Surfaces with big anti-canonical divisors and some related problems

Rikito Ohta (Osaka University)

usamarotokame@gmail.com

1 Background

• **k**: field of characteristic $p \ge 0$

• X: smooth projective surface over \mathbf{k} such that $-K_X$ is big **Theorem 1.1** (=[4, Theorem 1]). If X is rational, X is a Mori dream space. Question 1.2.

What if X is NOT rational?

 $\Rightarrow \kappa(X) = -\infty$

 $\Rightarrow X$ is obtained by repeatedly blowing up a geometrically ruled surface $\mathbb{P}_C(E) \to C$ with big anti-canonical line bundle.

– <u>AIM</u> —

- (1) Classify geometrically ruled surfaces with big anti-canonical bundle (Theorem 3.1).
- (2) Apply (1) to show that the *Picard group* of (weak) del Pezzo pairs (X, Δ) is finitely generated and torsion free (Corollary 4.2). This is a version of the base point free theorem.

2 Stability of vector bundles on curves

 ${\cal C}$: smooth projective curve

E : vector bundle on C

 $F\colon C\to C$: the Frobenius map

Definition 2.1.

E is *strongly semi-stable* if the vector bundle $(F^e)^*E$ over *C* is semi-stable for all $e \ge 0$.

Remark 2.2.

 $char(\mathbf{k}) = 0 \Rightarrow$ semi-stable = strongly semi-stable.

We use the following standard facts.

Proposition 2.3.

(1) E: strongly semi-stable \Rightarrow the n-th symmetric power $S^n(E)$ is also strongly semi-stable.

(2) If $g(C) \leq 1$, semi-stable = strongly semi-stable.

Proposition 2.4 (O-Okawa, [1]).

Suppose $g(C) \ge 1$, rank E = 2. If $-K_{\mathbb{P}_{C}(E)}$ is big, then E is not strongly semi-stable.

3 Geometrically ruled surface with big anticanonical divisor

Theorem 3.1 (O-Okawa, [1], Main theorem I).

Suppose $g(C) \ge 1$, rank E = 2, and E: unstable. Then $-K_X$ is big \iff there are line bundles L and M such that $E \simeq L \oplus M$ and $\deg L - \deg M > 2g - 2$.

Remark 3.2.

If $char(\mathbf{k}) = 0$ or g(C) = 1, we can omit the assumption that E is unstable by Proposition 2.3 and Proposition 2.4.

Corollary 3.3.

Suppose $g(C) \geq 1$ and rank E = 2. If $-K_{\mathbb{P}_{C}(E)}$ is big, there exists an integer $e \geq 0$ and line bundles L', M' such that $(F^{e})^{*} E \simeq L' \oplus M'$ and $\deg L' - \deg M' > 2g - 2$. The question below remains open.

Question 3.4.

Is E itself unstable under the same assumption?

4 Picard group of log del Pezzo surfaces via Theorem 3.1

Theorem 4.1 (O-Okawa, [1]).

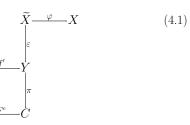
Let (X, Δ) be a pair of a normal projective surface and an effective \mathbb{R} -divisor such that $-(K_X + \Delta)$ is \mathbb{R} -Cartier. If either $\bullet \lfloor \Delta \rfloor = 0$ and $-(K_X + \Delta)$ is nef and big, or

• $\Delta \leq 1$ and $-(K_X + \Delta)$ is ample,

then there is a positive integer e = e(X) such that $L^{\otimes e} \simeq \mathcal{O}_X$ for any numerically trivial line bundle L on X.

- Rough sketch of the proof -

Take the minimal resolution $\widetilde{X} \to X$. One can easily check that $-K_{\widetilde{X}}$ is big, so that there is a birational morphism $\varepsilon \colon \widetilde{X} \to Y = \mathbb{P}_{C}(E)$ such that $-K_{Y}$ is also big. Then we can apply Corollary 3.3 to E and obtain the following diagram.



When \widetilde{X} is not rational, consider the section C' of π' corresponding to $(F^e)^* E \twoheadrightarrow M'$. It follows that $\varepsilon_*^{-1} f'(C')$ is contracted by φ . One easily sees that $\varphi^* L \simeq \varepsilon^* \pi^* L_C$ for some line bundle L_C on C, but the contractibility implies that $((f')^* \pi^* L_C)|_{C'} \simeq \mathcal{O}_{C'}$. Hence $L^{\otimes p^e} \simeq \mathcal{O}_X$.

Corollary 4.2 (O-Okawa, [1], Main theorem II).

Let (X, Δ) be as in Theorem 4.1. Then $\operatorname{Pic}(X)$ is a free abelian group of finite rank.

Remark 4.3.

Corollary 4.2 is shown in [3, Corollary 3.6] when X has only rational singularities. In fact, the rest is covered by [2]. Our proof as an application of Theorem 3.1 is more straightforward.

Remark 4.4.

The assumptions of Theorem 4.1 are optimal. In fact, consider a smooth cubic curve $C \subset \mathbb{P}^2$ and $X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(1))$. The projective cone over C is obtained from X by contracting the negative section E. Then $(X, \Delta = E)$ is a pair such that $-(K_X + \Delta)$ is nef and big, but Pic (X) is not a free abelian group.

References

- R. Ohta and S. Okawa. On ruled surfaces with big anticanonical divisor and numerically trivial line bundle on weak log fano surfaces. in preparation.
- [2] S. Schröer. Normal del Pezzo surfaces containing a nonrational singularity. Manuscripta Math., 104(2):257–274, 2001.
- [3] H. Tanaka. The X-method for klt surfaces in positive characteristic. J. Algebraic Geom., 24(4):605–628, 2015.
- [4] D. Testa, A. Várilly-Alvarado, and M. Velasco. Big rational surfaces. Math. Ann., 351(1):95– 107, 2011.