μ_n-actions on K3 surfaces in positive characteristic

Yuya Matsumoto (matsumoto.yuya@math.nagoya-u.ac.jp)

Kinosaki Algebraic Geometry Symposium 2017
2017/10/23–27
http://arxiv.org/abs/1710.07158

Abstract: We study μ_p-actions on K3 surfaces in char p.

NB: Smooth K3 surfaces admit no μ_p-actions since they admit no global derivations. However RDP K3 surfaces may admit some.

Preliminaries

• A *K3 surface* is a proper smooth (algebraic) surface X over a field with $\Omega^2_X \cong \mathcal{O}_X$ and $H^0(X, \mathcal{O}_X) = 0$.
• An *RDP K3 surface* is a proper surface X with only RDP (rational double point) singularities whose resolution $\tilde{X}$ is a K3.
Definition. An automorphism of a K3 X is **symplectic** if it acts on the (1-dim vector space) $H^0(X, \Omega^2_X)$ trivially.

Nikulin: actions in char 0

G: finite abelian group, X: K3.
$G \curvearrowright X$: a *symplectic* action.
Then • $\mathrm{Fix}(G)$ is isolated.
• X/G is an RDP K3.
• If $G = \mathbb{Z}/n\mathbb{Z}$ with $n > 1$, then $n \le 8$ and

$$\#\mathrm{Fix}(G) = \frac{24}{n} \prod_{l:\mathrm{prime}, l \mid n} \frac{l}{l+1}$$

$= 8, 6, 4, 4, 2, 3, 2 \quad (n = 2, 3, 4, 5, 6, 7, 8)$.

Non-symplectic quotients

are either birational to Enriques, or rational.

Actions in char p?

Nikulin's results hold in char $p > 0$ provided the order of G is prime to p.
However, **automorphisms of order p are automatically symplectic**, since there are no nontrivial p-th root of unity in char p.
• $\exists$ order p auto with 1-dim fixed locus,
• $\exists$ order p auto with non-K3 quotient.
Remark. $\exists$ order p auto only if $p \le 11$.
For more discussions see Dolgachev–Keum.

Keum: Orders of auto.

$S_{\mathrm{cyc}}(p) := \{n \mid \mathbb{Z}/n\mathbb{Z} \curvearrowright \exists X \text{ K3 in char } p\}$.
Keum determined this set for $p \ne 2, 3$.
• $S_{\mathrm{cyc}}(0) = \{n \mid \phi(n) \le 20\}$
$= \{1, \dots, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 36, 38, 40, 42, 44, 48, 50, 54, 60, 66\}$,
• $S_{\mathrm{cyc}}(p) = S_{\mathrm{cyc}}(0) \setminus E_p$ if $p \ge 5$, where

$$E_p = \begin{cases} \{p, 2p\} & \text{if } p = 13, 17, 19, \\ \{44\} & \text{if } p = 11, \\ \{25, 50, 60\} & \text{if } p = 5, \\ \emptyset & \text{if } p = 7 \text{ or } p \ge 23. \end{cases}$$

• Moreover the prime-to-p elements of $S_{\mathrm{cyc}}(p)$ coincide with those of $S_{\mathrm{cyc}}(0)$, for all $p \ge 2$.

Main Def: symplecticness

Action $\mu_n \curvearrowright \operatorname{Spec} B$ (affine scheme)
$\longleftrightarrow$ a decomposition $B = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} B_i$ of vector spaces satisfying $B_i B_j \subset B_{i+j}$.
$\longrightarrow$ a decomp. $\Omega^*_{B/k} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} (\Omega^*_{B/k})_i$.
We say that $\mathrm{wt}(b) = i$ if $b \in B_i$.
Similar for μ_n-actions on schemes.
Remark. If $p \nmid n$, then μ_n-action is equivalent to the action of the cyclic group $\mu_n(k)$, and B_i is the eigenspace for the $\mu_n(k)$-action with eigenvalue i: $\mu_n(k) \ni g \mapsto g^i \in k^*$.
Definition. We call an action $\mu_n \curvearrowright X$ on an RDP K3 to be **symplectic** if the decomposition of (1-dim vector space) $H^0(X^{\mathrm{sm}}, \Omega^2_X)$ is concentrated on $i = 0 \in \mathbb{Z}/n\mathbb{Z}$.
Remark. $H^0(X^{\mathrm{sm}}, \Omega^2_X) \cong H^0(\tilde{X}, \Omega^2_{\tilde{X}})$ for an RDP K3 X (hence 1-dim).
Remark. Equivalent to the classical definition if $p \nmid n$ (in which case $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$).

Theorem A

X: RDP K3, with an action $\mu_n \curvearrowright X$.
• symplectic $\implies X/\mu_n$: RDP K3.
• non-symplectic
$\implies X/\mu_n$: RDP Enriques or rational.
• $n = p$, non-symplectic, fixed-point-free
$\implies X/\mu_p$: RDP Enriques $\implies p = 2$.
• $n = p$, non-symplectic, not fixed-point-free
$\implies X/\mu_p$: rational surface.

Theorem B

X: RDP K3.
$\mu_n \curvearrowright X$: *symplectic* action, $n > 1$.
Then • $n \le 8$, • $\mathrm{Fix}(G)$ is isolated,
• $\#\mathrm{Fix}(G)$ = same as in Nikulin's theorem, when counted with suitable multiplicity.

Theorem C

$S_\mu(p) := \{n \mid \mu_n \curvearrowright \exists X \text{ RDP K3 in char } p\}$.
• $S_\mu(0) = S_{\mathrm{cyc}}(0) = \downarrow$.
• $S_\mu(p) = S_\mu(0) \setminus E'_p$, where

$$E'_p = \begin{cases} \{33, 66\} & \text{if } p = 11, \\ \{25, 40, 50\} & \text{if } p = 5, \\ \{27, 33, 48, 54, 66\} & \text{if } p = 3, \\ \{34, 40, 44, 48, 50, 54, 66\} & \text{if } p = 2, \\ \emptyset & \text{otherwise.} \end{cases}$$

• In particular, $\exists$ RDP K3 with a μ_p-action in char $p \iff p \le 19$.

Proof of A (symplectic case)

Suffices to consider (symplectic) μ_p-actions. Write $\pi \colon X \to Y = X/\mu_p$. Then $\mathcal{O}_Y = (\mathcal{O}_X)_0$.
Lemma. Suppose $z \in X$: a fixed point (smooth or RDP), and μ_p-action symplectic at z. Then z is an isolated fixed point, and $\pi(z)$ is an RDP.
Proof. If z is smooth, then $\exists x, y$: coordinate with weight $a, b \in \mathbb{Z}/p\mathbb{Z}$, $\ne 0$. We have $a + b = 0$ since symplectic. Then $\hat{\mathcal{O}}_{Y,\pi(z)} = k[[x^p, xy, y^p]]$: RDP of type A_{p-1}.
If z is an RDP, consider $\mu_p \curvearrowright \mathrm{Bl}_z X$ and reduce to the smooth case.
Cf: action of finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.
Lemma. Suppose $z \in X$: a non-fixed RDP. Then $\pi(z)$ is smooth or RDP.
Remark. This is a new feature.
Example. A_{pm-1} $(xy + z^{pm} = 0)$, with action $\mathrm{wt}(x, y, 1+z) = (0, 0, 1)$: quotient is A_{m-1} $(xy + Z^m = 0$ where $Z = z^p)$.
Lemma. Outside the fixed locus and the RDPs, $(\pi_*(\Omega^2_X))_0 \cong \Omega^2_Y$.

From the lemmas, Y has only RDPs as singularities, and $H^0(Y^{\mathrm{sm}}, \Omega^2_Y)$ is 1-dim.

Proof of C

If $p \nmid n$, this follows from Keum's result. We consider the case $p \mid n$. Write $n = p^e r$, $p \nmid r$. X has an RDP z (since a smooth K3 has no μ_p-action). We may assume it is not fixed by μ_{p^e} (otherwise, blow it up).
We classify all non-fixed actions on RDPs. and check that $n \in S_\mu(0) \setminus E'_p$ in all cases.
Typical case: $p \ne 2$, the μ_r-orbit of z has $r/2$ elements, each of type A_{p^e-1}, and μ_2 acts on z non-symplectically. In this case we deduce $(r/2)(p^e - 1) < b_2(\mathrm{K3}) = 22$. This almost implies $n \in S_\mu(0) \setminus E'_p$, and we kill few exceptions individually.

Examples for C: $n = 34$

$X(p)$: $y^2 = x^3 + t^7 x + t^2$ in char p.
• $X(0)$ is Kondo's example of an RDP K3 (A_1, A_2) with a μ_{34}-action $(\mathrm{wt}(x, y, t) = (4, 23, 6))$ or equivalently a $\mathbb{Z}/34\mathbb{Z}$-action.
• $X(17)$ is an RDP K3 (A_1, A_2, A_{16}) with a μ_{34}-action (but not with a $\mathbb{Z}/34\mathbb{Z}$-action).
• $X(2)$ is not an RDP K3 (a non-RDP singularity the at origin). The true char 2 reduction of $X(0)$ is $y'(y' - t') = x'^3 + t'^7 x'$, and does not admit a μ_{34}-action (but instead admits a $\mathbb{Z}/34\mathbb{Z}$-action).

References

[1] V. V. Nikulin, *Finite automorphism groups of Kähler K3 surfaces*, Trudy Moskov. Mat. Obshch. **38** (1979), 75–137 (Russian). English translation: Trans. Moscow Math. Soc. **1980**, no. 2, 71–135.

[2] Shigeyuki Kondō, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), no. 1, 75–98.

[3] Igor V. Dolgachev and JongHae Keum, *Finite groups of symplectic automorphisms of K3 surfaces in positive characteristic*, Ann. of Math. (2) **169** (2009), no. 1, 269–313.

[4] JongHae Keum, *Orders of automorphisms of K3 surfaces*, Adv. Math. **303** (2016), 39–87.

[5] Yuya Matsumoto, *μ_n-actions on K3 surfaces in positive characteristic* (2017), available at http://arxiv.org/abs/1710.07158.