cartier divisors and $a_i \geq 0$, $1 \leq i \leq k$. The local complexity of this decomposition is $n - d$, where n is the dimension of X and d is the sum of $a_1, \dots, a_k$.

Using the notion of local complexity, it is then not too hard to prove by induction on the dimension, using Bertini and the adjunction formula, that the local version of Question 0.1 has an affermative answer. The following result is a generalized adaptation of [12, 18.22].

Proposition 2.2. [3] *Let $(x \in X, \Delta)$ be the germ of a log canonical pair where X has dimension n and let $\sum a_i D_i \leq \Delta$ be a local decomposition. Assume that K_X and $D_1, \dots, D_k$ are Cartier.*

If $\gamma = n - \sum a_i = n - d$ is the local complexity then

(1) $\gamma \geq 0$.

(2) *If $\gamma < 1$ then, possibly re-ordering $D_1, \dots, D_k$, there is an integer $m \geq n - \lfloor 2\gamma \rfloor \geq 0$ such that*

$$(X, D_1 + \cdots + D_m)$$

is log smooth, and

$$\lfloor \Delta \rfloor \leq D_1 + \cdots + D_m.$$

(3) *If $\gamma < \frac{3}{2}$ then either X is smooth at x or has a cA_l singularity at x.*

More generally, when we consider a proper variety X and not just a local setting, it is not possible to find analogous bounds. In fact, already starting in dimension one and considering a pair $(C, \sum_1^n p_i)$, where C is a smooth curve and the p_i are n distinct points on C, then the sum of the coefficients tends to infinity with n. Nonetheless, if we are willing to bound the positivity of the divisor $K_C + \sum_1^n p_i$ then it is actually possible to find an immediate bound. Looking at the Kodaira dimension of $K_C + \sum_1^n p_i$ we get that:

$$\kappa(K_C+\sum_1^n p_i) = \begin{cases} -\infty & \text{if } g(C)=0 \text{ and } n<2, \\ 0 & \text{if } g(C)=0 \text{ and } \ n=2, \text{ or } g(X)=1 \text{ and } \ n=0, \\ 1 & \text{if else,} \end{cases}$$

where $g(C)$ denotes the genus of C.

Hence, we should restrict ourselves to consider pairs (X, Δ) whose Kodaira dimension is non-positive, at the very least, if we wish to have any hope to prove that some kind of bound exists.

Let us first turn to a toy example, which naturally extends the above remark for the one-dimensional case. Let (X, Δ) be a pair given by a smooth Fano manifold of dimension n with $\rho(X)=1$ and a log canonical boundary such that $-(K_X+\Delta)$ is nef. Let H be a generator of the Picard group of X. Writing $\Delta = \sum_i a_i D_i$, we know that for every i, $D_i \sim b_i H$ for some positive integer b_i. Hence, $\Delta \sim (\sum_i a_i b_i) H$ and the following classical estimate, due to Kobayashi and Ochiai, immediately implies that $\sum_i a_i \leq \dim X + 1$.

Theorem 2.3. [9] *Let X be a smooth Fano manifold. Let H be an ample Cartier divisor on X such that $-K_X \sim lH$. Then $l \leq n+1$ and equality holds if and only if X is isomorphic to $\mathbb{P}^n$.*

3. Shokurov's conjecture

The above argument is not completely satisfactory from our point of view, as we have not used at all the fact that (X, Δ) is log canonical. In fact, the very same proof that we just explained would work for any divisor $\Delta \geq 0$ such that $-(K_X+\Delta)$ is nef, without restriction on the singularities of Δ.

Can we use the extra piece of information on the singularities of (X, Δ) to come up with a different proof that may perhaps provide a strategy to approach also the case of higher Picard rank?

It turns out that it is actually possible to do that.

Going back to our toy example, let $H' \sim lH$, $l > 0$ b be a very ample Cartier divisor and let $Y = C(X, H')$ be the affine cone over the embedding $X \to \mathbb{P}(H^0(X, H'))$ given by $|H'|$. Let us denote by $v \in Y$ the vertex of the cone. Using the affine cones $C(D_i, H')$ over the components D_i of Δ, we also have a natural choice of a divisor $\Gamma = \sum a_i C(D_i)$ passing through v. By blowing up v, we see immediately that (Y, Γ) is log canonical at v. Hence, by using Proposition 2.2 we reprove that $\sum a_i \leq = \dim Y = n+1$, since each component of Γ is $\mathbb{Q}$-Cartier, as $\rho(X)=1$.

Modifying this argument a little, it is not hard to see that following statement holds.

Theorem 3.1. [12, Cor. 18.23] *Let* $(X,\Delta), \Delta = \sum_i a_i D_i$ *be a* $\mathbb{Q}$*-factorial log canonical pair with* $\rho(X) = 1$*. Assume that* $-(K_X + \Delta)$ *is nef.*

Then $\sum a_i \leq \dim X + 1$.

Shokurov, cf. the discussion before Corollary 6.3 on [16, pg 3923], conjectured that a similar result should hold in any dimension. For simplicity, we state in the

Conjecture 3.2 (Shokurov)**.** *Let* $(X,\Delta), \Delta = \sum_i a_i D_i$ *be a* $\mathbb{Q}$*-factorial log canonical pair. Assume that* $-(K_X + \Delta)$ *is nef. Then*

$$\sum_i a_i \leq \dim X + \rho(X).$$

Moreover, if equality holds, then X *is a toric variety and there exists a choice of an equivariant torus embedding* $\mathbb{G}_m^{\dim X} \subset X$ *such that the torus-invariant divisor* $D = X \setminus \mathbb{G}_m^{\dim X}$ *satisfies* $\lfloor \Delta \rfloor \leq D$.

It is an easy computation to show that every pair given by a proper toric variety X and the sum of all torus-invariant divisors with coefficient 1 satisfies the equality in Shokurov's conjecture, see [4, §5.1].

It is also not hard to see that, in the conjecture, it is crucial that (X,Δ) is log canonical.

Example 3.3. Take $X = \mathbb{F}_n$, the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a curve E_∞ of self-intersection $-n$. Let $\Delta = 2E_\infty + \sum F_i$, where $F_1, \dots, F_{n+2}$ are $n+2$ distinct fibres of the bundle. Then $K_X + \Delta \sim 0$ and the sum of the coefficients of Δ is arbitrarily large and negative while $\rho(X) = 2$. Note that if one contracts E_∞ then the image of Δ is a boundary, i.e., all the coefficients are not greater than 1, and the complexity is $c = 1 - n$.

It is also impossible to relax the assumption on the nefness of $-(K_X + \Delta)$.

Example 3.4. With the same notation of Example 3.3, if we replace Δ by $E_\infty + \sum F_i$ in (3.3) then (X,Δ) is log canonical and $-(K_X + \Delta)$ is pseudo-effective but the sum of the coefficients is again $1 - n$.

Unfortunately, the argument we explained above using affine cones does not work when $\rho(X) > 1$. Here the obstruction comes from the following simple remark. When $\rho(X) > 1$, using the same notation as above, then the cone $C(D, H')$ over a $\mathbb{Q}$-Cartier prime Weil divisor D on X is $\mathbb{Q}$-Cartier at

$v \in C(X, H')$ if and only if $D \sim_{\mathbb{Q}} \lambda H'$, for a very ample Cartier divisor H' on X. Hence, when $\rho(X) > 1$, it may not be possible to use Proposition 2.2 to prove the bound on the coefficients of Δ as before.

For certain classes of varieties, though, we can still use a similar strategy. For this purpose, let us recall the notion of Cox ring.

Let X be a projective variety satisfying $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$. Let $L_1, \dots, L_r$ be a choice of Cartier divisors that provide a $\mathbb{Z}$-basis for $\mathrm{Pic}(X)_{\mathbb{Q}}$ and whose affine hull contains the pseudoeffective cone. Then the Cox ring of X is the ring

$$\bigoplus_{(a_1,\dots,a_r)\in\mathbb{N}^r} H^0(X, \mathcal{O}_X(a_1L_1 + \cdots + a_rL_r)).$$

The Cox ring of a variety with finitely generated class group, as originally defined in [7], is unique up to isomorphism but it ignores torsion in the class group. Subsequently [6] gave a refined definition which takes into account torsion in the class group. As we would like to allow torsion, in [3] we actually use the latter definition of the Cox ring.

A normal projective $\mathbb{Q}$-factorial variety X is a Mori dream space if and only if the Cox ring is finitely generated. While this is not the original definition of Mori dream space given in [7], it is an equivalent characterization and it is the most useful one for our purpose.

Under the assumptions of Conjecture 3.2, when X is a Mori dream space, instead of working with a cone we can work with the affine variety Y given by the spectrum of the Cox ring of X. The Cox ring is naturally graded by the class group, the group of Weil divisors modulo linear equivalence. This grading corresponds to the action on Y of an algebraic group H, the spectrum of the group algebra associated to the class group, which is the product of a Y contains a special point p – analogous to the vertex of the cone – corresponding to the maximal ideal

$$\bigoplus_{(a_1,\dots,a_r)\in(\mathbb{N}^r\setminus(0,\dots,0))} H^0(X, \mathcal{O}_X(a_1L_1 + \cdots + a_rL_r))$$

torus and a finite abelian group. We can recover X as the GIT quotient of Y by H. In the case when the class group is isomorphic to $\mathbb{Z}$ (so that, in particular, the Picard number is one), Y is a cone and H is a one dimensional torus, acting in the usual way on the lines of the cone, hence this construction automatically includes our original toy example. The dimension of Y is equal to $\dim X + \rho(X)$.

As in the case of a cone, there is a natural log pair (Y, Γ) associated to (X, Δ), every component of Γ passing through the point p: in fact, every Weil divisor D on X corresponds to a Cartier divisor D_Y on Y, via its tautological section. Hence if $\Delta = \sum a_i D_i$, then $\Gamma = \sum a_i D_{i,Y}$.

The pair (Y, Γ) is log canonical if and only if (X, Δ) is log canonical by [2], [5], and [8]. Hence, using Proposition 2.2, it is immediate to see that for $\Delta = \sum_i a_i D_i$, $\sum_i a_i \leq \dim Y = \dim X + \rho(X)$. Moreover, if $\sum a_i > \dim Y - 1$, then Y is smooth at p and that implies that the Cox ring of X is polynomial, since the Cox ring is graded by the class group. Thus, since a GIT quotient of an affine space by a torus action is toric, X is a toric variety, cf. [7, Cor. 2.10]. Again using Proposition 2.2(2), it is also not hard to see that there exists a choice of a toric-invariant divisor containing $\lfloor \Delta \rfloor$.

4. Main results

In order to prove Conjecture 3.2, we need to introduce a global version of the local complexity.

Definition 4.1. *Let X be a proper variety of dimension n and let (X, Δ) be a log pair. A decompositions of Δ is an expression of the form*

$$\sum_{i=1}^{k} a_i S_i \leq \Delta,$$

where $S_i \geq 0$ are $\mathbb{Z}$-divisors and $a_i \geq 0$, $1 \leq i \leq k$. The complexity of this decomposition is $n + r - d$, where r is the rank of the vector space spanned by $S_1, \ldots, S_k$ in the space of Weil divisors modulo algebraic equivalence and d is the sum of $a_1, \ldots, a_k$.

The complexity $c = c(X, \Delta)$ of (X, Δ) is the infimum of the complexity of any decomposition of Δ.

Using the complexity, we can prove a refined version of Shokurov's Conjecture.

Theorem 4.2. [3] *Let X be a proper variety of dimension n and let (X, Δ) be a log canonical pair such that $-(K_X + \Delta)$ is nef. Then,*

1. *the complexity is non-negative;*
2. *if the complexity is less than one then the components of Δ span the Néron-Severi group;*

(3) *if* $\sum a_i S_i$ *is a decomposition of complexity* $c < 1$ *and there is a divisor* D *such that* (X, D) *is a toric pair, where* $D \geq \lfloor \Delta \rfloor$ *and all but* $\lfloor 2c \rfloor$ *components of* D *are elements of the set* $\{ S_i \,|\, 1 \leq i \leq k \}$.

Let us explain the proof of the theorem, at least in the case when X is projective.

The first step is to replace (X, Δ) by a divisorially log terminal model (Y, Γ). This means that Y is projective, $\mathbb{Q}$-factorial and (Y, Γ) is divisorially log terminal. There is a birational contraction map $\pi \colon Y \to X$ and the only exceptional divisors have log discrepancy zero. Then we can take π to be a morphism – by a result of Hacon, see [10, Theorem 3.1] – and since $c(X, \Delta) = c(Y, \Gamma)$ and X is toric if and only if Y is, we may then assume that X is projective, $\mathbb{Q}$-factorial and (X, Δ) is divisorially log terminal.

The next step is to proceed assuming that X is a Mori dream space. In order to use the argument summarized at the end of the previous section, we just need to show that if $c(X, \Delta) < 1$ then the components S_i of a decomposition of low discrepancy span the class group of X. This property is proven by induction on the number of components, using a very nice construction of Brown involving vector bundles over X, cf. [2, §4]. Once we know that the S_i span the class group of X, then the argument from Section 2 shows that part (1) and (3) of Theorem 4.2 hold true.

To reduce to the case when X is a Mori dream space we have to pass to a different model Y such that $-(K_Y + \Gamma)$ is ample for some kawamata log terminal pair (Y, Γ). In fact, by [1, Cor. 1.3.2], this condition guarantees that Y is a Mori dream space. Note that in this case $K_Y + B + \Gamma$ is numerically trivial, where $B = -(K_Y + \Gamma)$ is ample. So we look for divisors $0 \leq \Delta_0 \leq \Delta$ and ample divisors A such that $K_X + A + \Delta_0$ has numerical dimension zero. In this case Y is a log terminal model of $(X, A + \Delta_0)$. If the numerical dimension is not zero then there is a non-trivial fibration $Y \to Z$. Not every component of Δ dominates Z, since otherwise the complexity of the general fibre would be negative and by induction we would obtain a contradiction. On the other hand, it is not hard to decrease the numerical dimension if there is a component of D which does not dominate. To finish off, we replace $A + \Delta_0$ by a convex linear combination of $A + \Delta_0$ and $M + \Delta$, where $M = -(K_X + \Delta)$, and cancel off common components of Δ_0 and exceptional divisors of $f \colon X \dashrightarrow Y$ so that the complexity of $(X, A + \Delta_0)$ is close to the complexity of (X, Δ) and f does not contract any components of Δ.

To terminate the proof of the theorem, we can just show that if Y is toric then so is X, by opportunely modifying the divisors Γ and Δ_0.

Toric varieties are special as they are rational. We can further give a rationality criterion using a slightly different version of the complexity.

Definition 4.3. *Let X be a proper variety of dimension n and let (X, Δ) be a log pair. The absolute complexity $\gamma = \gamma(X, \Delta)$ of (X, Δ) is $n+\rho-d$, where ρ is the rank of the group of Weil divisors modulo algebraic equivalence and d is the sum of the coefficients of Δ.*

When X is $\mathbb{Q}$-factorial then ρ is nothing but Picard number.

Theorem 4.4. [3] *Let X be a proper variety. Suppose that (X, Δ) is log canonical and $-(K_X + \Delta)$ is nef.*

If $\gamma(X, \Delta) < \frac{3}{2}$ then there is a proper finite morphism $Y \to X$ of degree at most two, which is étale outside a closed subset of codimension at least two, such that Y is rational.

In particular if $A_{n-1}(X)$ contains no 2-torsion then X is rational.

Note that most rationality criteria are used to establish irrationality. There are relatively few criteria to show rationality.

It is easy to see that to prove rationality we need to work with the absolute complexity rather than with the complexity and that also Theorem 4.2 is sharp:

Example 4.5. If $X = E$ is an elliptic curve and we consider the pair $(X, 0)$ then $K_X \sim 0$ and $c(X, 0) = 1$, but E is not rational. In this case $\gamma(X, 0) = 2$.

Moreover, when working with a non-algebraically closed field, it is easy to see that we need to allow an extension of degree two for rationality:

Example 4.6. Consider the conic X given by the equation $x^2+y^2+z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. It is a smooth conic over $\mathbb{R}$ without a real point. Let D be the sum of two conjugated $\mathbb{C}$-valued closed points. Then D is a divisor defined over $\mathbb{R}$ such that $K_X + D \sim 0$ and the absolute complexity $\gamma(X, D)$ (over $\mathbb{R}$) is one. On the other hand C is irrational but C becomes rational if we replace $\mathbb{R}$ with $\mathbb{C}$.

The condition on torsion in the class group is necessary and we give an example of this in [3, §7]: we exhibit log canonical pairs (X, Δ) of absolute complexity one such that X is irrational. The idea is to start with a conic bundle of relative Picard number two over $\mathbb{P}^1 \times \mathbb{P}^1$ and take a $\mathbb{Z}/2\mathbb{Z}$-quotient to achieve relative Picard number one. The key observation is that the discriminant curve, the locus of reducible fibres, makes no contribution in Kawamata's canonical bundle formula. Thus we can arrange for the discriminant curve to have arbitrarily large genus, in which case X is irrational by a result of Shokurov, [15]. In [14], Okada has shown that $\frac{3}{2}$ is a sharp bound for the absolute complexity in the theorem.

As for the proof of Theorem 4.4, the first part of the argument in the sketch of the proof of Theorem 4.2 applies unchanged to show that we can reduce to the case that X is a Mori dream space. Performing the construction carried out at the end of the Section 2 and applying Proposition 2.2(3) we can prove that the Cox ring of X is of the form

$$k[x_1, \ldots, x_n]/(f)$$

where f is a polynomial whose quadratic part has rank two.

The action of H on Y, the spectrum of the Cox ring of X, extends to $\mathbb{A}^n$. Hence, via the GIT quotient of Y by the action of H, X is birational to the image of Y which is a hypersurface in a toric variety. If, after possibly reordering the variables, x_1x_2 is a monomial with non-zero coefficient in f, then there is a one dimensional torus whose general orbit intersects X in a single point. Thus X is birational to an invariant divisor and X is rational. Otherwise after rescaling we may assume that the quadratic part of f has the form $x_1^2 + X_2^2$. If x and y have the same multidegree then we may change variable and reduce to the previous case. Otherwise there must be torsion in the class group and there is a cover $Z \to X$ of degree two.

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