RELATIVE EMBEDDINGS OF DEL PEZZO FIBRATIONS OF DEGREE SIX

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Abstract. The aim of this report is to give an outline of my talk about sextic del Pezzo fibrations in Kinosaki Algebraic Symposium 2017.

1. Introduction

Let X be a smooth projective 3-fold over $\mathbb{C}$. By virtue of Mori theory, if K_X is not nef, then we have an extremal contraction

$$\varphi\colon X \to C,$$

that is, φ is a surjective morphism onto a normal projective variety C such that $\mathcal{O}_C \simeq \varphi_*\mathcal{O}_X$, $\rho(X/C) = 1$ and $-K_X$ is φ-ample. If $\dim C = 1$, then C is a smooth and we call φ a *del Pezzo fibration*. A general φ-fiber F is a del Pezzo surface and the *degree* of φ is defined to be $d = (-K_F)^2$.

It is well-known that every del Pezzo surface F is a complete intersection of a Fano variety. For example, if $d = 3$ (resp. 4), then F is a cubic surface in $\mathbb{P}^3$ (resp. a complete intersection of two quadrics in $\mathbb{P}^4$). To study del Pezzo fibrations, we want to relativize these description. The main problem of this report is the following.

Problem 1.1. *For every del Pezzo fibration $\varphi\colon X \to C$, construct another extremal contraction $\varphi_Y\colon Y \to C$ such that Y contains X as a relative (weighted) complete intersection.*

If φ is not of degree 6, this problem was mostly solved by using a part of works due to H. D'Souza, M. L. Fania, T. Fujita, S. Mori and K. Takeuchi. Some answers to this problem has been applied for classifying the singular fibers of φ [4], weak Fano 3-folds with del Pezzo fibrations [12], and so on.

In this report, we focus on *sextic del Pezzo fibrations*, which are del Pezzo fibrations of degree 6. Roughly speaking, the main results show that every sextic del Pezzo fibration is a relative hyperplane section of a $(\mathbb{P}^1)^3$-fibration and a relative codimension 2 linear section of a $(\mathbb{P}^2)^2$-fibration. As an application, we give a classification of singular fibers of sextic del Pezzo fibrations.

I am now writing a paper including the results on this report and I will submit it to ArXiv soon.

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2. Known results and main results

2.1. **Previous research.** We recall a part of previous research. Let $\varphi\colon X \to C$ be a del Pezzo fibration of degree d. In 1982, Mori established the following results in his paper [10].

- It holds that $d \in \{1, 2, 3, 4, 5, 6, 8, 9\}$.
- If $d = 9$, then φ is a $\mathbb{P}^2$-bundle.
- If $d = 8$, then every φ-fiber is a reduced irreducible quadric surface in $\mathbb{P}^3$. Moreover, there exists a $\mathbb{P}^3$-bundle $\mathbb{F} \to C$ containing X with a relation $X \sim_C 2\xi_{\mathbb{F}}$ as divisors on $\mathbb{F}$, where $\xi_{\mathbb{F}}$ denotes a tautological divisor on $\mathbb{F}$.

Let us assume that $d \leq 6$. D'Souza [1], D'Souza-Fania [2] and Fujita [4] proved that there exists a bundle $p\colon P \to C$ containing X. Here, P is the $\mathbb{P}^d$-bundle $\mathbb{P}_C(\varphi_*\mathcal{O}(-K_X))$ (resp. a $\mathbb{P}(1,1,1,2)$-bundle, a $\mathbb{P}(1,1,2,3)$-bundle) if $d \geq 3$ (resp. $d = 2$, $d = 1$). Moreover, if $d \in \{3, 4\}$ (resp. $d \in \{1, 2\}$), then X is a relative (resp. weighted) complete intersection in P.

Example 2.1. To make sense of the sentence "relative complete intersection", let us check the precise statement when $d = 4$. In this case, X is embedded in the $\mathbb{P}^4$-bundle P. Let $\mathcal{O}_P(1)$ denote the tautological line bundle. D'Souza and Fujita proved that there exists a rank 2 bundle $\mathcal{E}$ on C and a section $s \in H^0(P, \mathcal{O}_P(2) \otimes p^*\mathcal{E})$ such that $X = (s = 0)$. In particular, every fiber X_t is a complete intersection of two quadrics in $\mathbb{P}^4$. Note that if $C = \mathbb{P}^1$, then X is a scheme-theoretic complete intersection in P.

When $d \in \{5, 6\}$, X is not a relative complete intersection in P. Takeuchi however proved that if $d = 5$, then X is defined by Pfaffians in P relatively [12].

2.2. **Singular fibers of del Pezzo fibrations.** A classification of singular fibers of del Pezzo fibrations of degree $d \neq 6$ was given by Fujita [4]. Especially, he proved the following theorem.

Theorem 2.2 ([4, (4,10)]). *Let $\varphi\colon X \to C$ be a del Pezzo fibration of degree d. If $d \neq 6$, then every fiber of φ is normal.*

On the other hand, when $d = 6$, a classification of singular fibers was yet to be known. In particular, Fujita proposed a following question.

Question 2.3 ([4, (3,7)]). Does there exist sextic del Pezzo fibrations containing non-normal fibers?

2.3. **Main results.** Suppose that $d = 6$. It is known that every sextic del Pezzo surface S is a hyperplane section of $(\mathbb{P}^1)^3$ with respect to the Segre embedding $(\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7$ and also a codimension 2 linear section of $(\mathbb{P}^2)^2$ with respect to the Segre embedding $(\mathbb{P}^2)^2 \hookrightarrow \mathbb{P}^8$. The main results relativize these descriptions.

Theorem 2.4. *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration. Then there exists a smooth projective 4-fold Y and an extremal contraction $\varphi_Y\colon Y \to C$ satisfying the following properties.*

- *Every smooth fiber of φ_Y is isomorphic to $(\mathbb{P}^1)^3$.*
- *It holds that $K_Y + 2X \sim_C 0$ as divisors on Y.*

Theorem 2.5. *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration. Then there exists a smooth projective 5-fold Z and an extremal contraction $\varphi_Z\colon Z \to C$ satisfying the following properties.*

- *Every smooth fiber of φ_Z is isomorphic to $(\mathbb{P}^2)^2$.*
- *There exists a divisor H on Z such that $K_Z + 3H \sim_C 0$.*
- *There exists a rank 2 vector bundle $\mathcal{E}$ on C and a section $s \in H^0(Z, \varphi_Z^*\mathcal{E} \otimes \mathcal{O}_Z(H))$ such that $X = (s = 0)$.*

We can apply these theorems to classify singular fibers of sextic del Pezzo fibrations. As a result, we will give an affirmative answer to Question 2.3, i.e., there exists sextic del Pezzo fibrations containing non-normal fibers.

3. 2-ray games for sextic del Pezzo fibrations

3.1. **2-ray games.** A key point of our proof of Theorems 2.4 and 2.5 is a 2-ray game for a sextic del Pezzo fibration. This is a relativization of the following birational transformation.

Observation 3.1. Let S be a sextic del Pezzo surface and $x \in S$ a point. Since S is a hyperplane section of $(\mathbb{P}^1)^3$, we can find three conic bundles $p_i\colon S \to \mathbb{P}^1$. If we take $x \in S$ as a general point, then $l_i = p_i^{-1}(p_i(x))$ is a smooth conic for each i. Thus the blowing-up of S at x contains the proper transform of l_i as a (-1)-curve for each i. By blowing them down, we have a smooth quadric surface $\mathbb{Q}^2$, i.e., $\mathrm{Bl}_x S \simeq \mathrm{Bl}_{y_1,y_2,y_3} \mathbb{Q}^2$, where $y_1, y_2, y_3 \in \mathbb{Q}^2$ are three different points:

$$S \leftarrow \mathrm{Bl}_x S \simeq \mathrm{Bl}_{y_1,y_2,y_3} \mathbb{Q}^2 \to \mathbb{Q}^2. \tag{3.1}$$

The curve extracted by $\mathrm{Bl}_x S \to S$ is transformed into a conic on $\mathbb{Q}^2$ containing the three points y_1, y_2, y_3. Therefore, y_1, y_2, y_3 are non-colinear.

Next we recall the following theorem.

Theorem 3.2 ([9, Theorem (4.2)], [11], [6])**.** *Every del Pezzo fibration $\varphi\colon X \to C$ has a section.*

More precisely, the existence of sections of sextic del Pezzo fibrations was proved by Manin [9, Theorem (4.2)]. Using this theorem, we can relativize the geometry in Observation 3.1 over a curve.

Proposition 3.3 (2-ray game)**.** *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration and take a φ-section C_0. Let $\mu\colon \widetilde{X} := \mathrm{Bl}_{C_0} X \to X$ be the blowing-up along C_0 and $E = \mathrm{Exc}(\mu)$. Then there exists the following diagram*

$$\begin{array}{ccccccc} & & \mathrm{Bl}_{C_0} X = \widetilde{X} & \overset{\chi}{\dashrightarrow} & \widetilde{Q} = \mathrm{Bl}_T Q & & \\ & \swarrow^{\mu} & & & & \searrow^{\sigma} & \\ C_0 \subset X & & & & & & Q \supset T \\ & \searrow_{\varphi} & & & & \swarrow_{q} & \\ & & C & = & C, & & \end{array} \tag{3.2}$$

where

- *χ is an isomorphism or a flop over C,*

- *Q is smooth and σ is the blowing-up along a smooth curve T, and*
- *q is a quadric fibration and $q|_T\colon T \to C$ is a triple cover.*

When $E_Q \subset Q$ denotes the proper transform of E on Q, E_Q contains T and $K_Q + 2E_Q \sim_C 0$. Moreover, we have the following equality:

$$(-K_Q)^3 = \frac{1}{3}\left(4(-K_X)^3 + 16g(T) - 48g(C) + 32\right). \tag{3.3}$$

Remark 3.4. In this report, a quadric fibration $q\colon Q \to C$ is defined to be an extremal contraction whose general fibers are quadric surfaces.

4. Proof of Theorem 2.4

In this section, we see a sketch of a proof of Theorem 2.4. A main idea to prove this theorem is the following birational transformation.

As in Observation 3.1, let S be a sextic del Pezzo surface with a general point x. We consider an embedding $S \hookrightarrow (\mathbb{P}^1)^3$ and the blowing up $\mathrm{Bl}_x(\mathbb{P}^1)^3$. Then we can show that $\mathrm{Bl}_x(\mathbb{P}^1)^3$ flops into $\mathrm{Bl}_{y_1,y_2,y_3}\mathbb{P}^3$:

$$(\mathbb{P}^1)^3 \leftarrow \mathrm{Bl}_x(\mathbb{P}^1)^3 \overset{\text{flop}}{\dashrightarrow} \mathrm{Bl}_{y_1,y_2,y_3}\mathbb{P}^3 \to \mathbb{P}^3. \tag{4.1}$$

Now we obtain the birational transform (3.1) by chasing the proper transformations of a hyperplane section of $(\mathbb{P}^1)^3$ containing x. Therefore, we may regard this transformation (4.1) as an "extension" of (3.2). We will prove Theorem 2.4 by relativizing (4.1).

Sketch of a proof of Theorem 2.4. Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration and we fix a φ-section C_0. Then Proposition 3.3 produces a quadric fibration $q\colon Q \to C$ with a trisection T. Let E_Q be as in Proposition 3.3 and set $\mathbb{F} := \mathbb{P}_C(q_*\mathcal{O}_Q(E_Q))$, which is a $\mathbb{P}^3$-bundle over C. Note that $\mathbb{F}$ contains Q since E_Q is q-very ample.

To relativize (4.1) over the base C, we investigate a flop over C of the blowing-up $\mathrm{Bl}_T\mathbb{F}$. To make this flop explicitly, we consider a sub $\mathbb{P}^2$-bundle $\mathbb{E} \subset \mathbb{F}$ such that $\mathbb{E}_t$ is the linear span of T_t in $\mathbb{F}_t = \mathbb{P}^3$ for every $t \in C$. Let

- $\sigma_{\widetilde{\mathbb{F}}}\colon \widetilde{\mathbb{F}} = \mathrm{Bl}_T\mathbb{F} \to \mathbb{F}$ be the blowing-up of $\mathbb{F}$ along T,
- $\sigma_{\widetilde{\mathbb{E}}}\colon \widetilde{\mathbb{E}} = \mathrm{Bl}_T\mathbb{E} \to \mathbb{E}$ the blowing-up of $\mathbb{E}$ along T,
- $L_{\widetilde{\mathbb{F}}} := \sigma^*(2\xi_{\mathbb{F}}) - \mathrm{Exc}(\sigma_{\mathbb{F}})$, where $\xi_{\mathbb{F}}$ is a tautological divisor of $\mathbb{F}$, and
- $L_{\widetilde{\mathbb{E}}} := L_{\widetilde{\mathbb{F}}}|_{\widetilde{\mathbb{E}}}$.

Then we can check that $L_{\widetilde{\mathbb{F}}}$ and $L_{\widetilde{\mathbb{E}}}$ are free over C. Let

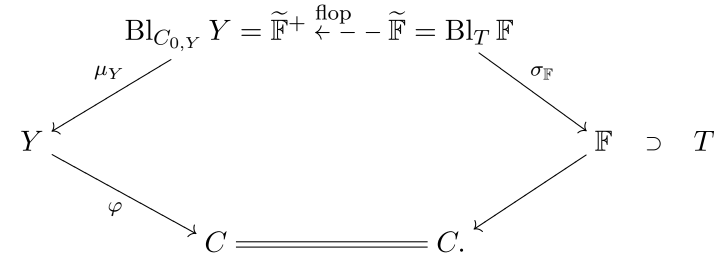
- $\psi_{\widetilde{\mathbb{F}}}\colon \widetilde{\mathbb{F}} \to \overline{\mathbb{F}}$ be the Stein factorization of the morphism over C given by $L_{\widetilde{\mathbb{F}}}$ and
- $\psi_{\widetilde{\mathbb{E}}}\colon \widetilde{\mathbb{E}} \to \overline{\mathbb{E}}$ the Stein factorization of the morphism over C given by $L_{\widetilde{\mathbb{E}}}$.

Then we can show that $\overline{\mathbb{E}}$ is also a $\mathbb{P}^2$-bundle over C and $\psi_{\widetilde{\mathbb{E}}}$ is the blowing up of $\overline{\mathbb{E}}$ along a smooth curve $\overline{T}$ such that $\overline{T} \to C$ is also of degree 3. In other words, $\overline{\mathbb{E}} \leftarrow \widetilde{\mathbb{E}} \to \mathbb{E}$ is nothing but a family of quadratic Cremona transformations.

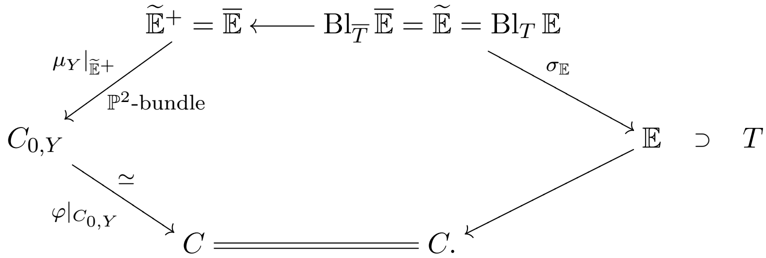
Since $L_{\widetilde{\mathbb{F}}} \sim_C \widetilde{\mathbb{E}} + \sigma_{\widetilde{\mathbb{F}}}^*\xi_F$, we have $\mathrm{Exc}(\psi_{\widetilde{\mathbb{F}}}) = \mathrm{Exc}(\psi_{\widetilde{\mathbb{E}}})$. Noting that $2L_{\widetilde{\mathbb{F}}} \sim_C -K_{\widetilde{\mathbb{F}}}$, we can show that the contraction $\psi_{\widetilde{\mathbb{F}}}$ is a family of Atiyah flopping contractions. Therefore, if $\widetilde{\mathbb{F}} \dashrightarrow \widetilde{\mathbb{F}}^+$ denotes the flop, then $\widetilde{\mathbb{F}}^+$ is also smooth.

On $\widetilde{\mathbb{F}}^+$, the proper transform $\widetilde{\mathbb{E}}^+$ of $\widetilde{\mathbb{E}}$ is isomorphic to $\overline{\mathbb{E}}$. Then there exists a morphism $\mu_Y\colon \widetilde{\mathbb{F}}^+ \to Y$ over C such that μ_Y blows $\widetilde{\mathbb{E}}^+$ down to $C_{0,Y}$, which is a section of $Y \to C$. In particular, Y is a smooth 4-fold.

(4.2)



(4.3)



The birational transformation (4.2) is nothing but a relativization of (4.1). Every smooth fiber of φ_Y is isomorphic to $(\mathbb{P}^1)^3$ and $\rho(Y) = 2$. This Y contains X which we took in the beginning of this argument. Moreover, under this embedding $X \hookrightarrow Y$, the section $C_{0,Y}$ coincides with C_0. This is a sketch of a proof of Theorem 2.4. □

5. Proof of Theorem 2.5

A main idea to Theorem 2.5 is similar to that of Theorem 2.4. Consider a sextic del Pezzo surface S with a general point x and an embedding $S \hookrightarrow (\mathbb{P}^2)^2$. Then the blowing up $\mathrm{Bl}_x(\mathbb{P}^2)^2$ flops into $\mathbb{P}_{\mathbb{Q}^2}(\mathcal{F}_{\mathbb{Q}^2})$, which is a $\mathbb{P}^2$-bundle over $\mathbb{Q}^2$:

$$(\mathbb{P}^2)^2 \leftarrow \mathrm{Bl}_x(\mathbb{P}^2)^2 \overset{\text{flop}}{\dashrightarrow} \mathbb{P}_{\mathbb{Q}^2}(\mathcal{F}_{\mathbb{Q}^2}) \to \mathbb{Q}^2. \tag{5.1}$$

Moreover, we have

$$\mathcal{F}_{\mathbb{Q}^2} = \mathcal{S}_{\mathbb{Q}^3}|_{\mathbb{Q}^2} \oplus \mathcal{O}_{\mathbb{Q}^2}(1),$$

where $\mathcal{S}_{\mathbb{Q}^3}$ denotes the Spinor bundle on $\mathbb{Q}^3$ and $\mathcal{O}_{\mathbb{Q}^2}(1)$ denotes the polarization with respect to $\mathbb{Q}^2 \hookrightarrow \mathbb{P}^3$. The transformation (5.1) is another "extension" of the transformation (3.1).

To prove Theorem 2.5, we want to relativize the transformation (5.1) over the base C. For this purpose, we need a family of this bundle on a quadric fibration $Q \to C$ with a trisection. We construct such a vector bundle $\mathcal{F}$ on Q by using the following proposition.

Proposition 5.1. *Let $q\colon Q \to C$ be a quadric fibration and $T \subset Q$ be a smooth curve such that $\deg(q|_T) = 3$. Then there exists a locally free sheaf $\mathcal{F}$ on Q such that $\mathcal{F}$ fits into the following exact sequence*

$$0 \to q^*(R^1 q_* \mathcal{I}_{T/Q}(-K_C))^\vee \to \mathcal{F} \to \mathcal{I}_{T/Q}(-K_Q) \to 0 \tag{5.2}$$

and there are no surjections $\mathcal{F}|_{Q_t} \twoheadrightarrow \mathcal{O}_{Q_t}$ for every $t \in C$.

The exact sequence (5.2) tells us that T is a zero locus of a certain locally free sheaf. For a proof, we do not use the Hartshorne-Serre correspondence since it does not give the extension (5.2), but use a "universal extension" of coherent sheaves. In general, for a coherent sheaf $\mathcal{E}$ on a projective variety V, we can define the *universal extension of* $\mathcal{E}$ *by* $\mathcal{O}$ as the exact sequence

$$0 \to \operatorname{Ext}^1(\mathcal{E}, \mathcal{O})^\vee \otimes \mathcal{O} \to \mathcal{F} \to \mathcal{E} \to 0$$

corresponding to the element

$$\operatorname{Ext}^1(\mathcal{E}, \operatorname{Ext}^1(\mathcal{E}, \mathcal{O})^\vee \otimes \mathcal{O}) \simeq \operatorname{Ext}^1(\mathcal{E}, \mathcal{O}) \otimes \operatorname{Ext}^1(\mathcal{E}, \mathcal{O})^\vee \ni \mathrm{id}\,.$$

The "universality" of $\mathcal{F}$ implies that there are no surjections $\mathcal{F} \twoheadrightarrow \mathcal{O}$.

In the setting of Proposition 5.1, we can obtain the exact sequence (5.2) as a "relative universal extension". Note that the restriction of the sequence (5.2) to Q_t is the universal extension of $\mathcal{I}_{T_t/Q_t}(-K_{Q_t})$ by $\mathcal{O}_{Q_t}$ for each $t \in C$.

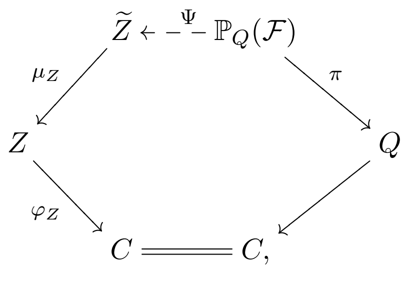
Let $\mathcal{F}$ be as in Proposition 5.1. Then we need to check that the vector bundle $\mathcal{F}|_{Q_t}$ is the desired one for each $t \in C$. To do this, we use the following characterization of the bundle $\mathcal{S}_{\mathbb{Q}^3}|_{Q_0} \oplus \mathcal{O}_{Q_0}(1)$ on a reduced irreducible quadric surface Q_0.

Proposition 5.2. *Let Q_0 be a reduced irreducible quadric surface and $\mathcal{F}_0$ a vector bundle of rank* 3*. If $\mathcal{F}_0$ is nef,* $\det \mathcal{F}_0 \simeq \mathcal{O}(-K_{Q_0})$*,* $c_2(\mathcal{F}_0) = 3$ *and* $h^0(\mathcal{F}_0^\vee) = 0$*, then* $\mathcal{F}_0 \simeq \mathcal{S}_{\mathbb{Q}^3}|_{Q_0} \oplus \mathcal{O}_{Q_0}(1)$.

We omit proofs of Propositions 5.1 and 5.2.

Sketch of a proof of Theorem 2.5. Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration and take a φ-section C_0. Let $q\colon Q \to C$ be the quadric fibration with the trisection T as in Proposition 3.3. By Proposition 5.1, we have the exact sequence (5.2). By Proposition 3.3, $-K_{\mathrm{Bl}_T Q}$ is globally generated over C. Thus $\mathcal{I}_{T/Q}(-K_Q)$ is also globally generated over C and so is $\mathcal{F}$. For each $t \in C$, $\mathcal{F}|_{Q_t}$ has no trivial bundles as quotients, which implies that $h^0(\mathcal{F}|_{Q_t}^\vee) = 0$ since $\mathcal{F}|_{Q_t}$ is globally generated. Therefore, by Proposition 5.2, we have $\mathcal{F}|_{Q_t} \simeq \mathcal{S}_{\mathbb{Q}^3}|_{Q_t} \oplus \mathcal{O}_{Q_t}(1)$ for each $t \in C$.

Set $\pi\colon \mathbb{P}_Q(\mathcal{F}) \to Q$. Note that $\mathbb{P}_Q(\mathcal{F})$ contains $\mathrm{Bl}_T Q$. In analogy with the proof of Theorem 2.4, we have the following commutative diagram:



where Ψ is a family of Atiyah flops and μ_Z is a blowing-up along a φ_Z-section $C_{0,Z}$. Hence Z is smooth projective 5-fold and general fibers of φ_Z are isomorphic to $(\mathbb{P}^2)^2$. We can show that Z contains X and $C_0 = C_{0,Z}$. This is a sketch of a proof of Theorem 2.5. $\square$

6. Singular fibers

Every fiber of a del Pezzo fibration $\varphi\colon X \to C$ is a Gorenstein del Pezzo surface. Roughly speaking, Gorenstein del Pezzo surfaces S with $(-K_S)^2 = 6$ are classified into the following 3 cases.

(1) S has only Du Val singularities. In this case, let $\widetilde{S} \to S$ denote the minimal resolution. Then $\widetilde{S}$ has a birational morphism $\varepsilon\colon \widetilde{S} \to \mathbb{P}^2$, where ε is the blowing up of $\mathbb{P}^2$ at three (possibly infinitely near) points. If $\Sigma \subset \mathbb{P}^2$ denotes the 0-dimensional subscheme of length 3 corresponding the three (possibly infinitely near) points, then the isomorphism classes of S was determined by Σ as follows:

Type	Σ	Is Σ colinear?	# of lines	Singularity
(2,3)	reduced	non-colinear	6	smooth
(2,2)	$\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2)$	non-colinear	4	A_1
(2,1)	$\operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^3)$	non-colinear	2	A_2
(1,3)	reduced	colinear	3	A_1
(1,2)	$\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2)$	colinear	2	$A_1 + A_1$
(1,1)	$\operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^3)$	colinear	1	$A_1 + A_2$

(2) S is non-normal and the normalization $\overline{S}$ is isomorphic to a Hirzebruch surface $\mathbb{F}_k$ with $k \in \{2, 4\}$. We say that S is *of type* (n_k) for such S.

(3) S is a cone over a curve $C \subset \mathbb{P}^5$ of degree 6 and arithmetic genus 1.

A smooth projective 3-fold never includes S belonging to case (3) since $\dim T_v S = 6$, where v denotes the vertex of S. In other words, every fiber of a sextic del Pezzo fibration belongs to (1) or (2).

In order to give a complete classification of singular fibers, we need to care whether each surface really appears as a fiber of a sextic del Pezzo fibration. The following theorem enables us to reduce this problem to a problem about quadric fibrations with trisections.

Theorem 6.1. *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration and take a φ-section C_0. Let $q\colon Q \to C$ be the quadric fibration with the trisection T as in Proposition 3.3. For any point $t \in C$, we set $X_t = \varphi^{-1}(t)$, $Q_t = q^{-1}(t)$ and $T_t = (q|_T)^{-1}(t)$.*

Then the following assertions hold for each $t \in C$.

(1) For $j \in \{1, 2, 3\}$, X_t is of type $(2, j)$ if and only if Q_t is smooth and $\#(T_t)_{\mathrm{red}} = j$.

(2) For $j \in \{1, 2, 3\}$, X_t is of type $(1, j)$ if and only if Q_t is singular, $\#(T_t)_{\mathrm{red}} = j$ and $\operatorname{Sing} Q_t \cap T_t = \varnothing$.

(3) X_t is of type (n_2) if and only if Q_t is singular, $\#(T_t)_{\mathrm{red}} = 2$ and the double point of T_t is supported at the vertex of Q_t.

(4) X_t is of type (n_4) if and only if Q_t is singular, $\#(T_t)_{\mathrm{red}} = 1$ and $\operatorname{Sing} Q_t = (T_t)_{\mathrm{red}}$.

Using this Theorem 6.1, we can deduce the following corollary by constructing such quadric fibrations $q\colon Q \to C$ with trisections T explicitly.

Corollary 6.2. *Let S be a Gorenstein sextic del Pezzo surface. If S is not a cone over a curve, then there exists a sextic del Pezzo fibration containing S as a fiber.*

In particular, in contrast with Theorem 2.2, non-normal sextic del Pezzo surfaces really appear as singular fibers of sextic del Pezzo fibrations.

Sketch of a proof of Theorem 6.1. We use Theorem 2.4 to prove Theorem 6.1. We proceed in 3 steps.

Step 1. Classify singular fibers of $\varphi_Y : Y \to C$ as in Theorem 2.4. As a result, every fiber Y_t is isomorphic to one of the following:

- $(\mathbb{P}^1)^3$,
- $\mathbb{P}^1 \times \mathbb{P}^{1,1}$, where $\mathbb{P}^{1,1}$ denotes the cone over a conic, or
- $\mathbb{P}^{1,1,1}$, which denotes the del Pezzo variety of type (si31i) in the sense of [3].

Step 2. Classify hyperplane sections of $(\mathbb{P}^1)^3$, $\mathbb{P}^1 \times \mathbb{P}^{1,1}$ and $\mathbb{P}^{1,1,1}$.

Step 3. Let $Y_t \dashrightarrow \mathbb{P}^3$ be the birational map in the diagram (4.2). Chasing the proper transform of a hyperplane section of Y_t on $\mathbb{P}^3$, we have Theorem 6.1. □

Remark 6.3. The classification of singular fibers of $\varphi_Z : Z \to C$ as in Theorem 2.4 was done by Fujita [4]. As a result, every singular fiber is isomorphic to $\mathbb{P}^{2,2}$, which is defined to be the anti-canonical model of the weak Fano manifold $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(2) \oplus \Omega_{\mathbb{P}^2}(2))$.

7. (2, 3)-coverings

Finally, we construct the coverings associated with a sextic del Pezzo fibration. The following lemma is well-known for experts:

Lemma 7.1. *Let $q : Q \to C$ be a quadric fibration. Then there exists a smooth curve B with a double covering structure $q_B : B \to C$ such that*

(1) q_B branches at $\{t \in C \mid Q_t = q^{-1}(t) \text{ is singular}\}$,
(2) the intermediate Jacobian $\mathcal{J}(Q)$ is isomorphic to the Jacobian $J(B)$ as complex tori, and
(3) it holds that $(-K_Q)^3 = 40 - (8g(B) + 32g(C))$.

We call this $q_B : B \to C$ the double covering associated with q.

By using Proposition 3.3 and Lemma 7.1, we define coverings associated with φ as follows:

Definition 7.2. Let $\varphi : X \to C$ be a sextic del Pezzo fibration. When we fix a φ-section C_0, Proposition 3.3 gives a quadric fibration $q : Q \to C$ with a trisection T. Set $\varphi_B : B \to C$ be the double covering associated with q and $\varphi_T = q|_T : T \to C$. We call the pair $(\varphi_B : B \to C, \varphi_T : T \to C)$ the *(2,3)-covering* associated with $\varphi : X \to C$.

As a corollary of Proposition 3.3 and Lemma 7.1, we have equalities for invariants of sextic del Pezzo fibrations.

Corollary 7.3. *The following assertions hold.*

(1) $\mathcal{J}(X) \times J(C) \simeq J(B) \times J(T)$ as complex tori.
(2) $(-K_X)^3 = 22 - (6g(B) + 4g(T) + 12g(C))$.

Proof. (1) follows from the diagram (3.2). (2) follows from the equality (3.3) and Lemma 7.1 (3). □

By definition, the $(2,3)$-covering may depend on the choice of φ-sections C_0. In fact, however, the following corollary implies the $(2,3)$-covering is independent of the choice of them.

Corollary 7.4. *For each $t \in C$, the pair of number*

$$(\#B_t, \#T_t) := (\#\varphi_B^{-1}(t), \#\varphi_T^{-1}(t))$$

is determined from the isomorphism class of X_t.

Moreover, if X_t has only Du Val singularities, then X_t is of type $(\#B_t, \#T_t)$.

Proof. This immediately follows from Theorem 6.1 and Lemma 7.1 (1). □

Remark 7.5. Corollary 7.3 has an application to classifying weak Fano sextic del Pezzo fibrations $\varphi\colon X \to \mathbb{P}^1$. In particular, we can easily enumerate the possible value of $((-K_X)^3, h^{1,2}(X))$ for such X.

For example, if $(-K_X)^3 = 2$, then it is obvious that $h^{1,2}(X) \in \{4,5\}$. In fact, there exist two weak Fano del Pezzo fibrations X_1 and X_2 such that $(-K_{X_1})^3 = (-K_{X_2})^3 = 2$, $h^{1,2}(X_1) = 4$ and $h^{1,2}(X_2) = 5$. For each i, X_i has a flopping contraction and the flop X_i^+ is also a sextic del Pezzo fibration over $\mathbb{P}^1$. Hence the classification result due to Jahnke-Peternell-Radloff [7],[8] and Fukuoka [5] can not distinguish X_1 from X_2. Therefore, the classification of them up to deformation equivalence is still open.

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