The moduli spaces of parabolic connections with a quadratic differential and isomonodromic deformations

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1. Introduction

Let $(C,\boldsymbol{t})$ ($\boldsymbol{t}=(t_1,\dots,t_n)$) be an n-pointed smooth projective curve of genus g over $\mathbb{C}$ where $t_1,\dots,t_n$ are distinct points. We take a positive integer r, and take an element $\boldsymbol{\nu}=(\nu_j^{(i)})_{0\le j\le r-1}^{1\le i\le n}\in\mathbb{C}^{nr}$ such that $\sum_{i,j}\nu_j^{(i)}=-d\in\mathbb{Z}$. Put $D=t_1+\cdots+t_n$.

Definition of parabolic connections

We say $(E,\nabla,\boldsymbol{l})$ ($\boldsymbol{l}=\{l_*^{(i)}\}_{1\le i\le n}$) is a *$\boldsymbol{\nu}$-parabolic connection* of rank r on $(C,\boldsymbol{t})$ if

- E is a rank r algebraic vector bundle of degree d on C,
- $\nabla\colon E\to E\otimes\Omega_C^1(t_1+\cdots+t_n)$ is a connection, and
- for each t_i, $l_*^{(i)}$ is a filtration $E|_{t_i}=l_0^{(i)}\supset l_1^{(i)}\supset\cdots\supset l_r^{(i)}=0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)})=1$ and $(\mathrm{res}_{t_i}(\nabla)-\nu_j^{(i)}\mathrm{id}_{E|_{t_i}})(l_j^{(i)})\subset l_{j+1}^{(i)}$ for $j=0,\dots,r-1$.

Take rational numbers

$$0<\alpha_1^{(i)}<\alpha_2^{(i)}<\cdots<\alpha_r^{(i)}<1$$

for $i=1,\dots,n$ satisfying $\alpha_j^{(i)}\neq\alpha_{j'}^{(i')}$ for $(i,j)\neq(i',j')$. We choose a sufficiently generic $\boldsymbol{\alpha}=(\alpha_j^{(i)})$.

$\boldsymbol{\alpha}$-stability of parabolic connections

A parabolic connection $(E,\nabla,\boldsymbol{l})$ is *$\boldsymbol{\alpha}$-stable* if for any proper nonzero subbundle $F\subset E$ satisfying $\nabla(F)\subset F\otimes\Omega_C^1(t_1+\cdots+t_n)$, the inequality

$$\frac{\deg F+\sum_{i=1}^n\sum_{j=1}^r\alpha_j^{(i)}\dim((F|_{t_i}\cap l_{j-1}^{(i)})/(F|_{t_i}\cap l_j^{(i)}))}{\mathrm{rank}F}<\frac{\deg E+\sum_{i=1}^n\sum_{j=1}^r\alpha_j^{(i)}\dim(l_{j-1}^{(i)}/l_j^{(i)})}{\mathrm{rank}E}$$

holds.

Moduli space of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections [2]

$$M^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu}):=\left\{((C,\boldsymbol{t}),(E,\nabla,\boldsymbol{l}))\;\middle|\;\begin{array}{l}(E,\nabla,\boldsymbol{l})\text{ is }\boldsymbol{\alpha}\text{-stable}\\ \boldsymbol{\nu}\text{-parabolic connection on }(C,\boldsymbol{t})\end{array}\right\}/\sim$$

Moduli space of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections with a quadratic differential

$$\begin{array}{ccc}\widehat{M}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu}) & \rightarrow & M^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})\\ {\scriptstyle\hat{\pi}}\downarrow & & {\scriptstyle\pi}\downarrow\\ T^*\tilde{M}_{g,n} & \rightarrow & \tilde{M}_{g,n}.\end{array}$$

where $\tilde{M}_{g,n}$ is a smooth algebraic scheme which is a certain covering of the moduli stack of n-pointed smooth projective curves of genus g over $\mathbb{C}$.

2. Deformation theory of parabolic connections with a quadratic differential

$((C,\boldsymbol{t},\psi),(E,\nabla,\boldsymbol{l}))\in\widehat{M}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$, where $\psi\in H^0(C,\Omega_C^{\otimes2}(D))$.

Definition of $\mathcal{F}_C^0$ and $\mathcal{F}_C^1$

$$\widetilde{\mathcal{F}}_C^0:=\left\{s\in\mathcal{E}nd(E)\;\middle|\;s|_{t_i}(l_j^{(i)})\subset l_j^{(i)}\text{ for any }i,j\right\}$$

$$0\longrightarrow\widetilde{\mathcal{F}}_C^0\longrightarrow\mathcal{F}_C^0\xrightarrow{\mathrm{symb}_1}\Theta_C(-D)\longrightarrow0$$

$$\widetilde{\mathcal{F}}_C^1:=\left\{s\in\mathcal{E}nd(E)\otimes\Omega_C^1(D)\;\middle|\;\mathrm{res}_{t_i}(s)(l_j^{(i)})\subset l_{j+1}^{(i)}\text{ for any }i,j\right\};$$

$$\mathcal{F}_C^1:=\widetilde{\mathcal{F}}_C^1\oplus\Omega_C^{\otimes2}(D)$$

Transformation of elements of $\mathcal{F}_C^0$

$$a_i\frac{\partial}{\partial z_i}+\eta_i\longmapsto a_i\frac{\partial z_j}{\partial z_i}\frac{\partial}{\partial z_j}+\theta_{ij}^{-1}\eta_i\theta_{ij}+\theta_{ij}^{-1}\frac{\partial\theta_{ij}}{\partial z_i},$$

Section of symb_1 for $\nabla(=d+A_idz_i)$

$$\begin{aligned}\iota(\nabla)\colon\Theta_C(-D)&\longrightarrow\mathcal{F}_C^0\\ a_i\frac{\partial}{\partial z_i}&\longmapsto a_i\frac{\partial}{\partial z_i}+a_iA_i\end{aligned}$$

The complex $\mathcal{F}_C^\bullet$

$$d_{\mathcal{F}_C^\bullet}:=(d_\nabla,d_\psi)\circ(\mathrm{Id}-\iota(\tilde{\nabla})\circ\mathrm{symb}_1,\mathrm{symb}_1)\colon\mathcal{F}_C^0\longrightarrow\mathcal{F}_C^1,$$

where $d_\nabla\colon\widetilde{\mathcal{F}}_C^0\to\widetilde{\mathcal{F}}_C^1;\ s\mapsto\nabla\circ s-s\circ\nabla$ and $d_\psi\colon\Theta_C(-D)\to\Omega_C^{\otimes2}(D)$ defined by

$$a_i\frac{\partial}{\partial z_i}\longmapsto\left(\frac{\partial\psi_i}{\partial z_i}a_i+2\psi_i\frac{\partial a_i}{\partial z_i}\right)dz_i^{\otimes2}$$

$\boldsymbol{H}^1(C,\mathcal{F}_C^\bullet)$ is the tangent space of $\widehat{M}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$ at $((C,\boldsymbol{t},\psi),(E,\nabla,\boldsymbol{l}))$.

Symplectic structure

For $v=[(\{u_{ij}\},\{(v_i,w_i)\})]\in\boldsymbol{H}^1(C,\mathcal{F}_C^\bullet)$ and $w=[(\{u'_{ij}\},\{(v'_i,w'_i)\})]\in\boldsymbol{H}^1(C,\mathcal{F}_C^\bullet)$, we put

$$\omega_1(v,w)=[(\{\mathrm{Tr}(\eta(u_{ij})\circ\eta(u'_{jk}))\},-\{\mathrm{Tr}(\eta(u_{ij})\circ v'_j)-\mathrm{Tr}(v_i\circ\eta(u'_{ij}))\})]\text{ and}$$
$$\omega_2(v,w)=[\{2\cdot\mathrm{symb}_1(u_{ji})\circ\mathrm{symb}_1(u'_{jk})\circ\psi_j\},-\{\mathrm{symb}_1(u_{ji})\circ w'_j+w_i\circ\mathrm{symb}_1(u'_{ij})\}].$$

where $\eta(s):=s-\iota(\nabla)\circ\mathrm{symb}_1(s)$.

$$\omega:=\omega_1+\omega_2\colon\boldsymbol{H}^1(C,\mathcal{F}_C^\bullet)\otimes\boldsymbol{H}^1(C,\mathcal{F}_C^\bullet)\longrightarrow\boldsymbol{H}^2(C,\Omega_C^\bullet)\cong\mathbb{C}$$

3. Cotangent bundle on moduli space of curves with a bundle

Let $\mathfrak{P}_{g,n}(r,d)$ be the moduli stack of pairs $((C,\boldsymbol{t}),(E,\boldsymbol{l}))$, where $(C,\boldsymbol{t})$ ($\boldsymbol{t}=(t_1,\dots,t_n)$) is an n-pointed smooth projective curve of genus g over $\mathbb{C}$ where $t_1,\dots,t_n$ are distinct points, and $(E,\boldsymbol{l})$ is a quasi-parabolic bundle of rank r and of degree d on $(C,\boldsymbol{t})$.
Let $\mathfrak{P}_{g,n}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$ be the substack defined by the condition where a quasi-parabolic bundle admits an $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connection.

Tangent space at $((C,\boldsymbol{t}),(E,\boldsymbol{l}))$

$H^1(C,\mathcal{F}_C^0)$ is the tangent space of $\mathfrak{P}_{g,n}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$ at $((C,\boldsymbol{t}),(E,\boldsymbol{l}))$.

Cotangent space at $((C,\boldsymbol{t}),(E,\boldsymbol{l}))$

$$0\longrightarrow\Omega_C^{\otimes2}(D)\longrightarrow\mathcal{H}_C^1\longrightarrow(\widetilde{\mathcal{F}}_C^0)^*\otimes\Omega_C^1\longrightarrow0.$$

Note that $(\widetilde{\mathcal{F}}^0)^*\otimes\Omega_C^1\cong\widetilde{\mathcal{F}}_C^1$.
$H^0(C,\mathcal{H}_C^1)$ is the cotangent space of $\mathfrak{P}_{g,n}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$ at $((C,\boldsymbol{t}),(E,\boldsymbol{l}))$.

Transformation of elements of $\mathcal{H}_C^1$

$$\begin{aligned}\left(\Phi_i(z_i)dz_i,\phi_i(z_i)dz_i^{\otimes2}\right)&\longmapsto(\Phi_j(z_i)dz_i,\phi_j(z_i)dz_i^{\otimes2})\\ &:=\left((\theta_{ij}^{-1}\Phi_i(z_i)\theta_{ij})dz_i,\phi_i(z_i)df_i^{\otimes2}+\mathrm{Tr}\left(\theta_{ij}^{-1}\Phi_i(z_i)\frac{\partial\theta_{ij}}{\partial z_i}\right)dz_i^{\otimes2}\right),\end{aligned}$$

Extended parabolic Higgs bundles

We call $((C,\boldsymbol{t}),(E,\boldsymbol{l}),\widehat{\Phi})$, where $\widehat{\Phi}\in H^0(C,\mathcal{H}_C^1)$, an *extended parabolic Higgs bundle*. We call $\widehat{\Phi}\in H^0(C,\mathcal{H}_C^1)$ an *extended parabolic Higgs field*.

4. Deformation theory of extended parabolic Higgs bundles

The complex $\mathcal{H}_C^\bullet$

For an extended parabolic Higgs bundle $((C,\boldsymbol{t}),(E,\boldsymbol{l}),\widehat{\Phi})$ ($\widehat{\Phi}=(\Phi_idz_i,\phi_idz_i^{\otimes2})$ on U_i),

$$\begin{aligned}d_{\mathcal{H}_C^\bullet}\colon\mathcal{F}_C^0&\longrightarrow\mathcal{H}_C^1\\ a_i\frac{\partial}{\partial z_i}+\eta_i&\longmapsto\Big(\Phi_idz_i\circ\eta_i-\eta_i\circ\Phi_idz_i-\frac{\partial(a_i\Phi_i)}{\partial z_i}dz_i,\\ &\qquad\mathrm{Tr}\left(\frac{\partial\eta_i}{\partial z_i}\Phi_idz_i^{\otimes2}\right)-a_i\frac{\partial\phi_i}{\partial z_i}dz_i^{\otimes2}-2\frac{\partial a_i}{\partial z_i}\phi_idz_i^{\otimes2}\Big).\end{aligned}$$

$\boldsymbol{H}^1(C,\mathcal{H}_C^\bullet)$ is the space of first order deformations of $((C,\boldsymbol{t}),(E,\boldsymbol{l}),\widehat{\Phi})$.

Symplectic structure

$$\omega_H\colon\boldsymbol{H}^1(C,\mathcal{H}_C^\bullet)\otimes\boldsymbol{H}^1(C,\mathcal{H}_C^\bullet)\longrightarrow\boldsymbol{H}^2(C,\Omega_C^\bullet)\cong\mathbb{C}$$

which is defined by

$$\begin{aligned}&[(\{a_{ij}\partial/\partial f_i+\eta_{ij}\},\{(\hat{v}_i,\hat{w}_i)\})]\otimes\left[\left(\left\{a'_{ij}\partial/\partial f_i+\eta'_{ij}\right\},\{(\hat{v}'_i,\hat{w}'_i)\}\right)\right]\\ \longmapsto&\left[\left(\left\{\mathrm{Tr}(\eta_{ji}(a'_{jk}\Phi_j))+\mathrm{Tr}((a_{ji}\Phi_j)\eta'_{jk})-2a_{ji}a'_{jk}\phi_j\right\},\right.\right.\\ &\quad\left.\left.-\left\{-\mathrm{Tr}(\eta_{ji}\hat{v}'_j)+(a_{ji}\hat{w}'_j)-\mathrm{Tr}(\hat{v}_i\eta'_{ij})+(\hat{w}_ia'_{ij})\right\}\right)\right].\end{aligned}$$

5. Twisted cotangent bundle on moduli space of curves with a bundle

Definition of twisted cotangent bundles [1]

Let $T^*=T^*(X)\to X$ be the cotangent bundle on X. A *twisted cotangent bundle* on X is a T^*-torsor $\pi_\phi\colon\phi\to X$ (i.e., π_ϕ is a fibration equipped with a simple transitive action of T^* along the fibers) together with a symplectic form ω_ϕ on ϕ such that π_ϕ is a polarization for ω_ϕ (i.e., $\dim\phi=2\dim X$ and the Poisson bracket $\{\cdot,\cdot\}$ vanished on $\pi_\phi^{-1}\mathcal{O}_X$) and for any 1-form ν one has $t_\nu^*(\omega_\phi)=\pi_\phi^*d\nu+\omega$. Here $t_\nu\colon\phi\to\phi;\ t_\nu(a)=a+\nu_{\pi(a)}$ is the translation by ν.

Let T^* be the moduli stack of pairs $((C,\boldsymbol{t}),(E,\boldsymbol{l}),\widehat{\Phi})$, where $(C,\boldsymbol{t}),(E,\boldsymbol{l})\in\mathfrak{P}_{g,n}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$ and $\widehat{\Phi}$ is an extended parabolic Higgs field on $((C,\boldsymbol{t}),(E,\boldsymbol{l}))$:

$$T^*\longrightarrow\mathfrak{P}_{g,n}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu}).$$

Let $\widehat{\mathfrak{M}}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$ be the moduli stack corresponding to $\widehat{M}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$:

$$\pi_{\widehat{\mathfrak{M}}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})}\colon\widehat{\mathfrak{M}}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})\longrightarrow\mathfrak{P}_{g,n}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu}).$$

T^*-torsor structure on $\pi_{\widehat{\mathfrak{M}}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})}$

Action of an extended parabolic Higgs field $\widehat{\Phi}=(\Phi_idz_i,\phi_idz_i^{\otimes2})$ on a parabolic connection with quadratic differential $(d+A_idz_i,\psi_idz_i^{\otimes2})$:

$$(d+A_idz_i,\psi_idz_i^{\otimes2})\longmapsto\left(d+(A_i+\Phi_i)dz_i,\psi_idz_i^{\otimes2}+\phi_idz_i^{\otimes2}-\mathrm{Tr}\left(\Phi_iA_i+\frac{1}{2}\Phi_i\Phi_i\right)dz_i^{\otimes2}\right),$$

Main result

$(\pi_{\widehat{\mathfrak{M}}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})},\omega)$ *is a twisted cotangent bundle on* $\mathfrak{P}_{g,n}^{\boldsymbol{\alpha}}(r,d,\boldsymbol{\nu})$.

References

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