Some Computations on Higher Nash Blowups of Toric Surfaces

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1 Background

Let X be a variety over $k = \overline{k}$. The Nash blowup (X^*, π) of X was defined in [N]. For the Nash blowup, the following problem is suggested and studied: Is X^* (or its normalization) a resolution of singularities of X?

Recently Higher Nash blowups was defined by A. Oneto & E. Zetini [OZ], T. Yasuda [Y1] independently, and analogous questions are studied.

Definition. (Yasuda's construction [Y1])

Let $n > 0$ be an integer. Let $X_{\mathrm{sm}} := X \setminus \mathrm{Sing}(X)$ and $\gamma_n : X_{\mathrm{sm}} \to \mathrm{Hilb}(X)$, $P \mapsto [P^{(n)}]$ where $P^{(n)}$ is the n-th fat point whose support is P, and its graph $\Gamma_n : X_{\mathrm{sm}} \to X \times \mathrm{Hilb}(X)$ with 1st projection $\pi_n : X \times \mathrm{Hilb}(X) \to X$.

Now let $\mathrm{Nash}_n(X) :=$ (the schematic closure of $\Gamma_n(X_{\mathrm{sm}})$) and $\pi_n := \pi_n|_{\mathrm{Nash}_n(X)}$. Then $(\mathrm{Nash}_n(X), \pi_n)$ is called the n-th Nash blowup of X. It is isomorphic to the classical Nash blowup when $n = 1$.

Regarding resolution of singularities of X, Yasuda suggested the following problem in [Y1]:

Problem.

Is $\mathrm{Nash}_n(X)$ non-singular for all $n \gg 0$ when char $k = 0$?

Known Results. (T. Yasuda [Y1])

1. If char $k = 2, 3$, then there exists X such that $\mathrm{Nash}_n(X)$ is singular for all $n > 0$.
2. If char $k = 0$ and X is a curve, then the answer is YES.

2 Main Result

Let $k = \mathbb{C}$ in what follows. Yasuda stated in [Y2] that A_3-singularity is probably a counter example for the problem, and here is my main result:

Main Result.

Let $X := (z^4 - xy = 0) \subset \mathbb{A}^3$, which is a toric surface with an A_3-singular point $P = (0, 0, 0)$. Then $\mathrm{Nash}_n(X)$ is singular for all $n > 0$.

3 Gröbner Bases in Subalgebras

The proof of main result is roughly explained below.

Let X be an affine normal toric variety over $\mathbb{C}$. Then $X = \mathrm{Spec}(S)$ for some subalgebra $S = \mathbb{C}[x^{a_1}, \ldots, x^{a_s}] \subset \mathbb{C}[x_1, \ldots, x_n]$ where $x^{a_i} = x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}$ is a monomial in $\mathbb{C}[x_1, \ldots, x_n]$ for $a_i = (a_{i,1}, \ldots, a_{i,n})$. A. Duarte showed the following theorem:

Theorem. (A. Duarte [D])

Let J_n be the ideal $\langle x^{a_1} - 1, \ldots, x^{a_s} - 1\rangle^{n+1}$ in S. Then the normalization $\overline{\mathrm{Nash}_n(X)}$ of $\mathrm{Nash}_n(X)$ is the toric variety whose fan is given by $\mathrm{GF}(J_n)$, the Gröbner fan of J_n.

Note that $\mathrm{GF}(J_n)$ is not a Gröbner fan of an ideal in the usual polynomial ring but monomial subalgebra S. Duarte [D] established a theory of Gröbner bases in monomial subalgebras, and an algorithm to compute a Gröbner basis of any ideal in S w. r. t. any monomial order on S.

Duarte's algorithm needs to be extended slightly in order to compute a reduced Gröbner basis and a Gröbner fan:

Proposition. (c.f. [D], Algorithm A.3.13.)

Let $<$ be any monomial order on $S = \mathbb{C}[x^{a_1}, \ldots, x^{a_s}]$ and I be any ideal in S. Then there exists a matrix M (obtained from $a_1, \ldots, a_s$ and $<$ explicitly) such that the following steps give the reduced Gröbner bases of I w. r. t. $<$:

1. Take the ring map $T : \mathbb{C}[y_1, \ldots, y_s] \to S$, $y_i \mapsto x^{a_i}$ and fix any monomial order $<'$ on $\mathbb{C}[y_1, \ldots, y_s]$.
2. Compute the reduced Gröbner basis G of $T^{-1}(I)$ w. r. t. M-weighted order $<'_M$.
3. $\widehat{G} := G \setminus \ker(T)$.
4. Take a subset $\widehat{G}_{\min}$ of $\widehat{G}$ satisfying the followings:
   - $\forall g \in \widehat{G}$, $\exists g' \in \widehat{G}_{\min}$; $\mathrm{lm}_<(T(g')) \mid \mathrm{lm}_<(T(g))$ in S.
   - $\forall$distinct $g', g'' \in \widehat{G}_{\min}$, $\mathrm{lm}_<(T(g')) \nmid \mathrm{lm}_<(T(g''))$ in S.
5. $T\left(\widehat{G}_{\min}\right) = \left\{ T(g') \;\middle|\; g' \in \widehat{G}_{\min} \right\}$ is the red. G. b. of I.

Using above algorithm, $\mathrm{GF}(J_n)$ was computed by Macaulay2 for $X = (z^4 - xy = 0) = \mathrm{Spec}(\mathbb{C}[u, u^3v^4, uv])$ and $n \leq 24$. Then certain regularity was observed, and the following proposition which Yasuda and Duarte suggested was proved by induction on n:

Proposition.

For $X = (z^4 - xy = 0)$, $\mathrm{GF}(J_n)$ contains a non-regular cone for all $n > 0$.

Hence $\overline{\mathrm{Nash}_n(X)}$ is singular for all $n > 0$, and so is $\mathrm{Nash}_n(X)$.

References

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