Reconstruction of one-punctured elliptic curves in positive characteristic by their geometric fundamental groups

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Abstract

The principal theme of anabelian geometry is the reconstruction of the geometry of algebraic varieties over number fields by their étale fundamental groups. The following theorem was proved by H.Nakamura, A.Tamagawa and S.Mochizuki.

Grothendieck conjecture for hyperbolic curves([M])

Let k be a sub-p-adic field (i.e. a subfield of a finitely generated field over $\mathbb{Q}_p$), U_1, U_2 hyperbolic curves (i.e. smooth connected curve s.t. $2g_U + n_U - 2 > 0$, where g_U and n_U stand for the genus of U^{cpt} and the cardinality of $(U \times \overline{k})^{cpt} \backslash U \times \overline{k}$ respectively) over k. The natural map $Isom_k(U_1, U_2) \to Isom_{G_k}(\pi_1(U_1), \pi_1(U_2)/Inn(\pi_1(U_2 \times \overline{k}))$ is bijective.

When the characteristic is positive, it is expected that the geometric fundamental groups (i.e. $\pi_1((-) \times \overline{k})$) have much information.

Theorem([T])

Let k be an algebraically closed field of characteristic $p > 0$, U_1, U_2 smooth connected curves over k s.t. $\pi_1(U_1) \simeq \pi_1(U_2)$. The following hold.

(1) $g_{U_1} = g_{U_2}$, $n_{U_1} = n_{U_2}$, there is a natural bijection $U_1^{cpt} \backslash U_1 \simeq U_2^{cpt} \backslash U_2$

(2) When $k \simeq \overline{\mathbb{F}_p}$ and $g_{U_1} = 0$, there is an isomorphism of schemes $U_1 \simeq U_2$.

The main result of this poster is the following generalization of the above theorem (2).

Main result(S.)

We use the same symbols as in the above theorem. When $k \simeq \overline{\mathbb{F}_p}$, $p \neq 2$, $g_{U_1} = 1$ and $n_{U_1} = 1$, there is an isomorphism of schemes $U_1 \simeq U_2$.

Addition on $\mathbb{P}^1(k) \backslash \{\infty\}$

Fix $P_0, P_\infty \in \mathbb{P}^1(k)$ s.t. $P_0 \neq P_\infty$. Then an isomorphism $\phi : \mathbb{P}^1 \simeq \mathbb{P}^1$ s.t. $\phi(P_0) = 0$ and $\phi(P_\infty) = \infty$ is unique up to scalar multiplication. The bijection $\mathbb{P}^1(k) \backslash \{P_\infty\} \simeq \mathbb{P}^1 \backslash \{\infty\} \simeq k$ given by such an isomorphism defines addition on $\mathbb{P}^1(k) \backslash \{P_\infty\}$.

The idea of proof of the main result

Let $(E_1, \mathcal{O}_1), (E_2, \mathcal{O}_2)$ be elliptic curves over $\overline{\mathbb{F}_p}$ defined by $y^2 = x(x-1)(x-\lambda_i)$ $(i = 1, 2)$ respectively s.t. $\pi_1(E_1 \backslash \{\mathcal{O}_1\}) \simeq \pi_1(E_2 \backslash \{\mathcal{O}_2\})$.

First we consider the mulplication-by-m map (i.e. $[m] : E_i \backslash E_i[m] \to E_i \backslash \{\mathcal{O}_i\}$) and the projection to x-line

$$\begin{array}{ccc} E_i \backslash E_i[m] & \to & \mathbb{P}^1 \backslash S \\ {\scriptstyle [m]}\downarrow & & \\ E_i \backslash \{\mathcal{O}_i\} & & \end{array}$$

(here, S stands for the image of $E_i[m]$), and we prove that the isomorphism $\pi_1(E_1 \backslash \{\mathcal{O}_1\}) \simeq \pi_1(E_2 \backslash \{\mathcal{O}_2\})$ preserve that linear relations of elements of S with respect to the addition on $\mathbb{P}^1 \backslash \{\infty\}$ corresponds to $P_0 = 0$ and $P_\infty = \infty$ (see the above block). Then we use linear relations and the addition formula of elliptic curves, and prove that the minimal polynomials of λ_1 and λ_2 on $\mathbb{F}_p$ are equal. Thus $E_1 \backslash \{\mathcal{O}_1\}$ and $(E_2 \backslash \{\mathcal{O}_2\}) \times_\phi \overline{\mathbb{F}_p}$ ($\phi : \overline{\mathbb{F}_p} \simeq \overline{\mathbb{F}_p}$ s.t. $\phi(\lambda_1) = \lambda_2$) are expressed by the same equation and the cusps are equal to ∞.

- [M] S.Mochizuki, The local pro-p anabelian geometry of curves, Invent. Math. 138 (1999), 319-423.
- [T] A.Tamagawa, On the Fundamental Groups of Curves over Algebraically Closed Fields of Characteristic > 0, Internat. Math. Res. Notices (1999), no.16, 853-873.