Tropical Ideals, genera of tropicalization of curves and the minimum finishing time of projective networks

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~Tropical Ideals~

●Background

- For a tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$, the tropical variety $\mathbf{V}(f) \subset \mathbb{T}^n$ is the support of a finite polyhedral complex in $\mathbb{T}^n$.

- If we define the tropical variety $\mathbf{V}(I)$ defined by an ideal I in $\mathbb{T}[x_1, \dots, x_n]$ as
$$\mathbf{V}(I) = \bigcap_{f \in I} \mathbf{V}(f),$$
then the variety $\mathbf{V}(I)$ is **not** always the support of a finite polyhedral complex (Example 5.14 in [1]).

- Maclagan and Rincón defined **tropical ideals** in [1]. The tropical variety defined by a tropical ideal is the support of a finite polyhedral complex.

●Motivations

It is difficult to treat tropical ideals like classical one because they are not closed under the addition, multiplication or intersection.
(We cannot even "generate" a tropical ideal from an arbitrary set of tropical polynomials.)

⇩

We want to make another "tropical ideals" such that
(a) The tropical variety defined by any of them is the support of a finite polyhedral complex,
(b) They are closed under the addition, multiplication and intersection.

●Notation

- $\mathbb{T}$: **Tropical semifield**.
  i.e. the semifield $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where
  $a \oplus b := \max\{a, b\}$ (addition)
  $a \odot b := a + b$ (multiplication).

- $\mathbb{T}[x_1, \dots, x_n]$: **Tropical polynomial semiring**

[1] D. Maclagan and F. Rincón, Tropical ideals, Discrete Mathematics & Theoretical Computer Science proc. **BC** (2016), pp. 803-814.

●Definition

The **tropical polynomial function semiring** is the quotient semiring $\mathbb{T}[x_1, \dots, x_n]/\sim$, where $\sim$ is defined as
$$f \sim g \Leftrightarrow f(\boldsymbol{a}) = g(\boldsymbol{a}) \quad \forall \boldsymbol{a} \in \mathbb{R}^n.$$

The **tropical variety** $\mathbf{V}(f)$ defined by a tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ is
$$\mathbf{V}(f) = \left\{ \mathbf{a} \in \mathbb{T}^n \,\middle|\, \begin{array}{l}\text{The maximum of } f(\mathbf{a}) \text{ is attained at} \\ \text{least twice or } f(\mathbf{a}) = -\infty.\end{array} \right\}.$$

We may define the tropical variety $\mathbf{V}(\varphi)$ just for a tropical polynomial function φ.

We denote by $[f]_{\mathbf{x^u}}$ the coefficient of the monomial $\mathbf{x^u}$ in a tropical polynomial f.

The **maximum representation** $\varphi^{\max}$ of a tropical polynomial function φ is the representation f such that
$$[f]_{\mathbf{x^u}} \geq [g]_{\mathbf{x^u}}$$
for any representation g of φ and any monomial $\mathbf{x^u}$.

Each tropical polynomial function φ has a unique maximum representation.

An ideal I in $\mathbb{T}[x_1, \dots, x_n]/\sim$ is a **tropical ideal** if for any $\varphi, \psi \in I$ and any monomial $\mathbf{x^u}$ with $[\varphi^{\max}]_{\mathbf{x^u}} = [\psi^{\max}]_{\mathbf{x^u}} \neq -\infty$, there is a tropical polynomial h such that,
(1) the class of h is in I,
(2) $[h]_{\mathbf{x^u}} = -\infty$,
(3) $[h]_{\mathbf{x^v}} \leq [\varphi^{\max}]_{\mathbf{x^v}} \oplus [\psi^{\max}]_{\mathbf{x^v}}$ for all $\mathbf{v}$, and
(4) $[h]_{\mathbf{x^v}} = [\varphi^{\max}]_{\mathbf{x^v}} \oplus [\psi^{\max}]_{\mathbf{x^v}}$ if $[\varphi^{\max}]_{\mathbf{x^v}} \neq [\psi^{\max}]_{\mathbf{x^v}}$.

· **Maclagan and Rincón's definition of tropical ideals**

An ideal I in $\mathbb{T}[x_1, \dots, x_n]$ is a **tropical ideal** if for any $f, g \in I$ and any monomial $\mathbf{x^u}$ with $[f]_{\mathbf{x^u}} = [g]_{\mathbf{x^u}} \neq -\infty$, there is a tropical polynomial h such that,
(1) $h \in I$,
(2) $[h]_{\mathbf{x^u}} = -\infty$,
(3) $[h]_{\mathbf{x^v}} \leq [f]_{\mathbf{x^v}} \oplus [g]_{\mathbf{x^v}}$ for all $\mathbf{v}$, and
(4) $[h]_{\mathbf{x^v}} = [f]_{\mathbf{x^v}} \oplus [g]_{\mathbf{x^v}}$ if $[f]_{\mathbf{x^v}} \neq [g]_{\mathbf{x^v}}$.

●Main Results

Theorem 1. (principal ⇒ tropical)
For any tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$, the set $\varphi \odot \mathbb{T}[x]/\sim \; := \{\varphi \odot \psi \mid \psi \in \mathbb{T}[x]/\sim\}$ is a tropical ideal in $\mathbb{T}[x]/\sim$.

Theorem 2. (like PID)
Every tropical ideal in $\mathbb{T}[x]/\sim$ is of the form $\varphi \odot \mathbb{T}[x]/\sim$ for some $\varphi \in \mathbb{T}[x]/\sim$.

Corollary 3. (intersect & generate)
Tropical ideals in $\mathbb{T}[x]/\sim$ are closed under the intersection. Hence for any set S of tropical polynomial functions, there is the minimum tropical ideal including S.

~ Genera of tropicalization of curves ~

●Motivations

Let C be a projective curve over a valued field K.

Q. If K has "many" valuations, can we find the tropicalization whose genus is equal to the one of C by varying valuations?

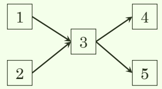
●Theorem

K : an algebraic function field
C : an elliptic curve on K with nonconstant j-invariant.

Theorem.
In this situation,
$\exists K'$: a finite extension of K,
$\exists C' \hookrightarrow \mathbb{P}^2_{K'}$: an embedding, where C' is the scalar extension of C to K', and
$\exists v$: a valuation on K'
such that the tropicalization of C' via v has the genus 1.

~The minimum finishing time of projective networks~

A **project network** consists of some activities, where each activity can be started after all the preceding activities have finished.



In the above project network , let t_i be the time to complete the activity. Then the **minimum finishing time** of this project network is
$$\max\{t_1 + t_3 + t_4,\; t_1 + t_3 + t_5,\; t_2 + t_3 + t_4,\; t_2 + t_3 + t_5\}$$
$$= t_1t_3t_4 \oplus t_1t_3t_5 \oplus t_2t_3t_4 \oplus t_2t_3t_5 \text{ (in tropical notation)},$$
which is a tropical polynomial of t_i's.

Q. What kind of tropical polynomials can be realized as the minimum finishing time of a project network?

Definition
A ***P*-polynomial** is a tropical polynomial $f(t)$ such that
(1) the degree on each variable is exactly one,
(2) the coefficient of each term is a unity,
(3) no term is divisible by any other terms.

Definition
A P-polynomial $f(t) = f(t_1, \dots, t_n)$ has **term extendability** if, for any subset $I \subset [n]$ such that $\forall i, j \in I$, there is a term of $f(t)$ divisible by t_it_j, there is a term of $f(t)$ divisible by $\prod_{i \in I} t_i$.

Theorem 1.
There is a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \dots, t_n)$ having term extendability and the set of simple graphs with the vertex set $[n]$.

We denote by $\mathrm{TG}(f)$ the simple graph corresponding to $f(t)$.

Theorem 2.
Let $f(t)$ be a P-polynomial of degree d having term extendability. Then $f(t)$ is realizable if and only if there is a vertex coloring of $\mathrm{TG}(f)$ with the color set $\{1, \dots, d\}$ such that if v_1, v_2, v_3 is colored by c_1, c_2, c_3 with $c_1 < c_2 < c_3$ and $\{v_1, v_2\}$, $\{v_2, v_3\}$ are adjacent respectively, then $\{v_1, v_3\}$ is also adjacent.

[1] Ito, T., A characterization for tropical polynomials being the minimum finishing time of project networks, (2017) Hokkaido Mathmatical Journal, in press.

[2] Kobayashi, M. and Odagiri, S., Tropical geometry of PERT, Journal of Math-for-Industry, Vol. 5 (2013B-8), pp. 145-149