Surfaces with big anti-canonical divisors and some related problems

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1 Background

• $\mathbf{k}$: field of characteristic $p \geq 0$

• X: smooth projective surface over $\mathbf{k}$ such that $-K_X$ *is big*

Theorem 1.1 (=[4, Theorem 1]).
If X is rational, X is a Mori dream space.

Question 1.2.
What if X is *NOT* rational?

$\Rightarrow \kappa(X) = -\infty$

$\Rightarrow X$ is obtained by repeatedly blowing up a geometrically ruled surface $\mathbb{P}_C(E) \to C$ *with big anti-canonical line bundle.*

AIM

(1) Classify geometrically ruled surfaces with big anti-canonical bundle (Theorem 3.1).

(2) Apply (1) to show that the *Picard group* of (weak) del Pezzo pairs (X, Δ) is finitely generated and torsion free (Corollary 4.2). This is a version of the base point free theorem.

2 Stability of vector bundles on curves

C : smooth projective curve
E : vector bundle on C
$F: C \to C$: the Frobenius map

Definition 2.1.
E is *strongly semi-stable* if the vector bundle $(F^e)^*E$ over C is semi-stable for all $e \geq 0$.

Remark 2.2.
$\mathrm{char}(\mathbf{k}) = 0 \Rightarrow$ semi-stable = strongly semi-stable.

We use the following standard facts.

Proposition 2.3.
(1) E: strongly semi-stable $\Rightarrow$ the n-th symmetric power $S^n(E)$ is also strongly semi-stable.
(2) If $g(C) \leq 1$, semi-stable = strongly semi-stable.

Proposition 2.4 (O-Okawa, [1]).
Suppose $g(C) \geq 1$, $\mathrm{rank}\, E = 2$. If $-K_{\mathbb{P}_C(E)}$ is big, then E is not strongly semi-stable.

3 Geometrically ruled surface with big anti-canonical divisor

Theorem 3.1 (O-Okawa, [1], Main theorem I).
Suppose $g(C) \geq 1$, $\mathrm{rank}\, E = 2$, and E: unstable. Then $-K_X$ is big $\iff$ there are line bundles L and M such that $E \simeq L \oplus M$ and $\deg L - \deg M > 2g - 2$.

Remark 3.2.
If $\mathrm{char}(\mathbf{k}) = 0$ or $g(C) = 1$, we can omit the assumption that E is unstable by Proposition 2.3 and Proposition 2.4.

Corollary 3.3.
Suppose $g(C) \geq 1$ and $\mathrm{rank}\, E = 2$. If $-K_{\mathbb{P}_C(E)}$ is big, there exists an integer $e \geq 0$ and line bundles L', M' such that $(F^e)^* E \simeq L' \oplus M'$ and $\deg L' - \deg M' > 2g - 2$.

The question below remains open.

Question 3.4.
Is E itself unstable under the same assumption?

4 Picard group of log del Pezzo surfaces via Theorem 3.1

Theorem 4.1 (O-Okawa, [1]).
Let (X, Δ) be a pair of a normal projective surface and an effective $\mathbb{R}$-divisor such that $-(K_X + \Delta)$ is $\mathbb{R}$-Cartier. If either
• *$\lfloor \Delta \rfloor = 0$ and $-(K_X + \Delta)$ is nef and big, or*
• *$\Delta \leq 1$ and $-(K_X + \Delta)$ is ample,*
then there is a positive integer $e = e(X)$ such that $L^{\otimes e} \simeq \mathcal{O}_X$ for any numerically trivial line bundle L on X.

Rough sketch of the proof

Take the minimal resolution $\widetilde{X} \to X$. One can easily check that $-K_{\widetilde{X}}$ is big, so that there is a birational morphism $\varepsilon\colon \widetilde{X} \to Y = \mathbb{P}_C(E)$ such that $-K_Y$ is also big. Then we can apply Corollary 3.3 to E and obtain the following diagram.

$$\begin{tikzcd}
 & \widetilde{X} \arrow[r, dash, "\varphi"] \arrow[d, dash, "\varepsilon"] & X \\
Y' \arrow[r, dash, "f'"] \arrow[d, dash, "\pi'"] & Y \arrow[d, dash, "\pi"] & \\
C \arrow[r, dash, "F^e"] & C &
\end{tikzcd} \tag{4.1}$$

When $\widetilde{X}$ is not rational, consider the section C' of π' corresponding to $(F^e)^* E \twoheadrightarrow M'$. It follows that $\varepsilon_*^{-1} f'(C')$ is contracted by φ. One easily sees that $\varphi^* L \simeq \varepsilon^* \pi^* L_C$ for some line bundle L_C on C, but the contractibility implies that $\left((f')^* \pi^* L_C\right)|_{C'} \simeq \mathcal{O}_{C'}$. Hence $L^{\otimes p^e} \simeq \mathcal{O}_X$.

Corollary 4.2 (O-Okawa, [1], Main theorem II).
Let (X, Δ) be as in Theorem 4.1. Then $\mathrm{Pic}(X)$ is a free abelian group of finite rank.

Remark 4.3.
Corollary 4.2 is shown in [3, Corollary 3.6] when X has only rational singularities. In fact, the rest is covered by [2]. Our proof as an application of Theorem 3.1 is more straightforward.

Remark 4.4.
The assumptions of Theorem 4.1 are optimal. In fact, consider a smooth cubic curve $C \subset \mathbb{P}^2$ and $X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(1))$. The projective cone over C is obtained from X by contracting the negative section E. Then $(X, \Delta = E)$ is a pair such that $-(K_X + \Delta)$ is nef and big, but $\mathrm{Pic}(X)$ is not a free abelian group.

References

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