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Some Computations on Higher Nash Blowups of Toric Surfaces

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1 Background

Let $X$ be a variety over $k = \mathbb{F}$. The Nash blowup $(X^*, \pi)$ of $X$ was defined in [N]. For the Nash blowup, the following problem is suggested and studied: Is $X^*$ (or its normalization) a resolution of singularities of $X$?

Recently Higher Nash blowups was defined by A. Oneto & E. Zetini [OZ], T. Yasuda [Y1] independently, and analogous questions are studied.

Definition. (Yasuda’s construction [Y1])

Let $n > 0$ be an integer. Let $X_{sm} := X \setminus \operatorname{Sing}(X)$ and $\gamma_n : X_{sm} \to \operatorname{Hilb}(X)$, $P \mapsto [P^{(n)}]$ where $P^{(n)}$ is the $n$-th fat point whose support is $P$, and its graph $\Gamma_n : X_{sm} \to X \times \operatorname{Hilb}(X)$ with 1st projection $\pi_n : X \times \operatorname{Hilb}(X) \to X$.

Now let $\text{Nash}_n(X) :=$ (the schematic closure of $\Gamma_n(X_{sm})$) and $\pi_n := \pi_n|_{\text{Nash}_n(X)}$. Then $(\text{Nash}_n(X), \pi_n)$ is called the $n$-th Nash blowup of $X$. It is isomorphic to the classical Nash blowup when $n = 1$.

Regarding resolution of singularities of $X$, Yasuda suggested the following problem in [Y1]:

Problem.

Is $\text{Nash}_n(X)$ non-singular for all $n \gg 0$ when char $k = 0$?

Known Results. (T. Yasuda [Y1])

1. If char $k = 2, 3$, then there exists $X$ such that $\text{Nash}_n(X)$ is singular for all $n > 0$.

2. If char $k = 0$ and $X$ is a curve, then the answer is YES.

2 Main Result

Let $k = \mathbb{C}$ in what follows. Yasuda stated in [Y2] that $A_3$-singularity is probably a counter example for the problem, and here is my main result:

Main Result.

Let $X := (z^4 - xy = 0) \subset A^3$, which is a toric surface with an $A_3$-singular point $P = (0, 0, 0)$. Then $\text{Nash}_n(X)$ is singular for all $n > 0$.

3 Gröbner Bases in Subalgebras

The proof of main result is roughly explained below.

Let $X$ be an affine normal toric variety over $\mathbb{C}$. Then $X = \operatorname{Spec}(S)$ for some subalgebra $S = \mathbb{C}[x^{a_1}, \ldots, x^{a_n}] \subset \mathbb{C}[x_1, \ldots, x_n]$ where $x^{a_i} = x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}$ is a monomial in $\mathbb{C}[x_1, \ldots, x_n]$ for $a_i = (a_{i,1}, \ldots, a_{i,n})$. A. Duarte showed the following theorem:

Theorem. (A. Duarte [D])

Let $J_n$ be the ideal $(z^{a_1} - 1, \ldots, z^{a_n} - 1)^{n+1}$ in $S$. Then the normalization $\text{Nash}_n(X)$ of $\text{Nash}_n(X)$ is the toric variety whose fan is given by $\text{GF}(J_n)$, the Gröbner fan of $J_n$.

Note that $\text{GF}(J_n)$ is not a Gröbner fan of an ideal in the usual polynomial ring but monomial subalgebra $S$. Duarte [D] established a theory of Gröbner bases in monomial subalgebras, and an algorithm to compute a Gröbner basis of any ideal in $S$ w. r. t. any monomial order on $S$.

Duarte’s algorithm needs to be extended slightly in order to compute a reduced Gröbner basis and a Gröbner fan:

Proposition. (c.f. [D], Algorithm A.3.13.)

Let $< \le \text{any monomial order on } S = C[x^{a_1}, \ldots, x^{a_n}] \text{ and } I \text{ be any ideal in } S$. Then there exists a matrix $M$ (obtained from $a_1, \ldots, a_n$ and $< \text{ explicitly}$) such that the following steps give the reduced Gröbner bases of $I$ w. r. t. $<$.:

1. Take the ring map $T : C[y_1, \ldots, y_s] \to S$, $y_i \mapsto x^{a_i}$ and fix any monomial order $<^*$ on $C[y_1, \ldots, y_s]$.

2. Compute the reduced Gröbner basis $G$ of $T^{-1}(I)$ w. r. t. $M$-weighted order $<^*_M$.

3. $\hat{G} := G \setminus \ker(T)$.

4. Take a subset $\hat{G}_{\min}$ of $\hat{G}$ satisfying the followings:

   • $\forall g \in \hat{G}$, $\exists g' \in \hat{G}_{\min}$ : $\operatorname{im}_{<}(T(g')) \setminus \operatorname{im}_{<}(T(g)) \in S$.

   • $\forall \text{distinct } g', g'' \in \hat{G}_{\min}$ : $\operatorname{im}_{<}(T(g')) \setminus \operatorname{im}_{<}(T(g'')) \in S$.

5. $T \left( \hat{G}_{\min} \right) := \left\{ T(g') \mid g' \in \hat{G}_{\min} \right\}$ is the reduced G. b. of $I$.

Using above algorithm, $\text{GF}(J_n)$ was computed by Macaulay2 for $X = (z^4 - xy = 0) = \operatorname{Spec}(C[u, w^4, uv])$ and $n \leq 24$. Then certain regularity was observed, and the following proposition which Yasuda and Duarte suggested was proved by induction on $n$:

Proposition.

For $X = (z^4 - xy = 0)$, $\text{GF}(J_n)$ contains a non-regular cone for all $n > 0$.

Hence $\text{Nash}_n(X)$ is singular for all $n > 0$, and so is $\text{Nash}_n(X)$.

References


