Relative cohomology for the sections of a complex of fine sheaves

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This is a summary of an article in preparation under the same title. In the Kinosaki AG Symposium 2017, I talked about relative Dolbeault cohomology and its application to the Sato hyperfunction theory, whose contents can be seen in [10], [18] and [19]. The idea is to represent the relative cohomology with coefficients in the sheaf of holomorphic forms by the relative Dolbeault cohomology, which enables us to express the relative cohomology in a simple explicit way. In this article we introduce a general theory of Čech cohomology for a complex of fine sheaves, which naturally leads to the notion of the relative cohomology of the associated complex of sections. In the case the complex gives a resolution of a certain sheaf, the relative cohomology of sections of the complex is canonically isomorphic with the relative cohomology of the sheaf.

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1 Introduction

Let $\mathscr{S}$ be a sheaf of Abelian groups on a topological space X. For an open set X' in X, the relative cohomology $H^q(X, X'; \mathscr{S})$ is defined, taking a flabby resolution $0 \to \mathscr{S} \to \mathscr{F}^\bullet$, as the cohomology of the complex $\mathscr{F}^\bullet(X, X')$ of sections that vanish on X'. Theoretically it works well as the flabbiness implies the exactness of the sequence

$$0 \longrightarrow \mathscr{F}^\bullet(X, X') \xrightarrow{j^*} \mathscr{F}^\bullet(X) \xrightarrow{i^*} \mathscr{F}^\bullet(X') \longrightarrow 0. \tag{1.1}$$

In practice we would like to have a concrete way of representing the cohomology. One possibility is to use the relative Čech cohomology, however this can be rather complicated. In this paper we present a systematical way of representing the cohomology using fine resolutions. Traditionally this has been done successfully in the absolute case where $X' = \emptyset$, as culminated in the de Rham and Dolbeault theorems. In the relative case, a fine resolution is not directly applicable, as the morphism i^* in (1.1) fails to be surjective. However it is possible to remedy the situation by incorporating the Čech philosophy.

In general let $\mathscr{K}^\bullet$ be a complex of fine sheaves on X. Letting $V_0 = X'$ and V_1 a neighborhood of $X \smallsetminus X'$, consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of X and X'. We replace $\mathscr{K}^\bullet(X)$ by the complex $\mathscr{K}^\bullet(\mathcal{V})$ of triples $\xi = (\xi_0, \xi_1, \xi_{01})$ with ξ_0, ξ_1 and ξ_{01} sections of $\mathscr{K}^\bullet$ on V_0, V_1 and $V_{01} = V_0 \cap V_1$, respectively, the differential being defined

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in an appropriate manner. The morphism i^* corresponds to the assignment $\xi \mapsto \xi_0$ and $\mathscr{K}^\bullet(X, X')$ is replaced by the subcomplex $\mathscr{K}^\bullet(\mathcal{V}, \mathcal{V}')$ of triples ξ with $\xi_0 = 0$ so that a cochain is a pair (ξ_1, ξ_{01}). Then we have the exact sequence (cf. (5.2) below)

$$0 \longrightarrow \mathscr{K}^\bullet(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} \mathscr{K}^\bullet(\mathcal{V}) \xrightarrow{i^*} \mathscr{K}^\bullet(X') \longrightarrow 0.$$

It is shown that the cohomology of $\mathscr{K}^\bullet(\mathcal{V}, \mathcal{V}')$ is determined uniquely modulo canonical isomorphisms, independently of the choice of V_1. It is denoted by $H^q_{D_{\mathscr{K}}}(X, X')$ and called the *relative cohomology of the sections of* $\mathscr{K}^\bullet$. In the case $\mathscr{K}^\bullet$ gives a resolution of a sheaf $\mathscr{S}$, $H^q_{D_{\mathscr{K}}}(X, X')$ is canonically isomorphic with $H^q(X, X'; \mathscr{S})$ (Theorem 5.7). We also note that the relative cohomology $H^q_{D_{\mathscr{K}}}(X, X')$ goes well with derived functors (cf. Section 6).

We present the theory so that the isomorphisms are canonical and the correspondences in them are trackable.

2 Cohomology of a complex of sheaves

In the sequel, by a sheaf we mean a sheaf with at least the structure of Abelian groups. For a sheaf $\mathscr{S}$ on a topological space X and an open set V in X, we denote by $\mathscr{S}(V)$ the group of sections of $\mathscr{S}$ on V. Also, for an open subset V' of V, we denote by $\mathscr{S}(V, V')$ the subgroup of $\mathscr{S}(V)$ consisting of sections that vanish on V'.

A complex $\mathscr{K}$ of sheaves is a collection $(\mathscr{K}^q, d^q_{\mathscr{K}})_{q \in \mathbb{Z}}$, where $\mathscr{K}^q$ is a sheaf on X and $d^q_{\mathscr{K}} : \mathscr{K}^q \to \mathscr{K}^{q+1}$ a morphism, called a differential, with $d^{q+1}_{\mathscr{K}} \circ d^q_{\mathscr{K}} = 0$. We omit the subscript or superscript on d if there is no fear of confusion. The complex is also denoted by $(\mathscr{K}^\bullet, d)$ or $\mathscr{K}^\bullet$. We only consider the case $\mathscr{K}^q = 0$ for $q < 0$. We say that $\mathscr{K}$ is a resolution of $\mathscr{S}$ if there is a morphism $\iota : \mathscr{S} \to \mathscr{K}^0$ such that the following sequence is exact:

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{K}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{K}^q \xrightarrow{d} \cdots .$$

2.1 Cohomology via flabby resolutions

As reference cohomology theory, we adopt the one via flabby resolutions (cf. [5],[13]). Recall that a sheaf $\mathscr{F}$ is *flabby* if the restriction $\mathscr{F}(X) \to \mathscr{F}(V)$ is surjective for any open set V in X.

Let $\mathscr{S}$ be a sheaf on X. We may use any flabby resolution of $\mathscr{S}$ to define the cohomology of $\mathscr{S}$, however we take the Godement resolution, to fix the idea:

$$0 \longrightarrow \mathscr{S} \longrightarrow \mathscr{C}^0(\mathscr{S}) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{C}^q(\mathscr{S}) \xrightarrow{d} \cdots .$$

The q-th cohomology $H^q(X; \mathscr{S})$ of X with coefficients in $\mathscr{S}$ is the q-th cohomology of the complex $(\mathscr{C}^\bullet(\mathscr{S})(X), d)$.

More generally, let X' be an open set in X. We denote by $H^q(X, X'; \mathscr{S})$ the cohomology of $(\mathscr{C}^\bullet(\mathscr{S})(X, X'), d)$. Note that $H^q(X, \emptyset; \mathscr{S}) = H^q(X; \mathscr{S})$. Setting $S = X \smallsetminus X'$, it will also be denoted by $H^q_S(X; \mathscr{S})$. This cohomology in the first expression is referred to as the *relative cohomology* of $\mathscr{S}$ on (X, X') (cf. [14]) and in the second expression the *local cohomology* of $\mathscr{S}$ on X with support in S (cf. [8]).

Remark 2.1 The cohomology $H^q(X, X'; \mathscr{S})$ is determined uniquely modulo canonical isomorphisms, independently of the flabby resolution. Although this fact is well-known, we indicate an outline below in order to make the correspondence explicit (cf. Corollary 2.5).

2.2 Cohomology of a complex of sheaves

Let $\mathscr{K} = (\mathscr{K}^q, d_{\mathscr{K}})$ be a complex of sheaves on a topological space X. For each q, we take the Godement resolution $0 \to \mathscr{K}^q \to \mathscr{C}^\bullet(\mathscr{K}^q)$ whose differential is denoted by δ^{G}. The differential $d_{\mathscr{K}} : \mathscr{K}^q \to \mathscr{K}^{q+1}$ induces a morphism of complexes $\mathscr{C}^\bullet(\mathscr{K}^q) \to \mathscr{C}^\bullet(\mathscr{K}^{q+1})$, which is also denoted by $d_{\mathscr{K}}$. Thus we have the double complex $(\mathscr{C}^\bullet(\mathscr{K}^\bullet), \delta^{\mathrm{G}}, (-1)^\bullet d_{\mathscr{K}})$. We consider the associated single complex $(\mathscr{C}(\mathscr{K})^\bullet, D^{\mathrm{G}}_{\mathscr{K}})$, where

$$\mathscr{C}(\mathscr{K})^q = \bigoplus_{q_1+q_2=q} \mathscr{C}^{q_1}(\mathscr{K}^{q_2}), \qquad D^{\mathrm{G}}_{\mathscr{K}} = \delta^{\mathrm{G}} + (-1)^{q_1} d_{\mathscr{K}}.$$

Then there is an exact sequence $0 \to \mathscr{K}^\bullet \to \mathscr{C}(\mathscr{K})^\bullet$ of complexes.

Definition 2.2 Let X' be an open set in X. The cohomology $H^q(X, X'; \mathscr{K}^\bullet)$ of $\mathscr{K}^\bullet$ on (X, X') is the cohomology of $(\mathscr{C}(\mathscr{K})^\bullet(X, X'), D^{\mathrm{G}}_{\mathscr{K}})$.

If $X' = \emptyset$, we denote $H^q(X, X'; \mathscr{K}^\bullet)$ by $H^q(X; \mathscr{K}^\bullet)$. We have $H^0(X, X'; \mathscr{K}^\bullet) = \mathscr{S}(X, X')$, where $\mathscr{S}$ is the kernel of $d : \mathscr{K}^0 \to \mathscr{K}^1$.

In the above situation, we have a complex $(\mathscr{K}^\bullet(X, X'), d_{\mathscr{K}})$, whose cohomology is denoted by $H^q_{d_{\mathscr{K}}}(X, X')$. Let $\mathscr{H}^q(\mathscr{K}^\bullet)$ denote the q-th cohomology sheaf of the complex $\mathscr{K}^\bullet$. Considering the two spectral sequences associated with the double complex $\mathscr{C}^\bullet(\mathscr{K}^\bullet)(X, X')$, we have:

Proposition 2.3 **1.** *Suppose $H^{q_2}(X, X'; \mathscr{K}^{q_1}) = 0$ for $q_1 \geq 0$ and $q_2 \geq 1$. Then the inclusion $\mathscr{K}^q(X, X') \hookrightarrow \mathscr{C}^0(\mathscr{K}^q)(X, X') \subset \mathscr{C}(\mathscr{K})^q(X, X')$ induces an isomorphism*

$$H^q_{d_{\mathscr{K}}}(X, X') \xrightarrow{\sim} H^q(X, X'; \mathscr{K}^\bullet).$$

2. *Suppose $\mathscr{H}^{q_2}(\mathscr{K}^\bullet)) = 0$ for $q_2 \geq 1$. Let $\mathscr{S}$ denote the kernel of $d : \mathscr{K}^0 \to \mathscr{K}^1$. Then the inclusion $\mathscr{C}^q(\mathscr{S})(X, X') \hookrightarrow \mathscr{C}^q(\mathscr{K}^0)(X, X') \subset \mathscr{C}(\mathscr{K})^q(X, X')$ induces an isomorphism*

$$H^q(X, X'; \mathscr{S}) \xrightarrow{\sim} H^q(X, X'; \mathscr{K}^\bullet).$$

In particular, the hypothesis of Proposition 2.3. 2 is satisfied if $0 \to \mathscr{S} \to \mathscr{K}^\bullet$ is a resolution of $\mathscr{S}$. Thus we have:

Theorem 2.4 *For a resolution $0 \to \mathscr{S} \to \mathscr{K}^\bullet$ with $H^{q_2}(X, X'; \mathscr{K}^{q_1}) = 0$, for $q_1 \geq 0$ and $q_2 \geq 1$, there is a canonical isomorphism:*

$$H^q_{d_{\mathscr{K}}}(X, X') \simeq H^q(X, X'; \mathscr{S}).$$

Corollary 2.5 *For any flabby resolution $0 \to \mathscr{S} \to \mathscr{F}^\bullet$, there is a canonical isomorphism:*

$$H^q_{d_{\mathscr{F}}}(X, X') \simeq H^q(X, X'; \mathscr{S}).$$

Remark 2.6 1. The cohomology $H^q(X, X'; \mathscr{K}^\bullet)$ in Definition 2.2 is sometimes referred to as the hypercohomology of $\mathscr{K}^\bullet$.

2. We may explicitly describe the correspondence in each of the above isomorphisms. For example, in Theorem 2.4 we think of a cocycle s in $\mathscr{K}^q(X, X')$ and a cocycle γ in $\mathscr{C}^q(\mathscr{S})(X, X')$ as being cocycles in $\mathscr{C}(\mathscr{K})^q(X, X')$. The classes $[s]$ and $[\gamma]$ correspond in the above isomorphism, if and only if s and γ define the same class in $H^q(X, X'; \mathscr{K}^\bullet)$, i.e., there exists a $(q-1)$-cochain χ in $\mathscr{C}(\mathscr{K})^{q-1}(X, X')$ such that

$$s - \gamma = D^{\mathrm{G}}_{\mathscr{K}} \chi,$$

see the remark after Theorem 3.4 and Remark 3.8 below.

3 Čech cohomology of a complex of sheaves

3.1 Čech cohomology of a sheaf

We briefly recall the usual Čech theory in order to fix the notation and conventions.

Let $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ be an open covering of X. We set $W_{\alpha_0 \dots \alpha_q} = W_{\alpha_0} \cap \dots \cap W_{\alpha_q}$ and consider the direct product

$$C^q(\mathcal{W}; \mathscr{S}) = \prod_{(\alpha_0, \dots, \alpha_q) \in I^{q+1}} \mathscr{S}(W_{\alpha_0 \dots \alpha_q}).$$

The q-th Čech cohomology $H^q(\mathcal{W}; \mathscr{S})$ of $\mathscr{S}$ on $\mathcal{W}$ is the q-th cohomology of the complex $(C^\bullet(\mathcal{W}; \mathscr{S}), \delta)$ with $\delta : C^q(\mathcal{W}; \mathscr{S}) \to C^{q+1}(\mathcal{W}; \mathscr{S})$ defined by

$$(\delta\sigma)_{\alpha_0 \dots \alpha_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu \sigma_{\alpha_0 \dots \widehat{\alpha_\nu} \dots \alpha_{q+1}}.$$

Let X' be an open set in X. Let $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ be a covering of X such that $\mathcal{W}' = \{W_\alpha\}_{\alpha \in I'}$ is a covering of X' for some $I' \subset I$. We set

$$C^q(\mathcal{W}, \mathcal{W}'; \mathscr{S}) = \{\, \sigma \in C^q(\mathcal{W}; \mathscr{S}) \mid \sigma_{\alpha_0 \dots \alpha_q} = 0 \ \text{ if } \ \alpha_0, \dots, \alpha_q \in I' \,\}$$

The operator δ restricts to $C^q(\mathcal{W}, \mathcal{W}'; \mathscr{S}) \to C^{q+1}(\mathcal{W}, \mathcal{W}'; \mathscr{S})$. The q-th Čech cohomology $H^q(\mathcal{W}, \mathcal{W}'; \mathscr{S})$ of $\mathscr{S}$ on $(\mathcal{W}, \mathcal{W}')$ is the q-th cohomology of $(C^\bullet(\mathcal{W}, \mathcal{W}'; \mathscr{S}), \delta)$.

3.2 Čech cohomology of a complex of sheaves

Let $(\mathscr{K}^\bullet, d_{\mathscr{K}})$ be a complex of sheaves on a topological space X and $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ an open covering of X. Also let X' be an open set in X and $\mathcal{W}'$ a subcovering of $\mathcal{W}$ as before. Then we have a double complex $(C^\bullet(\mathcal{W}, \mathcal{W}'; \mathscr{K}^\bullet), \delta, (-1)^\bullet d_{\mathscr{K}})$. We consider the associated single complex $(\mathscr{K}^\bullet(\mathcal{W}, \mathcal{W}'), D_{\mathscr{K}})$. Thus

$$\mathscr{K}^q(\mathcal{W}, \mathcal{W}') = \bigoplus_{q_1+q_2=q} C^{q_1}(\mathcal{W}, \mathcal{W}'; \mathscr{K}^{q_2}), \qquad D_{\mathscr{K}} = \delta + (-1)^{q_1} d_{\mathscr{K}}.$$

Definition 3.1 The Čech cohomology $H^q(\mathcal{W},\mathcal{W}';\mathscr{K}^\bullet)$ of $\mathscr{K}^\bullet$ on $(\mathcal{W},\mathcal{W}')$ is the cohomology of $(\mathscr{K}^\bullet(\mathcal{W},\mathcal{W}'),D_{\mathscr{K}})$.

It will also be denoted by $H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W})$. In the case $X'=\emptyset$, we take $\emptyset$ as I' and denote $H^q(\mathcal{W},\mathcal{W}';\mathscr{K}^\bullet)$ by $H^q(\mathcal{W};\mathscr{K}^\bullet)$ or by $H^q_{D_{\mathscr{K}}}(\mathcal{W})$.

We have $H^0_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}')=\mathscr{S}(X,X')$, where $\mathscr{S}$ is the kernel of $d_{\mathscr{K}}:\mathscr{K}^0\to\mathscr{K}^1$.

For a triple $(\mathcal{W},\mathcal{W}',\mathcal{W}'')$, we have the exact sequence

$$0\longrightarrow\mathscr{K}^\bullet(\mathcal{W},\mathcal{W}')\longrightarrow\mathscr{K}^\bullet(\mathcal{W},\mathcal{W}'')\longrightarrow\mathscr{K}^\bullet(\mathcal{W}',\mathcal{W}'')\longrightarrow 0$$

yielding an exact sequence

$$\cdots\longrightarrow H^{q-1}_{D_{\mathscr{K}}}(\mathcal{W}',\mathcal{W}'')\xrightarrow{\delta^*}H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}')\xrightarrow{j^*}H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}'')\xrightarrow{i^*}H^q_{D_{\mathscr{K}}}(\mathcal{W}',\mathcal{W}'')\longrightarrow\cdots. \tag{3.2}$$

Considering the two spectral sequences associated with $C^\bullet(\mathcal{W},\mathcal{W}';\mathscr{K}^\bullet)$, we have:

Proposition 3.3 **1.** *Suppose* $H^{q_2}(\mathcal{W},\mathcal{W}';\mathscr{K}^{q_1})=0$ *for* $q_1\ge 0$ *and* $q_2\ge 1$. *Then the inclusion* $\mathscr{K}^q(X,X')\hookrightarrow C^0(\mathcal{W},\mathcal{W}';\mathscr{K}^q)\subset\mathscr{K}^q(\mathcal{W},\mathcal{W}')$ *induces an isomorphism*

$$H^q_{d_{\mathscr{K}}}(X,X')\xrightarrow{\sim}H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}').$$

2. *Suppose* $H^{q_2}(\mathscr{K}^\bullet(W_{\alpha_0\dots\alpha_{q_1}}))=0$ *for* $q_1\ge 0$ *and* $q_2\ge 1$. *Let* $\mathscr{S}$ *denote the kernel of* $d_{\mathscr{K}}:\mathscr{K}^0\to\mathscr{K}^1$. *Then the inclusion* $C^q(\mathcal{W},\mathcal{W}';\mathscr{S})\hookrightarrow C^q(\mathcal{W},\mathcal{W}';\mathscr{K}^0)\subset\mathscr{K}^q(\mathcal{W},\mathcal{W}')$ *induces an isomorphism*

$$H^q(\mathcal{W},\mathcal{W}';\mathscr{S})\xrightarrow{\sim}H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}').$$

From Proposition 3.3 we have:

Theorem 3.4 *Let* $(\mathscr{K}^\bullet,d_{\mathscr{K}})$ *be a complex of sheaves on* X *and let* $\mathscr{S}$ *be the kernel of* $d_{\mathscr{K}}:\mathscr{K}^0\to\mathscr{K}^1$. *Suppose* $H^{q_2}(\mathcal{W},\mathcal{W}';\mathscr{K}^{q_1})=0$ *and* $H^{q_2}(\mathscr{K}^\bullet(W_{\alpha_0\dots\alpha_{q_1}}))=0$, *for* $q_1\ge 0$ *and* $q_2\ge 1$. *Then there is a canonical isomorphism*:

$$H^q_{d_{\mathscr{K}}}(X,X')\simeq H^q(\mathcal{W},\mathcal{W}';\mathscr{S}).$$

In the above, we think of a cocycle s in $\mathscr{K}^q(X,X')$ and a cocycle σ in $C^q(\mathcal{W},\mathcal{W}';\mathscr{S})$ as cocycles in $\mathscr{K}^q(\mathcal{W},\mathcal{W}')$. The classes $[s]$ and $[\sigma]$ correspond in the above isomorphism, if and only if s and σ define the same class in $H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}')$, i.e., there exists a $(q-1)$-cochain χ in $\mathscr{K}^{q-1}(\mathcal{W},\mathcal{W}')$ such that

$$s-\sigma=D\chi.$$

The above correspondence may be illustrated in the following diagram. For simplicity, we consider the absolute case $(\mathcal{W}'=\emptyset)$, the relative case being similar.

$$
\begin{array}{ccccccccc}
 & & & & & & & & S \\
 & & & & & & & & \cap \\
 & & \mathscr{K}^0(X) & \xrightarrow{d^0} & \mathscr{K}^1(X) & \xrightarrow{d^1} \cdots & \xrightarrow{d^{q-1}} & & \mathscr{K}^q(X) \xrightarrow{d^q} \cdots \\
 & & \big\downarrow & & \big\downarrow & & & & \big\downarrow \\
C^0(\mathcal{W};\mathscr{S}) & \hookrightarrow & C^0(\mathcal{W};\mathscr{K}^0) & \xrightarrow{d^0} & C^0(\mathcal{W};\mathscr{K}^1) & \xrightarrow{d^1} \cdots & \xrightarrow{d^{q-1}} & & C^0(\mathcal{W};\mathscr{K}^q) \xrightarrow{d^q} \cdots \\
\big\downarrow \delta^0 & & \big\downarrow \delta^0 & & \big\downarrow \delta^0 & & & & \big\downarrow \delta^0 \\
C^1(\mathcal{W};\mathscr{S}) & \hookrightarrow & C^1(\mathcal{W};\mathscr{K}^0) & \xrightarrow{-d^0} & C^1(\mathcal{W};\mathscr{K}^1) & \xrightarrow{-d^1} \cdots & \xrightarrow{-d^{q-1}} & & C^1(\mathcal{W};\mathscr{K}^q) \xrightarrow{-d^q} \cdots \\
\big\downarrow \delta^1 & & \big\downarrow \delta^1 & & \big\downarrow \delta^1 & & & & \big\downarrow \delta^1 \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\big\downarrow \delta^{q-1} & & \big\downarrow \delta^{q-1} & & \big\downarrow \delta^{q-1} & & & & \\
\sigma \in C^q(\mathcal{W};\mathscr{S}) & \hookrightarrow & C^q(\mathcal{W};\mathscr{K}^0) & \xrightarrow{(-1)^q d^0} & C^q(\mathcal{W};\mathscr{K}^1) & \xrightarrow{(-1)^q d^1} \cdots & & & \\
\big\downarrow \delta^q & & \big\downarrow \delta^q & & \big\downarrow \delta^q & & & & \\
\vdots & & \vdots & & \vdots & & & &
\end{array}
\quad . \tag{3.5}
$$

If we let $\mathscr{K}^\bullet = \mathscr{C}^\bullet(\mathscr{S})$, the Godement resolution of $\mathscr{S}$, in Theorem 3.4, noting that $H^{q_2}(\mathscr{C}^\bullet(\mathscr{S})(W_{\alpha_0\ldots\alpha_{q_1}})) = H^{q_2}(W_{\alpha_0\ldots\alpha_{q_1}}, \mathscr{S})$ we have:

Corollary 3.6 (Relative Leray theorem) *If* $H^{q_2}(W_{\alpha_0\ldots\alpha_{q_1}}, \mathscr{S}) = 0$ *for* $q_1 \geq 0$ *and* $q_2 \geq 1$*, there is a canonical isomorphism*

$$H^q(X, X'; \mathscr{S}) \simeq H^q(\mathcal{W}, \mathcal{W}'; \mathscr{S}).$$

Remark 3.7 We may use only "alternating cochains" in the above construction and the resulting cohomology is canonically isomorphic with the one defined above, as in the usual Čech theory.

Some special cases: **I.** In the case $\mathcal{W} = \{X\}$, we have $(\mathscr{K}^\bullet(\mathcal{W}), D_{\mathscr{K}}) = (\mathscr{K}^\bullet(X), d_{\mathscr{K}})$ so that

$$H^q_{D_{\mathscr{K}}}(\mathcal{W}) = H^q_{d_{\mathscr{K}}}(X).$$

II. In the case $\mathcal{W}$ consists of two open sets W_0 and W_1, we may write (cf. Remark 3.7)

$$\mathscr{K}^q(\mathcal{W}) = C^0(\mathcal{W}, \mathscr{K}^q) \oplus C^1(\mathcal{W}, \mathscr{K}^{q-1}) = \mathscr{K}^q(W_0) \oplus \mathscr{K}^q(W_1) \oplus \mathscr{K}^{q-1}(W_{01}).$$

Thus a cochain $\xi \in \mathscr{K}^q(\mathcal{W})$ is expressed as a triple $\xi = (\xi_0, \xi_1, \xi_{01})$ and the differential

$$D : \mathscr{K}^q(\mathcal{W}) \to \mathscr{K}^{q+1}(\mathcal{W}) \quad \text{is given by} \quad D(\xi_0, \xi_1, \xi_{01}) = (d\xi_0, d\xi_1, \xi_1 - \xi_0 - d\xi_{01}).$$

If $\mathcal{W}' = \{W_0\}$,

$$\mathscr{K}^q(\mathcal{W}, \mathcal{W}') = \{\, \xi \in \mathscr{K}^q(\mathcal{W}) \mid \xi_0 = 0 \,\} = \mathscr{K}^q(W_1) \oplus \mathscr{K}^{q-1}(W_{01}).$$

Thus a cochain $\xi \in \mathscr{K}^q(\mathcal{W}, \mathcal{W}')$ is expressed as a pair $\xi = (\xi_1, \xi_{01})$ and the differential

$$D : \mathscr{K}^q(\mathcal{W}, \mathcal{W}') \to \mathscr{K}^{q+1}(\mathcal{W}, \mathcal{W}') \quad \text{is given by} \quad D(\xi_1, \xi_{01}) = (d\xi_1, \xi_1 - d\xi_{01}).$$

The q-th cohomology of $(\mathscr{K}^\bullet(\mathcal{W},\mathcal{W}'),D)$ is $H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}')$.

If we set $\mathcal{W}''=\emptyset$, then $H^{q-1}_{D_{\mathscr{K}}}(\mathcal{W}',\mathcal{W}'')=H^{q-1}_{D_{\mathscr{K}}}(\mathcal{W}')=H^{q-1}_{d_{\mathscr{K}}}(W_0)$ and the connecting morphism δ^* in (3.2) assigns to the class of a $(q-1)$-cocycle ξ_0 on W_0 the class of $(0,-\xi_0)$ (restricted to W_1) in $H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}')$.

We discuss this case more in detail in the subsequent sections.

Remark 3.8 It is possible to establish an isomorphism as in Theorem 3.4 without introducing the Čech cohomology of a complex of sheaves, using the so-called Weil lemma instead (cf. [11, Lemma 5.2.7]). The latter amounts to performing the "ladder diagram chasing" in (3.5) with all the horizontal differentials with positive sign to find a correspondence. However this correspondence is different from the one in Theorem 3.4, the difference being the sign of $(-1)^{\frac{q(q+1)}{2}}$.

Similar remarks as above apply to the isomorphism of Theorem 2.4, with $\mathscr{K}^\bullet(\mathcal{W},\mathcal{W}')$ replaced by $\mathscr{C}(\mathscr{K})^\bullet(X,X')$.

4 Čech cohomology of a complex of fine sheaves

In this section we let X be a paracompact topological space and consider only locally finite coverings.

Recall that if $\mathscr{F}$ is a fine sheaf on X, then $H^q(X;\mathscr{F})=0$ for $q\geq 1$. Thus we see that $H^q(X,X';\mathscr{F})=0$ for $q\geq 2$. However $H^1(X,X';\mathscr{F})\neq 0$ in general. In fact we have the exact sequence

$$0\longrightarrow \mathscr{F}(X,X')\xrightarrow{j^*}\mathscr{F}(X)\xrightarrow{i^*}\mathscr{F}(X')\xrightarrow{\delta^*}H^1(X,X';\mathscr{F})\longrightarrow 0$$

and $H^1(X,X';\mathscr{F})$ is the obstruction to i^* being surjective.

We present various canonical isomorphisms that follow from the arguments in the previous sections.

First from Theorem 2.4 with $X'=\emptyset$, we have:

Theorem 4.1 (de Rham type theorem) *For a fine resolution* $0\to\mathscr{S}\to\mathscr{K}^\bullet$ *of* $\mathscr{S}$*, there is a canonical isomorphism:*

$$H^q_{d_{\mathscr{K}}}(X)\simeq H^q(X;\mathscr{S}).$$

Let $\mathscr{K}^\bullet$ be a complex of fine sheaves.

Definition 4.2 We say that a covering $\mathcal{W}=\{W_\alpha\}$ of X is *good* for $\mathscr{K}^\bullet$ if the hypothesis of Proposition 3.3. 2 holds, i.e., $H^{q_2}(\mathscr{K}^\bullet(W_{\alpha_0\dots\alpha_{q_1}}))=0$ for $q_1\geq 0$ and $q_2\geq 1$.

Let $\mathscr{S}$ be the kernel of $d_{\mathscr{K}}:\mathscr{K}^0\to\mathscr{K}^1$.

Theorem 4.3 *We have the following canonical isomorphisms:*

1. *For any covering* $\mathcal{W}$*,*

$$H^q_{d_{\mathscr{K}}}(X)\xrightarrow{\sim}H^q_{D_{\mathscr{K}}}(\mathcal{W}).$$

2. *For a good covering* $\mathcal{W}$ *for* $\mathscr{K}^\bullet$*,*

$$H^q_{D_{\mathscr{K}}}(\mathcal{W},\mathcal{W}')\xleftarrow{\sim}H^q(\mathcal{W},\mathcal{W}';\mathscr{S}).$$

3. *If $\mathcal{W}$ is good for $\mathscr{K}^\bullet$ and if $0 \to \mathscr{S} \to \mathscr{K}^\bullet$ is a resolution,*

$$H^q(\mathcal{W}, \mathcal{W}'; \mathscr{S}) \simeq H^q(X, X'; \mathscr{S}).$$

In the subsequent section, we introduce the missing piece, i.e., a relative version of the cohomology $H^q_{d_\mathscr{K}}(X)$ which is canonically isomorphic with $H^q(X, X'; \mathscr{S})$.

We finish this section by considering the following special case.

Case of coverings with two open sets: In the case $\mathcal{W} = \{W_0, W_1\}$,

$$\mathscr{K}^q(\mathcal{W}) = \mathscr{K}^q(W_0) \oplus \mathscr{K}^q(W_1) \oplus \mathscr{K}^{q-1}(W_{01}).$$

and the inclusion $\mathscr{K}^q(X) \hookrightarrow C^0(\mathcal{W}; \mathscr{K}^q) \subset \mathscr{K}^q(\mathcal{W})$ is given by $s \mapsto (s|_{W_0}, s|_{W_1}, 0)$. It induces the isomorphism in Theorem 4.3.1; $H^q_{d_\mathscr{K}}(X) \xrightarrow{\sim} H^q_{D_\mathscr{K}}(\mathcal{W})$.

Proposition 4.4 *The inverse of the above isomorphism is given by assigning to the class of $\xi = (\xi_0, \xi_1, \xi_{01})$ the class of s given by $\xi_0 + d(\rho_1 \xi_{01})$ on W_0 and by $\xi_1 - d(\rho_0 \xi_{01})$ on W_1.*

Note that the two sections above coincide on W_{01} by the cocycle condition.

5 Relative cohomology for the sections of a complex

Let $\mathscr{K}^\bullet$ be a complex of fine sheaves on a paracompact space X and X' an open set in X. Letting $V_0 = X'$ and V_1 a neighborhood of the closed set $S = X \smallsetminus X'$, consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of X and X'. We consider the cohomology $H^q_{D_\mathscr{K}}(\mathcal{V}, \mathcal{V}')$, which is the cohomology of the complex $(\mathscr{K}^\bullet(\mathcal{V}, \mathcal{V}'), D_\mathscr{K})$, where

$$\mathscr{K}^q(\mathcal{V}, \mathcal{V}') = \mathscr{K}^q(V_1) \oplus \mathscr{K}^{q-1}(V_{01}), \qquad V_{01} = V_0 \cap V_1, \tag{5.1}$$

and $D : \mathscr{K}^q(\mathcal{V}, \mathcal{V}') \to \mathscr{K}^{q+1}(\mathcal{V}, \mathcal{V}')$ is given by $D(\xi_1, \xi_{01}) = (d\xi_1, \xi_1 - d\xi_{01})$. Noting that $\mathscr{K}^q(\{V_0\}) = \mathscr{K}^q(X')$, we have the exact sequence

$$0 \longrightarrow \mathscr{K}^\bullet(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} \mathscr{K}^\bullet(\mathcal{V}) \xrightarrow{i^*} \mathscr{K}^\bullet(X') \longrightarrow 0, \tag{5.2}$$

where i^* assigns ξ_0 to $\xi = (\xi_0, \xi_1, \xi_{01})$. This gives rise to the exact sequence (cf. (3.2))

$$\cdots \longrightarrow H^{q-1}_{d_\mathscr{K}}(X') \xrightarrow{\delta^*} H^q_{D_\mathscr{K}}(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} H^q_{D_\mathscr{K}}(\mathcal{V}) \xrightarrow{i^*} H^q_{d_\mathscr{K}}(X') \longrightarrow \cdots. \tag{5.3}$$

By Theorem 4.3.1, we have a canonical isomorphism $H^q_{D_\mathscr{K}}(\mathcal{V}) \simeq H^q_{d_\mathscr{K}}(X)$. In the above, δ^* assigns to the class of θ the class of $(0, -\theta)$, j^* assigns to the class of (ξ_1, ξ_{01}) the class of $(0, \xi_1, \xi_{01})$ or the class of $\xi_1 - d(\rho_0 \xi_{01})$ (cf. Proposition 4.4). From (5.3), we have:

Proposition 5.4 *The cohomology $H^q_{D_\mathscr{K}}(\mathcal{V}, \mathcal{V}')$ is uniquely determined modulo canonical isomorphisms, independently of the choice of V_1.*

In view of the above proposition, we introduce the following:

Definition 5.5 We denote $H^q_{D_{\mathscr{K}}}(\mathcal{V}, \mathcal{V}')$ by $H^q_{D_{\mathscr{K}}}(X, X')$ and call it the *relative cohomology for the sections* of $\mathscr{K}^\bullet$ on (X, X').

In the case $X' = \emptyset$, it coincides with $H^q_{d_{\mathscr{K}}}(X)$.

Proposition 5.6 (Excision) *Let S be a closed set in X. Then, for any open set V in X containing S, there is a canonical isomorphism*

$$H^q_{D_{\mathscr{K}}}(X, X \smallsetminus S) \xrightarrow{\sim} H^q_{D_{\mathscr{K}}}(V, V \smallsetminus S).$$

In the case $\mathscr{K}^\bullet$ gives a resolution of a sheaf $\mathscr{S}$, the cohomology $H^q_{D_{\mathscr{K}}}(X, X')$ shares all the properties with the relative cohomology $H^q(X, X'; \mathscr{S})$. In fact we have:

Theorem 5.7 (Relative de Rham type theorem) *Suppose $0 \to \mathscr{S} \to \mathscr{K}^\bullet$ is a fine resolution. Then there is a canonical isomorphism:*

$$H^q_{D_{\mathscr{K}}}(X, X') \simeq H^q(X, X'; \mathscr{S}).$$

6 Relation with derived functors

After a brief review of basics on complexes, we introduce the notion of co-mapping cone. We then see that the complex introduced in Section 5 is given as a co-mapping cone, which leads to a statement of the isomorphism of Theorem 5.7 in terms of derived functors.

For derived categories and functors we refer to [12] and the literatures therein.

6.1 Category of complexes

Let $\mathcal{C}$ be an additive category. A complex K in $\mathcal{C}$ is a collection $(K^q, d^q_K)_{q\in\mathbb{Z}}$, where K^q is an object in $\mathcal{C}$ and $d^q_K : K^q \to K^{q+1}$ is a morphism with $d^{q+1} \circ d^q = 0$. Let $\mathsf{C}(\mathcal{C})$ denote the additive category of complexes in $\mathcal{C}$, where a morphism $f : K \to L$ is a collection (f^q) of morphisms $f^q : K^q \to L^q$ such that $d^q_L \circ f^q = f^{q+1} \circ d^q_K$. For a complex K and an integer k, we denote by $K[k]$ the complex with $K[k]^q = K^{k+q}$ and $d^q_{K[k]} = (-1)^k d^{k+q}_K$. Considering an object K in $\mathcal{C}$ as the complex given by $K^0 = K$, $K^q = 0$ for $q \neq 0$ and $d^q = 0$, we may think of $\mathcal{C}$ as a subcategory of $\mathsf{C}(\mathcal{C})$. Identifying two morphisms in $\mathsf{C}(\mathcal{C})$ that are “homotopic”, we have an additive category $\mathsf{K}(\mathcal{C})$.

Suppose $\mathcal{C}$ is an Abelian category. For a complex K in $\mathcal{C}$, its q-th cohomology is defined by

$$H^q(K) = \operatorname{Ker} d^q_K / \operatorname{Im} d^{q-1}_K.$$

Then it induces additive functors $H^q : \mathsf{C}(\mathcal{C}) \to \mathcal{C}$ and $H^q : \mathsf{K}(\mathcal{C}) \to \mathcal{C}$.

Proposition 6.1 *Let $0 \to J \xrightarrow{e} K \xrightarrow{f} L \to 0$ be an exact sequence in $\mathsf{C}(\mathcal{C})$. Then there exists an exact sequence*

$$\cdots \longrightarrow H^q(J) \xrightarrow{e} H^q(K) \xrightarrow{f} H^q(L) \xrightarrow{\delta} H^{q+1}(J) \longrightarrow \cdots,$$

where, e and f denotes $H^q(e)$ and $H^q(f)$, respectively, and δ assigns to the class of $c \in L^q$, $d(c) = 0$, the class of $a \in J^{q+1}$ such that $e(a) = d(b)$ for some $b \in K^q$ with $f(b) = c$.

6.2 Co-mapping cone

Let $\mathcal{C}$ be an additive category. For a morphism $f : K \to L$ of $\mathsf{C}(\mathcal{C})$, we define a complex $M^*(f)$ called the *co-mapping cone* of f. We set

$$M^*(f)^q = K^q \oplus L^{q-1}$$

and define the differential $d : M^*(f)^q = K^q \oplus L^{q-1} \to M^*(f)^{q+1} = K^{q+1} \oplus L^q$ by

$$d(x, y) = (dx, fx - dy).$$

We define morphisms $\alpha^* = \alpha^*(f) : M^*(f) \to K$ and $\beta^* = \beta^*(f) : L[-1] \to M^*(f)$ by

$$\begin{aligned} &\alpha^* : M^*(f)^q = K^q \oplus L^{q-1} \longrightarrow K^q, \qquad (x, y) \mapsto x \\ &\beta^* : L[-1]^q = L^{q-1} \longrightarrow M^*(f)^q = K^q \oplus L^{q-1}, \qquad y \mapsto (0, -y). \end{aligned}$$

Then we have a sequence of morphisms

$$L[-1] \xrightarrow{\beta^*} M^*(f) \xrightarrow{\alpha^*} K \xrightarrow{f} L.$$

We have $\alpha^* \circ \beta^* = 0$ in $\mathsf{C}(\mathcal{C})$. Moreover, we may prove that $\beta^* \circ f[-1]$ and $f \circ \alpha^*$ are homotopic to 0 so that $\beta^* \circ f[-1]$=0 and $f \circ \alpha^* = 0$ in $\mathsf{K}(\mathcal{C})$.

A *co-triangle* in $\mathsf{K}(\mathcal{C})$ is a sequence of morphisms

$$L[-1] \longrightarrow J \longrightarrow K \longrightarrow L.$$

The co-triangle is *distinguished* if it is isomorphic to

$$L'[-1] \xrightarrow{\beta^*} M^*(f) \xrightarrow{\alpha^*} K' \xrightarrow{f} L'$$

for some f in $\mathsf{C}(\mathcal{C})$.

Proposition 6.2 *Let $\mathcal{C}$ be an Abelian category. Then, for any distinguished cotriangle $L[-1] \to J \to K \to L$ in $\mathsf{K}(\mathcal{C})$, there is an exact sequence*

$$\cdots \longrightarrow H^{q-1}(L) \longrightarrow H^q(J) \longrightarrow H^q(K) \longrightarrow H^q(L) \longrightarrow \cdots.$$

Remark 6.3 The co-mapping cone defined above is dual to the mapping cone as defined in [12] in the following sense. Namely, while the mapping cone is a notion extracted from the complex of singular chains of the mapping cone of a continuous map of topological spaces, the co-mapping cone is the one corresponding to the complex of singular cochains. In this context, we may also think of a cotriangle as a notion dual to a triangle.

6.3 Derived categories and derived functors

Let $\mathcal{C}$ be an Abelian category. A morphism $f : K \to L$ in $\mathsf{K}(\mathcal{C})$ is a *quasi-isomorphism*, *qis* for short, if the induced morphism $H^q(K) \to H^q(L)$ is an isomorphism for all q. The derived category $\mathsf{D}(\mathcal{C})$ is the category obtained from $\mathsf{K}(\mathcal{C})$ by regarding a qis as an isomorphism. We have the functors

$$[k] : \mathsf{D}(\mathcal{C}) \longrightarrow \mathsf{D}(\mathcal{C}) \qquad \text{and} \qquad H^q : \mathsf{D}(\mathcal{C}) \longrightarrow \mathcal{C}.$$

Proposition 6.4 *Let $\mathcal{C}$ be an Abelian category and*

$$0 \longrightarrow J \xrightarrow{e} K \xrightarrow{f} L \longrightarrow 0 \tag{6.5}$$

an exact sequence in $\mathsf{C}(\mathcal{C})$. Let $M^*(f)$ be the co-mapping cone of f and let

$$\varphi^q : J^q \longrightarrow M^*(f)^q = K^q \oplus L^{q-1} \qquad \text{be defined by } \ z \mapsto (e(z), 0).$$

Then the following diagram is commutative and φ is a qis :

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & J & \xrightarrow{e} & K & \xrightarrow{f} & L \longrightarrow 0 \\ & & & & \wr\downarrow\varphi & \nearrow_{\alpha^*(f)} & & & \\ L[-1] & \xrightarrow{\beta^*(f)} & & & M^*(f). & & & & \end{array}$$

In the above situation, the distinguished cotriangle

$$L[-1] \xrightarrow{h} J \longrightarrow K \longrightarrow L$$

is called the distinguished cotriangle associated with (6.5), where $h = \beta^*(f) \circ \varphi^{-1}$. The above distinguished cotriangle gives rise to a long exact sequence (cf. Proposition 6.2)

$$\cdots \longrightarrow H^{n-1}(L) \xrightarrow{H^n(h)} H^n(J) \longrightarrow H^n(K) \longrightarrow H^n(L) \longrightarrow \cdots$$

and $H^n(h) = \delta$, δ being defined in Proposition 6.1.

For an Abelian category $\mathcal{C}$, we donote by $\mathsf{D}^+(\mathcal{C})$ the full subcategory of $\mathsf{D}(\mathcal{C})$ consisting of complexes bounded below.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of Abelian categories. If there exists an "F-injective" subcategory $\mathcal{I}$, we may define the right derived functor

$$\boldsymbol{R}F : \mathsf{D}^+(\mathcal{C}) \longrightarrow \mathsf{D}^+(\mathcal{C}') \qquad \text{by} \quad \boldsymbol{R}F(K) = F(I), \quad K \underset{\text{qis}}{\xrightarrow{\sim}} I.$$

We define a functor $R^qF : \mathcal{C} \to \mathcal{C}'$ as the composition

$$\mathcal{C} \longrightarrow \mathsf{D}^+(\mathcal{C}) \xrightarrow{\boldsymbol{R}F} \mathsf{D}^+(\mathcal{C}') \xrightarrow{H^q} \mathcal{C}', \quad \text{i.e.,} \quad R^qF(K) = H^q(\boldsymbol{R}F(K)) = H^q(F(I)).$$

Cohomology of sheaves : For a topological space X, we denote by $\mathcal{S}h(X)$ the category of sheaves of Abelian groups on X. We also denote by $\mathcal{A}$ the category of Abelian groups. For an open set X' in X, we have the functor

$$\varGamma(X, X';\ ) : \mathcal{S}h(X) \longrightarrow \mathcal{A}$$

defined by $\varGamma(X, X'; \mathscr{S}) = \mathscr{S}(X, X')$. The subcategory of flabby sheaves is injective for this functor. For $\mathscr{S}$ in $\mathcal{S}h(X)$,

$$\begin{aligned} \boldsymbol{R}\varGamma(X, X'; \mathscr{S}) &= \varGamma(X, X'; \mathscr{F}^\bullet) \qquad \text{and} \\ R^q\varGamma(X, X'; \mathscr{S}) &= H^q(\varGamma(X, X'; \mathscr{F}^\bullet)) \simeq H^q(X, X'; \mathscr{S}), \end{aligned}$$

where $\mathscr{S} \underset{\text{qis}}{\xrightarrow{\sim}} \mathscr{F}^\bullet$ is a flabby resolution.

6.4 Relative cohomology for sections as a co-mapping cone

Let $f : Y \to X$ be a continuous map of topological spaces and $\mathscr{K} = (\mathscr{K}^\bullet, d_{\mathscr{K}})$ a complex of sheaves on X. Then we have the complex of sheaves $(f^{-1}\mathscr{K}^\bullet, f^{-1}d_{\mathscr{K}})$ on Y. We define a complex $\mathscr{K}(f)$ as the co-mapping cone $M^*(f^{-1})$ of $f^{-1} : \mathscr{K}^\bullet(X) \to (f^{-1}\mathscr{K}^\bullet)(Y)$. Thus

$$\mathscr{K}^q(f) = \mathscr{K}^q(X) \oplus (f^{-1}\mathscr{K}^{q-1})(Y)$$

and the differential $d = d_{\mathscr{K}} : \mathscr{K}^q(f) \to \mathscr{K}^{q+1}(f)$ is given by

$$d(s,t) = (ds, f^{-1}s - (f^{-1}d)t).$$

Definition 6.6 The q-th cohomology $H^q_{d_{\mathscr{K}}}(f)$ of f for $\mathscr{K}$ is the q-th cohomology of $(\mathscr{K}^\bullet(f), d_{\mathscr{K}})$.

We have the exact sequence (cf. the proof of Proposition 6.2)

$$0 \longrightarrow \mathscr{K}^\bullet(Y)[-1] \xrightarrow{\beta^*} \mathscr{K}^\bullet(f) \xrightarrow{\alpha^*} \mathscr{K}^\bullet(X) \longrightarrow 0,$$

where $\beta^*(t) = (0, -t)$ and $\alpha^*(s,t) = s$. From this we have the exact sequence

$$\cdots \longrightarrow H^{q-1}_{f^{-1}d_{\mathscr{K}}}(Y) \xrightarrow{\beta^*} H^q_{d_{\mathscr{K}}}(f) \xrightarrow{\alpha^*} H^q_{d_{\mathscr{K}}}(X) \xrightarrow{\delta^*} H^q_{f^{-1}d_{\mathscr{K}}}(Y) \longrightarrow \cdots,$$

where $\delta^* = f^{-1}$.

Remark 6.7 A similar construction is done in [3] for the de Rham complex, except the morphism β^*, which is denoted by α there, is defined as $t \mapsto (0, t)$.

Suppose X is paracompact and $\mathscr{K}^\bullet$ is a complex of fine sheaves on X. In the above, let $Y = X'$ be an open set in X and $f = i : X' \hookrightarrow X$ the inclusion. Then setting $\mathcal{V} = \{V_0, V_1\}$, $\mathcal{V}' = \{V_0\}$, $V_0 = X'$ and $V_1 = X$, we have (cf. (5.1)):

$$\mathscr{K}^\bullet(i) = \mathscr{K}^\bullet(\mathcal{V}, \mathcal{V}') \quad \text{and} \quad H^q_{d_{\mathscr{K}}}(i) = H^q_{D_{\mathscr{K}}}(X, X'). \tag{6.8}$$

For a sheaf $\mathscr{S}$ on X and an open set X' in X, we have an exact sequence

$$0 \longrightarrow \boldsymbol{R}\Gamma(X, X'; \mathscr{S}) \longrightarrow \boldsymbol{R}\Gamma(X; \mathscr{S}) \longrightarrow \boldsymbol{R}\Gamma(X'; \mathscr{S}) \longrightarrow 0.$$

Suppose $0 \to \mathscr{S} \to \mathscr{K}^\bullet$ is a fine resolution. Then by Theorem 4.1, $\boldsymbol{R}\Gamma(X; \mathscr{S})$ and $\boldsymbol{R}\Gamma(X'; \mathscr{S})$ are quasi-isomorphic with $\Gamma(X; \mathscr{K}^\bullet)$ and $\Gamma(X'; \mathscr{K}^\bullet)$, respectively. By Proposition 6.4 and (6.8), we have:

Theorem 6.9 *In the above situation, it holds that*

$$\boldsymbol{R}\Gamma(X, X'; \mathscr{S}) \underset{\text{qis}}{\simeq} \mathscr{K}^\bullet(\mathcal{V}, \mathcal{V}') \quad \textit{and} \quad H^q(X, X'; \mathscr{S}) \simeq H^q_{D_{\mathscr{K}}}(X, X').$$

Remark 6.10 This way we recover an isomorphism as in Theorem 5.7 without referring to coverings of X. However, to find out the actual correspondence, we need to go through the Čech theory.

7 Some particular cases

The manifolds we consider below are assumed to have countable basis, thus paracompact. Also the coverings are assumed to be locally finite.

I. de Rham complex

Let X be a C^∞ manifold of dimension m and $\mathscr{E}^{(q)}$ the sheaf of C^∞ q-forms on X. By the Poincaré lemma, we have a fine resolution of the constant sheaf $\mathbb{C}$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{E}^{(0)} \xrightarrow{d} \mathscr{E}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{E}^{(m)} \longrightarrow 0.$$

The *de Rham cohomology* $H^q_d(X)$ of X is the cohomology of $(\mathscr{E}^{(\bullet)}(X), d)$. By Theorem 4.1, there is a canonical isomorphism (de Rham theorem):

$$H^q_d(X) \simeq H^q(X;\mathbb{C}). \tag{7.1}$$

Let X' be an open set in X and $(\mathcal{W}, \mathcal{W}')$ a pair of coverings for (X, X'). The *Čech-de Rham cohomology* $H^q_D(\mathcal{W}, \mathcal{W}')$ on $(\mathcal{W}, \mathcal{W}')$ is the cohomology of $(\mathscr{E}^{(\bullet)}(\mathcal{W}, \mathcal{W}'), D)$, $D = \delta + (-1)^\bullet d$ (cf. Definition 3.1).

We say that $\mathcal{W}$ is *good* if every non-empty finite intersection $W_{\alpha_0 \dots \alpha_q}$ is diffeomorphic with $\mathbb{R}^m$. If $\mathcal{W}$ is good, then it is good for $\mathscr{E}^{(\bullet)}$. From Theorem 4.3, we have the following canonical isomorphisms:

(1) For any covering $\mathcal{W}$, $H^q_d(X) \xrightarrow{\sim} H^q_D(\mathcal{W})$.

(2) For a good covering $\mathcal{W}$,

$$H^q_D(\mathcal{W}, \mathcal{W}') \xleftarrow{\sim} H^q(\mathcal{W}, \mathcal{W}';\mathbb{C}) \simeq H^q(X, X';\mathbb{C}).$$

Note that X always admits a good covering and the good coverings are cofinal in the set of coverings.

The relative de Rham cohomology $H^q_D(X, X')$ is defined as in Section 5 and, from Theorem 5.7, we have:

Theorem 7.2 (Relative de Rham theorem) *There is a canonical isomorphism*

$$H^q_D(X, X') \simeq H^q(X, X';\mathbb{C}).$$

For more about Čech-de Rham cohomology and applications, we refer to [3], [15] and [16] and references therein.

II. Dolbeault complex

Let X be a complex manifold of dimension n and $\mathscr{E}^{(p,q)}$ the sheaf of C^∞ (p,q)-forms on X. By the Dolbeault-Grothendieck lemma, we have a fine resolution of the sheaf $\mathscr{O}^{(p)}$ of holomorphic p-forms:

$$0 \longrightarrow \mathscr{O}^{(p)} \longrightarrow \mathscr{E}^{(p,0)} \xrightarrow{\bar\partial} \mathscr{E}^{(p,1)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{(p,n)} \longrightarrow 0.$$

The Dolbeault cohomology $H^{p,q}_{\bar\partial}(X)$ of X is the cohomology of the complex $(\mathscr{E}^{(p,\bullet)}(X),\bar\partial)$. By Theorem 4.1, there is a canonical isomorphism (Dolbeault theorem):

$$H^{p,q}_{\bar\partial}(X)\simeq H^q(X;\mathscr{O}^{(p)}). \tag{7.3}$$

Let $(\mathcal{W},\mathcal{W}')$ be as above. The Čech-Dolbeault cohomology $H^{p,q}_{\bar\vartheta}(\mathcal{W},\mathcal{W}')$ on $(\mathcal{W},\mathcal{W}')$ is the cohomology of $(\mathscr{E}^{(p,\bullet)}(\mathcal{W},\mathcal{W}'),\bar\vartheta)$, $\bar\vartheta=\delta+(-1)^{\bullet}\bar\partial$ (cf. Definition 3.1).

We say that $\mathcal{W}$ is *Stein* if every non-empty finite intersection $W_{\alpha_0\dots\alpha_q}$ is biholomorphic with a domain of holomorphy in $\mathbb{C}^n$ (cf. [6]). If $\mathcal{W}$ is Stein, then it is good for $\mathscr{E}^{(p,\bullet)}$. From Theorem 4.3, we have the following canonical isomorphisms:

(1) For any covering $\mathcal{W}$, $H^{p,q}_{\bar\partial}(X)\xrightarrow{\sim}H^{p,q}_{\bar\vartheta}(\mathcal{W})$.

(2) For a Stein covering $\mathcal{W}$,

$$H^{p,q}_{\bar\vartheta}(\mathcal{W},\mathcal{W}')\xleftarrow{\sim}H^q(\mathcal{W},\mathcal{W}';\mathscr{O}^{(p)})\simeq H^q(X,X';\mathscr{O}^{(p)}).$$

Note that X always admits a Stein covering and the Stein coverings are cofinal in the set of coverings.

The relative Dolbeault cohomology $H^{p,q}_{\bar\vartheta}(X,X')$ is defined as in Section 5 and, from Theorem 5.7, we have:

Theorem 7.4 (Relative Dolbeault theorem) *There is a canonical isomorphism*

$$H^{p,q}_{\bar\vartheta}(X,X')\simeq H^q(X,X';\mathscr{O}^{(p)}).$$

For more about Čech-Dolbeault cohomology we refer to [17] and [18]. Applications are given in [1] for localization of Atiyah classes, [2] for the Hodge decomposition problem and [10] for the Sato hyperfunction theory.

Remark 7.5 The seemingly standard proof in textbooks, e.g., [7], [9], of the isomorphism as in Theorem 4.1 (thus (7.1) and (7.3)) gives a correspondence same as the one given by the Weil lemma. Thus there is a sign difference as explained in Remark 3.8.

III. Mixed complex

Let X be a complex manifold. We set

$$\mathscr{E}^{(p,q)+1}=\mathscr{E}^{(p+1,q)}\oplus\mathscr{E}^{(p,q+1)}$$

and consider the complex

$$\cdots\xrightarrow{d}\mathscr{E}^{(p-2,q-2)+1}\xrightarrow{\bar\partial+\partial}\mathscr{E}^{(p-1,q-1)}\xrightarrow{\bar\partial\partial}\mathscr{E}^{(p,q)}\xrightarrow{d}\mathscr{E}^{(p,q)+1}\xrightarrow{\bar\partial+\partial}\mathscr{E}^{(p+1,q+1)}\xrightarrow{\bar\partial\partial}\cdots. \tag{7.6}$$

From this we have the Bott-Chern, Aeppli and third cohomologies and their relative versions. For details and applications to the localization problem of Bott-Chern classes, we refer to [4].

IV. Others

Another type of complex is considered in [10].

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